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The H^p corona theorem in analytic polyhedra

Jörgen Boo(1)

Abstract. The H^p corona problem is the following: Let $g_1, \dots, g_m$ be bounded holomorphic functions with $0<\delta\leq\sum|g_i|$. Can we, for any H^p function φ, find H^p functions $u_1, \dots, u_m$ such that $\sum g_i u_i=\varphi$? It is known that the answer is affirmative in the polydisc, and the aim of this paper is to prove that it is in non-degenerate analytic polyhedra. To prove this, we construct a solution using a certain integral representation formula. The H^p estimate for the solution is then obtained by localization and some harmonic analysis results in the polydisc.

1. Introduction and statement of the result

The H^p corona problem is the following: Given $g=(g_1, \dots, g_m)\in H^\infty$ such that $0<\delta\leq\sum|g_i|$, can we for all $\varphi\in H^p$ find $u=(u_1, \dots, u_m)\in H^p$ such that $g_1u_1+\dots+g_mu_m=\varphi$? If $p=\infty$, this is the true corona problem; to find bounded functions u_i. If the corona problem is solvable, then the H^p corona problem is solvable, too, because if we solve the corona problem $g_1v_1+\dots+g_mv_m=1$, then the functions $u_i=v_i\varphi$ will solve the H^p corona problem $g_1u_1+\dots+g_mu_m=\varphi$. In one variable, the converse is also true—if we can solve the H^p corona problem, then the corona problem is also solvable, see [An3]. Hence the corona problem and the H^p corona problem are equivalent when $n=1$. This is not true in higher dimensions.

In the unit disc, $n=1$, the corona problem is solvable; this is the classical result of Carleson. When $n>1$, the corona problem is in general not possible to solve, not even in smooth pseudoconvex domains, see for example [FS] or [S]. It is not known if the corona problem is solvable in strictly pseudoconvex domains, or even in the unit ball of $\mathbf{C}^n$. This leads to studying the H^p corona problem for $p<\infty$ instead, and this was originally done in [An1]. For $p<\infty$, some positive results are known. In [Am], [An1], [An2], [An3], [AC1], [AC2], the H^p corona problem is solved under

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different conditions. The problem is solved for $0<p\leq 2$ in a large class of weakly pseudoconvex domains and for all $p<\infty$ in strictly pseudoconvex domains. The subject of this paper is to solve the H^p corona problem in non-degenerate analytic polyhedra when $1<p<\infty$; this is a generalization of the results in [Li], [L2], where the problem is solved in the polydisc.

Definition 1. A bounded domain $\Omega\subset\mathbf{C}^n$ is an *analytic polyhedron* with N defining functions f_i if

$$\Omega=\{z\in\mathbf{C}^n:|f_i(z)|<1,\ i=1,\dots,N\},$$

where the defining functions are holomorphic in some neighbourhood of $\bar{\Omega}$. Its skeleton σ is the part

$$\sigma=\bigcup_{1\leq I_1<\dots<I_n\leq N}\sigma_I$$

of $\partial\Omega$, where

$$\sigma_I=\{z\in\bar{\Omega}:|f_i(z)|=1,\ i\in I\}.$$

That Ω is a *non-degenerate* analytic polyhedron means that $\partial f_{I_1}\wedge\dots\wedge\partial f_{I_k}\neq 0$ on $\{|f_{I_1}|=\dots=|f_{I_k}|=1\}$. $\square$

Note that the polydisc D^n in $\mathbf{C}^n$ is a non-degenerate analytic polyhedron with n defining functions. Its skeleton is the torus T^n.

Near each point on the boundary of a non-degenerate analytic polyhedron, the non-degeneracy condition assures that we have a holomorphic change of variables, where the functions defining that part of the boundary are (some of) the new coordinates. In particular, each point on the skeleton has a neighbourhood that is biholomorphically equivalent to a neighbourhood of some point on the torus in $\mathbf{C}^n$ in such a way that points in the polyhedron correspond to points in the polydisc.

In $\mathbf{C}^n$, an analytic polyhedron is degenerate if more than n edges $\{z:|f_i|=1\}$ intersect. For example, a non-trivial intersection of two discs in the plane is a degenerate analytic polyhedron. Hence, the only non-degenerate analytic polyhedra in $\mathbf{C}$ are the ones that locally are defined by one single defining function. When $n>1$, we have the possibility of non-degenerate analytic polyhedra with more than n defining functions such that the edges intersect in a more complicated way.

Example 1. In $\mathbf{C}^2$, consider the following analytic polyhedron:

$$\Omega=\{z\in\mathbf{C}^2:|f_i(z)|<1,\ i=1,2,3\},\quad \begin{cases} f_1(z)=z_1,\\ f_2(z)=z_2,\\ f_3(z)=4z_1z_2-2;\end{cases}$$

in other words, Ω is the intersection of D^2 with the set $\{z\in\mathbf{C}^2:|4z_1z_2-2|<1\}$. If $|f_1(z)|=|f_2(z)|=1$, then the point z does not belong to $\bar{\Omega}$; $|f_3(z)|\geq 2$. Assume that $|f_1(z)|=|f_3(z)|=1$. (Such points z exist, for example $\left(1,\frac{3}{4}\right)$.) Then $|f_2(z)|=|z_2|\leq\frac{3}{4}$, so $z\in\sigma$. Furthermore, $\partial f_1\wedge\partial f_3=4z_1dz_1\wedge dz_2\neq 0$ there. Thus Ω is non-degenerate. □

Example 2. Of course, it is possible to have non-trivial non-degenerate analytic polyhedra with more than one defining function in $\mathbf{C}$, as long as any two of those functions never have modulus 1 at the same time. Consider the annulus

$$A=\{z\in\mathbf{C}:1<|z|<2\}.$$

This is a non-degenerate analytic polyhedron defined by the functions $f_1(z)=\frac{1}{2}z$ and $f_2(z)=1/z$, both holomorphic in a neighbourhood of $\bar{A}$. □

The spaces H^p in analytic polyhedra are defined as follows.

Definition 2. For a non-degenerate analytic polyhedron Ω, let

$$\Omega_\varepsilon=\{z\in\mathbf{C}^n:|f_i(z)|<1-\varepsilon,\ i=1,\dots,N\}.$$

Then Ω_ε are non-degenerate analytic polyhedra for small $\varepsilon>0$. Let σ_ε be the skeleton of Ω_ε. We define $H^p(\Omega)$ to be the set of $\phi\in\mathcal{O}(\Omega)$ such that the H^p-norm of ϕ, $\|\phi\|_{H^p(\Omega)}=\sup_\varepsilon\|\phi\|_{L^p(\sigma_\varepsilon)}$, is finite. The space $H^\infty(\Omega)$ is the space of bounded holomorphic functions in Ω. □

The main result of this paper is the following.

Theorem 1.1. *Let Ω be a non-degenerate analytic polyhedron. Assume that $g_i\in H^\infty(\Omega)$, $1\leq i\leq m$, satisfies $0<\delta\leq\sum_{i=1}^m|g_i|$ for some δ. Whenever $\varphi\in H^p(\Omega)$, $1<p<\infty$, there are $u_i\in H^p(\Omega)$, $1\leq i\leq m$, such that $\varphi=\sum_{i=1}^m g_iu_i$.*

The H^p corona problem has already been solved in the polydisc. In this paper we will solve the problem in the more general case of the polyhedron. To do this, we will use a certain integral representation formula for holomorphic functions (see [B]) generalizing the Weil formula; this will yield an explicit solution. The idea is to write φ as an integral where we can make a factorization in the kernel to obtain u, and methods related to those in Wolff's proof of the corona theorem will then be used to get the H^p estimate. The method will be different from that of [L2], where the problem was solved by studying the Koszul complex and solving certain $\bar{\partial}$-equations, but the core of the H^p estimation, some nontrivial estimates in product domains, will essentially be the same.

Remark 1. We think that the theorem probably is true even for the case $p=1$, but this is not considered here. The method of proof used in this paper would not carry over to that case without major modifications. In the proof we use duality between L^p and L^q, so to go along the same lines one would have to use duality between L^1 and BMO. For instance, a version of Lemma 3.2, saying that the weighted Cauchy operator maps BMO to BMO, would have to be proved; but it is not obvious, at least not to the author, that this result is true.

