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Compactness-like operator properties preserved by complex interpolation

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Some history

The well-known Riesz–Thorin interpolation theorem states that if T is a bounded linear operator from $L^{p_0}(\mu)$ to $L^{q_0}(\nu)$ with norm M_0 and if T is a bounded linear operator from $L^{p_1}(\mu)$ to $L^{q_1}(\nu)$ with norm M_1 (where the four exponents are contained in the closed interval $[1,\infty]$), then T is a bounded linear operator from $L^p(\mu)$ to $L^q(\nu)$ with norm $M \le M_0^{1-\theta} M_1^{\theta}$ provided that $0<\theta<1$ and

$$\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}, \quad \frac{1}{q}=\frac{1-\theta}{q_0}+\frac{\theta}{q_1}.$$

In the early 1960's Calderón [6] extended the ideas of Thorin's 1938 proof of this theorem, creating, for any pair of Banach spaces satisfying certain conditions, a continuous scale of intermediate spaces. Let us denote the endpoint spaces by X_0 and X_1, and the so constructed intermediate spaces by X_t, for $t\in(0,1)$. Calderón showed, among other things, that if T is a bounded linear operator on X_0 and T is a bounded linear operator on X_1 then T is a bounded linear operator on X_t for all $t\in(0,1)$. Boundedness is thus a *two-sided interpolation property.* What further properties of bounded linear operators are two-sided interpolation properties? What properties are *one-sided interpolation properties*? One-sided interpolation properties are properties such that if T has the property on X_0, then T has the property on X_t for all $t\in[0,1)$; generally, these properties do not extend to the entire closed interval $[0,1]$. Of course, these questions are also interesting when asked about the operators on the spaces constructed via other interpolation methods. One positive answer that is of interest here is due to Cwikel. In 1992 [8] he proved that compactness is a one-sided interpolation property for the real method of Lions and Peetre [17]; this improved the 1969 result of Hayakawa [13] that compactness is a two-sided interpolation property for the real method.

The starting point for this paper is: it is unknown as to whether or not compactness is a one-sided interpolation property for the complex method, or even whether or not it is a two-sided interpolation property for the complex method.

This question has been studied by several authors, and positive partial results have been established. The first such result is due to Krasnosel'skiĭ [15] who proved that compactness is a one-sided interpolation property if $X_0=L^{p_0}$ and $X_1=L^{p_1}$. More recently, M^cCarthy [19] has proved that compactness is a one-sided interpolation property with the strong restriction that both endpoint spaces are separable Hilbert spaces. The way he does this is by showing that the intermediate spaces that arise in this case via the complex method on the one hand, and via the real method (with second parameter set equal to 2) on the other hand, are in fact isomorphic; he then applies Cwikel's real method result. M^cCarthy's result also follows from the work found in [20]. For an up to date account of what weaker conditions on the endpoint spaces lead to compactness being preserved under complex interpolation, we refer the reader to [9], [10] and [18]. Finally, we point out that Cwikel (again in [8]) has proved the related, and very interesting, complex extrapolation result which says that if T acts compactly on any single intermediate space then T must act compactly on every intermediate space.

The positive results about Calderón interpolation of compact operators mentioned in the preceding paragraph all have been established for Banach spaces satisfying certain conditions. One reason that a compact operator is tractable, and hence appealing, is because of the nature of its spectrum; it is either finite or it is a sequence tending to zero. (This result about spectra is part of F. Riesz's classical theory of compact operators; for a modern account see, for example, p. 301 of [16].) We let the Banach spaces be general, and instead consider operators that have the spectral properties of compact operators (but are not necessarily compact). Our approach is to view questions of operator interpolation in a Banach algebraic setting.

