ON THE REPRESENTATION OF NUMBERS IN THE FORM $ax^2 + by^2 + cz^2 + dt^2$.[1]

BY

H. D. KLOOSTERMAN

of THE HAGUE.

1. Introduction.

1. 1. The object of the present paper is to treat the problem of the representation of large positive integers in the form $ax^2+by^2+cz^2+dt^2$ (where a, b, c, d are given positive integers) by means of the method introduced into the analytic theory of numbers by G. H. HARDY and J. E. LITTLEWOOD.[2] In my dissertation[3] I have proved an asymptotic formula for the number $r(n)$ of representations of a positive integer n in the form $a_1x_1^2+a_2x_2^2+\cdots+a_sx_s^2$, if $s \geq 5$. The proof of this formula is merely a direct application of the method mentioned above without any new idea. The result is

$$(1.\ 11) \qquad r(n) = \frac{1}{\Gamma\left(\frac{1}{2}s\right)} \frac{\pi^{\frac{1}{2}s}}{\sqrt{a_1 a_2 \ldots a_s}} n^{\frac{1}{2}s-1} S(n) + O\left(n^{\frac{1}{4}s+\varepsilon}\right) + O\left(n^{\frac{1}{2}s-1-\frac{1}{4}+\varepsilon}\right)$$

for every positive ε. Here $S(n)$ is *the singular series*. Obviously this formula is of no use for the form $ax^2+by^2+cz^2+dt^2$, where $s=4$, so that in this case the approximation of the error term must be improved, if possible. The principal

---

[1] An account of the principal results of this paper has been published in the *'Verslagen van de Koninklijke Akademie van Wetenschappen'*, Amsterdam, 31 Oct. '25.

[2] For the litterature on this subject I refer to the article of BOHR-CRAMÉR (Die neuere Entwicklung der analytischen Zahlentheorie) in the *'Enzyklopaedie der Mathematischen Wissenschaften'*.

[3] 'Over het splitsen van geheele positieve getallen in een som van kwadraten', Groningen, 1924.

result of this paper is, that this improvement is possible. The proof is difficult and a very deep analysis is necessary.

1. 2. A great number of special cases of the form $ax^2+by^2+cz^2+dt^2$ have been considered by LEGENDRE, JACOBI, LIOUVILLE, EISENSTEIN and others.[1] In some simple cases it has been possible to express the number of representations in terms of the sum of the divisors of the number in consideration or in terms of other simple arithmetical functions. A great number of results of this kind has been obtained by LIOUVILLE.[2] The principal object of these writers was the solution of the following problem: to determine, whether a given positive integer is representable in a given form or not. This can also be expressed in such a way, that they distinguished between two classes of forms, namely

1° forms, that represent all positive integers;

2° forms, that do not represent all positive integers.

Another classification is the following:

A. forms, that represent all positive integers with a finite number of exceptions at most;

B. forms, for which there is an infinite number of positive numbers which can not be represented.

The latter classification is arithmetically more essential than the first. Thus, the form $x^2+y^2+5z^2+5t^2$ does not represent the number 3. But this is not a consequence of any important arithmetical property of the form $x^2+y^2+5z^2+5t^2$, but merely a consequence of the facts, that 3 is <5 and is not a sum of two squares. Now LIOUVILLE has proved, that all other positive integers can be represented in the form $x^2+y^2+5z^2+5t^2$. Therefore, if we neglect the trivial exception 3, we may say, that the form $x^2+y^2+5z^2+5t^2$ is capable of representing positive integers.

From the asymptotic formula for the number $r(n)$ of representations of n in the form $ax^2+by^2+cz^2+dt^2$, that will be obtained in this paper, a solution can be derived of the following

Problem P. To determine which forms $ax^2+by^2+cz^2+dt^2$ belong to class A and which forms belong to class B.

It has been proved by RAMANUJAN[3], that there are only 55 forms, which

---

[1] L. E. DICKSON, 'History of the theory of numbers', Vol. III (1923), Ch. X.

[2] In my paper 'On the representation of numbers in the form $ax^2+by^2+cz^2+dt^2$', *Proc. London Math. Soc.*, 25 (1926), 143—173, I have proved some of LIOUVILLE's formulae and some new formulae by means of methods due to HARDY and MORDELL.

[3] *Proc. Camb. Phil. Soc.*, 19 (1917), 11—21.

belong to class 1°, that is to say, represent all positive integers. In the same paper he also determined all values a and d for which $a(x^2+y^2+z^2)+dt^2$ belongs to class A, that is to say, represents all positive integers with a finite number of exceptions.

1. 3. The first object is the proof of the following

Main theorem. If $r(n)$ is the number of representations of n in the form $ax^2+by^2+cz^2+dt^2$, then

$$r(n)=\frac{\pi^2}{\sqrt{abcd}}\,n\,S(n)+O\left(n^{\frac{17}{18}+\varepsilon}\right) \tag{1. 31}$$

for every positive ε, where

$$S(n)=\sum_{q=1}^{\infty}A_q,\quad A_q=q^{-4}\sum_p{}' S_{ap,q}\,S_{bp,q}\,S_{cp,q}\,S_{dp,q}\,e^{-\frac{2n\pi ip}{q}},\quad A_1=1,$$

and where p runs through all positive integers, less than and prime to q.

The proof of this theorem is given in section 3. A large number of lemma's, leading up to what is called the 'fundamental lemma' is necessary for the proof. I have collected these lemma's in section 2, which is the most difficult part of the paper.

The ideas which lead to a proof of (1. 31) can be explained as follows. A straightforward application of the Hardy-Littlewood method would give (1. 31) with the error term $O(n^{1+\varepsilon})$ (see 1. 11), which is not sufficient. The approximation of this error term must therefore be improved. Now this error term appears in the form of a series

$$\sum_q\sum_p{}' u_{p,q}, \tag{1. 32}$$

where p runs through the positive integers, less than and prime to q. If we write

$$\left|\sum_q\sum_p{}' u_{p,q}\right|\leq\sum_q\sum_p{}'\left|u_{p,q}\right|$$

we obtain the error term $O(n^{1+\varepsilon})$. It may therefore be expected, that, if we write

$$\left|\sum_q\sum_p{}' u_{p,q}\right|\leq\sum_q\left|\sum_p{}' u_{p,q}\right|$$

something better can be obtained. For this it is necessary to find an approximation to the sum

$$(1.33) \qquad \sum_p{}' u_{p,q}$$

which is better, than the approximation, given by

$$\Big|\sum_p{}' u_{p,q}\Big| \leq \sum_p{}' |u_{p,q}|,$$

or, as we shall say, it is necessary to find a non-trivial approximation for the sum (1. 33). This non trivial approximation is given by the fundamental lemma, proved in 2. 6. For the proof of this lemma the method of section 2. 43 is very important. A similar method has already been used by HARDY and LITTLEWOOD who applied it to obtain non trivial results about the corresponding sums which occur in the general Waring's problem. They refer to these results in their first memoir on Waring's problem[1], but, having been unable to apply them in the manner which they desired, have never published their analysis. I am much indebted to Messrs. HARDY and LITTLEWOOD for the suggestion that a similar method might prove valuable in the present problem.

1. 4. In order to draw any conclusions from (1. 31) it is necessary to investigate the *singular series* $S(n)$ first. This investigation is given in section 4. By combining the results of section 4 with elementary arguments, I study the solution of problem P in section 5.

1. 5. Notation. The notation, introduced in this section, remains valid throughout the paper. Other notations to be introduced afterwards are only valid in the section, where they are introduced, if it is not explicitly stated otherwise.

n is a positive integer.

a, b, c, d are the positive integral coefficients (≥ 1) of the quadratic form $ax^2+by^2+cz^2+dt^2$ (x, y, z, t integers, positive, negative or zero).

$r(n)$ denotes the number of different sets of values of x, y, z, t, for which $n=ax^2+by^2+cz^2+dt^2$.[2]

The ordinary Hardy-Littlewood machinery of the Farey-dissection of order

---

[1] A new solution of Waring's problem, *Quarterly J. of pure and applied math.*, vol. 48 (1919), p. 272—293.

[2] Two representations $n=ax_1^2+by_1^2+cz_1^2+dt_1^2$ and $n=ax_2^2+by_2^2+cz_2^2+dt_2^2$ will be considered as the same if and only if $x_1=x_2$, $y_1=y_2$, $z_1=z_2$, $t_1=t_2$.

$$N = [\sqrt{n}] \tag{1.51}$$

will be used. Let Γ denote the circle

$$|w| = e^{-\frac{1}{n}}$$

in the complex w-plane. Then we divide Γ into Farey-arcs $\xi_{p,q}$ in the following manner. If $\frac{p}{q}$ is a term of the Farey-series and $\frac{p'}{q'}$, $\frac{p''}{q''}$ are the adjacent terms to the right and left, then the intervals $(q > 1)$

$$\frac{p}{q} - \frac{1}{q(q+q'')}, \frac{p}{q} + \frac{1}{q(q+q')} \tag{1.52}$$

will be denoted by $j_{p,q}$. The intervals $\left(0, \frac{1}{N+1}\right)$ and $\left(1-\frac{1}{N+1}, 1\right)$ will be denoted by $j_{0,1}$ and $j_{1,1}$. We now obtain the Farey-dissection of Γ into the arcs $\xi_{p,q}$ if the intervals $j_{p,q}$ are considered as intervals of variation of $\frac{\theta}{2\pi}$, where $\theta = \arg w$, and if the two extreme intervals are joined into one.

On $\xi_{p,q}$ we write

$$w = e^{\frac{2p\pi i}{q}} W = \exp\left(\frac{2p\pi i}{q} - \frac{1}{n} + i\theta\right). \tag{1.53}$$

If w describes $\xi_{p,q}$, then θ varies between two numbers $-\theta'_{p,q}$ and $\theta_{p,q}$. Then

$$\frac{2\pi}{q(q+N)} \leq \theta_{p,q} < \frac{2\pi}{qN}, \quad \frac{2\pi}{q(q+N)} \leq \theta'_{p,q} < \frac{2\pi}{qN}. \tag{1.54}$$

We have

$$1 + \sum_{n=1}^{\infty} r(n) w^n = \vartheta(w^a)\,\vartheta(w^b)\,\vartheta(w^c)\,\vartheta(w^d),$$

where

$$\vartheta(w) = \sum_{\nu=-\infty}^{+\infty} w^{\nu^2} \qquad (|w| < 1).$$

ε stands for an arbitrary positive number, not always the same.

K is a constant, depending on a, b, c, d, ε only, not always the same constant, where it occurs.

$O(f)$ denotes a number, whose absolute value is $< Kf$.

B is a number depending on q, a, b, c, d only, which is bounded for all values of q. It does not always represent the same function of q, a, b, c, d.

If L and M are two integers, we denote by (L, M) the greatest common divisor of L and M.

Wherever the letter p occurs, it will always denote a positive integer, such that $(p, q) = 1$. We denote by Σ' a summation, where p runs through all integers, for which

(1. 55) $$0 < p \leq q - 1, \qquad (p, q) = 1,$$

if $q > 1$. For $q = 1$ the only value of p is 1. A summation, where p is subject to other restrictions, except (1. 55) will be denoted by the same symbol Σ', but the additional conditions will be written explicitly under the symbol Σ'.

For $s = a, b, c, d$ only (not for other letters) I write

$$s = (s, q)\, s_q, \quad q = (s, q)\, q_s, \quad q_s = 2^{\mu_s} Q_s \quad (Q_s \text{ odd}).$$

If M is an odd positive number and $(L, M) = 1$, then $\left(\frac{L}{M}\right)$ is the symbol of LEGENDRE-JACOBI $\left(\left(\frac{L}{M}\right) = 1\right.$, if L is a quadratic residu of M; $\left(\frac{L}{M}\right) = -1$ if L is not a quadratic residu of M; $\left.\left(\frac{L}{1}\right) = 1\right)$.

$\delta | n$ means: δ is a divisor of n; $\delta \nmid n$ means: δ is not a divisor of n.

ϖ, also, when a suffix is attached to it, is a prime number.

The RAMANUJAN sum[1] is defined by

$$c_q(n) = \Sigma' e^{\frac{2 n p \pi i}{q}} = \Sigma' e^{-\frac{2 n p \pi i}{q}}.$$

If $(q, q') = 1$, then

$$c_q(n)\, c_{q'}(n) = c_{q q'}(n).$$

Also

(1. 56) $$c_q(n) = \sum_{\delta | (n, q)} \delta \mu\left(\frac{q}{\delta}\right),$$

where μ denotes the arithmetical function of MÖBIUS.

[1] 'On certain trigonometrical sums and their applications in the theory of numbers', *Trans. Camb. Phil. Soc.* 22 (1918), 259—276. The formula (1. 56) has already been given by J. C. KLUYVER, 'Eenige formules aangaande de getallen kleiner dan n en ondeelbaar met n', *Versl. Kon. Akad. v. Wetensch., Amsterdam,* 1906.

Further, we write (if ν is an integer)

$$S_{p,q,\nu} = \sum_{j=0}^{q-1} \exp\left(\frac{2p\pi i j^2}{q} + \frac{2\nu\pi i j}{q}\right).$$

For $\nu \equiv 0 \pmod q$, this is the GAUSSian sum $S_{p,q}$.

For abbreviation I write

$$\{S_q^p\} = S_{ap,q}\, S_{bp,q}\, S_{cp,q}\, S_{dp,q}.$$

2. Preliminary lemmas.

2. 1. **Lemma 1.** *If s is a positive integer, then the sum $S_{sp,q,\nu}$ vanishes identically or a positive integer ν'' can be found, which is independent of p, such that either*

(2. 11) $$S_{sp,q,\nu} = \exp\frac{2\pi i p'\nu''}{q} \cdot S_{sp,q}, \qquad pp'+1 \equiv 0 \pmod q$$

or

(2. 12) $$S_{sp,q,\nu} = \frac{(s,2)}{2(s,8)} \exp\frac{2\pi i p'\nu''}{4q} \cdot S_{sp,4q}, \qquad pp'+1 \equiv 0 \pmod{4q}.$$

For we have

$$S_{sp,q,\nu} = \sum_{j=0}^{q-1} \exp\left(\frac{2\pi i s p j^2}{q} + \frac{2\pi i \nu j}{q}\right).$$

Now write

$$j = j_1 + \mu q_s, \quad j_1 = 0, 1, 2, \ldots q_s - 1; \quad t = 0, 1, 2, \ldots, (s,q) - 1.$$

Then

(2. 13) $$S_{sp,q,\nu} = \sum_{j_1=0}^{q_s-1} \exp\left(\frac{2\pi i s_q p j_1^2}{q_s} + \frac{2\pi i \nu j_1}{q}\right) \sum_{\mu=0}^{(s,q)-1} \exp\frac{2\pi i \nu \mu}{(s,q)}.$$

This is 0, if $(q,s) \nmid \nu$. Therefore we may suppose further, that $(q,s) \mid \nu$. Writing

$$\nu = (q,s)\,\nu',$$

we find from (2. 13), that

(2. 14) $$S_{sp,q,\nu} = (s,q)\, S_{s_q p,\, q_s,\, \nu'}.$$

For any integer p'' we have

$$S_{s_q p, q_s, \nu'} = \sum_{j=0}^{q_s-1} \exp\left(\frac{2\pi i s_q p (j+p'')^2}{q_s} + \frac{2\pi i \nu' (j+p'')}{q_s}\right) =$$

$$= \exp\left(\frac{2\pi i s_q p p''^2}{q_s} + \frac{2\pi i \nu' p''}{q_s}\right) \sum_{j=0}^{q_s-1} \exp\left(\frac{2\pi i s_q p j^2}{q_s} + \frac{2\pi i j(\nu' + 2 s_q p p'')}{q_s}\right).$$

We now consider a few cases separately.

1°. q_s is odd. Then let p'' be such that

$$\nu' + 2 s_q p p'' \equiv 0 \pmod{q_s}.$$

Then we have

$$s_q p p'' + \frac{\nu'}{2} \equiv 0 \pmod{q_s} \quad \text{or} \quad s_q p p'' + \frac{\nu' + q_s}{2} \equiv 0 \pmod{q_s}$$

according as ν' is even or odd. Hence

$$S_{s_q p, q_s, \nu'} = \exp \frac{\pi i p'' \nu'}{q_s} \cdot S_{s_q p, q_s} \quad \text{or} \quad S_{s_q p, q_s, \nu'} = \exp \frac{\pi i p'' (\nu' + q_s)}{q_s} \cdot S_{s_q p, q_s}$$

according as ν' is even or odd and

$$S_{s_q p, q_s, \nu'} = (-1)^{p''\nu'} \exp \frac{\pi i p'' \nu'}{q_s} \cdot S_{s_q p, q_s} \tag{2. 15}$$

in both cases.