Some words on the case $0<p<1$: H^p for $p<1$ is very different from the case $p>1$, and to study the problem in that case one would probably have to rely on some other technique, for instance atomic decompositions.

A major part of the proof of Theorem 1.1 is the localization, and a study of so called Hefer forms. We will here give an example, where we make considerations analogous to those in the proof.

Recall Weil's integral representation formula (see, for instance, [A]). Let Ω be a non-degenerate analytic polyhedron, say $\Omega=\{|f_i|<1,\ i=1,\dots,N\}$. Since the defining functions are holomorphic across the boundary of Ω, we can find Hefer functions, that is holomorphic functions H_i^k satisfying

$$\sum_{k=1}^{n} H_i^k(\zeta,z)(\zeta_k-z_k)=f_i(\zeta)-f_i(z).$$

Define the Hefer forms h_i by $h_i(\zeta,z)=\sum_{k=1}^n H_i^k(\zeta,z)\,d\zeta_k$. For any $\varphi\in\mathcal{O}(\bar\Omega)$,

$$\varphi(z)=\sum_I \frac{1}{(2\pi i)^n}\int_{\sigma_I}\varphi(\zeta)\bigwedge_{i\in I}\frac{h_i(\zeta,z)}{f_i(\zeta)-f_i(z)},$$

where the summation is performed over strictly increasing multiindices. Consider the $n\times n$-matrix $H_I(\zeta,z)=[H_i^k(\zeta,z)]_{i\in I}^{1\le k\le n}$. As $\bigwedge_{i\in I}h_i=\det H_I\,d\zeta_1\wedge\ldots\wedge d\zeta_n$, Weil's formula can also be written

$$\varphi(z)=\sum_I \frac{1}{(2\pi i)^n}\int_{\sigma_I}\frac{\varphi(\zeta)\det H_I(\zeta,z)\,d\zeta_1\wedge\ldots\wedge d\zeta_n}{\prod_{i\in I}(f_i(\zeta)-f_i(z))}.$$

Let us consider the simplest case, where Ω is a biholomorphic image of the polydisc D^n; Ω is defined by the n functions $f_i\colon\bar\Omega\to\bar D$ which form a global, holomorphic change of variables. Introduce the new variables $\xi_i=f_i(\zeta)$ and $x_i=f_i(z)$, from this $\xi\in T^n$ and $x\in D^n$. Furthermore, if we by $\partial\zeta/\partial\xi$ denote the Jacobian matrix of the change of variables, then

$$d\zeta_1\wedge\ldots\wedge d\zeta_n=\det\frac{\partial\zeta}{\partial\xi}\,d\xi_1\wedge\ldots\wedge d\xi_n$$

so the Weil formula transforms to

$$\varphi(x)=\frac{1}{(2\pi i)^n}\int_{T^n}\frac{\varphi(\xi)\det H(\xi,x)\det(\partial\zeta/\partial\xi)(\xi)\,d\xi_1\wedge\ldots\wedge d\xi_n}{(\xi_1-x_1)\ldots(\xi_n-x_n)}.$$

Since

$$\det H(x,x)\det\frac{\partial\zeta}{\partial\xi}(x)=1,$$

this is nothing but Cauchy's formula applied to the holomorphic function

$$\xi\longmapsto\varphi(\xi)\det H(\xi,x)\det\frac{\partial\zeta}{\partial\xi}(\xi)$$

and evaluated at the point x. Hence, in the biholomorphic case, Weil's formula may be viewed as a variant of Cauchy's formula in the other coordinate system.

In the general case, near each p on any σ_I we have a local change of variables, and by compactness σ can be covered by a finite number of sets, in each of which we have a change of variables. Introducing a partition of unity and comparing with the biholomorphic case, we may thus write the Weil formula as a finite sum of Cauchy-like formulas. Local properties of φ may then be studied by looking at the corresponding properties in polydiscs. Contrary to the biholomorphic case, we get singularities even if z is so far from the corner where ζ is located that they do not lie in the domain of any common change of variables.

Remark 2. By the methods used in the proof of Theorem 1.1, we see that the integral operator corresponding to Weil's integral formula,

$$\phi\longmapsto\mathcal{W}\phi,\quad \mathcal{W}\phi(z)=\sum_I\frac{1}{(2\pi i)^n}\int_{\sigma_I}\phi(\zeta)\bigwedge_{i\in I}\frac{h_i(\zeta,z)}{f_i(\zeta)-f_i(z)},$$

is an operator $\mathcal{W}\colon L^p(\sigma)\to H^p(\Omega)$; this is obtained by reducing to the known polydisc case.

The paper is organized like this: In Section 2 we construct the division formula for solving the problem. In Section 3 we look at the integral formula in the special case of the polydisc, and prove the desired estimate. In Section 4 we study the general integral formula, and use localization to reduce to the polydisc case. Section 5 contains the definition and estimates of certain Hefer functions needed in Section 4.

2. A division formula in the polyhedron

From now on, assume that Ω is a non-degenerate analytic polyhedron with N defining functions f_i. To construct the division formula that will solve the H^p

corona problem, as indicated in the introduction, we need some preliminaries. First some notes on notation. We will often use ζ as variable in Ω and z as variable on σ, and when a function depends only on ζ, we will (mostly) omit the argument. All differential forms will contain differentials of ζ only. We let $a\cdot b=\sum_{i=1}^m a_ib_i$. Thus the division problem may be formulated by saying that we want to find $u\in H^p$ such that $g\cdot u=\varphi$.

Consider the (column) matrix g of functions. Let ${}_gH_j^k$, $j=1,\dots,m$, $k=1,\dots,n$ be Hefer functions for g, i.e. holomorphic functions satisfying

$$ {}_gH(\zeta,z)(\zeta-z)=g(\zeta)-g(z), $$

where we consider ${}_gH$ to be an $m\times n$ matrix of functions. We will make explicit choices of these Hefer functions for g later on. Define the corresponding holomorphic Hefer forms ${}_gh$ by

$$ {}_gh_j(\zeta,z)=\sum_{i=1}^n {}_gH_j^i(\zeta,z)\,d\zeta_i, $$

or, in matrix notation, ${}_gh={}_gH\,d\zeta$. With a similar notation, we will denote the Hefer functions for the defining functions by ${}_fH$ and let ${}_fh={}_fH\,d\zeta$.

We will use the following integral formula of Berndtsson (see [B]):

Consider the kernel

$$ K_0(\zeta,z)=\sum_{\substack{\alpha=(\alpha_0,\dots,\alpha_N)\\ |\alpha|=n}} c_\alpha \bigwedge_{i=0}^N G_i^{(\alpha_i)}(Q_i(\zeta,z)(\zeta-z))(\bar\partial q_i)^{\alpha_i}, \tag{2.1} $$

where c_α are constants, G_i, $i=0,\dots,N$, are one-variable holomorphic functions with $G_i(0)=1$, Q_i^j, $i=0,\dots,N$ and $j=1,\dots,n$, are mappings from $\mathbf{C}^n\times\mathbf{C}^n$ to $\mathbf{C}$ and the $(1,0)$-forms q_i are defined as below.

If these mappings are chosen such that everything make sense, then

$$ \varphi(z)=\int_\Omega K_0(\zeta,z)\varphi. \tag{2.2} $$

To solve the problem, we must choose the mappings G_i and Q_i^j such that we can factor out $g(z)$ from the kernel so that the remainder still is holomorphic, and in a way that enables us to get H^p-estimates.

First we define the $N+1$ one-variable functions G_i. Let $G_0(t)=(1+t)^\nu$, where ν is a sufficiently large integer ($>n$ will do). For $i=1,\dots,N$, let $G_i(t)=1/(1+t)^\varepsilon$, where $\varepsilon>0$. Modulo constants, the k-th derivative will be $G_0^{(k)}(t)=(1+t)^{\nu-k}$ when $k<\nu$ and $G_i^{(k)}(t)=\varepsilon/(1+t)^{k+\varepsilon}$ for $i,k\geq 1$.

For $i=0,\dots,N$ and $j=1,\dots,n$ we shall define mappings Q_i^j; that is, we shall define the $(N+1)\times n$ matrix Q of functions. Then we set $q=Q\,d\zeta$, i.e. $q_i(\zeta,z)=\sum_j Q_i^j(\zeta,z)\,d\zeta_j$.