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The complex interpolation algebra

In this section we introduce what we refer to as the *interpolation algebra* $\mathcal{I}[X_0, X_1]$. The basic idea of this algebra is simple and natural and has been defined in the L^p-space situation by Barnes in [3]. For the remainder of this paper, X_0 and X_1 will denote Banach spaces. Each of these is continuously embedded in the complex Hausdorff topological vector space $X\equiv X_0+X_1$, when X is endowed with the usual norm

$$\|x\|_+ \equiv \inf\{\|x\|_0+\|x\|_1 : x=x_0+x_1,\ x_0\in X_0,\ x_1\in X_1\}.$$

In other words, $[X_0, X_1]$ is an interpolation pair in the sense of Calderón (see [6]). Since the embeddings into X are continuous, there are positive constants M_0 and M_1 such that $\|\cdot\|_X \le M_0\|\cdot\|_{X_0}$ and $\|\cdot\|_X \le M_1\|\cdot\|_{X_1}$. We assume that $X_0\cap X_1$ is dense in X_j, $j=0,1$. For an arbitrary Banach space Y, let $\mathcal{B}(Y)$ denote the Banach algebra of all bounded linear operators on Y. We say that an operator is *j-continuous* if it is continuous in the norm on X_j. Notice that $X_0\cap X_1$ is a Banach space with respect to the norm

$$\|x\|_\cap \equiv \max\{\|x\|_{X_0},\ \|x\|_{X_1}\}.$$

Let $\mathcal{I}[X_0, X_1]$ denote the set of all linear operators $T: X_0\cap X_1\to X_0\cap X_1$ that are both 0-continuous and 1-continuous. If T is an element of $\mathcal{I}[X_0, X_1]$ then T has unique extensions $T_0\in\mathcal{B}(X_0)$ and $T_1\in\mathcal{B}(X_1)$. Let $X_t=[X_0, X_1]_t$ be the interpolation space obtained by Calderón's complex method. Then $T\in\mathcal{I}[X_0, X_1]$ induces $T_t\in \mathcal{B}(X_t)$ with

$$\|T_t\|_{\mathcal{B}(X_t)} \le \|T_0\|_{\mathcal{B}(X_0)}^{1-t}\|T_1\|_{\mathcal{B}(X_1)}^{t}, \quad t\in(0,1).$$

Our first observation is one about the structure of the collection of interpolation operators; it says that it forms a Banach algebra. This algebra first appeared in the context of our study of the function $t\mapsto\sigma(T_t)$ (see [4] and [14]). For example, as we shall see, it leads to a nice bound for the intermediate operators' spectra in terms of the endpoint operators' spectra. This problem, of finding bounds for the intermediate spectra in terms of the endpoint spectra, has been studied by several authors; see, for example, [21] and [23].

Proposition 1. *The set $\mathcal{I}[X_0, X_1]$ is a Banach algebra in the norm*

$$\|T\|_{\mathcal{I}[X_0,X_1]} \equiv \max\{\|T_0\|_{\mathcal{B}(X_0)},\ \|T_1\|_{\mathcal{B}(X_1)}\}.$$

Proof. The proof is elementary. To show some work, we prove that the norm is complete. Suppose that $\{T_n\}_{n=1}^\infty$ is a Cauchy sequence in $\mathcal{I}[X_0, X_1]$. Then, for $\varepsilon>0$,

$$\|T_n-T_m\|_{\mathcal{I}[X_0,X_1]} \equiv \max\{\|(T_n)_0-(T_m)_0\|_{\mathcal{B}(X_0)},\ \|(T_n)_1-(T_m)_1\|_{\mathcal{B}(X_1)}\}<\varepsilon$$

for sufficiently large n and m. Thus there exist $T\in\mathcal{B}(X_0)$ and $S\in\mathcal{B}(X_1)$ such that

$$\|(T_n)_0-T\|_{\mathcal{B}(X_0)}<\varepsilon \quad\text{and}\quad \|(T_n)_1-S\|_{\mathcal{B}(X_1)}<\varepsilon$$

for sufficiently large n. Let $x\in X_0\cap X_1$. Then Tx and Sx both make sense and, since X_0 and X_1 are each continuously embedded in X,

$$\begin{aligned}\|Tx-Sx\|_X &\le \|Tx-T_nx\|_X+\|T_nx-Sx\|_X\\ &\le M_0\|Tx-T_nx\|_{X_0}+M_1\|T_nx-Sx\|_{X_1}\\ &\le M_0\|x\|_{X_0}\varepsilon+M_1\|x\|_{X_1}\varepsilon\end{aligned}$$

for sufficiently large n. Since ε was arbitrary, $Tx=Sx$. We conclude that $T=S: X_0\cap X_1\to X_0\cap X_1$ is both 0-continuous and 1-continuous, hence is in $\mathcal{I}[X_0,X_1]$, and that $\|T_n-T\|_{\mathcal{I}[X_0,X_1]}\to 0$ as $n\to\infty$. □