Now let ν'' and p' be such that

$$\nu'^2 \equiv 4 \frac{\nu''}{(s, q)} s_q \pmod{q_s}, \quad 1 + p p' \equiv 0 \pmod{q}.$$

Then

$$2 s_q p'' \equiv \nu' p' \pmod{q_s}$$

and therefore

$$4 p' \frac{\nu''}{(s, q)} s_q \equiv p' \nu'^2 \equiv 2 \nu' s_q p'' \equiv 4 s_q p'' \nu' \frac{1 + q_s}{2} \pmod{q_s},$$

$$p' \frac{\nu''}{(s, q)} \equiv p'' \nu' \frac{1 + q_s}{2} \pmod{q_s},$$

so that we find from (2. 14) and (2. 15), that

$$S_{sp,q,\nu} = (s, q) \exp \frac{2\pi i p' \nu''}{q} S_{s_q p, q_s} = \exp \frac{2\pi i p' \nu''}{q} \cdot S_{sp,q}.$$

2°. q_s is even and ν' is even. Then let p'' be such, that

$$s_q p p'' + \frac{\nu'}{2} \equiv 0 \pmod{q_s}.$$

Then

$$S_{s_q p, q_s, \nu'} = \exp \frac{\pi i \nu' p''}{q_s} \cdot S_{s_q p, q_s}.$$

Now let ν'' and p' be such that

$$\frac{\nu'^2}{4} \equiv \frac{\nu''}{(q, s)} s_q \pmod{q_s}, \quad 1 + pp' \equiv 0 \pmod{q}.$$

Then

$$p' \frac{\nu'}{2} \equiv s_q p'' \pmod{q_s}$$

and therefore

$$p' \frac{\nu''}{(s, q)} s_q \equiv p' \frac{\nu'^2}{4} \equiv s_q p'' \frac{\nu'}{2} \pmod{q_s},$$

$$p' \frac{\nu''}{(q, s)} \equiv p'' \frac{\nu'}{2} \pmod{q_s},$$

so that

$$S_{sp,q,\nu} = \exp \frac{2\pi i p' \nu''}{q} \cdot S_{sp,q}.$$

3°. q_s is even and ν' is odd. Then let p'' be such that

$$2 s_q p p'' + \nu' \equiv s_q p \pmod{4 q_s}.$$

Then

$$S_{s_q p, q_s, \nu'} = \exp \left(\frac{2\pi i s_q p p''^2}{q_s} + \frac{2\pi i \nu' p''}{q_s} \right) \cdot \sum_{j=0}^{q_s - 1} \exp \frac{2\pi i s_q p (j^2 + j)}{q_s}.$$

But

$$\sum_{j=0}^{q_s-1} \exp \frac{2\pi i s_q p(j^2+j)}{q_s} = \exp\left(-\frac{2\pi i s_q p}{4q_s}\right) \cdot \sum_{j=0}^{q_s-1} \exp \frac{2\pi i s_q p(2j+1)^2}{4q_s} =$$

$$= \exp\left(-\frac{2\pi i s_q p}{4q_s}\right) \cdot \left\{ \sum_{j=0}^{2q_s-1} \exp \frac{2\pi i s_q p j^2}{4q_s} - \sum_{j=0}^{q_s-1} \exp \frac{2\pi i s_q p j^2}{q_s} \right\} =$$

$$= \exp\left(-\frac{2\pi i s_q p}{4q_s}\right) \cdot \left(\frac{1}{2} S_{s_q p, 4q_s} - S_{s_q p, q_s}\right).$$

This is 0 if $q_s \not\equiv 2 \pmod 4$. But if $q_s \equiv 2 \pmod 4$, we have

$$S_{s_q p, q_s, \nu'} = \frac{1}{2} \exp\left(\frac{2\pi i (2p''-1) s_q p + 2\pi i \,.\, 2p''\nu'}{4q_s}\right) \cdot S_{s_q p, 4q_s}$$

Now let ν'' and p' be such, that

$$\nu'^2 \equiv s_q \frac{\nu''}{(s, q)} \pmod{4q_s}, \quad 1 + pp' \equiv 0 \pmod{4q}.$$

Then

$$(2p''-1)\, s_q \equiv \nu' p' \pmod{4q_s}$$

and therefore

$$\nu'(2p''-1)s_q \equiv \nu'^2 p' \equiv p' s_q \frac{\nu''}{(s, q)} \pmod{4q_s},$$

$$(2p-1)\nu' \equiv p' \frac{\nu''}{(s, q)} \pmod{4q_s},$$

so that we find

$$S_{s_q p, q_s, \nu'} = \frac{1}{2} \exp \frac{2\pi i p' \nu''}{4q} \cdot S_{s_q p, 4q_s}$$

and

$$S_{sp, q, \nu} = \frac{1}{2} \exp \frac{2\pi i p' \nu''}{4q} \cdot S_{sp, 4q} \frac{(s, q)}{(s, 4q)} = \frac{(s, 2)}{2(s, 8)} \exp \frac{2\pi i p' \nu''}{4q} \cdot S_{sp, 4q}.$$

This completes the proof of the lemma.

2. 2. Let μ be an integer such that

$$0 \leq \mu \leq q - 1$$

and let $\nu_1, \nu_2, \nu_3, \nu_4$ be integers. Let p_1 be determined by

(2. 21) $$p(p_1+N)+1\equiv 0 \pmod q, \quad 0<p_1\leq q.$$

Then there is one and only one p_1 to every p. We write

(2. 22) $$\sigma_1=\sum_{p_1\leq\mu}{}' S_{ap,q,\nu_1}S_{bp,q,\nu_2}S_{cp,q,\nu_3}S_{dp,q,\nu_4}\exp\left(-\frac{2n\pi ip}{q}\right).$$

Lemma 2. *If $\sigma_1\neq 0$, then it is always possible to find an integer v (depending on $\nu_1, \nu_2, \nu_3, \nu_4$, a, b, c, d, q, but not on p or P), such that either*

$$\sigma_1=\sum_{p_1\leq\mu}{}'\{S_q^p\}\exp\left(\frac{2\pi iup}{q}+\frac{2\pi ivp'}{q}\right), \qquad 1+pp'\equiv 0 \pmod q$$

(where we have written $u=-n$), or

$$\sigma_1=K\sum_{P_1\leq\mu}\{S_{4q}^P\}\exp\left(\frac{2\pi iuP}{4q}+\frac{2\pi ivP'}{4q}\right), \quad 1+PP'\equiv 0 \pmod{4q},$$

(where we have written $u=-4n$) and where in the second sum the summation over P is defined by

$$(P, 4q)=1, \quad 0\leq P\leq 4q-1, \quad P_1\leq\mu,$$

and where P_1 is determined by

$$P(P_1+N)+1\equiv 0 \pmod{4q}, \quad 0<P_1\leq 4q.$$

Consider first the case, that none of the numbers q_a, q_b, q_c, q_d is $\equiv 2 \pmod 4$. Then it follows from the preceding section, that either $\sigma_1=0$, or there are integers $\nu_a, \nu_b, \nu_c, \nu_d$, such that

$$S_{sp,q,\nu_j}=\exp\frac{2\pi ip'\nu_s}{q}\cdot S_{sp,q} \qquad (s=a, b, c, d)$$

where $j=1, 2, 3, 4$ according as $s=a, b, c, d$. Then we have

$$\sigma_1=\sum_{p_1\leq\mu}{}'\{S_q^p\}\exp\left(\frac{2\pi iup}{q}+\frac{2\pi ip'(\nu_a+\nu_b+\nu_c+\nu_d)}{q}\right),$$

which is the statement of the lemma with $v=\nu_a+\nu_b+\nu_c+\nu_d$.

A similar result is true if one (or several) of the numbers q_a, q_b, q_c, q_d is (are) $\equiv 2 \pmod 4$, and (all) the corresponding $\nu'_j = \frac{\nu_j}{(s, q)}$ is (are) even. If however one (or more) of the numbers q_a, q_b, q_c, q_d is (are) $\equiv 2 \pmod 4$ and (all) the corresponding ν'_j is (are) odd, then we first make the following remark: In the sum $\sum'_{p_1 \leq \mu}$ the variable of summation is p. However, we may also regard p_1 as the variable of summation. For this we let p_1 run through the numbers $1, 2, \ldots \mu$ and for those values of p_1, for which this is possible, we determine p by

$$p(p_1 + N) + 1 \equiv 0 \pmod q, \qquad 0 < p < q,$$

and sum over the values of p, obtained in this way. We now determine, if possible, to every $p_1 \leq \mu$ the number P by the conditions

$$P(p_1 + N) + 1 \equiv 0 \pmod{4q}, \qquad 0 < P < 4q.$$

Then we have

$$P \equiv p \pmod q$$

and therefore (writing P_1 instead of p_1)

$$\sigma_1 = \sum_{P_1 \leq \mu} S_{aP, q, \nu_1} S_{bP, q, \nu_2} S_{cP, q, \nu_3} S_{dP, q, \nu_4} \exp\left(-\frac{2n\pi i P}{q}\right).$$

But it follows from lemma 1, that one of the three following equations is always true ($s = a, b, c, d$):

$$S_{sP, q, \nu_j} = 0; \quad S_{sP, q, \nu_j} = \exp\frac{2\pi i P'' \nu_s}{q} \cdot S_{sP, q} = \frac{1}{2}\exp\frac{2\pi i P'' \nu_s}{q} S_{sP, 4q},$$
$$[1 + PP'' \equiv 0 \pmod q];$$

$$S_{sP, q, \nu_j} = K \exp\frac{2\pi i P' \nu_s}{4q} \cdot S_{sP, 4q}, \qquad [1 + PP' \equiv 0 \pmod{4q}].$$

Since $P' \equiv P'' \pmod q$, we have always

$$S_{sP, q, \nu_j} = 0 \quad \text{or} \quad S_{sP, q, \nu_j} = K \exp\frac{2\pi i P' \nu_s}{4q} \cdot S_{sP, 4q},$$

from which the statement of the lemma follows.

The lemma can also be expressed in the following form:

Lemma 2*. *We have always*

$$\sigma_1 = K\sigma_2,$$

where σ_2 *is a sum of the type*

$$\sigma_2 = \sum_{p_1 < \mu}{}' \{S_q^p\} \exp\left(\frac{2\pi i u p}{q} + \frac{2\pi i v p'}{q}\right), \tag{2. 23}$$

where

$$1 + pp' \equiv 0 \pmod q, \quad u = -n,$$

and where the q*, occurring on the right hand side of* (2. 23) *is either the same as that, occurring in* σ_1 *or it is four times the* q *occurring in* σ_1.

Therefore, if we want to calculate σ_1 for large values of q, we need only consider σ_2.

2. 3. Let $\eta(p, q, s)$ be defined by

$$\begin{aligned}
\eta(p, q, s) &= 1 && \text{if } q_s = \text{odd} = Q_s; \\
&= 0 && \text{if } q_s \equiv 2 \pmod 4; \\
&= \exp(\tfrac{1}{4} s_q p Q_s \pi i) && \text{if } q_s = 2^{\mu_s} Q_s \text{ and } \mu_s \text{ is odd } > 2; \\
&= 1 + \exp(\tfrac{1}{2} s_q p Q_s \pi i) && \text{if } q_s = 2^{\mu_s} Q_s \text{ and } \mu_s \text{ is even } \geq 2,
\end{aligned}$$

and $\zeta(p, q)$ by

$$\zeta(p, q) = \zeta(p, q, a, b, c, d) = \eta(p, q, a)\,\eta(p, q, b)\,\eta(p, q, c)\,\eta(p, q, d).$$

Lemma 3. *We have*

$$\{S_q^p\} = B\left(\frac{p}{Q_a\, Q_b\, Q_c\, Q_d}\right) \zeta(p, q)\, q^2.$$

This follows from the well known values of the Gaussian sums (See: Bachmann, Die analytische Zahlentheorie 2 (1894), 146—187).

Now let $q = 2^\mu Q$ (Q odd) and let G be the smallest multiple of (a, Q), (b, Q), (c, Q), (d, Q). Then we define the number Δ as being $8G$; $4G$; $2G$; G, according as $8|q$; $4|q$, $8 \nmid q$; $2|q$, $4 \nmid q$; q odd. Then obviously we have

$$\Delta | q \quad \text{and} \quad \Delta < K. \tag{2. 31}$$

As an immediate consequence of lemma 3, we have

Lemma 3*. *We have*

$$|\sigma_2| \leq K q^2 \sum_{\lambda=1}^{\Lambda} \left| \sum_{\substack{p_1 \leq \mu \\ p \equiv \lambda \pmod{\Lambda}}}' \exp\left(\frac{2\pi i u p}{q} + \frac{2\pi i v p'}{q}\right) \right|.$$

For we have

$$Q = (Q, s)\, Q_s$$

and therefore

$$\left(\frac{p}{Q_s}\right) = \left(\frac{p}{Q}\right) \left(\frac{p}{(Q, s)}\right) \qquad (s = a, b, c, d).$$

Hence

$$\left(\frac{p}{Q_a Q_b Q_c Q_d}\right) = \left(\frac{p}{(Q, a)}\right) \left(\frac{p}{(Q, b)}\right) \left(\frac{p}{(Q, c)}\right) \left(\frac{p}{(Q, d)}\right)$$

and therefore

$$\left(\frac{p + \Lambda}{Q_a Q_b Q_c Q_d}\right) = \left(\frac{p}{Q_a Q_b Q_c Q_d}\right).$$

Also we have

$$\zeta(p + \Lambda, q) = \zeta(p, q)$$

and therefore it follows from lemma 3, that

$$\sigma_2 = B q^2 \sum_{\lambda=1}^{\Lambda} \left(\frac{\lambda}{Q_a Q_b Q_c Q_d}\right) \zeta(\lambda, q) \sum_{\substack{p_1 \leq \mu \\ p \equiv \lambda \pmod{\Lambda}}}' \exp\left(\frac{2\pi i u p}{q} + \frac{2\pi i v p'}{q}\right),$$

from which the statement follows.

2. 4. The sum $S(u, v; \lambda, \Lambda; q)$.

We shall show afterwards, that the approximation for large values of q of the sum occurring on the right hand side of the formula of lemma 3*, can be reduced to the calculation for large values of q of the sum

$$S(u, v; \lambda, \Lambda; q) = \sum_{p \equiv \lambda \pmod{\Lambda}}' \exp\left(\frac{2\pi i u p}{q} + \frac{2\pi i v p'}{q}\right).$$

But before performing the reduction, we shall first consider this sum S. The object of this section is the proof of lemma 4. The lemmas 4 b—4 e are special cases of lemma 4, from which the general lemma 4 will be deduced.

2. 41. **Lemma 4 a.** *If* $\Delta_1|q_1,\ \Delta_2|q_2,\ (q_1,q_2)=1$, *then*

$$S(u, v_1;\lambda_1, \Delta_1; q_1)\, S(u, v_2; \lambda_2, \Delta_2; q_2) = S(u, v_1 q_2^2 + v_2 q_1^2; \lambda_1 q_2 + \lambda_2 q_1, \Delta_1\Delta_2; q_1 q_2).$$

For we have

$$(2.\ 411)\quad S(u, v_1; \lambda_1, \Delta_1, q_1)\, S(u, v_2; \lambda_2, \Delta_2, q_2) =$$

$$= \sum_{p_1\equiv\lambda_1 (\mathrm{mod}\, \Delta_1)}' \sum_{p_2\equiv\lambda_2(\mathrm{mod}\,\Delta_2)}' \exp\left(\frac{2\pi i u(p_1q_2+p_2q_1)}{q_1q_2} + \frac{2\pi i(v_1p_1'q_2+v_2p_2'q_1)}{q_1q_2}\right)^{1},$$

where the summation must be extended over those p_1 and p_2 for which

$$(p_1, q_1)=1,\ 0\leq p_1<q_1,\ p_1\equiv\lambda_1\,(\mathrm{mod}\,\Delta_1);\ (p_2,q_2)=1,\ 0\leq p_2<q_2,\ p_2\equiv\lambda_2\,(\mathrm{mod}\,\Delta_2).$$

(This has been denoted by dashes, just like the analogous summations over the letter p. The same will be done for summations over P).

Now let

$$P = p_1 q_2 + p_2 q_1.$$

Then P runs through all numbers for which (since $\Delta_1|q_1$, $\Delta_2|q_2$ and $(q_1, q_2)=1$)

$$(2.\ 412)\qquad 0\leq P<q_1q_2,\ \ (P, q_1q_2)=1,\ \ P\equiv\lambda_1q_2+\lambda_2q_1\,(\mathrm{mod}\,\Delta_1\Delta_2).$$

Further, let P' be determined mod q_1q_2 by

$$1+PP'\equiv 0\ (\mathrm{mod}\ q_1q_2).$$

Then

$$-1\equiv PP'\equiv P'(p_1q_2+p_2q_1)\ (\mathrm{mod}\ q_1q_2)$$

and therefore

$$p_1p_1'\equiv -1\equiv P'p_1q_2\ (\mathrm{mod}\ q_1),\quad p_2p_2'\equiv -1\equiv P'p_2q_1\ (\mathrm{mod}\ q_2);$$

or

$$p_1'\equiv P'q_2\ (\mathrm{mod}\ q_1),\ \ p_2'\equiv P'q_1\ (\mathrm{mod}\ q_2).$$

Hence

$$v_1p_1'q_2+v_2p_2'q_1\equiv P'(v_1q_2^2+v_2q_1^2)\ (\mathrm{mod}\ q_1q_2).$$

This, together with (2. 411) and (2. 412) proves the lemma.

---

[1] Of course the p_1 occurring here and the p_1 of the lemma's 2, 2*, 3* have quite a different meaning.

2. 42. **Lemma 4 b.** *Let*

$$q = \varpi_1^{\xi_1} \varpi_2^{\xi_2} \dots \varpi_r^{\xi_r},$$

so that $\varpi_1, \varpi_2, \dots \varpi_r$ *are the different primes, which divide* q. *Further let*

$$(u, q) = 1, \quad (v, q) = 1, \quad \Lambda = \varpi_1^{\zeta_1} \varpi_2^{\zeta_2} \dots \varpi_r^{\zeta_r}$$

(where the ζ_j *may also be* 0, *but are* $\leq$ *the corresponding* ξ_j*). Then there are integers* v_j, λ_j, *such that*

$$(v_j, \varpi_j^{\xi_j}) = 1 \qquad (j = 1, 2, \dots r)$$

and

$$S(u, v; \lambda, \Lambda; q) = \prod_{j=1}^{r} S(u, v_j; \lambda_j, \varpi_j^{\zeta_j}; \varpi_j^{\xi_j}). \tag{2. 421}$$

For the proof write

$$q = \varpi_1^{\xi_1} A_1.$$

Let the numbers $v_1 \pmod{\varpi_1^{\xi_1}}$ and $V_1 \pmod{A_1}$ be determined by

$$v \equiv v_1 A_1^2 + V_1 \varpi_1^{2\xi_1} \pmod{q}\,[1]$$

and let $\lambda_1 \pmod{\varpi_1^{\xi_1}}$ and $\varrho_1 \pmod{A_1}$ be determined by

$$\lambda \equiv \lambda_1 A_1 + \varrho_1 \varpi_1^{\xi_1} \pmod{q}.$$

Further, write

$$\Lambda = \varpi_1^{\zeta_1} \Lambda_1.$$

Then we have from lemma 4 a:

$$S(u, v; \lambda, \Lambda; q) = S(u, v_1; \lambda_1, \varpi_1^{\zeta_1}; \varpi_1^{\xi_1})\, S(u, V_1; \varrho_1, \Lambda_1; A_1).$$

Since

$$(v_1, \varpi_1^{\xi_1}) = 1, \quad (V_1, A_1) = 1,$$

the same argument can be repeated, which proves (2. 421).