Put

$$\gamma=\frac{\bar{g}}{|g|^2}$$

and let

$$Q_0(\zeta,z)=-{}_gH(\zeta,z)\cdot\gamma,$$

whence

$$Q_0(\zeta,z)(\zeta-z)=-{}_gH(\zeta,z)(\zeta-z)\cdot\gamma=(g(z)-g)\cdot\gamma=g(z)\cdot\gamma-1$$

and therefore

$$G_0^{(k)}(Q_0(\zeta,z)(\zeta-z))=(g(z)\cdot\gamma)^{\nu-k}.$$

Since $\nu>n$, we can write

$$G_0^{(k)}(Q_0(\zeta,z)(\zeta-z))=g(z)\cdot\omega(\zeta,z),$$

which is the desired factorization. Note that ω depends on k; this is not indicated in the notation, since we only consider such properties of ω that are independent of k. For further reference we note that ω can be decomposed as a sum of products;

$$\omega(\zeta,z)=\sum_i\omega_i(\zeta)\widetilde{\omega}_i(z). \tag{2.3}$$

One important property of $\omega_i(\zeta)$ is that we can estimate it and its derivatives by derivatives of g;

$$\left|\frac{\partial^k\omega_i}{\partial\zeta_{j_1}\dots\partial\zeta_{j_k}}\right|\lesssim\left|\frac{\partial^k g}{\partial\zeta_{j_1}\dots\partial\zeta_{j_k}}\right|+\text{ lower order derivatives.} \tag{2.4}$$

For $i=1,\dots,N$, let

$$Q_i(\zeta,z)=\bar{f}_i\frac{{}_fH_i}{1-|f_i|^2},$$

whence

$$Q_i(\zeta,z)(\zeta-z)=\frac{1-\bar{f}_if_i(z)}{1-|f_i|^2}-1$$

and therefore

$$G_i(Q_i(\zeta,z)(\zeta-z))=\left(\frac{1-|f_i|^2}{1-\bar{f}_if_i(z)}\right)^{\varepsilon}$$

and

$$G_i^{(k)}(Q_i(\zeta,z)(\zeta-z))=\varepsilon\left(\frac{1-|f_i|^2}{1-\bar{f}_i f_i(z)}\right)^{k+\varepsilon}\quad\text{when } k\geq 1.$$

From the definition of q, we have

$$\bar{\partial}q_0={}_gh\cdot\bar{\partial}\gamma=\sum {}_gh_j\wedge\bar{\partial}\gamma_j,$$

so

$$(\bar{\partial}q_0)^k=\sum_{\beta=(\beta_1,\dots,\beta_k)}\bigwedge_{j\in\beta}{}_gh_j\wedge\bar{\partial}\gamma_j,$$

where the summation is performed over strictly increasing multiindices β; $1\leq\beta_1<\dots<\beta_k\leq m$. For $i\geq 1$,

$$\bar{\partial}q_i={}_fh_i\wedge\bar{\partial}\frac{\bar{f}_i}{1-|f_i|^2}={}_fh_i\wedge\frac{d\bar{f}_i}{(1-|f_i|^2)^2},$$

and in particular $(\bar{\partial}q_i)^k=0$ if $k>1$.

Use the factorization of $G_0^{(k)}$ to rewrite the kernel (2.1) (depending on ε) as

$$K_0^\varepsilon(\zeta,z)=g(z)\cdot K^\varepsilon(\zeta,z),$$

where (modulo constants)

$$\begin{aligned}
K^\varepsilon(\zeta,z)&=\sum_{|\alpha|=n}\omega(\bar{\partial}q_0)^{\alpha_0}\bigwedge_{i=1}^N G_i^{(\alpha_i)}(Q_i(\zeta,z)(\zeta-z))(\bar{\partial}q_i)^{\alpha_i}\\
&=\sum_{k=0}^n\sum_{\substack{\alpha=(\alpha_1,\dots,\alpha_N)\\|\alpha|=n-k}}\omega(\bar{\partial}q_0)^k\bigwedge_{i=1}^N G_i^{(\alpha_i)}(Q_i(\zeta,z)(\zeta-z))(\bar{\partial}q_i)^{\alpha_i}\\
&=\sum_{k=0}^n\sum_{\substack{\alpha=(\alpha_1,\dots,\alpha_N)\\|\alpha|=n-k\\\alpha_i\in\{0,1\}}}\omega(\bar{\partial}q_0)^k\bigwedge_{i=1}^N G_i^{(\alpha_i)}(Q_i(\zeta,z)(\zeta-z))(\bar{\partial}q_i)^{\alpha_i}\\
&=\sum_{k=0}^n\sum_{\substack{\alpha=(\alpha_1,\dots,\alpha_{n-k})\\1\leq\alpha_1<\dots<\alpha_{n-k}\leq N}}\omega(\bar{\partial}q_0)^k\bigwedge_{i\in\alpha}G_i'(Q_i(\zeta,z)(\zeta-z))\bar{\partial}q_i\prod_{i\notin\alpha}G_i(Q_i(\zeta,z)(\zeta-z)).
\end{aligned}$$

Let

$$u^\varepsilon(z)=\int_\Omega K^\varepsilon(\zeta,z)\varphi,$$

then, by (2.2), u^ε will be a holomorphic solution to $g\cdot u^\varepsilon=\varphi$.

Inserting the calculated quantities, we see that (still modulo constants, that we may include in ω)

$$K^\varepsilon(\zeta,z)=\sum_{k,\alpha,\beta}\omega\biggl(\bigwedge_{j\in\beta}{}_gh_j\wedge\bar\partial\gamma_j\biggr)\biggl(\bigwedge_{i\in\alpha}\varepsilon\frac{(1-|f_i|^2)^{\varepsilon-1}}{(1-\bar f_if_i(z))^{\varepsilon+1}}{}_fh_i\wedge d\bar f_i\biggr)\prod_{i\notin\alpha}\biggl(\frac{1-|f_i|^2}{1-\bar f_if_i(z)}\biggr)^\varepsilon,$$

where the summation is performed over $k=0,\dots,n$, $1\le\alpha_1<\dots<\alpha_{n-k}\le N$ and $1\le\beta_1<\dots<\beta_k\le m$. The final step in the construction of the division formula will be to let $\varepsilon\to0$. When we do that, we arrive at the following proposition.

Proposition 2.1. *Let Ω be a non-degenerate analytic polyhedron with N defining functions f_i. Let $g=(g_1,\dots,g_m)\in H^\infty(\Omega)$, $0<\delta\le\sum|g_i|$. Let ${}_fh$ be Hefer forms for the defining functions and ${}_gh$ be Hefer forms for g. Assume that $\varphi\in\mathcal{O}(\Omega)$. With γ and ω as defined above the function u given by*

$$u(z)=\sum_{\substack{0\le k\le n\\ 1\le\alpha_1<\dots<\alpha_{n-k}\le N\\ 1\le\beta_1<\dots<\beta_k\le m}}\int_{\left\{\substack{|f_i|=1,i\in\alpha,\\ |f_i|<1,i\notin\alpha}\right\}}K_{k,\alpha,\beta}(\zeta,z)\varphi,\tag{2.5}$$

where

$$K_{k,\alpha,\beta}(\zeta,z)=\omega\biggl(\bigwedge_{j\in\beta}{}_gh_j\wedge\bar\partial\gamma_j\biggr)\bigwedge_{i\in\alpha}\frac{{}_fh_i}{f_i-f_i(z)},\tag{2.6}$$

is a holomorphic solution to the division problem

$$g\cdot u=\varphi.$$

Remark 3. Note that we integrate over all possible σ_I, including $\sigma_\emptyset=\Omega$ and the corners constituting the skeleton. It would be nice to be able to construct a solution formula in which we (as in Weil's formula) only integrate over the skeleton.