For $T\in\mathcal{I}[X_0,X_1]$ let $\sigma(T)\equiv\sigma_{\mathcal{I}[X_0,X_1]}(T)$ and $\sigma(T_t)\equiv\sigma_{\mathcal{B}(X_t)}(T_t)$. Also, $T_{0,1}$ will denote the operator considered as an element of $\mathcal{B}(X_0\cap X_1)$, and $\sigma(T_{0,1})\equiv\sigma_{\mathcal{B}(X_0\cap X_1)}(T_{0,1})$.

For a compact set $K\subseteq\mathbf{C}$, we let $K\hat{}$ denote its polynomially convex hull. That is,

$$K\hat{}\equiv\left\{z\in\mathbf{C}:|p(z)|\le\sup_{\xi\in K}|p(\xi)| \text{ for all polynomials } p\right\}.$$

This set is exactly the complement of the (unique) unbounded component of $\mathbf{C}\setminus K$; that is, $K\hat{}$ is K together with all of its 'holes'. Note that if K and J are compact subsets of $\mathbf{C}$ with $\partial K\subseteq J\subseteq K$, then $K\hat{}=J\hat{}$. For a discussion of basic properties of polynomially convex hulls one can read [11], starting on p. 39.

Theorem 2. *For $T\in\mathcal{I}[X_0,X_1]$,*

$$\bigcup_{t\in[0,1]}\sigma(T_t)\subseteq\sigma(T)=\sigma(T_0)\cup\sigma(T_1)\cup\sigma(T_{0,1}). \tag{1}$$

Further,

$$\partial\sigma(T_{0,1})\subseteq\sigma(T_0)\cup\sigma(T_1) \tag{2}$$

and hence

$$\left[\bigcup_{t\in[0,1]}\sigma(T_t)\right]\hat{}=\sigma(T)\hat{}=[\sigma(T_0)\cup\sigma(T_1)]\hat{}. \tag{3}$$

Proof. Since $\mathcal{I}[X_0,X_1]$ can be viewed as a subalgebra of $\mathcal{B}(X_t)$, $\sigma(T_t)\subseteq\sigma(T)$ for all $t\in[0,1]$. Next we show that $\sigma(T)=\sigma(T_0)\cup\sigma(T_1)\cup\sigma(T_{0,1})$. This fact and the following proof of it are generalizations of what is found in [3, Theorem 5.1]. Assume that $\lambda\notin\sigma(T)$. We assume further that $\lambda=0$. Then T has an inverse $T^{-1}\in\mathcal{I}[X_0,X_1]$

with extensions $(T^{-1})_0 \in \mathcal{B}(X_0)$ and $(T^{-1})_1 \in \mathcal{B}(X_1)$. For $x \in X_0 \cap X_1$, $T_0 x \in X_0 \cap X_1$ and so

$$(T^{-1})_0 T_0 x = T^{-1} T x = x.$$

So T_0 is invertible; that is, $\lambda \notin \sigma(T_0)$. Likewise, $\lambda \notin \sigma(T_1)$. Also, T^{-1} is a set inverse for $T_{0,1}$ and so we just need to see that it is continuous with respect to the norm on $X_0 \cap X_1$. Well,

$$\begin{aligned}\|T^{-1}x\|_{X_0\cap X_1} &\equiv \max\{\|(T^{-1})_0 x\|_{X_0},\ \|(T^{-1})_1 x\|_{X_1}\}\\ &\le \max\{\|(T^{-1})_0\|\,\|x\|_{X_0},\ \|(T^{-1})_1\|\,\|x\|_{X_1}\}\\ &\le \max\{\|(T^{-1})_0\|,\ \|(T^{-1})_1\|\}\max\{\|x\|_{X_0},\ \|x\|_{X_1}\}\\ &\equiv \|T^{-1}\|\,\|x\|_{X_0\cap X_1}.\end{aligned}$$

Thus T^{-1} is continuous with respect to the norm on $X_0 \cap X_1$ and hence $\lambda \notin \sigma(T_{0,1})$. This shows one inclusion.