---

[1] It can be proved as follows, that v_1, V_1 exist. Consider the system of numbers $v_1 A_1^2 + V_1 \varpi_1^{2\xi_1}$, if v_1 runs through all numbers, less than and prime to $\varpi_1^{\xi_1}$ and V_1 through all numbers, less than and prime to A_1. Then these numbers are all incongruent mod q and they are prime to q. Further the system consists of $\varphi(\varpi_1^{\xi_1})\,\varphi(A_1) = \varphi(q)$ numbers. Therefore one of them must be $\equiv v \pmod{q}$.

2. 43. **Lemma 4 c.** *If* $q=\varpi^{\xi}$; $\Delta=\varpi^{\zeta}$ $(\zeta\leq\xi)$, $(u,\varpi)=1$, $(v,\varpi)=1$, *then*

$$|S(u,v;\lambda,\Delta;q)|<Kq^{\frac{3}{4}}.$$

Consider the expression

$$\sigma_3=\sum_{\lambda}{}'\sum_{u}{}'|S(u,v;\lambda,\Delta;q)|^4,$$

where λ runs through all positive integers, less than and prime to Δ and u runs through all positive integers, less than and prime to q. (This has again been denoted by dashes, just like analogous summations over p. If $\Delta=1$, then $\lambda=1$ only).

σ_3 *is independent of* v. To prove this, we write

$$up\equiv P\ (\text{mod}\ q),\quad 1+PP'\equiv 0\ (\text{mod}\ q)$$

in the expression, which defines $S(u,v;\lambda,\Delta;q)$. Then

$$P'u\equiv p'\ (\text{mod}\ q),\quad P\equiv u\lambda\ (\text{mod}\ \Delta).$$

Hence

$$\sigma_3=\sum_{\lambda}{}'\sum_{u}{}'\Big|\sum_{P\equiv u\lambda\,(\text{mod}\,\Delta)}{}'\exp\Big(\frac{2\pi iP}{q}+\frac{2\pi iuvP'}{q}\Big)\Big|^4=\sum_{u}{}'\sum_{\lambda}{}'\Big|\sum_{P\equiv u\lambda\,(\text{mod}\,\Delta)}{}'\Big|^4.$$

Now we have $(u,q)=1$, so that also $(u,\Delta)=1$. Therefore, if λ runs through all positive integers, less than and prime to Δ, then $(u\lambda)$[1] does the same, so that

$$\sigma_3=\sum_{u}{}'\sum_{\lambda}{}'\Big|\sum_{P\equiv\lambda\,(\text{mod}\,\Delta)}{}'\Big|^4=\sum_{\lambda}{}'\sum_{u}{}'|S(1,uv;\lambda,\Delta;q)|^4=$$

$$=\sum_{\lambda}{}'\sum_{u}{}'|S(1,u;\lambda,\Delta;q)|^4,$$

since, if u runs through all positive integers, less than and prime to q, then (uv) does the same, v being prime to q.

Now we have also

$$\sigma_3=\sum_{\lambda}{}'\sum_{u}{}'\sum_{p_1,p_2,\pi_1,\pi_2}\exp\Big(\frac{2\pi iu(p_1+p_2-\pi_1-\pi_2)}{q}+\frac{2\pi iv(p_1'+p_2'-\pi_1'-\pi_2')}{q}\Big),$$

[1] We denote by (M) the number which is $\equiv M\,(\text{mod}\ q)$ and for which $0\leq(M)<q$.

where p_1[1], p_2, π_1, π_2 run through all positive integers, less than and prime to q which are $\equiv \lambda \pmod{\Delta}$ and

$$1 + p_j p_j' \equiv 0 \pmod{q}, \quad 1 + \pi_j \pi_j' \equiv 0 \pmod{q}, \quad (j = 1, 2).$$

Therefore, summing over u and writing

$$H = p_1 + p_2 - \pi_1 - \pi_2, \quad H' = p_1' + p_2' - \pi_1' - \pi_2',$$

we have

$$\sigma_3 = \sum_{\lambda}{}' \sum_{p_j, \pi_j}{}' \exp \frac{2\pi i v H'}{q} \cdot c_q(H) =$$

$$= -\varpi^{\xi-1} \sum_{\lambda}{}' \sum_{\substack{p_j, \pi_j \\ H \equiv 0 \, (\mathrm{mod}\, \varpi^{\xi-1}),\ \not\equiv 0 \, (\mathrm{mod}\, q)}}{}' \exp \frac{2\pi i v H'}{q} + \varphi(q) \sum_{\lambda}{}' \sum_{\substack{p_j, \pi_j \\ H \equiv 0 \, (\mathrm{mod}\, q)}}{}' \exp \frac{2\pi i v H'}{q}.$$

We now sum over all positive integers v, less than and prime to q. Since σ_3 is independent of v, we get

$$\varphi(q) . \sigma_3 = -\varpi^{\xi-1} \sum_{\lambda}{}' \sum_{\substack{p_j, \pi_j \\ H \equiv 0 \, (\mathrm{mod}\, \varpi^{\xi-1}),\ \not\equiv 0 \, (\mathrm{mod}\, q)}} c_q(H') + \varphi(q) \sum_{\lambda}{}' \sum_{\substack{p_j, \pi_j \\ H \equiv 0 \, (\mathrm{mod}\, q)}}{}' c_q(H') =$$

$$= \varpi^{2\xi-2} N_1 - \varpi^{\xi-1} \varphi(q) N_2 - \varpi^{\xi-1} \varphi(q) N_3 + (\varphi(q))^2 N_4,$$

where

$N_1 = \sum_{\lambda}' N_1^{(\lambda)}$; $N_1^{(\lambda)} =$ number of solutions of $H \equiv 0 \pmod{\varpi^{\xi-1}}$; $H' \equiv 0 \pmod{\varpi^{\xi-1}}$; $H \not\equiv 0 \pmod{q}$; $H' \not\equiv 0 \pmod{q}$; $p_1, p_2, \pi_1, \pi_2 \equiv \lambda \pmod{\Delta}$.

$N_2 = \sum_{\lambda}' N_2^{(\lambda)}$; $N_2^{(\lambda)} =$ number of solutions of $H \equiv 0 \pmod{\varpi^{\xi-1}}$; $H \not\equiv 0 \pmod{q}$; $H' \equiv 0 \pmod{q}$; $p_1, p_2, \pi_1, \pi_2 \equiv \lambda \pmod{\Delta}$.

$N_3 = \sum_{\lambda}' N_3^{(\lambda)}$; $N_3^{(\lambda)} =$ number of solutions of $H \equiv 0 \pmod{q}$; $H' \equiv 0 \pmod{\varpi^{\xi-1}}$; $H' \not\equiv 0 \pmod{q}$; $p_1, p_2, \pi_1, \pi_2 \equiv \lambda \pmod{\Delta}$.

$N_4 = \sum_{\lambda}' N_4^{(\lambda)}$; $N_4^{(\lambda)} =$ number of solutions of $H \equiv 0 \pmod{q}$; $H' \equiv 0 \pmod{q}$; $p_1, p_2, \pi_1, \pi_2 \equiv \lambda \pmod{\Delta}$.

---

[1] See footnote [1] on p. 421.

Therefore

(2. 431) $$\varphi(q)\,.\,\sigma_3 \leq \varpi^{2\xi-2} N_1 + (\varphi(q))^2 N_4.$$

We shall prove

$$N_4 = O(q^2), \quad N_1 = O(\varpi^{2\xi+2}).$$

In the first place, we have

$$N_1 \leq N_1', \quad N_4 \leq N_4',$$

where

N_1' = number of solutions of $H \equiv 0 \pmod{\varpi^{\xi-1}}$, $H' \equiv 0 \pmod{\varpi^{\xi-1}}$, $H \not\equiv 0 \pmod q$, $H' \not\equiv 0 \pmod q$;

N_4' = number of solutions of $H \equiv 0 \pmod q$, $H' \equiv 0 \pmod q$.

Consider first N_4', that is to say, the number of solutions of

$$p_1 + p_2 \equiv \pi_1 + \pi_2 \pmod q, \quad p_1' + p_2' \equiv \pi_1' + \pi_2' \pmod q.$$

The second congruence relation gives

$$\pi_1 \pi_2 (p_1 + p_2) \equiv p_1 p_2 (\pi_1 + \pi_2) \pmod q$$

and the first

$$\pi_1 \pi_2 (p_1 + p_2) \equiv \pi_1 \pi_2 (\pi_1 + \pi_2) \pmod q.$$

Therefore

$$(p_1 p_2 - \pi_1 \pi_2)(\pi_1 + \pi_2) \equiv 0 \pmod q, \quad (p_1 p_2 - \pi_1 \pi_2)(p_1 + p_2) \equiv 0 \pmod q.$$

Therefore we must have either

$$p_1 + p_2 \equiv 0 \pmod q \quad \text{and} \quad \pi_1 + \pi_2 \equiv 0 \pmod q$$

or

$$p_1 p_2 \equiv \pi_1 \pi_2 \pmod q.$$

In the first case p_1 and π_1 are determined, if p_2 and π_2 are given, so that there are at most $O(q^2)$ solutions. In the second case, we have

$$(p_1 - p_2)^2 \equiv (\pi_1 - \pi_2)^2 \pmod q$$

and

$$p_1 - p_2 \equiv \pm (\pi_1 - \pi_2) \pmod q.$$

Hence, if π_1, π_2 are given, then only two sets of solutions p_1, p_2 are possible, which gives again $O(q^2)$ solutions at most. Therefore $N_4 = O(q^2)$.

In the same way, considering N_1', we find, that there are at most $O(\varpi^{2\xi-2})$ solutions mod $\varpi^{\xi-1}$, or $O(\varpi^{2\xi+2})$ solutions mod $q\,(=\varpi^{\xi})$. Hence $N_1 = O(\varpi^{2\xi+2})$.

The inequality (2. 431) now becomes

$$\varphi(q) \,.\, \sigma_3 \leq K\varpi^{2\xi-2} \,.\, \varpi^{2\xi+2} + K \,.\, q^2 \,.\, q^2 \leq Kq^4.$$

Since

$$\varphi(q) = \varpi^{\xi-1}(\varpi - 1),$$

this gives $\sigma_3 \leq Kq^3$ and *à fortiori*:

$$|\,S(u, v;\, \lambda, \Delta;\, q)\,| < Kq^{\frac{3}{4}}.$$

2. 44. **Lemma 4 d.** *If* $\Delta \mid q$, $(u, q) = 1$, $(v, q) = 1$, *then*

$$S(u, v;\, \lambda, \Delta;\, q) = O(q^{\frac{3}{4}+\varepsilon}).$$

For it follows from lemma 4 b and lemma 4 c, that

$$|\,S(u, v;\, \lambda, \Delta;\, q)\,| \leq K^r q^{\frac{3}{4}}.$$

Now

$$K^r < 2^{Kr} \leq \{(1+\xi_1)(1+\xi_2)\cdots(1+\xi_r)\}^K.$$

But

$$(1+\xi_1)(1+\xi_2)\cdots(1+\xi_r)$$

is the number of divisors of q and is therefore $O(q^\varepsilon)$. Hence $K^r = O(q^\varepsilon)$ and therefore

$$S(u, v;\, \lambda, \Delta;\, q) = O(q^{\frac{3}{4}+\varepsilon}).$$

2. 45. **Lemma 4 e.** *If* $\Delta \mid q$, $(u, q) = 1$, *then*

$$S(u, v;\, \lambda, \Delta;\, q) = O(q^{\frac{3}{4}+\varepsilon}).$$

We write again

$$q = \varpi_1^{\xi_1}\varpi_2^{\xi_2}\cdots\varpi_r^{\xi_r} = \varpi_1^{\xi_1}A_1, \quad \Delta = \varpi_1^{\zeta_1}\varpi_2^{\zeta_2}\cdots\varpi_r^{\zeta_r} = \varpi_1^{\zeta_1}\Delta_1.$$

Then

$$(v, q) = (v, \varpi_1^{\xi_1})(v, \varpi_2^{\xi_2}) \cdots (v, \varpi_r^{\xi_r}) = (v, \varpi_1^{\xi_1})(v, A_1).$$

It is possible to determine numbers v_1, V_1 and λ_1, ϱ_1 by the congruences

$$v \equiv v_1 A_1^2 + V_1 \varpi_1^{2\xi_1} \pmod q, \quad \lambda \equiv \lambda_1 A_1 + \varrho_1 \varpi_1^{\xi_1} \pmod q.$$

Then

$$(v_1, \varpi_1^{\xi_1}) = (v, \varpi_1^{\xi_1}), \quad (V_1, A_1) = (v, A_1)$$

and (lemma 4 a)

$$S(u, v; \lambda, \Lambda; q) = S(u, v_1; \lambda_1, \varpi_1^{\xi_1}; \varpi_1^{\xi_1}) S(u, V_1; \varrho_1, \Lambda_1; A_1).$$

Repeating the same argument, we find, that there are integers v_j, λ_j ($j = 1, 2, \ldots r$) such that

$$S(u, v; \lambda, \Lambda; q) = \prod_{j=1}^{r} S(u, v_j; \lambda_j, \varpi_j^{\xi_j}; \varpi_j^{\xi_j}) \tag{2. 451}$$

and

$$(v, q) = \prod_{j=1}^{r} (v_j, \varpi_j^{\xi_j}).$$

We now write

$$(v, q) = \varpi_1^{\xi'_1} \varpi_2^{\xi'_2} \ldots \varpi_r^{\xi'_r}$$

(where the numbers ξ_j' may also be 0), so that

$$(v_j, \varpi_j^{\xi_j}) = \varpi_j^{\xi'_j} \quad (j = 1, 2, \ldots r).$$

We first consider those factors of the product (2. 451) (if there are any), for which $\xi'_j = 0$. Then $(v_j, \varpi_j^{\xi_j}) = 1$, so that we have in consequence of lemma 4 c

$$| S(u, v_j; \lambda_j, \varpi_j^{\xi_j}; \varpi_j^{\xi_j}) | < K \varpi_j^{\frac{3}{4}\xi_j}. \tag{2. 452}$$

In the second place, we consider those factors of the product (2. 451) (if there are any), for which $\xi_j' = \xi_j$. Then

$$(v_j, \varpi_j^{\xi_j}) = \varpi_j^{\xi_j}, \quad \text{or} \quad v_j \equiv 0 \pmod{\varpi_j^{\xi_j}}.$$

Therefore

$$S = S(u, v_j;\ \lambda_j, \varpi_j^{\zeta_j};\ \varpi_j^{\xi_j}) = \sum_{p \equiv \lambda_j \pmod{\varpi_j^{\zeta_j}}} \exp \frac{2\pi i u p}{\varpi_j^{\xi_j}}.$$

If $\zeta_j = 0$, this is $\mu(\varpi_j^{\xi_j})$, so that we have again (2. 452). If $\zeta_j \neq 0$, we may write

$$p = \lambda_j + \nu \varpi_j^{\zeta_j} \ (\nu = 0, 1, 2, \ldots, \varpi_j^{\xi_j - \zeta_j} - 1)^1,$$

so that

$$S = \sum_{\nu=0}^{\varpi_j^{\xi_j - \zeta_j} - 1} \exp \frac{2\pi i u \lambda_j}{\varpi_j^{\xi_j}} \cdot \exp \frac{2\pi i u \nu}{\varpi_j^{\xi_j - \zeta_j}}.$$

This is 0, unless $\xi_j = \zeta_j$, in which case

$$S = \exp \frac{2\pi i u \lambda_j}{\varpi_j^{\xi_j}},$$

so that still (2. 452) is true.