Proof. In view of the above work, we only need to show that $u=\lim_{\varepsilon\to0}u^\varepsilon$. But since

$$\lim_{\varepsilon\to0}\prod_{i\in\alpha}\frac{1}{(1-\bar f_if_i(z))^\varepsilon}\prod_{i\notin\alpha}\biggl(\frac{1-|f_i|^2}{1-\bar f_if_i(z)}\biggr)^\varepsilon=1$$

and

$$\frac{\varepsilon(1-|f_i|^2)^{\varepsilon-1}d\bar f_i}{1-\bar f_if_i(z)}=\frac{-\bar\partial(1-|f_i|^2)^\varepsilon}{f_i-|f_i|^2f_i(z)}$$

we will have

$$\begin{aligned}\lim_{\varepsilon\to 0} u^\varepsilon(z) &= \lim_{\varepsilon\to 0}\sum_{k,\alpha,\beta}\int_{\{|f_i|<1, i=1,\dots,N\}} \omega\varphi\Big(\bigwedge_{j\in\beta} {}_g h_j\wedge\bar\partial\gamma_j\Big)\\ &\quad\times\Big(\bigwedge_{i\in\alpha}\frac{\bar\partial(1-|f_i|^2)^\varepsilon\wedge{}_f h_i}{(f_i-|f_i|^2 f_i(z))(1-\bar f_i f_i(z))^\varepsilon}\Big)\prod_{i\notin\alpha}\Big(\frac{1-|f_i|^2}{1-\bar f_i f_i(z)}\Big)^\varepsilon\\ &= \sum_{k,\alpha,\beta}\int_{\left\{\substack{|f_i|=1, i\in\alpha\\ |f_i|<1, i\notin\alpha}\right\}}\omega\varphi\Big(\bigwedge_{j\in\beta} {}_g h_j\wedge\bar\partial\gamma_j\Big)\bigwedge_{i\in\alpha}\frac{{}_f h_i}{f_i-f_i(z)}\end{aligned}$$

as desired. The limiting process is justified by localizing to the polydisc, performing the corresponding operation there and then going back. The limiting process in the polydisc is a generalization of the one-variable result

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_D \Phi(\zeta,z)\bar\partial(1-|\zeta|^2)^\varepsilon\wedge d\zeta &= \lim_{\varepsilon\to 0}\int_D \bar\partial(\Phi(\zeta,z)(1-|\zeta|^2)^\varepsilon)\wedge d\zeta\\ &\quad-\lim_{\varepsilon\to 0}\int_D (1-|\zeta|^2)^\varepsilon\bar\partial\Phi(\zeta,z)\wedge d\zeta\\ &= \Big(\lim_{\varepsilon\to 0} -\int_T \Phi(\zeta,z)(1-|\zeta|^2)^\varepsilon\, d\zeta\Big)-\int_D\bar\partial\Phi(\zeta,z)\wedge d\zeta\\ &= \int_T \Phi(\zeta,z)\, d\zeta.\end{aligned}$$

This motivates the above calculation. □

This solves the part of the problem to find a holomorphic u such that $g\cdot u=\varphi$; what we need to do to solve the H^p corona problem is to show that $u\in H^p$ whenever $\varphi\in H^p$.

Remark 4. The term in (2.5) corresponding to $k=0$ is

$$\sum_{1\le\alpha_1<\dots<\alpha_n\le N}\int_{\left\{\substack{|f_i|=1, i\in\alpha,\\ |f_i|<1, i\notin\alpha}\right\}}\omega\varphi\bigwedge_{i\in\alpha}\frac{{}_f h_i}{f_i(\zeta)-f_i(z)};$$

a Weil integral.

Example 3. When $n=N=1$, the solution u from Proposition 2.1 is

$$u(z)=\int_{\partial\Omega}\omega\varphi\frac{{}_f h}{f(\zeta)-f(z)}+\sum_{\beta=1}^m\int_\Omega \omega\varphi_g h_\beta\wedge\bar\partial\gamma_\beta.$$

If we use $\nu=2$ in the definition of G_0, the factor ω in the last integral will simply be (still modulo constants) $\omega(\zeta,z)=\gamma(\zeta)$, and in the first integral it will be

$$\omega(\zeta,z)=\gamma g(z)\cdot\gamma=\sum_{i=1}^{m} g_i(z)\gamma(\zeta)\gamma_i(\zeta).$$

Further, note that we may choose

$${}_fH(\zeta,z)=\frac{f(\zeta)-f(z)}{\zeta-z}$$

and

$${}_gH_i(\zeta,z)=\frac{g_i(\zeta)-g_i(z)}{\zeta-z},$$

so we have that

$$u(z)=c\int_{|f(\zeta)|=1}\frac{\gamma(\zeta)\varphi(\zeta)}{\zeta-z}+\sum_{j=1}^{m} c'\int_{|f(\zeta)|<1}\frac{\gamma(\zeta)g(z)\cdot\gamma(\zeta)\varphi(\zeta)(g_j(\zeta)-g_j(z))\,d\zeta\wedge\bar\partial\gamma_j}{\zeta-z}. \qquad \square$$

3. Solution of the H^p-corona problem in the polydisc

In this section we will consider the model case, where $\Omega=D^n$. In the solution (2.5) we make choices of Hefer functions in the polydisc to obtain an explicit solution. We will then show that this solution is in H^p.

Choose Hefer functions for the coordinate functions ζ_i, that is, for the defining functions, as the Kronecker delta, ${}_\zeta H_i^j(\zeta,z)=\delta_i^j$. Then the Hefer forms ${}_\zeta h$ just become the differentials

$${}_\zeta h_i(\zeta,z)=d\zeta_i.$$

Then choose Hefer functions for g as

$$\tag{3.1} {}_gH^k=\frac{g(z_1,\dots,z_{k-1},\zeta_k,\dots,\zeta_n)-g(z_1,\dots,z_k,\zeta_{k+1},\dots,\zeta_n)}{\zeta_k-z_k}.$$

Remark 5. Given the above choice of Hefer functions for ζ, if we go through the procedure in Section 5 for choosing Hefer functions for g, we will return at (3.1).

With these choices, the solution (2.5) will be

$$u(z)=\sum_{\substack{0\leq k\leq n\\ 1\leq\alpha_1<\ldots<\alpha_{n-k}\leq n\\ 1\leq\beta_1<\ldots<\beta_k\leq m}} \int_{\left\{\substack{\zeta_i\in T, i\in\alpha,\\ \zeta_i\in D, i\notin\alpha}\right\}} K_{k,\alpha,\beta}(\zeta,z)\varphi, \tag{3.2}$$

where the kernel $K_{k,\alpha,\beta}$ is

$$K_{k,\alpha,\beta}(\zeta,z)=\omega\bigwedge_{j\in\beta}{}_g h_j\wedge\bar\partial\gamma_j\bigwedge_{i\in\alpha}\frac{d\zeta_i}{\zeta_i-z_i}.$$

The remainder of this section contains the proof of the following theorem, that by a normal families argument will prove Theorem 1.1 when $\Omega=D^n$.

Theorem 3.1. *Let $u(z)$ be as in (3.2), where $0<\delta\leq\sum|g_i|\leq 1$ and the Hefer forms ${}_g h_j$ are defined by (3.1). If we assume that the functions g_i and φ are holomorphic on $\bar D^n$, then $g\cdot u=\varphi$ and $\|u\|_{H^p(D^n)}\leq C\|\varphi\|_{H^p(D^n)}$.*

Define Γ_i^j by

$$\Gamma_i^j=\frac{\partial\gamma_i}{\partial\bar\zeta_j},$$

i.e. $\bar\partial\gamma_i=\Gamma_i^1 d\bar\zeta_1+\ldots+\Gamma_i^n d\bar\zeta_n$. By a simple calculation,

$$\Gamma_i^j=\left(|g|^2\overline{\left(\frac{\partial g_i}{\partial\zeta_j}\right)}-\bar g_i g\cdot\overline{\left(\frac{\partial g}{\partial\zeta_j}\right)}\right)/|g|^4,$$

that is, Γ^j is a sum of terms $\omega'\overline{\partial g/\partial\zeta_j}$, where ω' can be estimated by g in the same way as ω, see (2.4). Thus

$$\bigwedge_{j\in\beta}\bar\partial\gamma_j=\sum_{J=(j_1,\ldots,j_k)}\bigwedge_{j\in J}\omega'\overline{\left(\frac{\partial g}{\partial\zeta_j}\right)}d\bar\zeta_j.$$