To see the other inclusion, assume that T_0, T_1, and $T_{0,1}$ have inverses. Then for each $x \in X_0 \cap X_1$,

$$(T_0)^{-1}x = (T_1)^{-1}x = (T_{0,1})^{-1}x.$$

So $(T_{0,1})^{-1}$ is a map from $X_0 \cap X_1$ into itself that is 0-continuous and 1-continuous and is therefore the inverse of T in $\mathcal{I}[X_0, X_1]$. This completes the proof of (1).

It is a standard result that $\partial\sigma(T_{0,1})$ is contained in the approximate point spectrum of $T_{0,1}$ (see [16, Theorem 4.1] for example). Now assume that λ is an element of the approximate point spectrum of $T_{0,1}$. Then there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in $X_0 \cap X_1$ such that $\|x_n\|_{X_0\cap X_1} = 1$ and $\|(\lambda - T_{0,1})x_n\|_{X_0\cap X_1} \to 0$ as $n \to \infty$. We may assume that $\|x_n\|_{X_0} = 1$ for all n (since either $\|x_n\|_{X_0} = 1$ for an infinite number of n's, or $\|x_n\|_{X_1} = 1$ for an infinite number of n's). Then

$$\|(\lambda - T_0)x_n\|_{X_0} = \|(\lambda - T_{0,1})x_n\|_{X_0} \le \|(\lambda - T_{0,1})x_n\|_{X_0\cap X_1} \to 0,$$

as $n \to \infty$, showing that λ is an element of the approximate point spectrum of T_0 and hence is an element of $\sigma(T_0) \subseteq \sigma(T_0) \cup \sigma(T_1)$. This proves (2).

From (2) and (1) we see that

$$\partial\sigma(T) \subseteq \sigma(T_0) \cup \sigma(T_1) \subseteq \bigcup_{t\in[0,1]} \sigma(T_t) \subseteq \sigma(T)$$

and, from the note directly preceding the statement of the theorem, (3) follows. □

We end this section by pointing out that in

$$\bigcup_{t\in[0,1]} \sigma(T_t) \subseteq \sigma(T) = \sigma(T_0) \cup \sigma(T_1) \cup \sigma(T_{0,1}),$$

the set $\sigma(T_{0,1})$ is necessary. There are several examples in the literature which show this; for example we direct the reader's attention to the now classic Cesaro example of Boyd given in [5].

Applications of the interpolation algebra to operator interpolation

As stated before, it is unknown whether or not compactness is a two-sided interpolation property for the complex method. However, as our next result will show, it does follow from Theorem 2, that if T_0 and T_1 are compact operators then the spectrum of T_t for all values of $t\in[0,1]$ has the same properties that the spectrum of a compact operator is known to have; it is either finite or it is a sequence tending to zero.

Theorem 3. *Suppose that T_0 and T_1 are operators in $\mathcal{B}(X_0)$ and $\mathcal{B}(X_1)$ respectively, and that each of these operators has spectrum that is either finite or is a sequence tending to zero. Then the spectrum of the operator T_t in $\mathcal{B}(X_t)$, for all values of $t\in[0,1]$, is also of this type (is either finite or is a sequence tending to zero).*

Proof. Because $\sigma(T_0)$ and $\sigma(T_1)$ are both countable, so by (2) is $\partial\sigma(T_{0,1})$. Consequently, $\partial\sigma(T_{0,1})=\sigma(T_{0,1})$ and thus

$$\sigma(T_{0,1})\subseteq\sigma(T_0)\cup\sigma(T_1).$$

The first part of Theorem 2 in turn implies that

$$\bigcup_{t\in[0,1]}\sigma(T_t)\subseteq\sigma(T_0)\cup\sigma(T_1).$$

The result is now immediate. □

The same reasoning of proof of Theorem 3 clearly shows that quasi-nilpotency (T is *quasi-nilpotent* if $\|T^n\|^{1/n}\to 0$ as $n\to\infty$) is a two-sided interpolation property; Corollary 5 will improve on this.