In the third place, we consider those factors of the product (2. 451) (if there are any), for which $0 < \xi'_j < \xi_j$. Write $v_j = \varpi_j^{\xi'_j} v'_j$. Then

$$S = \sum_{p \equiv \lambda_j \pmod{\varpi_j^{\zeta_j}}}' \exp \left(\frac{2\pi i u p}{\varpi_j^{\xi_j}} + \frac{2\pi i v'_j p'}{\varpi_j^{\xi_j - \xi'_j}} \right).$$

In this formula the number p' must be determined from

$$1 + pp' \equiv 0 \pmod{\varpi_j^{\xi_j}},$$

but the value of S is not altered, if we determine it from

$$1 + pp' \equiv 0 \pmod{\varpi_j^{\xi_j - \xi'_j}}.$$

We now consider three cases separately. Let first $\zeta_j = \xi_j - \xi'_j$. Then we may write

$$p = \lambda_j + \nu \varpi_j^{\xi_j - \xi'_j} \quad (\nu = 0, 1, 2, \ldots, \varpi_j^{\xi'_j} - 1),$$

so that, if

(2. 453) $$1 + \lambda_j \lambda'_j \equiv 0 \pmod{\varpi_j^{\xi_j - \xi'_j}},$$

[1] If $\lambda_j \equiv 0 \pmod{\varpi}$, then S would be 0.

we have

$$S = \exp\left(\frac{2\pi i u \lambda_j}{\varpi_j^{\xi_j}} + \frac{2\pi i v_j \lambda'_j}{\varpi_j^{\xi_j}}\right) \cdot \sum_{\nu=0}^{\varpi_j^{\xi'_j}-1} \exp \frac{2\pi i u \nu}{\varpi_j^{\xi'_j}} = 0.$$

Secondly, let $\xi_j - \xi'_j > \zeta_j$. Then we may write

$$p = p_1 + \nu \varpi_j^{\xi_j - \xi'_j} \quad (\nu = 0, 1, 2, \ldots, \varpi_j^{\xi'_j - 1}),$$

where

$$p_1 \equiv \lambda_j \pmod{\varpi_j^{\zeta_j}}.$$

Writing

$$1 + p_1 p'_1 \equiv 0 \pmod{\varpi_j^{\xi_j - \xi'_j}},$$

we find

$$S = \sum_{p_1 < \varpi_j^{\xi_j - \xi'_j};\ p_1 \equiv \lambda_j \pmod{\varpi_j^{\zeta_j}}} \exp\left(\frac{2\pi i u p_1}{\varpi_j^{\xi_j}} + \frac{2\pi i v'_j p'_1}{\varpi_j^{\xi_j - \xi'_j}}\right) \cdot \sum_{\nu=0}^{\varpi_j^{\xi'_j}-1} \exp \frac{2\pi i u \nu}{\varpi_j^{\xi'_j}} = 0.$$

Thirdly let $\xi_j - \xi'_j < \zeta_j$. Then we write

$$p = \lambda_j + \nu \varpi_j^{\zeta_j} \quad (\nu = 0, 1, 2, \ldots, \varpi_j^{\xi_j - \zeta_j}).$$

Hence, if λ'_j is determined from (2. 453), we find

$$S = \exp\left(\frac{2\pi i u \lambda_j}{\varpi_j^{\xi_j}} + \frac{2\pi i v'_j \lambda'_j}{\varpi_j^{\xi_j - \xi'_j}}\right) \cdot \sum_{\nu=0}^{\varpi_j^{\xi_j - \zeta_j}-1} \exp \frac{2\pi i u \nu}{\varpi_j^{\xi_j - \zeta_j}} = 0.$$

Therefore (2. 452) is true in any case, so that we get from (2. 451)

$$|S(u, v; \lambda, \Delta; q)| < K^r q^{\frac{3}{4}} = O\left(q^{\frac{3}{4}+\epsilon}\right).$$

2. 46. **Lemma 4.** *If $\Delta \mid q$, then*

$$S(u, v;\ \lambda, \Delta;\ q) = O\left(q^{\frac{3}{4}+\epsilon} (u, q)^{\frac{1}{4}}\right).$$

$$S(u, v;\ \lambda, \Delta;\ q) = O\left(q^{\frac{3}{4}+\epsilon} (v, q)^{\frac{1}{4}}\right).$$

As in 2. 45 we find

$$(2.\ 461)\qquad S(u, v;\ \lambda, \Lambda;\ q) = \prod_{j=1}^{r} S(u, v_j;\ \lambda_j, \varpi_j^{\xi_j};\ \varpi_j^{\xi_j}),$$

where

$$q = \prod_{j=1}^{r} \varpi_j^{\xi_j},\ \Lambda = \prod_{j=1}^{r} \varpi_j^{\xi_j},\ (v, q) = \prod_{j=1}^{r} (v_j,\ \varpi_j^{\xi_j}),\ (u, q) = \prod_{j=1}^{r} (u, \varpi_j^{\xi_j}).$$

For those factors of the product (2. 461), for which both v_j and u are prime to ϖ_j, we have from section 2. 43

$$|S| < K \varpi_j^{\frac{3}{4}\xi_j} = K (u, \varpi_j^{\xi_j})^{\frac{1}{4}} \varpi_j^{\frac{3}{4}\xi_j} < K (v_j, \varpi_j^{\xi_j})^{\frac{1}{4}} \varpi_j^{\frac{3}{4}\xi_j}.$$

The same result is true (section 2. 45) if only u is prime to ϖ_j. If v_j is prime to ϖ_j, but not u, we observe that

$$S(u, v;\ \lambda, \Lambda; q) = S(v, u;\ \lambda', \Lambda; q)$$

if

$$1 + \lambda\lambda' \equiv 0 \pmod{\Lambda}.$$

Hence (section 2. 45)

$$|S| < K \varpi_j^{\frac{3}{4}\xi_j} < K (u,\ \varpi_j^{\xi_j})^{\frac{1}{4}} \varpi_j^{\frac{3}{4}\xi_j}.$$

$$|S| < K \varpi_j^{\frac{3}{4}\xi_j} < K (v,\ \varpi_j^{\xi_j})^{\frac{1}{4}} \varpi_j^{\frac{3}{4}\xi_j}.$$

It remains to consider those factors of (2. 461) for which

$$(v_j, \varpi_j^{\xi_j}) \neq 1, (u, \varpi_j^{\xi_j}) \neq 1.$$

Consider first the case, that

$$(u, \varpi_j^{\xi_j}) \geq (v_j, \varpi_j^{\xi_j}).$$

Then, writing

$$(v_j, \varpi_j^{\xi_j}) = \varpi_j^{\xi'_j},\ (u, \varpi_j^{\xi_j}) = \varpi_j^{\xi''_j},$$

we have

$$\xi'_j > 0,\ \xi''_j > 0,\ \xi''_j \geq \xi'_j.$$

Further, let

$$v_j = v'_j \varpi_j^{\xi'_j},\ u = u' \varpi_j^{\xi''_j}.$$

Then, if $\xi'_j = \xi_j$, we have

$$|S| < K\varpi_j^{\xi_j} = K\varpi_j^{\frac{1}{4}\xi'_j}\varpi_j^{\frac{3}{4}\xi'_j} = K(v_j, \varpi_j^{\xi_j})^{\frac{1}{4}}\varpi_j^{\frac{3}{4}\xi_j} \leq K(u, \varpi_j^{\xi_j})^{\frac{1}{4}}\varpi_j^{\frac{3}{4}\xi_j}.$$

Secondly, if $\xi'_j < \xi_j$, we consider three cases separately. In the first place, if $\zeta_j < \xi_j - \xi'_j$, we have

$$S = \sum_{\substack{p \equiv \lambda_j \pmod{\varpi_j^{\zeta_j}} \\ p < \varpi_j^{\xi_j}}}' \exp\left(\frac{2\pi i u' p}{\varpi_j^{\xi_j - \xi'_j}} + \frac{2\pi i v'_j p'}{\varpi_j^{\xi_j - \xi'_j}}\right) = \varpi_j^{\xi'_j} \sum_{\substack{p \equiv \lambda_j \pmod{\varpi_j^{\zeta_j}} \\ p < \varpi_j^{\xi_j - \xi'_j}}}' \exp\left(\frac{2\pi i u' p}{\varpi_j^{\xi_j - \xi'_j}} + \frac{2\pi i v'_j p'}{\varpi_j^{\xi_j - \xi'_j}}\right),$$

and therefore (since $(v'_j, \varpi_j^{\xi_j - \xi'_j}) = 1$):

$$|S| < K\varpi_j^{\xi'_j}\varpi_j^{\frac{3}{4}(\xi_j - \xi'_j)} = K\varpi_j^{\frac{1}{4}\xi'_j}\varpi_j^{\frac{3}{4}\xi_j} = K(v_j, \varpi_j^{\xi_j})^{\frac{1}{4}}\varpi_j^{\frac{3}{4}\xi_j} \leq K(u, \varpi_j^{\xi_j})^{\frac{1}{4}}\varpi_j^{\frac{3}{4}\xi_j}.$$

In the second place, if $\zeta_j = \xi_j - \xi'_j$ and

$$1 + \lambda_j\lambda'_j \equiv 0 \pmod{\varpi_j^{\zeta_j}},$$

we have

$$S = \sum_{\substack{p \equiv \lambda_j \pmod{\varpi_j^{\zeta_j}} \\ p < \varpi_j^{\xi_j}}}' \exp\left(\frac{2\pi i u' p}{\varpi_j^{\zeta_j}} + \frac{2\pi i v'_j p'}{\varpi_j^{\zeta_j}}\right) = \exp\left(\frac{2\pi i u' \lambda_j}{\varpi_j^{\zeta_j}} + \frac{2\pi i v'_j \lambda'_j}{\varpi_j^{\zeta_j}}\right)\cdot \varpi_j^{\xi_j - \zeta_j}$$

and therefore

$$|S| < K\varpi_j^{\xi_j - \zeta_j} = K\varpi_j^{\xi'_j} < K\varpi_j^{\frac{1}{4}\xi'_j}\varpi_j^{\frac{3}{4}\xi_j} = K(v_j, \varpi_j^{\xi_j})\varpi_j^{\frac{3}{4}\xi_j} \leq K(u, \varpi_j^{\xi_j})\varpi_j^{\frac{3}{4}\xi_j}.$$

In the third place, if $\zeta_j > \xi_j - \xi'_j$, we have

$$S = \exp\left(\frac{2\pi i u' \lambda_j}{\varpi_j^{\xi_j - \xi'_j}} + \frac{2\pi i v'_j \lambda'_j}{\varpi_j^{\xi_j - \xi'_j}}\right)\cdot \varpi_j^{\xi_j - \zeta_j}$$

$$|S| \leq K\varpi_j^{\xi'_j} < K(v_j, \varpi_j^{\xi_j})^{\frac{1}{4}}\varpi_j^{\frac{3}{4}\xi_j} \leq K(u, \varpi_j^{\xi_j})^{\frac{1}{4}}\varpi_j^{\frac{3}{4}\xi_j}.$$

At last, if $(u, \varpi_j^{\xi_j}) < (v_j, \varpi_j^{\xi_j})$, we write

$$v_j = v''_j\varpi_j^{\xi''_j}, \quad u = u''\varpi_j^{\xi''_j}$$

and proceed in the same way. Hence, we have in any case

$$|S| < K(u, \varpi_j^{\xi_j})^{\frac{1}{4}} \varpi_j^{\frac{3}{4}\xi_j},$$

$$|S| < K(v_j, \varpi_j^{\xi_j})^{\frac{1}{4}} \varpi_j^{\frac{3}{4}\xi_j},$$

from which the results of the lemma follow by multiplication.

2. 5. In this section, we return to the sum σ_1, defined by 2. 22. The object of this section is the proof of

Lemma 5. *If* $\Delta | q$, $\mu < q$ *and*

$$\sigma_4 = \sum_{\substack{p_1 \leq \mu \\ p \equiv \lambda \pmod{\Delta}}}' \exp\left(\frac{2\pi i u p}{q} + \frac{2\pi i v p'}{q}\right).$$

where

$$1 + pp' \equiv 0 \pmod{q},\ p' \equiv p_1 + N \pmod{q},$$

then

$$|\sigma_4| < K q^{\frac{7}{8}+\varepsilon} (u, q)^{\frac{1}{4}}.$$

In order to prove this, we shall consider the square

$$0 < \xi \leq 1,\ 0 \leq \eta < 1 \tag{2. 51}$$

of a $\xi\eta$-plane. On the ξ-axis we take the points

$$\xi = \frac{1}{q}, \frac{2}{q}, \ldots, \frac{q-1}{q}, 1.$$

In those points $\frac{\nu_1}{q}$ $(\nu_1 = 1, 2, \ldots, q)$ for which $(\nu_1 + N, q) = 1$, we erect an ordinate

$$\eta = \frac{(u\nu + v\nu')}{q},$$

where

$$\nu' \equiv \nu_1 + N \pmod{q},\ 1 + \nu\nu' \equiv 0 \pmod{q}.$$

We thus get a number $\varphi(q)$ of points, whose coordinates are (p running through all positive numbers, less than and prime to q)

$$\xi = \frac{p_1}{q},\ \eta = \frac{(up + vp')}{q}.$$

All these points P are situated in the square (2. 51).

Let M_m be the number of $p's$, for which

$$0 < p_1 \leq \mu,\ p \equiv \lambda \pmod{\Delta},\ \frac{m}{M} \leq \frac{(up + vp')}{q} < \frac{m+1}{M},$$

where M is a positive integer and $m = 0, 1, 2, \ldots, M-1$. Then

$$\sum'_{\substack{p_1 \leq \mu;\ p \equiv \lambda \pmod{\Delta};\\ \frac{m}{M} \leq \frac{(up+vp')}{q} < \frac{m+1}{M}}} \exp\left(\frac{2\pi iup}{q} + \frac{2\pi ivp'}{q}\right) = M_m e^{\frac{2\pi im}{M}} +$$

$$+ \sum'_{\substack{p_1 \leq \mu;\ p \equiv \lambda \pmod{\Delta};\\ \frac{m}{M} \leq \frac{(up+vp')}{q} < \frac{m+1}{M}}} \left(\exp\frac{2\pi i(up+vp')}{q} - \exp\frac{2\pi im}{M}\right) = M_m e^{\frac{2\pi im}{M}} + O\left(\frac{M_m}{M}\right),$$

so that

$$\sigma_4 = \sum_{p_1 \leq \mu;\ p \equiv \lambda \pmod{\Delta}} \exp\left(\frac{2\pi iup}{q} + \frac{2\pi ivp'}{q}\right) = \sum_{m=0}^{M-1} M_m \exp\frac{2\pi im}{M} + O\left(\frac{\mu}{M}\right).$$

It remains to calculate M_m. For this purpose we consider the function $f(\xi, \eta)$, defined by

1°. $f(\xi, \eta) = 1$, if $0 < \xi < \frac{\mu}{q}$, $\frac{m}{M} < \eta < \frac{m+1}{M}$;

2°. $f(\xi, \eta) = \frac{1}{2}$, if (ξ, η) lies on the boundary of the rectangle $0 < \xi < \frac{\mu}{q}$, $\frac{m}{M} < \eta < \frac{m+1}{M}$;

3°. $f(\xi, \eta) = 0$, in every other point of the square $0 < \xi \leq 1$, $0 \leq \eta < 1$ (if $m = M-1$: in every other point of the square $0 < \xi \leq 1$, $0 < \eta \leq 1$).

4°. $f(\xi, \eta)$ is periodic in ξ and in η with periods *one*.

Then (since the number of the points P, which lie on the boundary of the rectangle $0 < \xi < \frac{\mu}{q}$, $\frac{m}{M} < \eta < \frac{m+1}{M}$ is at most 4)

$$M_m = \sum'_{p \equiv \lambda \pmod{\Delta}} f\left(\frac{p_1}{q}, \frac{up+vp'}{q}\right) + O(1).$$

Now we have for all reel values of ξ and η:

$$f(\xi, \eta) = \sum_{h=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} a_{h,k} e^{2\pi i \xi h} e^{2\pi i \eta k},$$

where

$$a_{h,k} = \int_0^1 \int_0^1 f(\xi, \eta) e^{-2\pi i \xi h} e^{-2\pi i \eta k} d\xi \, d\eta$$

or explicitly

$$a_{0,0} = \frac{\mu}{qM}; \quad a_{h,0} = -\frac{1}{2\pi i h M}\left(e^{-\frac{2\pi i h \mu}{q}} - 1\right) (h \neq 0);$$

$$a_{0,k} = -\frac{\mu}{2\pi i k q}\left(e^{-\frac{2\pi i k(m+1)}{M}} - e^{-\frac{2\pi i k m}{M}}\right) \quad (k \neq 0);$$

$$a_{h,k} = -\frac{1}{4\pi^2 h k}\left(e^{-\frac{2\pi i h \mu}{q}} - 1\right)\left(e^{-\frac{2\pi i k(m+1)}{M}} - e^{-\frac{2\pi i k m}{M}}\right) \quad (h \neq 0, \; k \neq 0).$$

Hence

$$M_m = \sum_{p \equiv \lambda \, (\mathrm{mod}\, A)}' \sum_{h=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} a_{h,k} \exp\left(\frac{2\pi i h p_1}{q} + \frac{2\pi i k(up + vp')}{q}\right) + O(1).$$

For this sum we write (H being a large positive integer)

$$M_m = \sum_{h=-H}^{+H} \sum_{k=-H}^{+H} a_{h,k} \exp\left(-\frac{2\pi i h N}{q}\right) \sum_{p \equiv \lambda \, (\mathrm{mod}\, A)}' \exp\left(\frac{2\pi i k u p}{q} + \frac{2\pi i p'(h + kv)}{q}\right) +$$

$$+ \sum_{p \equiv \lambda \, (\mathrm{mod}\, A)}' \sum_{h=-\infty}^{+\infty} \sum_{|k|>H} + \sum_{p \equiv \lambda \, (\mathrm{mod}\, A)}' \sum_{|h|>H} \sum_{k=-H}^{+H} + O(1) = \Sigma_1 + \Sigma_2 + \Sigma_3 + O(1).$$

We shall consider these three sums separately.

2. 51. The term $h = 0$, $k = 0$ of Σ_1 is

$$a_{0,0} \sum_{p \equiv \lambda \, (\mathrm{mod}\, A)}' 1 = \frac{\mu}{qM} \sum_{p \equiv \lambda \, (\mathrm{mod}\, A)}' 1 = \frac{\mu}{qM} \varphi_\lambda(q)$$

say.