Observe, that we integrate over $\zeta_i\in D$ when $i\notin\alpha$ and over $\zeta_i\in T$ when $i\in\alpha$, so when we integrate $\bigwedge_{j\in\beta}\bar\partial\gamma_j$, the only multiindex that will be relevant is $J=\{1,\ldots,n\}\setminus\alpha$. Further, H^j is a sum of terms

$$\frac{g'}{\zeta_j-z_j},$$

where the crucial property of g' is that it depends on the variables $z_1,\ldots,z_l$ and $\zeta_{l+1},\ldots,\zeta_n$ only (for some $0\leq l\leq n$ depending on the function g'). Thus

$$\bigwedge_{j\in\beta}{}_g h_j\bigwedge_{j\in\alpha}d\zeta_j=\sum\prod_{j\notin\alpha}\frac{g'}{\zeta_j-z_j}\cdot d\zeta_1\wedge\ldots\wedge d\zeta_n.$$

This means that u is a sum of terms of the type

$$u_\alpha(z)=\int \omega B(\zeta,z)\varphi\prod_{j=1}^{n}\frac{1}{\zeta_j-z_j}\bigwedge_{j\notin\alpha}\overline{\left(\frac{\partial g}{\partial\zeta_j}\right)}\,d\bar\zeta_j\bigwedge_{j=1}^{n}d\zeta_j,$$

where we let ω include the functions ω', and let the function B be the product of the functions g'. Assume, without loss of generality, that $\alpha=(k+1,\dots,n)$, so that

$$\begin{aligned}u_\alpha(z)&=\int_{D^k\times T^{n-k}}\omega B(\zeta,z)\varphi\bigwedge_{i=1}^{n}\frac{d\zeta_i}{\zeta_i-z_i}\bigwedge_{j=1}^{k}\overline{\left(\frac{\partial g}{\partial\zeta_j}\right)}\,d\bar\zeta_j\\&=\int_{D^k\times T^{n-k}}\omega B(\zeta,z)\varphi\bigwedge_{i=1}^{k}\overline{\left(\frac{\partial g}{\partial\zeta_i}\right)}\frac{d\zeta_i\wedge d\bar\zeta_i}{\zeta_i-z_i}\bigwedge_{i=k+1}^{n}\frac{d\zeta_i}{1-\bar\zeta_i z_i},\end{aligned}\tag{3.3}$$

where we let ω include the "missing" factors $\bar\zeta_i$ from the last equality.

In order to show that $u\in H^p$, we estimate the H^p-norm of the typical term (3.3). Since φ and g are holomorphic on $\overline{D}^n$,

$$\|u\|_{H^p(D^n)}=\|u\|_{L^p(T^n)},$$

and we estimate $\|u_\alpha\|_{H^p(D^n)}$ by duality; we integrate u_α against a function $\psi\in L^q(T^n)$, where q is dual to p, and show that

$$\left|\int_{T^n}u_\alpha\psi\right|\lesssim\|\varphi\|_{L^p(T^n)}\|\psi\|_{L^q(T^n)}.$$

Looking at (3.3) and referring back to (2.3), we are to show that we can estimate

$$\begin{aligned}&\int_{T^n}\int_{D^k\times T^{n-k}}\omega(\zeta)\varphi\prod_{i=1}^{k}\overline{\left(\frac{\partial g}{\partial\zeta_i}\right)}\frac{B(\zeta,z)\widetilde\omega(z)\psi(z)}{\prod_{i=1}^{k}(\zeta_i-z_i)\prod_{i=k+1}^{n}(1-\bar\zeta_i z_i)}\\&\quad=\int_{D^k\times T^{n-k}}\omega(\zeta)\varphi\prod_{i=1}^{k}\overline{\left(\frac{\partial g}{\partial\zeta_i}\right)}\int_{T^n}\frac{B(\zeta,z)\widetilde\omega(z)\psi(z)}{\prod_{i=1}^{k}(\zeta_i-z_i)\prod_{i=k+1}^{n}(1-\bar\zeta_i z_i)}\\&\quad=\int_{D^k\times T^{n-k}}\omega(\zeta)\varphi\prod_{i=1}^{k}\overline{\left(\frac{\partial g}{\partial\zeta_i}\right)}\int_{T^n}\frac{B(\zeta,z)\widetilde\omega(z)\psi(z)}{\prod_{i=1}^{k}(1-\zeta_i\bar z_i)\prod_{i=k+1}^{n}(1-\bar\zeta_i z_i)}\\&\quad=\int_{D^k\times T^{n-k}}\omega\varphi\prod_{i=1}^{k}\overline{\left(\frac{\partial g}{\partial\zeta_i}\right)}\mathcal{T}\psi\end{aligned}\tag{3.4}$$

(where the second equality includes a modification of $\widetilde\omega$, and where we define the integral operator $\mathcal{T}$ in the obvious way) by $\|\varphi\|_{L^p(T^n)}\|\psi\|_{L^q(T^n)}$.

Remark 6. The change of order of integration in (3.4) is a formal calculation, motivated by the following. In the first k variables, the validity of the change of order of integration is not in doubt since the factors $1/(\zeta_i - z_i)$ are integrable. In the last $n-k$ variables, though, some extra argument is needed. In one variable, by definition,

$$\|\phi\|_{H^p(D)} = \lim_{r \uparrow 1} \|\phi_r\|_{L^p(T)},$$

where $\phi_r(z) = \phi(rz)$. In our situation, this means that we can estimate $\|u_\alpha\|_{H^p(D^n)}$ by estimating

$$\left| \int_{T^n} u_\alpha(z_1, \dots, z_k, rz_{k+1}, \dots, rz_n)\psi(z) \right| \tag{3.5}$$

for any $r<1$. The factors $1/(1-\bar{\zeta}_i r z_i)$ (for $i>k$) are then integrable, and the change of order of integration is justified. Now, the only place where the parameter r occurs is inside the integral operator $\mathcal{T}$. The L^p-estimate for $\mathcal{T}$ can be done uniformly in r. Thus, carrying on as below, we get an estimate for (3.5) that is independent of r, and from this the H^p-estimate for u_α follows.

Note that the operator $\mathcal{T}$ is a weighted Cauchy integral operator, holomorphic in ζ_i for $i \le k$, and in order to perform the estimates we need a certain generalization of the fact that the Cauchy integral is bounded on L^p. Obviously, the factor $\widetilde{\omega}$ will do no harm, since it is bounded and only depends on z. What remains to take care of is the factor B. However, B is constructed as a product of factors g', where each g' (for some l) only depends on the variables $z_1, \dots, z_l, \zeta_{l+1}, \dots, \zeta_n$. If we split B into its factors, ordered in a suitable way, and use iteration, we arrive at the desired result $\|\mathcal{T}\phi\|_{L^p(T^n)} \lesssim \|\phi\|_{L^p(T^n)}$. This is a special case of the following lemma.

Lemma 3.2. *Let $g_i(\zeta, z)$ for $i=1, \dots, n$, be bounded, $|g_i| \le 1$, and depend only on the i first z_k-variables and the $n-i$ last ζ_k-variables; that is $g_i(\zeta, z) = g_i(z_1, \dots, z_i, \zeta_{i+1}, \dots, \zeta_n)$. Then the weighted Cauchy operator $\mathcal{T}$ given by*

$$\mathcal{T}\psi(\zeta) = \int_{z \in T^n} \frac{\prod g_i(\zeta, z)\psi(z)}{\prod(\zeta_i - z_i)}$$

is bounded on $L^p(T^n)$, $1<p<\infty$.