The next theorem was first proved for l^p-spaces by Halberg [12] in 1956. We make minor modifications to Halberg's proof and present this theorem for any interpolation couple of Banach spaces.

We recall the basics about 'reiteration' of interpolation. If $0\leq x<t<y\leq 1$ let $\alpha\in[0,1]$ be such that $t=(1-\alpha)x+\alpha y$. The reiteration theorem states that

$$[X_x,X_y]_\alpha=X_t$$

with equality of norms. This theorem was proved by Calderón [6] in 1964 with an assumption on the spaces; he required that $X_0 \cap X_1$ be dense in $X_x \cap X_y$. In 1978 Cwikel [7] showed that this hypothesis was unnecessary. By equality of norms, we have

$$\|T_t\|_{\mathcal{B}(X_t)} \leq \|T_x\|_{\mathcal{B}(X_x)}^{1-\alpha} \|T_y\|_{\mathcal{B}(X_y)}^{\alpha}, \quad T \in \mathcal{I}[X_0, X_1], \tag{4}$$

whenever t, x, y and α are related as above. Then, for each n, we have that

$$\|(T^n)_t\| \leq \|(T^n)_x\|^{1-\alpha} \|(T^n)_y\|^{\alpha}, \quad T \in \mathcal{I}[X_0, X_1], \tag{5}$$

and hence that

$$r_{\mathcal{B}(X_t)}(T_t) \leq (r_{\mathcal{B}(X_x)}(T_x))^{1-\alpha} (r_{\mathcal{B}(X_y)}(T_y))^{\alpha}. \tag{6}$$

In this paper, $r_{\mathcal{A}}(a)$ denotes the spectral radius of an element a in a Banach algebra $\mathcal{A}$.

Theorem 4. *For* $T \in \mathcal{I}[X_0, X_1]$ *the function* $t \mapsto r_{\mathcal{B}(X_t)}(T_t)$ *is continuous on the open interval* $(0,1)$.

Proof. Case 1. Suppose that $r_{\mathcal{B}(X_x)}(T_x)=0$ for some $x \in [0,1]$. Let $t \in (0,1)$, $t \neq x$ be arbitrary. We may assume that $x<t$. Let y be such that $0<t<y\leq 1$. Let $\alpha \in (0,1)$ be such that $t=(1-\alpha)x+\alpha y$. By inequality (6)

$$r_{\mathcal{B}(X_t)}(T_t) \leq (r_{\mathcal{B}(X_x)}(T_x))^{1-\alpha} (r_{\mathcal{B}(X_y)}(T_y))^{\alpha} = 0.$$

We conclude that if $r_{\mathcal{B}(X_x)}(T_x)=0$ for some $x \in [0,1]$ then $r_{\mathcal{B}(X_t)}(T_t)$ is constantly zero and hence continuous on $(0,1)$.

Case 2. Now suppose that $r_{\mathcal{B}(X_t)}(T_t) \neq 0$ for all $t \in (0,1)$. Fix n and define

$$\phi_n(t) \equiv \frac{1}{n} \log(\|(T^n)_t\|_{\mathcal{B}(X_t)})$$

for $t \in (0,1)$. From (5) we see that ϕ_n is convex. Since $r_{\mathcal{B}(X_t)}(T_t) \neq 0$ for all $t \in (0,1)$,

$$\phi(t) \equiv \lim_{n\to\infty} \phi_n(t) = \log(r_{\mathcal{B}(X_t)}(T_t))$$

is finite valued for all $t \in (0,1)$. It follows that ϕ is convex and hence continuous on $(0,1)$. This shows that $r_{\mathcal{B}(X_t)}(T_t)=e^{\phi(t)}$ is continuous on $(0,1)$. □