The terms $k = 0$, $h \neq 0$ of Σ_1 together are

$$\frac{1}{2\pi i M} \sum_{\substack{h=-H \\ h \neq 0}}^{+H} \frac{1}{h}\left(e^{-\frac{2\pi i h \mu}{q}} - 1\right) \exp\left(-\frac{2\pi i h N}{q}\right) \cdot \sum_{p \equiv \lambda \, (\mathrm{mod}\, A)}' \exp\frac{2\pi i p' h}{q},$$

the absolute value of which, as follows from lemma 4, is

$$\leq \frac{K}{M}\sum_{h=1}^{H}\frac{(h,q)^{\frac{1}{4}}}{h}q^{\frac{3}{4}+\varepsilon} \leq K q^{\frac{3}{4}+\varepsilon}\sum_{\delta\mid q}\delta^{\frac{1}{4}}\sum_{\substack{(h,q)=\delta\\ h\leq H}}\frac{1}{h} \leq K q^{\frac{3}{4}+\varepsilon}\sum_{\delta\mid q}\delta^{-\frac{3}{4}}\sum_{h_1\leq\frac{H}{\delta}}\frac{1}{h_1}$$

$$\leq K q^{\frac{3}{4}+\varepsilon}\log H\sum_{\delta\mid q}1 = O\left(q^{\frac{3}{4}+\varepsilon}\log H\right).$$

The terms $k \neq 0$, $h = 0$ of Σ_1 together are

$$\frac{\mu}{2\pi i q}\sum_{\substack{k=-H\\ k\neq 0}}^{+H}\frac{1}{k}\left(e^{-\frac{2\pi i k(m+1)}{M}} - e^{-\frac{2\pi i k m}{M}}\right)\sum_{p\equiv\lambda \pmod{\Delta}}' \exp\left(\frac{2\pi i k u p}{q} + \frac{2\pi i k v p'}{q}\right),$$

the absolute value of which (as follows from lemma 4), is

$$\leq K\frac{\mu}{q}\sum_{k=1}^{H}\frac{(ku,q)^{\frac{1}{4}}}{k}q^{\frac{3}{4}+\varepsilon} \leq K q^{\frac{3}{4}+\varepsilon}(u,q)^{\frac{1}{4}}\sum_{k=1}^{H}\frac{(k,q)^{\frac{1}{4}}}{k} = O\left(q^{\frac{3}{4}+\varepsilon}(u,q)^{\frac{1}{4}}\log H\right).$$

The terms $k \neq 0$, $h \neq 0$ of Σ_1 together are absolutely (as follows from lemma 4)

$$\leq K\sum_{h=1}^{H}\sum_{k=1}^{H}\frac{1}{hk}(ku,q)^{\frac{1}{4}}q^{\frac{3}{4}+\varepsilon} \leq K q^{\frac{3}{4}+\varepsilon}(u,q)^{\frac{1}{4}}\sum_{h=1}^{H}\frac{1}{h}\sum_{k=1}^{H}\frac{(k,q)^{\frac{1}{4}}}{k} =$$

$$= O\left(q^{\frac{3}{4}+\varepsilon}(u,q)^{\frac{1}{4}}\log^2 H\right).$$

Collecting the results, we find

$$\sum\nolimits_1 = \frac{\mu}{qM}\varphi_\lambda(q) + O\left(q^{\frac{3}{4}+\varepsilon}(u,q)^{\frac{1}{4}}\log^2 H\right). \qquad (2.511)$$

2. 52. In order to make an estimation of Σ_2 and Σ_3, we observe, that there corresponds a point P of the square $0 < \xi \leq 1$, $0 \leq \eta < 1$ to every term of Σ_2 or Σ_3. We now take a small positive number ψ. Then we define the region $R_1(\psi)$ as follows:

$R_1(\psi)$ consists of the following strips of the square $0 < \xi \leq 1$, $0 \leq \eta < 1$:

1°. $0 \leq \xi \leq \psi$; 2°. $\frac{\mu}{q} - \psi \leq \xi \leq \frac{\mu}{q} + \psi$; 3°. $1 - \psi \leq \xi \leq 1$; 4°. $0 \leq \eta \leq \psi$;

5°. $\frac{m}{M} - \psi \leq \xi \leq \frac{m}{M} + \psi$; 6°. $\frac{m+1}{M} - \psi \leq \xi \leq \frac{m+1}{M} + \psi$; 7°. $1 - \psi \leq \eta \leq 1$.

We shall denote by $R_2(\psi)$ that part of the square $0 < \xi \leq 1$, $0 \leq \eta < 1$, which remains, if $R_1(\psi)$ is taken away from it, so that $R_2(\psi)$ consists of six rectangles. Then, if (ξ, η) belongs to $R_2(\psi)$, we have

$$\xi > \psi;\quad \left|\xi - \frac{\mu}{q}\right| > \psi;\quad 1 - \xi > \psi;\quad \eta > \psi;\quad \left|\eta - \frac{m}{M}\right| > \psi;\quad \left|\eta - \frac{m+1}{M}\right| > \psi;\quad 1 - \eta > \psi.$$

Further it is easy to see, that the number of points P, which are lying in $R_1(\psi)$ is $O(\psi q)$.

Writing for abbreviation

$$\xi = \frac{p_1}{q},\quad \eta = \frac{(up + vp')}{q},$$

we have

$$\left|\sum\nolimits_2\right| \leq K \sum_{p \equiv \lambda \,(\mathrm{mod}\, \Delta)}' \left| \sum_{\substack{h=-\infty \\ h \neq 0}}^{+\infty} \frac{1}{h} \left(e^{2\pi i h \left(\xi - \frac{\mu}{q}\right)} - e^{2\pi i h \xi}\right) \right| \left| \sum_{|k| > H} \frac{1}{k} \left(e^{2\pi i k \left(\eta - \frac{m+1}{M}\right)} - \right.\right.$$

$$\left.\left. - e^{2\pi i k \left(\eta - \frac{m}{M}\right)}\right) \right| + K \sum_{p \equiv \lambda \,(\mathrm{mod}\, \Delta)}' \left| \sum_{|k| > H} \frac{1}{k} \left(e^{2\pi i k \left(\eta - \frac{m+1}{M}\right)} - e^{2\pi i k \left(\eta - \frac{m}{M}\right)}\right) \right| \leq$$

$$\leq K \sum_{p \equiv \lambda \,(\mathrm{mod}\, \Delta)}' \left| \sum_{k > H} \frac{\sin 2\pi k \left(\eta - \frac{m+1}{M}\right) - \sin 2\pi k \left(\eta - \frac{m}{M}\right)}{k} \right|.$$

For those terms of this sum, for which the corresponding point P is inside $R_2(\psi)$, we have

$$\left| \sum_{k > H} \frac{\sin 2\pi k \left(\eta - \frac{m+1}{M}\right) - \sin 2\pi k \left(\eta - \frac{m}{M}\right)}{k} \right| < \frac{K}{H\psi}.$$

For the other points the same expression is $< K$. Therefore, if we take $\psi = H^{-\frac{1}{2}}$:

$$\sum\nolimits_2 = O\left(\frac{q}{H\psi}\right) + O(q\psi) = O\left(\frac{q}{\sqrt{H}}\right).$$

In the same way, we find also $\sum_3 = O\left(\frac{q}{\sqrt{H}}\right)$. Hence

(2. 521) $$\sum_2 + \sum_3 = O\left(\frac{q}{\sqrt{H}}\right).$$

2. 53. It is now easy to complete the proof of lemma 5. For we have from (2. 511) and (2. 521) that, if we take $H=q$:

$$M_m = \frac{\mu}{q\,M}\,\varphi_\lambda(q) + O\left(q^{\frac{3}{4}+\varepsilon}(u, q)^{\frac{1}{4}}\right).$$

Hence

$$\sigma_4 = \frac{\mu}{q\,M}\,\varphi_\lambda(q)\sum_{m=0}^{M-1} e^{\frac{2\pi i m}{M}} + O\left(M q^{\frac{3}{4}+\varepsilon}(u, q)^{\frac{1}{4}}\right) + O\left(\frac{q}{M}\right).$$

We take

$$M = \left[q^{\frac{1}{8}}\right].$$

Then it follows, that

$$\sigma_4 = O\left(q^{\frac{7}{8}+\varepsilon}(u, q)^{\frac{1}{4}}\right) + O\left(q^{\frac{7}{8}}\right) = O\left(q^{\frac{7}{8}+\varepsilon}(u, q)^{\frac{1}{4}}\right) \quad \text{q. e. d.}$$

2. 6. A combination of all results obtained now gives

Lemma 6. (Fundamental lemma). *We have* (see 2. 2)

$$\sum_{p_1 \leqq \mu}{}' S_{ap, q, \nu_1} S_{bp, q, \nu_2} S_{cp, q, \nu_3} S_{dp, q, \nu_4} \exp\left(-\frac{2\pi i n p}{q}\right) = O\left(q^{2+\frac{7}{8}+\varepsilon}(n, q)^{\frac{1}{4}}\right).$$

For this sum has been denoted by σ_1 formerly. Therefore the result follows from the lemmas 2*, 3* and 5 in connection wich (2. 31).

3. Proof of the main theorem.

3. 1. **Lemma 7.** *On the arc* $\xi_{p,q}$ *we have* $(s = a, b, c, d)$

$$\vartheta(w^s) = \varphi_s + \Phi_s,$$

where

$$\varphi_s = \sqrt{\frac{\pi}{s}}\,\frac{S_{sp,q}}{q}\left(\frac{1}{n} - i\theta\right)^{-\frac{1}{2}},$$

$$\Phi_s = \frac{2}{q}\sqrt{\frac{\pi}{s}}\left(\frac{1}{n} - i\theta\right)^{-\frac{1}{2}}\sum_{\nu=1}^{\infty} S_{sp,q,\nu}\exp\left(-\frac{\pi^2\nu^2}{s\,q^2\left(\frac{1}{n} - i\theta\right)}\right).$$

For, using the transformation-formula for the ϑ-function, we find

$$\vartheta(w^s) = \sum_{\nu=-\infty}^{+\infty} w^{s\nu^2} = \sum_{\nu=-\infty}^{+\infty} \exp\left(\frac{2\pi i p \nu^2 s}{q} - \nu^2 s\left(\frac{1}{n} - i\theta\right)\right) =$$

$$= \sum_{j=0}^{q-1} \exp\frac{2\pi i p s j^2}{q} \cdot \sum_{l=-\infty}^{+\infty} \exp\left\{-(lq+j)^2 s\left(\frac{1}{n} - i\theta\right)\right\} =$$

$$= \frac{1}{q}\sqrt{\frac{\pi}{s}}\left(\frac{1}{n} - i\theta\right)^{-\frac{1}{2}} \cdot \sum_{j=0}^{q-1} \exp\frac{2\pi i p s j^2}{q} \cdot$$

$$\cdot\left\{1 + 2\sum_{\nu=1}^{\infty} \cos\frac{2j\pi\nu}{q} \cdot \exp\left(-\frac{\pi^2\nu^2}{sq^2\left(\frac{1}{n} - i\theta\right)}\right)\right\}.$$

Since

$$\sum_{j=0}^{q-1} \exp\frac{2\pi i p s j^2}{q} \cdot \cos\frac{2j\pi\nu}{q} = \frac{1}{2}\sum_{j=0}^{q-1} \exp\left(\frac{2\pi i p s j^2}{q} + \frac{2j\pi i\nu}{q}\right) +$$

$$+ \frac{1}{2}\sum_{j=0}^{q-1} \exp\left(\frac{2\pi i p s j^2}{q} - \frac{2j\pi i\nu}{q}\right) = S_{sp,\,q,\,\nu}\,,$$

the result of the lemma follows.

3. 2. We have

$$1 + \sum_{n=1}^{\infty} r(n)\, w^n = \vartheta(w^a)\,\vartheta(w^b)\,\vartheta(w^c)\,\vartheta(w^d),$$

so that

$$r(n) = \frac{1}{2\pi i}\int_{\Gamma} \vartheta(w^a)\vartheta(w^b)\vartheta(w^c)\vartheta(w^d) w^{-n-1}\, dw =$$

$$= \frac{1}{2\pi i}\sum_{q=1}^{N}\sum_{p}{}' \int_{\xi_{p,q}} \vartheta(w^a)\,\vartheta(w^b)\,\vartheta(w^c)\,\vartheta(w^d)\, w^{-n-1}\, dw.$$

Therefore, in consequence of lemma *7*:

$$r(n) = \frac{1}{2\pi i}\sum_{q=1}^{N}\sum_{p}{}' \int_{\xi_{p,q}} \varphi_a\,\varphi_b\,\varphi_c\,\varphi_d\, w^{-n-1}\, dw +$$

$$+ \frac{1}{2\pi i}\sum_{q=1}^{N}\sum_{p}{}' \int_{\xi_{p,q}} (\Sigma\,\varphi_a\,\varphi_b\,\varphi_c\,\Phi_d + \Sigma\,\varphi_a\,\varphi_b\,\Phi_c\,\Phi_d + \Sigma\,\varphi_a\,\Phi_b\,\Phi_c\,\Phi_d +$$

$$+ \Phi_a\,\Phi_b\,\Phi_c\,\Phi_d)\, w^{-n-1}\, dw = J_1 + J_2.$$

Here we have written

$$\sum \varphi_a \varphi_b \varphi_c \Phi_d = \varphi_a \varphi_b \varphi_c \Phi_d + \varphi_a \varphi_b \Phi_c \varphi_d + \varphi_a \Phi_b \varphi_c \varphi_d + \Phi_a \varphi_b \varphi_c \varphi_d.$$

The other sums in the second integrand have similar meanings.

3. 21. Writing $\Delta = abcd$, we have

$$J_1 = \frac{\pi^2}{\sqrt{\Delta}} \cdot \frac{1}{2\pi i} \sum_{q=1}^{N} \sum_p{}' q^{-4} \left\{ S\frac{p}{q} \right\} \int_{\xi_{p,q}} \left(\frac{1}{n} - i\theta \right)^{-2} w^{-n-1}\, dw.$$

Further, writing

$$\int_{\xi_{p,q}} = \int_{\theta=-\frac{2\pi}{q(q+q'')}}^{\theta=-\frac{2\pi}{q(q+N)}} + \int_{\theta=-\frac{2\pi}{q(q+N)}}^{\theta=\frac{2\pi}{q(q+N)}} + \int_{\theta=\frac{2\pi}{q(q+N)}}^{\theta=\frac{2\pi}{q(q+q')}},$$

we have

(3. 211) $$J_1 = J_{1,1} + J_{1,2} + J_{1,3}.$$

3. 211. In $J_{1,2}$ we write

$$\left(\frac{1}{n} - i\theta \right)^{-2} = F\left(w e^{-\frac{2\pi i p}{q}} \right) + O(1),$$

where

$$F(w) = \sum_{\nu=1}^{\infty} \nu w^{\nu} = \frac{w}{(1-w)^2}.$$

Therefore, if η is the complementary arc on Γ of

$$-\frac{2\pi}{q(q+N)} \le \theta \le \frac{2\pi}{q(q+N)},$$

then

$$J_{1,2} = \frac{\pi^2}{\sqrt{\Delta}} \cdot \frac{1}{2\pi i} \sum_{q=1}^{N} \sum_p{}' q^{-4} \left\{ S\frac{p}{q} \right\} \int_{\theta=-\frac{2\pi}{q(q+N)}}^{\theta=\frac{2\pi}{q(q+N)}} F\left(w e^{-\frac{2\pi i p}{q}} \right) w^{-n-1}\, dw +$$

$$+ O\left(\sum_{q=1}^{N} \sum_p{}' \frac{1}{q^2} \cdot \frac{1}{qN} \right) =$$

$$= \frac{\pi^2}{\sqrt{\Delta}} \cdot \frac{1}{2\pi i} \sum_{q=1}^{N} \sum_p{}' q^{-4} \left\{ S\frac{p}{q} \right\} \int_{\Gamma} F\left(w e^{-\frac{2\pi i p}{q}} \right) w^{-n-1}\, dw +$$

$$+ K\sum_{q=1}^{N} q^{-4}\int_{\eta} \frac{e^{-\frac{1}{n}+i\theta}}{\left(1-e^{-\frac{1}{n}+i\theta}\right)^2} e^{1-ni\theta}\sum_{p}'\left\{S\frac{p}{q}\right\}e^{-\frac{2\pi i n p}{q}} + O\left(\frac{1}{N}\right) =$$

$$= \frac{\pi^2}{\sqrt{\Delta}} n\,S(n) + O\left(n\sum_{q=N+1}^{\infty} q^{-4}\left|\sum_{p}'\left\{S\frac{p}{q}\right\}e^{-\frac{2n\pi i p}{q}}\right|\right) +$$

$$+ O\left(\sum_{q=1}^{N} q^{-4}\int_{\frac{\pi}{qN}}^{\infty}\frac{d\theta}{\frac{1}{n^2}+\theta^2}\left|\sum_{p}'\left\{S\frac{p}{q}\right\}e^{-\frac{2n\pi i p}{q}}\right|\right) + O\left(\frac{1}{N}\right).$$

Hence, using the fundamental lemma with $\mu = q-1$, $\nu_1, \nu_2, \nu_3, \nu_4 \equiv 0 \pmod q$:

$$J_{1,2} = \frac{\pi^2}{\sqrt{\Delta}} n\,S(n) + O\left(n\sum_{q=N+1}^{\infty}\frac{(n,q)^{\frac{1}{4}}}{q^{1+\frac{1}{8}-\varepsilon}}\right) + O\left(N\sum_{q=1}^{N}\frac{(n,q)^{\frac{1}{4}}}{q^{\frac{1}{8}}}q^{\varepsilon}\right) + O\left(\frac{1}{N}\right) =$$

$$= \frac{\pi^2}{\sqrt{\Delta}} n\,S(n) + O\left(n\sum_{\delta\mid n}\frac{\delta^{\frac{1}{4}}}{\delta^{1+\frac{1}{8}-\varepsilon}}\sum_{q_1\geq\frac{N+1}{\delta}}\frac{1}{q_1^{1+\frac{1}{8}-\varepsilon}}\right) +$$

$$+ O\left(n^{\frac{1}{2}+\varepsilon}\sum_{\delta\mid n}\delta^{\frac{1}{4}}\delta^{-\frac{1}{8}}\sum_{q_1\leq\frac{N}{\delta}}\frac{1}{q_1^{\frac{1}{8}}}\right) + O\left(\frac{1}{N}\right) =$$

$$= \frac{\pi^2}{\sqrt{\Delta}} n\,S(n) + O\left(n^{1+\varepsilon}n^{-\frac{1}{16}}\sum_{\delta\mid n}\delta^{-\frac{3}{4}}\right) + O\left(n^{\frac{1}{2}+\varepsilon}n^{\frac{7}{16}}\sum_{\delta\mid n}\delta^{-\frac{3}{4}}\right) + O\left(\frac{1}{N}\right)$$

or

$$(3.\ 2111)\qquad J_{1,2} = \frac{\pi^2}{\sqrt{abcd}} n\,S(n) + O\left(n^{\frac{15}{16}+\varepsilon}\right).$$

3. 212. If $0 < N_1 < N$, we have

$$J_{1,3} = K\sum_{q=1}^{N}\sum_{p}' q^{-4}\left\{S\frac{p}{q}\right\}\int_{\frac{2\pi}{q(q+N)}}^{\frac{2\pi}{q(q+q')}}\left(\frac{1}{n}-i\theta\right)^{-2}e^{-\frac{2\pi i n p}{q}}e^{-ni\theta}\,d\theta =$$

$$= K\sum_{q=1}^{N_1} + K\sum_{q=N_1+1}^{N} = \sum\nolimits_1 + \sum\nolimits_2.$$