Proof. Write $\mathcal{T}$ as an iterated integral:

$$\mathcal{T}\psi(\zeta) = \int_{z_1 \in T} \frac{g_1(\zeta, z)}{\zeta_1 - z_1} \cdots \int_{z_i \in T} \frac{g_i(\zeta, z)}{\zeta_i - z_i} \cdots \int_{z_n \in T} \frac{g_n(\zeta, z)\psi(z)}{\zeta_n - z_n}.$$

Observe that
$$\int_{z_{i+1}\in T}\frac{g_{i+1}(\zeta,z)}{\zeta_{i+1}-z_{i+1}}\cdots\int_{z_n\in T}\frac{g_n(\zeta,z)\psi(z)}{\zeta_n-z_n}$$
is a function of $z_1,\dots,z_i,\zeta_{i+1},\dots,\zeta_n$ only; just as g_i is. Therefore, with
$$\psi_i=g_i(\zeta,z)\int_{z_{i+1}\in T}\frac{g_{i+1}(\zeta,z)}{\zeta_{i+1}-z_{i+1}}\cdots\int_{z_n\in T}\frac{g_n(\zeta,z)\psi(z)}{\zeta_n-z_n},$$
we have the usual Cauchy integral operator estimate
$$\begin{aligned}\int_{\zeta_i\in T}\left|\int_{z_i\in T}\frac{\psi_i(z_1,\dots,z_i,\zeta_{i+1},\dots,\zeta_n)}{\zeta_i-z_i}\right|^p &\lesssim \int_{z_i\in T}|\psi_i(z_1,\dots,z_i,\zeta_{i+1},\dots,\zeta_n)|^p\\ &\le \int_{z_i\in T}\left|\int_{z_{i+1}\in T}\frac{g_{i+1}(\zeta,z)}{\zeta_{i+1}-z_{i+1}}\cdots\int_{z_n\in T}\frac{g_n(\zeta,z)\psi(z)}{\zeta_n-z_n}\right|^p\end{aligned}$$
and iteration gives the estimate for $\mathcal{T}$. □

Remark 7. The need to have the variables separated in this particular way is the reason for some technical statements (such as Proposition 4.1) to appear.

We do not know whether the lemma is true when one drops the condition on the order of the variables, i.e. if the Cauchy operator with bounded holomorphic weight still will map L^p to L^p.

As a consequence, we get the following result, that we will need later on.

Corollary 3.3. *Let $C(\zeta,z)\in C^\infty(\overline{D}^n)$ and let $g_i(\zeta,z)$ for $i=1,\dots,n$, be as in Lemma 3.2. Then the weighted Cauchy operator $\mathcal{T}$ given by*
$$\mathcal{T}\psi(\zeta)=\int_{z\in T^n}\frac{\prod g_i(\zeta,z)C(\zeta,z)\psi(z)}{\prod(\zeta_i-z_i)}$$
is bounded on $L^p(T^n)$, $1<p<\infty$.

Proof. Since we can write
$$C(\zeta,z)=C(\zeta;z_1,\dots,z_i,\dots,z_n)-C(\zeta;z_1,\dots,\zeta_i,\dots,z_n)+C(\zeta;z_1,\dots,\zeta_i,\dots,z_n),$$
where $(C(\zeta;z_1,\dots,z_i,\dots,z_n)-C(\zeta;z_1,\dots,\zeta_i,\dots,z_n))/(\zeta_i-z_i)$ is a bounded function, we may assume that for all i, either C does not depend on z_i or $C(\zeta,z)/(\zeta_i-z_i)$ is bounded. Let I be the set of all i such that $C(\zeta,z)/(\zeta_i-z_i)$ is bounded; then C does not depend on z_i for $i\in I^c$. Let
$$B(\zeta,z_I)=\frac{C(\zeta,z)}{\prod_{i\in I}(\zeta_i-z_i)},$$

where z_I denotes the variables z_i with $i \in I$. Further, note that

$$\prod_{i=1}^{n} g_i(\zeta, z) = \prod_{i \in I^c} \tilde{g}_i(\zeta_{I^c}, z_{I^c}; \zeta_I, z_I),$$

where the functions $\tilde{g}_i$ has the same structure in the variables z_{I^c} and ζ_{I^c} as in the lemma, and depends on the variables z_I and ζ_I as parameters. Now, $\mathcal{T}$ can be written

$$\mathcal{T}\psi(\zeta) = \int_{z_I} B(\zeta, z_I) \int_{z_{I^c}} \frac{\prod_{i \in I^c} \tilde{g}_i(\zeta_{I^c}, z_{I^c}; \zeta_I, z_I)\psi(z)}{\prod_{i \in I^c}(\zeta_i - z_I)},$$

and an application of the lemma proves the claim. $\square$

After these results, we return to our main track. In the estimation of (3.4), the last $n-k$ variables will not make matters worse (since there are no singularities involved, and we just can integrate in these variables); the most difficult term is the one with $k=n$, so we focus our attention on that one. Hence, we must estimate

$$\int_{D^n} \omega\varphi\mathcal{T}\psi \overline{\prod_{i=1}^{n} \left(\frac{\partial g}{\partial \zeta_i}\right)}$$

by $\|\varphi\|_{L^p(T^n)} \|\psi\|_{L^q(T^n)}$. When this is done, the general result follows.

Let $\varrho_i = 1 - |\zeta_i|^2$ and $\varrho = \prod_{i=1}^{n} \varrho_i$. Use Green's formula to obtain (in one variable at a time)

$$\begin{aligned}
\int_D F &= \int_D |\zeta|^2 F + \int_D (1-|\zeta|^2)F = -\int_D \zeta F \frac{\partial}{\partial \zeta}(1-|\zeta|^2) + \int_D (1-|\zeta|^2)F \\
&= \int_D (1-|\zeta|^2)\frac{\partial}{\partial \zeta}(\zeta F) - \int_D \frac{\partial}{\partial \zeta}((1-|\zeta|^2)\zeta F) + \int_D (1-|\zeta|^2)F \\
&= \int_D \varrho \frac{\partial}{\partial \zeta}(\zeta F) + \int_D \varrho F.
\end{aligned}$$

This yields that it is sufficient to estimate integrals like

$$\int_{D^n} \varrho \left(\overline{\prod_{i=1}^{n} \left(\frac{\partial g}{\partial \zeta_i}\right)} \right) \frac{\partial}{\partial \zeta_1} \cdots \frac{\partial}{\partial \zeta_n} (\omega\varphi\mathcal{T}\psi). \tag{3.6}$$

Recall the estimates (2.4) for derivatives of $\omega(\zeta)$ by derivatives of g. Since lower-order derivatives will be easier to handle, we will estimate any derivative of ω by the corresponding derivative of g. Letting $h = \varphi\mathcal{T}\psi$ this leads to (just modulo permutations of indices) estimating integrals like

$$\int_{D^n} \varrho \left|\frac{\partial g}{\partial \zeta_1}\right| \cdots \left|\frac{\partial g}{\partial \zeta_n}\right| \left|\frac{\partial}{\partial \zeta_1} \cdots \frac{\partial}{\partial \zeta_l} g\right| \left|\frac{\partial}{\partial \zeta_{l+1}} \cdots \frac{\partial}{\partial \zeta_n} h\right|.$$

The proof is completed by invoking the following two lemmas. The proof of Lemma 3.4 uses some standard tent space techniques, see e.g. the proof of Lemma 1 in [L1], and Lemma 3.5 first appeared as Theorem 2 in [C], see also e.g. [F].

Lemma 3.4. *If h is holomorphic, then*

$$\int_{D^n} \varrho \left| \frac{\partial g}{\partial \zeta_1} \cdots \frac{\partial g}{\partial \zeta_n} \right| \left| \frac{\partial}{\partial \zeta_1} \cdots \frac{\partial}{\partial \zeta_n} h \right| \lesssim \int_{T^n} |h|.$$

Lemma 3.5. *If h is holomorphic, then*

$$\int_{D^n} \varrho \left| \frac{\partial}{\partial \zeta_1} \cdots \frac{\partial}{\partial \zeta_n} g \right|^2 |h| \lesssim \int_{T^n} |h|.$$

First note that Lemma 3.5 together with the well-known fact that

$$\varrho \left| \frac{\partial g}{\partial \zeta_1} \cdots \frac{\partial g}{\partial \zeta_n} \right|^2$$

is a Carleson measure yields the estimate

$$\int_{D^n} \varrho \left| \frac{\partial}{\partial \zeta_1} \cdots \frac{\partial}{\partial \zeta_n} g \right| \left| \frac{\partial g}{\partial \zeta_1} \cdots \frac{\partial g}{\partial \zeta_n} \right| |h| \lesssim \int_{T^n} |h|.$$