Corollary 5. *Quasi-nilpotentency is a one-sided interpolation property. In fact, a stronger result holds, that if T_x is quasi-nilpotent for some $x \in [0,1]$ then T_t is quasi-nilpotent for all $t \in (0,1)$.*

Proof. The corollary follows immediately from the proof of the first part of Theorem 4. □

This property of quasi-nilpotentency may be useful when considering interpolation of spectral operators. It also may be of interest for the connection between quasi-nilpotent operators and invariant subspace theorems on Banach lattices (as a good reference see [1]).

We are interested in the still open question of continuity of the function $t \mapsto \sigma(T_t)$ from $(0,1)$ to the collection of all compact subsets of the plane endowed with the Hausdorff metric topology (see [14]). We thus include the following corollary to Theorem 4.

Corollary 6. *Assume that $T \in \mathcal{I}[X_0, X_1]$ and that $\lambda - T$ is invertible in the Banach algebra $\mathcal{I}[X_0, X_1]$ for some value of $\lambda \in \mathbf{C}$. Then the function $t \mapsto d(\lambda, \sigma(T_t))$ is continuous on the open interval $(0,1)$.*

Proof. In any Banach algebra $\mathcal{A}$ the spectral mapping theorem implies that

$$d(\mu, \sigma_{\mathcal{A}}(a)) = \frac{1}{r_{\mathcal{A}}[(\mu - a)^{-1}]}$$

for any $a \in \mathcal{A}$ and $\mu \notin \sigma_{\mathcal{A}}(a)$ (see, for example, [2, Theorem 3.3.5]). By hypothesis and the fact that $\sigma(T_t) \subseteq \sigma(T)$ for all $t \in [0,1]$, $\lambda \notin \sigma(T_t)$ for all $t \in (0,1)$. Therefore, for $t \in (0,1)$,

$$d(\lambda, \sigma(T_t)) = \frac{1}{r_{\mathcal{B}(X_t)}[(\lambda - T_t)^{-1}]}.$$

Since $(\lambda - T_t)^{-1} = [(\lambda - T)^{-1}]_t$, we have that

$$d(\lambda, \sigma(T_t)) = \frac{1}{r_{\mathcal{B}(X_t)}(S_t)}$$

where $S \equiv (\lambda - T)^{-1}$ in $\mathcal{I}[X_0, X_1]$. Since $t \mapsto r_{\mathcal{B}(X_t)}(S_t)$ is continuous (and never zero) on $(0,1)$, the result follows. □

We end with an observation related to a question in [23]. In that paper, in an effort to provide bounds for the intermediate operators' spectra in terms of the endpoint operators' spectra, Stafney defines the set

$$H \equiv \{\lambda \in \mathbf{C} : (\lambda - T_0)^{-1} \in \mathcal{B}(X_0),\ (\lambda - T_1)^{-1} \in \mathcal{B}(X_1) \\ \text{and } (\lambda - T_0)^{-1}|_{X_0 \cap X_1} = (\lambda - T_1)^{-1}|_{X_0 \cap X_1}\},$$

and shows that each intermediate spectrum is contained in the complement of H. He further proves that the complement of this set ($\mathbf{C}\backslash H$) consists of $\sigma(T_0)\cup\sigma(T_1)$ together with (perhaps) a subcollection of bounded components of the complement ('holes') of $\sigma(T_0)\cup\sigma(T_1)$, and asks whether or not $\mathbf{C}\backslash H$ is in fact always equal to $\sigma(T_0)\cup\sigma(T_1)$. Our Theorem 2 gives information about $\mathbf{C}\backslash H$. First, it says that $\mathbf{C}\backslash H$ is in fact a Banach algebra spectrum (of T in the Banach algebra $\mathcal{I}[X_0, X_1]$); second, that the missing holes are contained in the spectrum of the operator considered as an element of $\mathcal{B}(X_0\cap X_1)$. As Boyd's example shows (see the comment following Theorem 2), the answer to Stafney's question is hence negative.

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