(3.2121) $$\left|\sum\nolimits_1\right| \leq K \sum_{q=1}^{N_1} \sum_p{}' q^{-4} q^2 \int_{\frac{\pi}{qN}}^{\infty} \frac{d\theta}{\frac{1}{n^2}+\theta^2} \leq K N \sum_{q=1}^{N_1} \frac{\varphi(q)}{q} = O(NN_1).$$

$$\sum\nolimits_2 = K \sum_{q=N_1+1}^{N} \sum_p{}' q^{-4} \left\{S\frac{p}{q}\right\} \sum_{\mu=q'+q-N}^{\mu=q-1} \int_{\frac{2\pi}{q(N+\mu+1)}}^{\frac{2\pi}{q(N+\mu)}} \left(\frac{1}{n}-i\theta\right)^{-2} e^{-ni\theta} e^{-\frac{2\pi i n p}{q}} d\theta =$$

$$= K \sum_{q=N_1+1}^{N} q^{-4} \sum_{\mu=1}^{q-1} \int_{\frac{2\pi}{q(N+\mu+1)}}^{\frac{2\pi}{q(N+\mu)}} \left(\frac{1}{n}-i\theta\right)^{-2} e^{-ni\theta} d\theta \sum_{q'+q-N\leqq\mu} \left\{S\frac{p}{q}\right\} e^{-\frac{2\pi i n p}{q}}.$$

Now we have $p'q-pq'=1$ or

$$(q'+q)p+1 \equiv 0 \pmod{q},$$

and

$$0 < q'+q-N \leq q,$$

so that $q'+q-N$ is the number p_1, defined in section 2. Therefore we can apply the fundamental lemma and we find

$$\left|\sum\nolimits_2\right| \leq K \sum_{q=N_1+1}^{N} q^{-4} \sum_{\mu=1}^{q-1} \int_{\frac{2\pi}{q(N+\mu+1)}}^{\frac{2\pi}{q(N+\mu)}} \frac{d\theta}{\frac{1}{n^2}+\theta^2} q^{2+\frac{7}{8}+\varepsilon} (n,q)^{\frac{1}{4}}$$

$$\leq K \sum_{q=N_1+1}^{N} \frac{n^{\varepsilon}}{q^{1+\frac{1}{8}}} (n,q)^{\frac{1}{4}} \int_{\frac{2\pi}{q(N+q)}}^{\frac{2\pi}{q(N+1)}} \frac{d\theta}{\frac{1}{n^2}+\theta^2}$$

$$\leq K n^{1+\varepsilon} \sum_{q=N_1+1}^{\infty} \frac{(n,q)^{\frac{1}{4}}}{q^{1+\frac{1}{8}}} \leq K n^{1+\varepsilon} \sum_{\delta\mid n} \frac{\delta^{\frac{1}{4}}}{\delta^{1+\frac{1}{8}}} \sum_{q_1>\frac{N_1}{\delta}} \frac{1}{q_1^{1+\frac{1}{8}}}$$

$$\leq K n^{1+\varepsilon} \sum_{\delta\mid n} \delta^{-\frac{7}{8}} \delta^{\frac{1}{8}} N_1^{-\frac{1}{8}} = O\left(n^{1+\varepsilon} N_1^{-\frac{1}{8}}\right).$$

Combining this result with (3. 2121) we find

$$J_{1,3} = O(N N_1) + O\left(n^{1+\varepsilon} N_1^{-\frac{1}{8}}\right).$$

Taking

$$N_1 = \left[n^{\frac{4}{9}}\right],$$

we find

$$J_{1,3} = O\left(n^{\frac{17}{18}+\varepsilon}\right). \tag{3. 2122}$$

In exactly the same way, we find also

$$J_{1,1} = O\left(n^{\frac{17}{18}+\varepsilon}\right). \tag{3. 2123}$$

3. 213. From (3. 211), (3. 2111), (3. 2122), (3. 2123) it follows that

$$J_1 = \frac{\pi^2}{\sqrt{a b c d}} n\, S(n) + O\left(n^{\frac{17}{18}+\varepsilon}\right). \tag{3. 2131}$$

3. 22. The calculation of J_2 does not differ essentially from that of J_1. It consists of a number of terms, which are of the same form as J_1, with the only difference, that one or more of the functions φ_a, φ_b, φ_c, φ_d are replaced by the corresponding Φ_a, Φ_b, Φ_c or Φ_d. All these terms of J_2 can be treated in exactly the same way. I give the complete proof for one of then only, viz.

$$I = \frac{1}{2\pi i}\sum_{q=1}^{N} \sum_{p}{}' \int_{\xi_{p,q}} \varphi_a \varphi_b \Phi_c \Phi_d\, w^{-n-1}\, dw.$$

Writing again

$$\int_{\xi_{p,q}} = \int_{\theta=-\frac{2\pi}{q(q+q'')}}^{\theta=-\frac{2\pi}{q(q+N)}} + \int_{\theta=-\frac{2\pi}{q(q+N)}}^{\theta=\frac{2\pi}{q(q+N)}} + \int_{\theta=\frac{2\pi}{q(q+N)}}^{\theta=\frac{2\pi}{q(q+q')}},$$

we have

$$I = I_1 + I_2 + I_3. \tag{3. 221}$$

3. 221. We have, l being a positive number, which can be taken arbitrary small:

$$I_2 = K\sum_{q=1}^{N} q^{-4}\int_{-\frac{2\pi}{q(q+N)}}^{\frac{2\pi}{q(q+N)}}\left(\frac{1}{n}-i\theta\right)^{-2} e^{ni\theta}\,d\theta \sum_{\nu_3=1}^{\infty}\sum_{\nu_4=1}^{\infty}$$

$$\left(\sum_p{}' S_{ap,\,q}\, S_{bp,\,q}\, S_{cp,\,q,\,\nu_3}\, S_{dp,\,q,\,\nu_4}\, e^{-\frac{2\pi inp}{q}}\right)\exp\left(-\frac{\pi^2\left(\frac{\nu_3^2}{c}+\frac{\nu_4^2}{d}\right)}{q^2\left(\frac{1}{n}-i\theta\right)}\right).$$

$$|I_2| \leq K\sum_{q=1}^{N} q^{-4}\int_0^{\frac{2\pi}{q(q+N)}}\frac{d\theta}{\frac{1}{n^2}+\theta^2}\sum_{\nu_3=1}^{\infty}\sum_{\nu_4=1}^{\infty}$$

$$\left|\sum_p{}' S_{ap,\,q}\, S_{bp,\,q}\, S_{cp,\,q,\,\nu_3}\, S_{dp,\,q,\,\nu_4}\, e^{-\frac{2\pi inp}{q}}\right|\exp\left(-\frac{\pi^2 n}{q^2(1+n^2\theta^2)}\left(\frac{\nu_3^2}{c}+\frac{\nu_4^2}{d}\right)\right) =$$

$$= K\sum_{0<q\leq n^{\frac{1}{2}-l}} q^{-4}\int_0^{\frac{1}{qn^{\frac{1}{2}+l}}} + K\sum_{0<q\leq n^{\frac{1}{2}-l}} q^{-4}\int_{q^{-1}n^{-\frac{1}{2}+l}}^{\frac{2\pi}{q(q+N)}} + K\sum_{n^{\frac{1}{2}-l}<q\leq N} q^{-4}\int_0^{\frac{2\pi}{q(q+N)}} =$$

$$= \Sigma_1 + \Sigma_2 + \Sigma_3 \text{ say.}$$

Applying the fundamental lemma with $\mu = q-1$, $\nu_1, \nu_2 \equiv 0 \pmod q$, we find

$$\Sigma_1 \leq K\sum_{0<q\leq n^{\frac{1}{2}-l}}\frac{q^{\varepsilon}(n,q)^{\frac{1}{4}}}{q}\,\frac{n^3}{q(q+N)}\sum_{\nu_3=1}^{\infty}\sum_{\nu_4=1}^{\infty}\exp\left(-Kn^{2l}(\nu_3^2+\nu_4^2)\right) =$$

$$= O\left(n^3\exp\left(-Kn^{2l}\right)\right) = O\left(n^{\frac{17}{18}+\varepsilon}\right).$$

$$\Sigma_2 \leq K\sum_{0<q\leq n^{\frac{1}{2}-l}}\frac{q^{\varepsilon}(n,q)^{\frac{1}{4}}}{q^{1+\frac{1}{8}}}\, n\int_{\frac{n^{\frac{1}{2}-l}}{q}}^{\infty}\frac{dt}{1+t^2}\sum_{\nu_3=1}^{\infty}\sum_{\nu_4=1}^{\infty}\exp\left(-K(\nu_3^2+\nu_4^2)\right)$$

$$\leq Kn^{\frac{1}{2}+l+\varepsilon} \sum_{0<q\leq n^{\frac{1}{2}-l}} \frac{(n,q)^{\frac{1}{4}}}{q^{\frac{1}{8}}} \leq Kn^{\frac{1}{2}+l+\varepsilon} \sum_{\delta\mid n} \delta^{\frac{1}{4}} \delta^{-\frac{1}{8}} \sum_{0<q_1\leq \frac{n^{\frac{1}{2}-l}}{\delta}} q_1^{-\frac{1}{8}}$$

$$\leq Kn^{\frac{1}{2}+l+\varepsilon} n^{\left(\frac{1}{2}-l\right)\frac{7}{8}} = O\left(n^{\frac{15}{16}+\varepsilon}\right) = O\left(n^{\frac{17}{18}+\varepsilon}\right),$$

since l can be taken arbitrary small.

$$\Sigma_3 \leq K \sum_{n^{\frac{1}{2}-l}<q\leq N} \frac{q^{\varepsilon}(n,q)^{\frac{1}{4}}}{q^{1+\frac{1}{8}}} \int_0^{\frac{2\pi}{q(q+N)}} \frac{d\theta}{\frac{1}{n^2}+\theta^2} \sum_{\nu_3=1}^{\infty} \sum_{\nu_4=1}^{\infty} \exp\left(-K(\nu_3^2+\nu_4^2)\right)$$

$$\leq Kn^{1+\varepsilon} \sum_{n^{\frac{1}{2}-l}<q} \frac{(n,q)^{\frac{1}{4}}}{q^{1+\frac{1}{8}}} \int_0^{\frac{2\pi n}{q(q+N)}} \frac{dt}{1+t^2} \leq Kn^{1+\varepsilon} \sum_{q>n^{\frac{1}{2}-l}} \frac{(n,q)^{\frac{1}{4}}}{q^{1+\frac{1}{8}}} =$$

$$= O\left(n^{\frac{15}{16}+\varepsilon}\right) = O\left(n^{\frac{17}{18}+\varepsilon}\right).$$

Collecting the results of this section, we find

$$I_2 = O\left(n^{\frac{17}{18}+\varepsilon}\right). \tag{3.2211}$$

3. 222. We have

$$I_3 = K\sum_{q=1}^{N} q^{-4} \sum_{p}{}' S_{ap,q}\, S_{bp,q}\, e^{-\frac{2\pi i n p}{q}} \int_{\frac{2\pi}{q(q+N)}}^{\frac{2\pi}{q(q+q')}} \left(\frac{1}{n}-i\theta\right)^{-2} e^{ni\theta}\, d\theta$$

$$\sum_{\nu_3=1}^{\infty} \sum_{\nu_4=1}^{\infty} S_{cp,q,\nu_3}\, S_{dp,q,\nu_4} \exp\left\{-\frac{\pi^2}{q^2\left(\frac{1}{n}-i\theta\right)}\left(\frac{\nu_3^2}{c}+\frac{\nu_4^2}{d}\right)\right\} =$$

$$= K\sum_{q=1}^{N_1} + K\sum_{q=N_1+1}^{N} = \sum\nolimits_1 + \sum\nolimits_2 \text{ say.}$$

$$\Sigma_2 = K \sum_{n=N_1+1}^{N} q^{-4} \sum_{p}' S_{ap,q}\, S_{bp,q}\, e^{-\frac{2\pi i n p}{q}} \sum_{\mu=q'+q-N}^{\mu=q-1} \int_{\frac{2\pi}{q(N+\mu+1)}}^{\frac{2\pi}{q(N+\mu)}} \left(\frac{1}{n} - i\theta\right)^{-2} e^{ni\theta}\, d\theta$$

$$\sum_{\nu_3=1}^{\infty} \sum_{\nu_4=1}^{\infty} S_{cp,q,\nu_3}\, S_{dp,q,\nu_4} \exp\left\{ - \frac{\pi^2}{q^2\left(\frac{1}{n} - i\theta\right)} \left(\frac{\nu_3^2}{c} + \frac{\nu_4^2}{d}\right)\right\}.$$

Therefore, changing the order of summation and applying the fundamental lemma as before:

$$\left|\sum\nolimits_2\right| \leq K \sum_{q=N_1+1}^{N} q^{-4} \sum_{\mu=1}^{q-1} \int_{\frac{2\pi}{q(N+\mu+1)}}^{\frac{2\pi}{q(N+\mu)}} \frac{d\theta}{\frac{1}{n^2}+\theta^2} q^{2+\frac{7}{8}+\varepsilon} (n, q)^{\frac{1}{4}} \sum_{\nu_3=1}^{\infty} \sum_{\nu_4=1}^{\infty} \exp\left(-K(\nu_3^2 + \nu_4^3)\right)$$

$$\leq K \sum_{q=N_1+1}^{N} \frac{(n, q)^{\frac{1}{4}}}{q^{1+\frac{1}{8}}} q^{\varepsilon} \int_{\frac{2\pi}{q(N+q)}}^{\frac{2\pi}{q(N+1)}} \frac{d\theta}{\frac{1}{n^2} + \theta^2}$$

$$\leq K n^{1+\varepsilon} \sum_{q=N_1+1}^{\infty} \frac{(n, q)^{\frac{1}{4}}}{q^{1+\frac{1}{8}}} = O\left(n^{1+\varepsilon} N_1^{-\frac{1}{8}}\right).$$

It is easily seen, that

$$\Sigma_1 = O(N N_1).$$

Therefore

$$I_3 = O(N N_1) + O\left(n^{1+\varepsilon} N_1^{-\frac{1}{8}}\right) = O\left(n^{\frac{17}{18}+\varepsilon}\right). \tag{3. 2221}$$

In the same way we find:

$$I_1 = O\left(n^{\frac{17}{18}+\varepsilon}\right). \tag{3. 2222}$$

3. 223. From (3. 221), (3. 2211), (3. 2221), (3. 2222), we find

$$I = O\left(n^{\frac{17}{18}+\varepsilon}\right).$$

The same arguments are valid for the other terms of J_2. Therefore

$$J_2 = O\left(n^{\frac{17}{18}+\varepsilon}\right). \tag{3. 2231}$$

3. 3. Main theorem. If $r(n)$ is the number of representations of the positive integer n in the form $ax^2 + by^2 + cz^2 + dt^2$, then

$$r(n) = \frac{\pi^2}{\sqrt{abcd}}\, n\, S(n) + O\left(n^{\frac{17}{18}+\varepsilon}\right).$$

The proof follows from (3. 2131) and (3. 2231).

4. The singular series.

4. 1. In order to draw any conclusions from the main-theorem a detailed discussion of the singular series is necessary. This discussion is very complicated. A large number of cases must be considered separately. However, the calculation does not present any essential difficulty. Therefore I shall indicate the general lines only, and the results to which they lead. I shall begin by making some remarks, to which the calculations have lead me.

Let $n_j\,(j = 1, 2, 3, \ldots)$ be a sequence of increasing positive integers, tending to infinity, if $j \to \infty$. Then there are three possibilities:

1°. There is a number $K > 0$, such that

$$S(n_j) > K \tag{4. 11}$$

if n_j is sufficiently large, or at any rate

$$n^{\varepsilon} S(n_j) > K \tag{4. 12}$$

(for every positive ε) if n_j is sufficiently large.

2°. We have

$$S(n_j) = 0 \tag{4. 13}$$

for an infinity of integers, belonging to the sequence n_j.

3°. We have

(4. 14) $$S(n_j) \sim \frac{K}{n_j}$$

for an infinity of integers belonging to the sequence n_j.[1]

In the case 1°. the main theorem gives

$$r(n_j) \sim \frac{\pi^2}{\sqrt{\Delta}} n_j S(n_j) \quad (j \to \infty),$$

where we have written

$$\Delta = abcd.$$

In particular, if the condition 1° is satisfied for all positive integers, we may conclude, that there is only a finite number of integers, which cannot be represented in the form $ax^2+by^2+cz^2+dt^2$. Such a conclusion is not possible in the cases 2°. and 3°. It might be expected, that there is an infinite number of exceptions, if 2°. or 3°. is true. Simple arguments, which are almost trivial, will show, that this conjecture is true in the case 2°. In the case 3°. the conjecture will appear to be generally true, but not always, and the proofs are not as trivial as the analogous proofs in the case 2°. However, it may occur (as it will appear in the following pages), that a sequence of integers n_j can be found, for which (4. 14) is true and yet there is only a finite number of integers (or even: no integer) which cannot be represented in the form.

4. 2. The calculation of the sum of the singular series is effected by the methods, given by Hardy and Littlewood. It depends on the fact, that

$$A_q A_{q'} = A_{qq'} \text{ if } (q, q') = 1.$$

From this property it results, that

(4. 21) $$S(n) = \prod_{\varpi} \chi_{\varpi},$$

where

$$\chi_{\varpi} = 1 + A_{\varpi} + A_{\varpi^2} + A_{\varpi^3} + \cdots$$

and the product must be extended over all prime numbers.

[1] Of course it is also possible, that $S(n_j)$ tends to zero, but not as quickly as $\frac{1}{n_j}$, if $n_j \to \infty$. But the discussion of the singular series shows, that in this case, we can always find another sequence, for which the condition 3° holds.