Then we have

$$\begin{aligned}
&\int_{D^n} \varrho \left| \frac{\partial g}{\partial \zeta_1} \right| \cdots \left| \frac{\partial g}{\partial \zeta_n} \right| \left| \frac{\partial}{\partial \zeta_1} \cdots \frac{\partial}{\partial \zeta_l} g \right| \left| \frac{\partial}{\partial \zeta_{l+1}} \cdots \frac{\partial}{\partial \zeta_n} h \right| \\
&= \int_{D^n} \varrho_1 \cdots \varrho_l \left| \frac{\partial g}{\partial \zeta_1} \cdots \frac{\partial g}{\partial \zeta_l} \right| \left| \frac{\partial}{\partial \zeta_1} \cdots \frac{\partial}{\partial \zeta_l} g \right| \left(\varrho_{l+1} \cdots \varrho_n \left| \frac{\partial g}{\partial \zeta_{l+1}} \cdots \frac{\partial g}{\partial \zeta_n} \right| \left| \frac{\partial}{\partial \zeta_{l+1}} \cdots \frac{\partial}{\partial \zeta_n} h \right| \right) \\
&\lesssim \int_{T_1 \times \ldots \times T_l} \int_{D_{l+1} \times \ldots \times D_n} \varrho_{l+1} \cdots \varrho_n \left| \frac{\partial g}{\partial \zeta_{l+1}} \cdots \frac{\partial g}{\partial \zeta_n} \right| \left| \frac{\partial}{\partial \zeta_{l+1}} \cdots \frac{\partial}{\partial \zeta_n} h \right| \\
&\lesssim \int_{T^n} |h| \le \|\varphi\|_{L^p(T^n)} \|\mathcal{T}\psi\|_{L^q(T^n)} \lesssim \|\varphi\|_{L^p(T^n)} \|\psi\|_{L^q(T^n)}
\end{aligned}$$

and this completes the proof of Theorem 3.1.

Remark 8. These lemmas hold even if h is replaced by $h\chi$, where $\chi \in C^\infty(\bar{D}^n)$. To see this, just observe that, for example,

$$\left| \frac{\partial}{\partial \zeta_1} \cdots \frac{\partial}{\partial \zeta_n} \chi h \right| \le \|\chi\|_{C^n} \sum_I \left| \frac{\partial}{\partial \zeta_{I_1}} \cdots \frac{\partial}{\partial \zeta_{I_k}} h \right|.$$

This means in particular that we could get the same $L^p(T^n)$-estimate for a function u given by $u = \int K\varphi\chi$, where K is the kernel of the integral in (3.3) and $\chi \in C_0^\infty$. (This observation will be needed when we are to prove the general case.)

4. Solution of the H^p-corona problem in the analytic polyhedron

Now we will work through the scheme in Section 3 to prove the analogue to Theorem 3.1 in the case of the polyhedron, that is, we will prove that $\|u\|_{H^p(\Omega)} \leq C\|\varphi\|_{H^p(\Omega)}$ if u is the solution given by Proposition 2.1 and we choose Hefer functions for g in a suitable way. The interesting things happen at the "edges" or at the "corners" of $\partial\Omega$, where some of the defining functions has modulus 1. The idea of the proof is to fix an edge, perform a suitable change of variables and see that the polyhedron kernel in the new variables looks like the polydisc kernel. Then we apply the polydisc proof of the H^p-estimate (with some minor modifications) and arrive at the desired result.

Fix one term $K_{k,\alpha,\beta}$ in the kernel (2.6); denote this term by K. Let u denote the corresponding term $u(z) = \int K(\zeta, z)\varphi(\zeta)$. (The sum of all such u will then be our solution (2.5) to $g \cdot u = \varphi$.) Look for possible singularities.

Remark 9. For a singularity to occur in the polydisc, we would have $\zeta_i = z_i$, but in the general polyhedron singularities occur whenever $f_i(\zeta) = f_i(z)$, which may happen as soon as ζ and z are on or near the same edge.

Some possible singularities in K are the $n-k$ factors

$$\frac{1}{f_i - f_i(z)}.$$

These are *de facto* singularities if and only if $f_i(\zeta)$ is close to $f_i(z)$, that is, if ζ and z are near or on the same edge $|f_i| = 1$. Thus, we have such singularities just at edges corresponding to the indices in α. The only way for other, "new", singularities to appear is to be a part of some ${}_g h_j$. To see what singularities may come from there, we use the definition of Hefer functions (5.3) in Section 5, and we will as a tool use Lemma 5.1 from there. For easy reference, we collect the needed facts in the following proposition.

Proposition 4.1. *There are functions A^j and a $(1,0)$-form b such that ${}_g h$ defined by*

$$ {}_g h(\zeta, z) = \sum_{j=1}^{N} A^j(\zeta, z) {}_f h_j(\zeta, z) + b(\zeta, z)$$

is a Hefer form for g. The functions $A^j(\zeta, z)(f_j(\zeta) - f_j(z))$ and the coefficients of b are holomorphic and bounded. Furthermore, if we perform local changes of variables to ξ and x as below, they can be decomposed as sums of functions

$$D_{k,\alpha}(\xi, x) \prod_{i\in\alpha} (\xi_i - x_i),$$

where $1 \le k \le n+1$, $\alpha \subset \{1, \dots, n\}$, *the function* $D_{k,\alpha}$ *is holomorphic and bounded and depends only on the variables* ξ_i *and* x_i *with* $i \in \alpha$, x_i *with* $i \notin \alpha$ *and* $i < k$ *and* ξ_i *with* $i \notin \alpha$ *and* $i \ge k$.

Remark 10. The properties of the functions $D_{k,\alpha}$ should be compared to what happened in the polydisc case, where ${}_gH$ was a sum of functions g', depending on some z_i and some ζ_i variables in a certain order, combined with singularities.

The remainder of this section is devoted to the proof of the following analogue to Theorem 3.1:

Theorem 4.2. *Let* $u(z)$ *be as in* (2.5), *where* $0 < \delta \le \sum |g_i| \le 1$ *and the Hefer forms* ${}_gh_j$ *are defined as in Proposition* 4.1. *If we assume that the functions* g_i *and* φ *are holomorphic on* $\bar{\Omega}$, *then* $g \cdot u = \varphi$ *and* $\|u\|_{H^p(\Omega)} \le C \|\varphi\|_{H^p(\Omega)}$.

By Proposition 4.1,

$$ {}_gh_j = \sum_{k=1}^{N} A_j^k {}_fh_k + b_j, $$

where the forms b_j have bounded coefficients. Therefore they will not contribute with any singularities, and with respect to singularities we will have

$$ {}_gh_j \sim \sum_{k=1}^{N} A_j^k {}_fh_k, $$

where each A_j^k will contribute with a singularity $1/(f_k(\zeta) - f_k(z))$.

Remark 11. With some modification, this is really the general case, where we take b into account. The difference is that we will have some factors b_j instead of factors $A_j^k {}_fh_k$, but noting (see below) that any such factor will be replaced by a bounded holomorphic function divided by $f_k(\zeta) - f_k(z)$, we can instead replace b_j by such a construction.

From this (leaving out irrelevant subscripts on A),

$$ \bigwedge_{j \in \beta} {}_gh_j \bigwedge_{i \in \alpha} {}_fh_i \sim \bigwedge_{j \in \beta} \left(\sum_{i=1}^{N} A_j^i {}_fh_i \right) \bigwedge_{i \in \alpha} {}_fh_i = \sum_{\substack{J = (j_1, \dots, j_k) \\ J \cap \alpha = \emptyset}} \bigwedge_{i \in J} A^i {}_fh_i \bigwedge_{i \in \alpha} {}_fh_i. $$

In this product, we will have k factors A^i, where $i \notin \alpha$, and hence we get k new possible singularities from there; none of these with respect to any defining function, with respect to which we already had a possible singularity. We conclude, that as a total we will have n possible singularities, and they will all be with respect to *different* defining functions.

From now on, we may (without loss of generality) assume that we have the possible singularities in the defining functions f_1 to f_n, and that $\alpha=(1,\dots,n-k)$. Then we integrate over the set

$$S=\{|f_i|=1,\ i=1,\dots,n-k,\ |f_i|<1,\ i=n-k+1,\dots,N\}$$

and the kernel we study looks like this:

$$K(\zeta,z)=\omega C(\zeta,z)\prod_{i=1}^{n-k}\frac{1}{f_i(\zeta)-f_i(z)}\prod_{i=n-k+1}^{n}A^i(\zeta,z)\bigwedge_{j\in\beta}\bar\partial\gamma_j\bigwedge_{i=1}^{n}d\zeta_i, \tag{4.1}$$

where the function C is holomorphic across the boundary of S and is made up from the coefficients in the forms ${}_fh_i$.