4. 3. First let ϖ be an odd prime, which does not divide any of the numbers a, b, c, d. Then, writing

$$n = \varpi^{\xi} n',$$

we find

$$\chi_\varpi = \left\{1 - \left(\frac{\Delta}{\varpi}\right)\frac{1}{\varpi^2}\right\}\left\{1 + \left(\frac{\Delta}{\varpi}\right)\frac{1}{\varpi} + \left(\frac{\Delta}{\varpi^2}\right)\frac{1}{\varpi^2} + \cdots + \left(\frac{\Delta}{\varpi^{\xi}}\right)\frac{1}{\varpi^{\xi}}\right\}. \tag{4. 31}$$

We now write

$$n = lm,$$

where l contains the factors 2 and those odd prime divisors of n, which divide one of the numbers a, b, c, d at least and m is odd and prime to Δ. Then, writing $\chi^{(1)}$ for the product of χ_2 and those factors χ_ϖ of (4. 21) for which ϖ divides one of the numbers a, b, c, d at least and $\chi^{(2)}$ for the product of those factors χ_ϖ of (4. 21), for which ϖ is odd and prime to Δ, we have

$$S(n) = \chi^{(1)}\chi^{(2)} \tag{4. 32}$$

and, as follows from (4. 31)

$$\chi^{(2)} = \left\{\sum_{\delta \mid m} \frac{1}{\delta}\left(\frac{\Delta}{\delta}\right)\right\} \cdot \prod_{\varpi}{}' \left\{1 - \left(\frac{\Delta}{\varpi}\right)\frac{1}{\varpi^2}\right\},$$

where ϖ runs through all odd prime numbers, which are prime to Δ, which has been denoted by $\prod'$. The product $\prod'$ does not depend on n. Further, it is easy to determine the behaviour of the sum

$$\sum_{\delta \mid m} \frac{1}{\delta}\left(\frac{\Delta}{\delta}\right)$$

for large values of m. For we have

$$\sum_{\delta \mid m} \frac{1}{\delta}\left(\frac{\Delta}{\delta}\right) = \prod_{\varpi \mid m} \frac{1 - \left(\frac{\Delta}{\varpi^{\xi+1}}\right)\frac{1}{\varpi^{\xi+1}}}{1 - \left(\frac{\Delta}{\varpi}\right)\frac{1}{\varpi}},$$

where ξ is the exponent of the highest power of ϖ, which divides m, so that $\xi \geq 1$. Hence

$$\left|\sum_{\delta \mid m} \frac{1}{\delta}\left(\frac{\Delta}{\delta}\right)\right| \geq \prod_{\varpi \mid m}\left(1-\frac{1}{\varpi}\right)=\frac{\varphi(m)}{m},$$

where $\varphi(m)$ is the number of positive integers, less than and prime to m. Now it is well known, that

$$\frac{\varphi(m)}{m} > \frac{K}{\log\log m} > \frac{K}{\log\log n}.$$

Therefore

$$|\chi^{(2)}| > \frac{K}{\log\log n}. \tag{4.33}$$

4.4 It remains to consider χ_2 and those factors χ_ϖ, for which ϖ is not prime to Δ.

Let first ϖ be an odd prime, which divides Δ. Then I write

$$a=\varpi^{\mu_a} a_1,\ b=\varpi^{\mu_b} b_1,\ c=\varpi^{\mu_c} c_1,\ d=\varpi^{\mu_d} d_1, \tag{4.41}$$

where a_1, b_1, c_1, d_1 are prime to ϖ. I suppose that

$$\mu_a \leq \mu_b \leq \mu_c \leq \mu_d, \tag{4.42}$$

which is not an essential restriction.

Then there is plainly an infinite number of integers, which cannot be represented in the form

$$n=ax^2+by^2+cz^2+dt^2 \tag{4.43}$$

if $\mu_a \geq 1$. For it follows from (4.43) that

$$n \equiv 0 \pmod{\varpi}.$$

There is also an infinite number of integers, which cannot be represented in the form (4.43) if

$$\mu_a=0,\ \mu_b \geq 1.$$

For then it follows from (4.43), that

$$n \equiv ax^2 \pmod{\varpi},$$

so that the integer na', where

$$aa' \equiv 1 \pmod{\varpi},$$

must be a quadratic residu of ϖ.

Therefore we may suppose

$$\mu_a = \mu_b = 0, \ \mu_c \leq \mu_d.$$

We now substitute in the expression for

$$A_{\varpi^\lambda} = \varpi^{-4\lambda} \sum_p{}' S_{ap,\,\varpi^\lambda} S_{bp,\,\varpi^\lambda} S_{cp,\,\varpi^\lambda} S_{dp,\,\varpi^\lambda} e^{-\frac{2n\pi ip}{\varpi^\lambda}}$$

the explicit values of the Gaussian sums. Then, summing over λ, the following results can be obtained by straightforward calculation:

1°. If

$$\mu_c \geq 1, \ \mu_d \geq 2, \ \left(\frac{ab}{\varpi}\right) = (-1)^{\frac{\varpi+1}{2}},$$

the factor χ_ϖ vanishes for an infinity of values of n, in particular for

$$n = \varpi n_1, \ (n_1, \varpi) = 1$$

if $\mu_c \geq 2$ and for

$$n = \varpi n_1, \ (n_1, \varpi) = 1, \ \left(\frac{c_1 n_1}{\varpi}\right) = -1$$

if $\mu_c = 1$.

2°. If

$$\mu_c = \mu_d = 1, \ \left(\frac{ab}{\varpi}\right) = \left(\frac{c_1 d_1}{\varpi}\right) = (-1)^{\frac{\varpi+1}{2}},$$

we have

$$\chi_\varpi \sim \frac{K}{n},$$

if n runs through all powers of ϖ.

3°. For sets of values of a, b, c, d, different from those, mentioned in 1°. and 2°., we have

$$\chi_\varpi > K$$

for all values of n.

I shall not give the proofs, but I shall work out a proof only in a very special case. Let us suppose for example, that

$$\mu_c = 1, \ \mu_d = 2, \ \left(\frac{ab}{\varpi}\right) = (-1)^{\frac{\varpi+1}{2}}, \ n = \varpi n_1, \ (n_1, \varpi) = 1, \ \left(\frac{c_1 n_1}{\varpi}\right) = -1.$$

Then the wellknown formulae for the Gaussian sums give the following results (p is prime to ϖ).

$$S_{ap,\varpi} = \sqrt{\varpi}\left(\frac{ap}{\varpi}\right) i^{\frac{(\varpi-1)^2}{4}}, \ S_{bp,\varpi} = \sqrt{\varpi}\left(\frac{bp}{\varpi}\right) i^{\frac{(\varpi-1)^2}{4}},$$

$$S_{cp,\varpi} = \varpi, \ S_{dp,\varpi} = \varpi,$$

and therefore

$$(4.44) \qquad A_\varpi = \frac{1}{\varpi}\left(\frac{ab}{\varpi}\right)(-1)^{\frac{\varpi-1}{2}} c_\varpi(-n) = -\frac{1}{\varpi} c_\varpi(-n).$$

Again

$$S_{ap,\varpi^2} = \varpi, \ S_{bp,\varpi^2} = \varpi,$$

$$S_{cp,\varpi^2} = \varpi^{\frac{3}{2}}\left(\frac{c_1 p}{\varpi}\right) i^{\frac{(\varpi-1)^2}{4}}, \ S_{dp,\varpi^2} = \varpi^2.$$

Hence

$$(4.45) \qquad A_{\varpi^2} = \varpi^{-\frac{5}{2}}\left(\frac{c_1}{\varpi}\right) i^{\frac{(\varpi-1)^2}{4}} \sigma_{\varpi^2}(n),$$

if for positive integral values of α, we define

$$(4.46) \qquad \sigma_q(n) = \sum_p{}' \left(\frac{p}{\varpi}\right) \exp\left(-\frac{2n\pi i p}{q}\right), \ q = \varpi^\alpha.$$

In the same way we find for α odd ≥ 3

$$(4.47) \qquad A_{\varpi^\alpha} = -\varpi^{-2\alpha+\frac{3}{2}}\left(\frac{d_1}{\varpi}\right) i^{\frac{(\varpi-1)^2}{4}} \sigma_{\varpi^\alpha}(n)$$

and for α even ≥ 4

$$(4.48) \qquad A_{\varpi^\alpha} = \varpi^{-2\alpha+\frac{3}{2}}\left(\frac{d_1}{\varpi}\right) i^{\frac{(\varpi-1)^2}{4}} \sigma_{\varpi^\alpha}(n).$$

In (4.46) we write

$$p = p' + \nu\varpi \quad (p = 1, 2, \ldots \varpi - 1; \ \nu = 0, 1, 2, \ldots \varpi^{\alpha-1} - 1).$$

Then

$$\sigma_{\varpi^\alpha}(n) = \sum_{p'=1}^{\varpi-1} \left(\frac{p'}{\varpi}\right) e^{-\frac{2n\pi i p'}{\varpi^\alpha}} \sum_{\nu=0}^{\varpi^{\alpha-1}-1} e^{-\frac{2n\pi i\nu}{\varpi^{\alpha-1}}}$$

This is 0, unless $\alpha = 2$, in which case

$$\sigma_{\varpi^2}(n) = \varpi \sum_{p'=1}^{\varpi-1} \left(\frac{p'}{\varpi}\right) e^{-\frac{2n\pi i p'}{\varpi}} = \varpi\, S_{-n_1,\,\varpi} = \varpi^{\frac{3}{2}} \left(\frac{-n_1}{\varpi}\right) i^{\frac{(\varpi-1)^2}{4}}.$$

Therefore, if we combine this result with (4. 44), (4. 45), (4. 47) and (4. 48), we have

$$\chi_\varpi = 1 - \frac{1}{\varpi} c_\varpi(-n) + \frac{1}{\varpi}\left(\frac{-c_1 n_1}{\varpi}\right)(-1)^{\frac{\varpi-1}{2}} = 1 - \frac{\varpi - 1}{\varpi} - \frac{1}{\varpi} = 0.$$

The other results can be obtained in very much the same sort of way.

4. 45. The calculation of χ_2 is still more elaborate, than that of χ_ϖ (ϖ odd). I write

$$a = 2^{\mu_a} a_1,\ b = 2^{\mu_b} b_1,\ c = 2^{\mu_c} c_1,\ d = 2^{\mu_d} d_1 \quad (a_1,\ b_1,\ c_1,\ d_1 \text{ odd})$$

and I suppose

$$\mu_a \leq \mu_b \leq \mu_c \leq \mu_d,$$

which is not an essential restriction. Then, if $\mu_a \geq 1$ the form $ax^2 + by^2 + cz^2 + dt^2$ represents even integers only. Further if

$$\mu_a = 0,\ \mu_b \geq 2,$$

we have

$$ax^2 + by^2 + cz^2 + dt^2 \equiv ax^2 \pmod 4,$$

so that in this case the form $ax^2 + by^2 + cz^2 + dt^2$ does not represent integers which are $\equiv a + 2 \pmod{4}$. Therefore we may suppose

$$\mu_a = 0,\ \mu_b \leq 1.$$

Then we have the following results.

1°. The factor χ_2 vanishes for an infinity of values of n, if

$$\mu_a,\ \mu_b,\ \mu_c,\ \mu_d = 0,\ 1,\ 1,\ \geq 3;$$
$$0,\ 1,\ 2,\ \geq 4;$$
$$0,\ 1,\ \geq 3,\ \geq 3;$$
$$0,\ 0,\ \geq 2,\ \geq 2;$$
$$0,\ 0,\ 0,\ \geq 3 \text{ and } a \equiv b \equiv c \pmod 4.$$

2°. The factor χ_2 behaves for

$$n = 2^{\xi} n_1 \quad (n_1 \text{ odd})$$

as

$$\chi_2 \sim \frac{K}{2^{\xi}}$$

in the following cases:

$$\mu_a,\ \mu_b,\ \mu_c,\ \mu_d = 0,\ 1,\ 1,\ 2 \text{ and } a + d_1 \equiv b_1 + c_1 \equiv 4 \pmod 8 \text{ or}$$
$$b_1 + c_1 + 2a \equiv a + d_1 + 2b_1 \equiv 4 \pmod 8;$$
$$0,\ 1,\ 2,\ 3 \text{ and } b_1 + d_1 \equiv a + c_1 \equiv 4 \pmod 8 \text{ or}$$
$$b_1 + d_1 + 2a \equiv a + c_1 + 2b_1 \equiv 4 \pmod 8;$$
$$0,\ 0,\ 1,\ \text{odd and } a + b \equiv c_1 + d_1 \equiv 4 \pmod 8 \text{ or}$$
$$a + b + 2c_1 \equiv c_1 + d_1 + 2a \equiv 4 \pmod 8;$$
$$\left.\begin{matrix} 0,\ 0,\ 0,\ 0 \\ 0,\ 0,\ 0,\ 2 \end{matrix}\right\} \text{ and } a \equiv b \equiv c \equiv d_1 \pmod 4 \text{ and } a + b + c + d_1 \equiv 4 \pmod 8.$$

3°. In all other cases we have

$$\chi_2 > K > 0$$

for all values of n.

4. 6. If we now collect the results of the sections 4. 2, 4. 4, 4. 5 and combine them with the main-theorem, we find the following result.

If the set of positive integers a, b, c, d, is such that

1°. *It is not of the type stated in* 1°. *or* 2°. *of section 4. 5;*

2°. *There is no prime for which* 1°. *or* 2°. *of section 4. 4 is satisfied;*

3°. *There is no odd prime which divides three or four of the numbers a, b, c, d;*

4°. *At least one of the numbers a, b, c, d is odd;*

5°. *At least two of the numbers a, b, c, d are not divisable by 4;*

then

$$S(n) > \frac{K}{\log \log n} > 0$$

for sufficiently large values of n, so that we arrive at the conclusion that

$$r(n) \sim \frac{\pi^2}{\sqrt{\Delta}} n S(n).$$

In particular, there is only a finite number of integers, which cannot be represented in the form

$$ax^2 + by^2 + cz^2 + dt^2.$$

5. Problem P.

5. 1. It is now natural to ask, what can be said of the representation of integers by forms, which do not satisfy the conditions of the theorem just obtained, in particular whether there is an infinite or only a finite number of integers which can not be represented. One might expect, that there is an infinite number of exceptions in these cases. But that this can not always be true, is already shown by the simple remark, that the form $x^2 + y^2 + z^2 + t^2$, which represents all positive integers, falls under 2°. in section 4. 5. Yet a more detailed examination shows (as will appear later on) that generally there is an infinite number of exceptions in the cases still to be considered and that there is only a limited number of forms, which do not satisfy the conditions of the theorem of section 4. 6 and yet represent all positive integers with a finite number of exceptions at most.

Though the methods, by which these results can be obtained, are quite different from the analytical methods of this paper, I shall give a short account of them.

5. 2. If the coëfficients a, b, c, d are such, that they satisfy the conditions 1°. of section 4. 4 for some prime ϖ or if they satisfy one of the conditions 1°. of section 4. 5, then there is an infinite number of integers, which cannot be represented in the form $ax^2 + by^2 + cz^2 + dt^2$.

The proofs of these statements are quite simple. Let us suppose for instance, that c and d are divisable by ϖ^2, that a and b are prime to ϖ, and that

$$\left(\frac{ab}{\varpi}\right) = (-1)^{\frac{\varpi+1}{2}}.$$

Then the numbers

$$n_1\varpi,\ (n_1,\ \varpi)=1$$

cannot be represented in the form $ax^2+by^2+cz^2+dt^2$. To prove this, I shall show, that the supposition

(5. 21) $$n_1\varpi = ax^2 + by^2 + cz^2 + dt^2$$

leads to a contradiction.

In the first place it would follow from (5. 21), that y is prime to ϖ. For if y were not prime to ϖ, it would follow, that ax^2 must be divisable by ϖ and therefore also by ϖ^2, since $(a,\ \varpi)=1$. Therefore $ax^2+by^2+cz^2+dt^2$ would be divisable by ϖ^2, which is not true. Therefore y (and also x) must be prime to ϖ.

I now consider three cases separately.

1°. $\varpi\equiv 1 \pmod 4$. Then exactly one of the numbers a and b must be a quadratic residu of ϖ. We may suppose

(5. 22) $$\left(\frac{a}{\varpi}\right)=1,\ \left(\frac{b}{\varpi}\right)=-1.$$

Then we can determine numbers m and v by

$$m^2\equiv a \pmod{\varpi},\ vy\equiv mx \pmod{\varpi}.$$

Then

$$0\equiv ax^2 + by^2\equiv m^2x^2 + by^2\equiv y^2(v^2 + b) \pmod{\varpi}$$

and therefore, since $(y,\ \varpi)=1$

$$-b\equiv v^2 \pmod{\varpi}.$$

Therefore $-b$ would be a quadratic residu of ϖ. But then $+b$ would also be a quadratic residu of ϖ, since $\varpi\equiv 1 \pmod 4$, contrary to (5. 22).

2°. $\varpi\equiv 3 \pmod 4$ and

$$\left(\frac{a}{\varpi}\right)=\left(\frac{b}{\varpi}\right)=1.$$

Since a and b are quadratic residus of ϖ, they are also quadratic residus of ϖ^2. Therefore we can find integers m_1 and m_2 such that

$$a \equiv m_1^2 \pmod{\varpi^2}, \quad b \equiv m_2^2 \pmod{\varpi^2}.$$

Then

(5. 23) $$ax^2 + by^2 \equiv (m_1 x)^2 + (m_2 y)^2 \pmod{\varpi^2}.$$

The left hand side is divisable by ϖ (as follows from (5. 21)). Hence also the right hand side is divisable by ϖ. But a sum of two squares which is divisable by ϖ, is also divisable by ϖ^2 (since $\varpi \equiv 3 \pmod 4$). Therefore it would follow from (5. 23), that

$$ax^2 + by^2 \equiv 0 \pmod{\varpi^2}$$

and this is in contradiction with the supposition $(n_1, \varpi) = 1$.

3°. $\varpi \equiv 3 \pmod 4$ and

$$\left(\frac{a}{\varpi}\right) = \left(\frac{b}{\varpi}\right) = -1.$$

Then

$$\left(\frac{-a}{\varpi}\right) = \left(\frac{-b}{\varpi}\right) = 1.$$

Therefore we can apply the same argument as in the preceding case, if we determine numbers m_1 and m_2 such that

$$-a \equiv m_1^2 \pmod{\varpi^2}, \quad -b \equiv m_2^2 \pmod{\varpi^2}.$$

The other statements can be proved by similar methods. The precise results are as follows.

If (in the notation of section 4. 4)

$$c = \varpi c_1, \ (c_1, \varpi) = 1, \ \varpi^2 \mid d, \ (a, \varpi) = (b, \varpi) = 1, \ \left(\frac{ab}{\varpi}\right) = (-1)^{\frac{\varpi+1}{2}},$$

then the numbers

$$n_1 \varpi, \ (n_1, \varpi) = 1, \ \left(\frac{c_1 n_1}{\varpi}\right) = -1,$$

cannot be represented.

For the cases, mentioned in 1°. of section 4. 5, we have (in the notation of that section):

1°. If $\mu_a, \mu_b, \mu_c, \mu_d = 0, 1, 1 \geq 3$, then the numbers

$$n \equiv a + 4 \pmod 8 \quad \text{or} \quad n \equiv a + 2b_1 + 4 \pmod 8$$

can not be represented, according as

$$b_1 + c_1 \equiv 0 \pmod 4 \quad \text{or} \quad b_1 + c_1 \equiv 2 \pmod 4.$$

2°. If $\mu_a,\ \mu_b,\ \mu_c,\ \mu_d = 0,\ 1,\ 2, \geq 4$, then the numbers $2n_1$ can not be represented, where

$$n_1 \equiv b_1 + 4 \pmod 8 \quad \text{or} \quad n_1 \equiv b_1 + 2a + 4 \pmod 8,$$

according as

$$a + c_1 \equiv 0 \pmod 4 \quad \text{or} \quad a + c_1 \equiv 2 \pmod 4.$$

3°. If $\mu_a,\ \mu_b,\ \mu_c,\ \mu_d = 0,\ 1,\ \geq 3,\ \geq 3$, then the numbers

$$n \equiv a + 2b_1 + 4 \pmod 8$$

can not be represented.

4°. If $\mu_a,\ \mu_b,\ \mu_c,\ \mu_d = 0,\ 1,\ \geq 2,\ \geq 2$, then the numbers

$$2n_1 (n_1 \text{ odd}) \quad \text{or} \quad n \equiv a + 2 \pmod 4$$

can not be represented, according as

$$a + b \equiv 0 \pmod 4 \quad \text{or} \quad a + b \equiv 2 \pmod 4.$$

5°. If $\mu_a,\ \mu_b,\ \mu_c,\ \mu_d = 0,\ 0,\ 0,\ \geq 3$, and $a \equiv b \equiv c \pmod 4$, then the numbers

$$n \equiv a + b + c + 4 \pmod 8$$

can not be represented.

5. 3. We now consider the case 2°. of section 4. 4. Then we can prove the following result.

If $a,\ b,\ c_1,\ d_1$ are prime to ϖ,

$$c_1 > 1,\ d_1 > 1,\ \left(\frac{a\,b}{\varpi}\right) = \left(\frac{c_1\,d_1}{\varpi}\right) = (-1)^{\frac{\varpi+1}{2}},$$

and ξ is an odd positive integer, then ϖ^{ξ} can not be represented in the form

$$ax^2 + by^2 + \varpi(c_1 z^2 + d_1 t^2).$$

For the proof we shall require the following lemma, the proof of which can be left to the reader:

Lemma: *Let ϖ^λ be the highest power of ϖ, which divides $AX^2 + BY^2$, where A, B, X, Y are integers, A and B are prime to ϖ and*

$$\left(\frac{AB}{\varpi}\right) = (-1)^{\frac{\varpi+1}{2}}.$$

Then λ is even.

By means of this lemma we shall show, that the supposition

$$\varpi^\xi = ax^2 + by^2 + \varpi(c_1 z^2 + d_1 t^2) \tag{5. 31}$$

leads to a contradiction.

In the first place, if (5. 31) is true, it follows from the lemma, that

$$c_1 z^2 + d_1 t^2 \neq 0, \tag{5. 32}$$

since ξ is odd.

In the second place we shall prove

$$ax^2 + by^2 \neq 0. \tag{5. 33}$$

For if

$$ax^2 + by^2 = 0,$$

we would have

$$\varpi^{\xi-1} = c_1 z^2 + d_1 t^2.$$

Now let ϖ^μ be the highest power of ϖ, which divides z^2. Then $\xi - 1 > \mu$ (since $z \neq 0$, $t \neq 0$ in consequence of $c_1 > 1$, $d_1 > 1$). Hence

$$\varpi^{\xi-1-\mu} = c_1 z_1^2 + d_1 t_1^2 \equiv 0 \pmod{\varpi}$$

where

$$z_1^2 = \frac{z^2}{\varpi^\mu} \text{ and } t_1^2 = \frac{t^2}{\varpi^\mu}$$

are prime to ϖ. It is now easily proved, that the relation

$$c_1 z_1^2 + d_1 t_1^2 \equiv 0 \pmod{\varpi}$$

is in contradiction with the supposition

$$\left(\frac{c_1 d_1}{\varpi}\right) = (-1)^{\frac{\varpi+1}{2}}.$$

Hence (5. 33) is proved.

Now let ϖ^λ be the highest power of ϖ which divides $ax^2 + by^2$, so that λ is even. Then (5. 32) and (5. 33) give

$$\xi > \lambda.$$

Therefore we find from (5. 31)

$$\varpi^{\xi-\lambda} = \frac{ax^2 + by^2}{\varpi^\lambda} + \frac{c_1 z^2 + d_1 t^2}{\varpi^{\lambda-1}}.$$

This equation would imply, that $\varpi^{\lambda-1}$ were the highest power of ϖ which divides $c_1 z^2 + d_1 t^2$ and this is in contradiction with the lemma, since $\lambda - 1$ is odd. Hence the result, stated at the beginning of this section, is proved.

5. 4. There are similar results, if the conditions $c_1 > 1$, $d_1 > 1$ of the statement of 5. 3 are replaced by the following conditions:

1°. $c_1 = 1$, $d_1 \neq 1$, $d_1 \neq 2$.
2°. $c_1 = 1$, $d_1 = 2$, $\varpi \neq 5$.
3°. $c_1 = 1$, $d_1 = 2$, $\varpi = 5$.
4°. $c_1 = 1$, $d_1 = 1$, $\varpi \neq 3$.
5°. $c_1 = 1$, $d_1 = 1$, $\varpi = 3$, $a > 1$, $b > 1$.
6°. $c_1 = 1$, $d_1 = 1$, $\varpi = 3$, $a = 1$, $b > 1$.

In these six cases the numbers

$$2 \cdot \varpi^\xi,\ 5 \cdot \varpi^\xi,\ 7 \cdot 5^\xi,\ 3 \cdot \varpi^\xi,\ 3^{\xi+1},\ 2 \cdot 3^{\xi+1}$$

respectively (where ξ is an arbitrary positive odd integer) can not be represented in the form (5. 31), which can be proved by arguments, similar to those of section 5. 3.

If however

$$c_1 = 1,\ d_1 = 1,\ \varpi = 3,\ a = 1,\ b = 1,$$

we have the form

$$x^2 + y^2 + 3z^2 + 3t^2$$

and it has already been proved by LIOUVILLE, that this form represents all positive integers.

Hence:

If for some prime ϖ the condition 2°. of section 4. 4 is satisfied, then there is always an infinite system of integers, which can not be represented in the form $ax^2 + by^2 + cz^2 + dt^2$, unless this form is $x^2 + y^2 + 3z^2 + 3t^2$, in which case every positive integer can be represented.

5. 5. I now proceed to the case 2°. of section 4. 5 and I shall first consider the form (a, b, c_1, d_1 odd)

$$(5.\ 51) \qquad ax^2 + by^2 + 2(c_1 z^2 + d_1 t^2),\ a + b \equiv c_1 + d_1 \equiv 4 \pmod 8.$$

Here we have:

If α is odd ≥ 3, then 2^α can not be represented in the form (5. 51) *if* $c_1 > 1$, $d_1 > 1$.

The proof depends on the following lemma:

If A and B are odd, $A + B \equiv 4$ (mod 8) and 2^μ is the highest power of 2, which divides $AX^2 + BY^2$, then μ is even.

Further we have

If $c_1 = 1$; $d_1 \neq 3$, 11, 19; α odd ≥ 3, then $5 \cdot 2^\alpha$ can not be represented in the form (5. 51).

If $a > 1$, $b > 1$, α even ≥ 4, then 2^α can not be represented in the form (5. 51).

If $a = 1$; $b \neq 3$, 11, 19, α even ≥ 4, then $5 \cdot 2^\alpha$ can not be represented in the form (5. 51).

The proofs of these results are consequences of the lemma, stated at the beginning of this section.

We thus have eliminated all forms of type (5. 51) with the exception of the following nine forms.

$$x^2 + 3y^2 + 2z^2 + 6t^2,\quad x^2 + 11y^2 + 2z^2 + 6t^2,\quad x^2 + 19y^2 + 2z^2 + 6t^2,$$
$$x^2 + 3y^2 + 2z^2 + 22t^2,\ x^2 + 11y^2 + 2z^2 + 22t^2,\ x^2 + 19y^2 + 2z^2 + 22t^2,$$
$$x^2 + 3y^2 + 2z^2 + 38t^2,\ x^2 + 11y^2 + 2z^2 + 38t^2,\ x^2 + 19y^2 + 2z^2 + 38t^2.$$

Now it is well known, that $x^2 + 3y^2 + 2z^2 + 6t^2$ represents all positive integers.

Further it can be proved, that there is only a finite number of non-representable integers in the case of

$$x^2 + 3y^2 + 2z^2 + 22t^2,\ x^2 + 3y^2 + 2z^2 + 38t^2,$$
$$x^2 + 11y^2 + 2z^2 + 6t^2,\ x^2 + 19y^2 + 2z^2 + 6t^2.$$

Let us take as a typical case the form

$$x^2 + 2z^2 + 6t^2 + 11y^2.$$

We shall first prove, that all odd numbers, except 5, can be represented in this form. For every odd number, which is not of the form $8\mu + 5$, can be represented in the form $x^2 + 2z^2 + 6t^2$.[1] If $N = 8\mu + 5$, and $\mu \neq 0$, we take $y = 1$. Then $N - 11 = 8\mu - 6$, and this can again be represented in the form $x^2 + 2z^2 + 6t^2$. Hence, every odd number $N \neq 5$ can be represented in the form $x^2 + 2z^2 + 6t^2 + 11y^2$ and therefore also all numbers of the form $2^\alpha . N$ (for $2^\alpha . N$ is of the form $x^2 + 2z^2 + 6t^2$ if α is odd). Since $20 = 3^2 + 11 . 1^2$ it now follows, that 5 is the only number which is not of the form $x^2 + 2z^2 + 6t^2 + 11y^2$.

By means of a result, obtained by G. HUMBERT[2], it can be proved, that there is only a finite number of integers, which can not be represented in the form $x^2 + 11y^2 + 2z^2 + 22t^2$.

However, I have not been able to solve Problem P for the forms

(5. 52) $x^2 + 11y^2 + 2z^2 + 38t^2$, $x^2 + 19y^2 + 2z^2 + 38t^2$, $x^2 + 19y^2 + 2z^2 + 22t^2$.

The solution of problem P in the cases

$$\begin{aligned} \mu_a,\ \mu_b,\ \mu_c,\ \mu_d = {} & 0,\ 1,\ 1,\ 2 \text{ and } a + d_1 \equiv b_1 + c_1 \equiv 4 \pmod 8; \\ & 0,\ 1,\ 2,\ 3 \text{ and } b_1 + d_1 \equiv a + c_1 \equiv 4 \pmod 8; \\ & 0,\ 0,\ 1, \text{odd and } a + b \equiv c_1 + d_1 \equiv 4 \pmod 8) \end{aligned}$$

can be studied in the same way, but differs in no point from that of (5. 51).

5. 6. Next we consider the forms (a, b, c_1, d_1 odd)

(5. 61) $ax^2 + by^2 + 2(c_1z^2 + d_1t^2)$, $a + b + 2c_1 \equiv c_1 + d_1 + 2a \quad 4 \pmod 8$.

Then, if

$$2^\xi n_1 = ax^2 + by^2 + 2(c_1z^2 + d_1t^2),\ \xi \geq 4,\ n_1 \text{ odd},$$

then also $2^{\xi-2}n_1$ can be represented in the same form. From this property the results of the following table can be deduced. In the second column I have written non-representable numbers, if the conditions of the first column are satisfied.

[1] S. RAMANUJAN, On the expression of a number in the form $ax^2 + by^2 + cz^2 + dt^2$, Proc. Camb. Phil. Soc. 19 (1917), footnote on p. 14.

[2] Comptes Rendus, Paris, 170 (1920), 354.

$a > 1;\ b > 1;\ c_1 > 1;\ d_1 > 1,\qquad 2^\xi (\xi \text{ even});$

$a = 1;\ c_1 \neq 1,\ 3,\ 5;\ d_1 \neq 1,\ 3,\ 5,\qquad 3.\ 2^\xi\ (\xi \text{ even});$

$a = 1;\ c_1 = 5;\ b \neq 1,\qquad 3.\ 2^\xi\ (\xi \text{ even});$

$a = 1;\ c_1 = 5;\ b = 1;\ d_1 \neq 5,\qquad 3.\ 2^\xi\ (\xi \text{ odd});$

$a = 1;\ c_1 = 3,\qquad 2^\xi\ (\xi \text{ odd});$

$a = 1;\ c_1 = 1;\ d_1 \neq 1,\ 9;\ b \neq 1,\ 9,\ 17,\qquad 5.\ 2^\xi (\xi \text{ even});$

$a = 1;\ c_1 = 1;\ d_1 = 1,\ 9;\ b \neq 1,\ 9,\ 17,\ 25,\qquad 7.\ 2^\xi\ (\xi \text{ even});$

$a = 1;\ c_1 = 1;\ b = 1, 9, 17;\ d_1 \neq 1, 9, 17, 25,\qquad 7.\ 2^\xi\ (\xi \text{ odd}).$

We thus have eliminated all forms of the type (5. 61) with the exception of the following 15 forms:

$\{a,\ b,\ c,\ d\} = \{1,\ 1,\ 10,\ 10\},\ \{1,\ 2,\ 2,\ 9\},$
$\{1,\ 2,\ 2,\ 17\},\ \{1,\ 2,\ 2,\ 25\},\ \{1,\ 1,\ 2,\ 18\},$
$\{1,\ 2,\ 9,\ 18\},\ \{1,\ 2,\ 17,\ 18\},\ \{1,\ 2,\ 18,\ 25\},$
$\{1,\ 1,\ 2,\ 34\},\ \{1,\ 1,\ 2,\ 50\},\ \{1,\ 2,\ 9,\ 34\},$
$\{1,\ 2,\ 9,\ 50\},\ \{1,\ 2,\ 17,\ 50\};$
$\{1,\ 1,\ 2,\ 2\};$
$\{1,\ 2,\ 17,\ 34\}.$

It can be proved by the method used by Ramanujan in his paper already referred to, that the first 13 of these forms represent all positive integers. However, I have not been able to solve Problem P for the form

$$x^2 + 2y^2 + 17z^2 + 34t^2. \qquad (5.\ 61)$$

The solution of Problem P for the cases

$\mu_a,\ \mu_b,\ \mu_c,\ \mu_d = 0,\ 1,\ 1,\ 2$ and $b_1 + c_1 + 2a \equiv a + d_1 + 2b_1 \equiv 4 \pmod 8;$
$0,\ 1,\ 2,\ 3$ and $b_1 + d_1 + 2a \equiv a + c_1 + 2b_1 \equiv 4 \pmod 8;$
$0,\ 0,\ 1$ *odd* and $a + b + 2c_1 \equiv c_1 + d_1 + 2a \equiv 4 \pmod 8$

differs in no point from that of (5. 61).

5. 7. The remaining forms to be considered are (a, b, c, d odd)

(5. 71) $ax^2 + by^2 + cz^2 + dt^2$, $a \equiv b \equiv c \equiv d \pmod 4$, $a + b + c + d \equiv 4 \pmod 8$,

and (a, b, c, d_1 odd)

(5. 72) $ax^2+by^2+cz^2+4d_1t^2$, $a \equiv b \equiv c \equiv d_1 \pmod 4$, $a + b + c + d_1 \equiv 4 \pmod 8$.

I shall first consider the form (5. 71). Then if

$$2^{\xi} n_1 = ax^2 + by^2 + cz^2 + dt^2, \ \xi \geq 3, \ n_1 \text{ odd},$$

then also $2^{\xi-2}n_1$ is representable in the same form. From this property the results of the following table can be deduced. In the second column I have written, as before, non representable numbers, if the conditions of the first column are satisfied.

$a > 1;\ b > 1;\ c > 1;\ d > 1;$	2^{ξ} (ξ odd);
$a = 1;\ b > 1;\ c > 1;\ d > 1;$	2^{ξ} (ξ odd);
$a = 1;\ b = 1;\ c \neq 1,\ 5;\ d \neq 1,\ 5;$	$3.\ 2^{\xi}$ (ξ odd);
$a = 1;\ b = 1;\ c = 1;\ d \neq 1, 9, 17, 25$	$7.\ 2^{\xi}$ (ξ even);
$a = 1;\ b = 1;\ c = 5;\ d \neq 5;$	$3.\ 2^{\xi}$ (ξ even).

We thus have eliminated all forms of the type (5. 71) with the exception of the following forms:

$$x^2 + y^2 + z^2 + dt^2, \ (d = 1,\ 9,\ 17,\ 25), \ x^2 + y^2 + 5z^2 + 5t^2.$$

The form $x^2 + y^2 + z^2 + t^2$ represents all integers and it can easily be proved that the others have a finite number of exceptions only.

Similarly, considering the case (5. 72), the forms

$$x^2 + y^2 + z^2 + dt^2 \ (d = 36,\ 68,\ 100)$$
$$x^2 + y^2 + 4z^2 + dt^2 \ (d = 9,\ 17,\ 25)$$
$$x^2 + y^2 + 5z^2 + 20t^2$$

have a finite number of exceptions and the form $x^2 + y^2 + 4z^2 + 4t^2$ represents all integers. The remaining forms of the type (5. 72) have an infinite number of exceptions.

5. 8. Final remarks.

5. 81. The preceding pages contain the solution of Problem P for all forms $ax^2 + by^2 + cz^2 + dt^2$ with the exception of (5. 52) an (5. 61) and some other forms, related to these.

5. 82. It has been stated by WARING[1], that $ax^2 + by^2 + cz^2 + dt^2$ represents every integer exceeding an assignable one, if a, b, c and d are relatively prime. The preceding pages show that this statement is incorrect.

[1] Meditationes algebraicae, Cambridge, ed. 3, 1782, 349.