To show that we have an H^p-solution to the division problem, it suffices to show that every $p\in\sigma$ has a neighbourhood $U\subset\mathbf{C}^n$ such that $u\in L^p(\sigma\cap U)$. Therefore, fix $p\in\sigma$. (We shall let z be on σ near p.) By a partition of unity argument, it is enough to show that any point $q\in\bar S$ has a neighbourhood $V\subset\mathbf{C}^n$ such that whenever $\chi\in C_0^\infty(V)$,

$$z\longmapsto\int_S K(\zeta,z)\chi(\zeta)\varphi(\zeta)\in L^p(\sigma\cap U). \tag{4.2}$$

Therefore, fix $q\in\bar S$; we shall let ζ be near q. By compactness, it suffices to choose U and V at the same time, such that (4.2) is valid.

Let J be the set of indices i such that $i\le n$, $|f_i(p)|=1$ and $f_i(p)=f_i(q)$. Choose $U\ni p$ and $V\ni q$ so small that the n functions f_i such that $|f_i(p)|=1$ constitute a local change of variables in U and the functions f_i such that $|f_i(q)|=1$ constitute a local change of variables in V. (Note, in particular, that the functions $f_1,\dots,f_{n-k}$ are among the functions that constitute the change of variables in V, and that the functions f_i with $i\in J$ are involved in the changes of variables both in U and in V.) In addition, U and V should be so small that the three following conditions be satisfied. If $|f_i(p)|<1$ and $z\in\bar U$, then $|f_i(z)|<1$. If $|f_i(q)|<1$ and $\zeta\in\bar V$, then $|f_i(\zeta)|<1$. If $f_i(p)\ne f_i(q)$, $z\in\bar U$ and $\zeta\in\bar V$, then $f_i(z)\ne f_i(\zeta)$. With these choices, the only remaining singularities will be the ones with respect to the functions f_i, $i\in J$. To see this, first remember that, by our very choice of term in the kernel, the only possible singularities was with respect to f_i, $i\le n$. Then, since every singularity could be expressed in terms of $1/(f_i(\zeta)-f_i(z))$, we will have no singularity for functions f_i such that $f_i(p)\ne f_i(q)$. Finally, we must see that we have no singularity with respect to f_i if $|f_i(p)|<1$, but then $|f_i(z)|<1$ for $z\in\overline{U}$. If $1\le i\le n-k$, we are integrating over $|f_i(\zeta)|=1$, so $f_i(p)\ne f_i(q)$; this is an excluded case. It remains to study the case where $n-k<i\le n$. Then the possible singularity is of the type $A^i(\zeta,z)$. Assuming that $f_i(p)=f_i(q)$, we have $(z,\zeta)\in U\times V$, and for such (z,ζ) the function A^i is bounded; see the definition (5.1) and compare with the proof for boundedness in Lemma 5.1.

Let x be the new variables in U (corresponding to z), and ξ be the new variables in V (corresponding to ζ). In particular, we will have $x_i=f_i(z)$ and $\xi_i=f_i(\zeta)$ when $i\in J$; furthermore $\xi_i=f_i(\zeta)$ for $i\le n-k$. When we perform these changes of variables, for suitable choices of the remaining functions in the change of variables in V we will have $\Omega\cap U\simeq D^n\cap\widetilde{U}$ and $\Omega\cap V\simeq D^n\cap\widetilde{V}$, where $\widetilde{U}$ is a neighbourhood of the point $x(p)\in T^n$ and $\widetilde{V}$ is a neighbourhood of the point $\xi(q)\in T^{n-k}\times\overline{D}^k$.

Let us perform the indicated changes of variables, expand the functions A^i and the forms $\bar\partial\gamma$. Then a calculation (that we omit) reveals that the kernel is

$$\omega B(\xi,x)C(\xi,x)\chi(\xi)\bigwedge_{i=1}^{n-k}\frac{\overline{\partial g/\partial\xi_i}d\bar\xi_i}{\xi_i-x_i}\prod_{i=n-k+1}^{n}\frac{1}{\xi_i-x_i}\bigwedge_{i=1}^{n}d\xi_i,$$

where the function B is a product of functions $D\prod(\xi_i-x_i)$ from Proposition 4.1, and we let C include all functions (holomorphic across the boundary) coming from the change of variables, from the "quasisingularities" $1/(f_i(\zeta)-f_i(z))$ where $i\notin J$ and from the fact that we (for notational convenience) have introduced factors $1/(\xi_i-x_i)$ even for $i\notin J$.

The argument for H^p in Section 3 may now be repeated to prove that the solution function in the polyhedron belongs to H^p (by showing that (4.2) is valid for it).

In short, we complete the proof in the following way: We integrate

$$\int_{T^{n-k}\times D^k}\omega B(\xi,x)C(\xi,x)\chi(\xi)\bigwedge_{i=1}^{n-k}\frac{\overline{\partial g/\partial\xi_i}\,d\bar\xi_i}{\xi_i-x_i}\prod_{i=n-k+1}^{n}\frac{1}{\xi_i-x_i}\bigwedge_{i=1}^{n}d\xi_i$$

against an L^q-function ψ, disposing of the singularities by introducing the integral operator $\mathcal{T}$ given by

$$\mathcal{T}\psi(\xi)=\int_x\frac{\widetilde\omega(x)B(\xi,x)C(\xi,x)\psi(x)}{\prod(\xi_i-x_i)}.$$

This is holomorphic, furthermore it is bounded on L^q as required, which we soon will see. If we split B into its parts and study each part separately, the integral operator we must estimate is

$$\mathcal{T}\psi(\xi)=\int_x\frac{\prod(D_{k,\alpha}(\xi;x)\prod(\xi_j-x_j))C(\xi,x)\widetilde\omega(x)\psi(x)}{\prod(\xi_i-x_i)}.$$

The first thing to notice, is that we can forget about $\widetilde{\omega}$, since it is bounded and only depending on the variable of integration. Next, we can forget about all variables with numbers j such that a factor $\xi_j - x_j$ occurs in the product B; this factor will then cancel a corresponding singularity. We may thus assume that the factor $\prod(D_{k,\alpha}(\xi;x)\prod(\xi_j-x_j))$ just is a product $\prod D_k(\xi;x)$, where D_k depends only on $x_1, \dots, x_{k-1}, \xi_k, \dots, \xi_n$. Since the operator

$$\psi \longmapsto \int_x \frac{\prod D_k(\xi;x)C(\xi,x)\psi(x)}{\prod(\xi_i - x_i)}$$

maps L^q to L^q by Corollary 3.3, this yields the L^q-boundedness for our original $\mathcal{T}$.

Then we integrate by parts and end up with the correspondence to (3.6). Finally the lemmas from Section 3 in conjunction with Remark 8 yield the desired estimate; Theorem 4.2 is proved.

Remark 12. To be more precise, we are only studying z in the set U, so the integral operator $\mathcal{T}$ would be defined by integration only over $\widetilde{U} \cap T^n$. This will, however, not cause any trouble, since we may choose a cutoff function $\chi'(z)$ with $\chi'=1$ on U and in whose support we still have the change of variables. When we change the variables, we get the same integral as before with the difference of the χ' occuring in $\mathcal{T}$; this does not disturb the L^q-boundedness. In addition we get an integral over the set $\operatorname{supp}\chi' \setminus \{\chi'=1\}$. But on that set there will be no singularities, so the estimate is still valid.

5. Choice of Hefer functions in the polyhedron

We want to choose Hefer functions for g such that the argument in the preceding section is valid. By Weil's integral formula (where we ignore the factor $(2\pi i)^{-n}$),

$$g(\zeta) - g(z) = \sum_I \int_{\sigma_I} g(w) \left(\frac{D_I(w,\zeta)}{\prod(f_{I_i}(w) - f_{I_i}(\zeta))} - \frac{D_I(w,z)}{\prod(f_{I_i}(w) - f_{I_i}(z))} \right),$$

where D_I is the determinant of the Hefer matrix ${}_{f_I}H$;

$$D_I(w,\cdot\,) = \det[{}_f H_i^j(w,\cdot\,)]_{i\in I}^{j=1,\dots,n}.$$

To obtain a Hefer decomposition of g, we rewrite the integral:

$$\begin{aligned} g(\zeta) - g(z) &= \sum_I \int_{\sigma_I} g(w) D_I(w,\zeta) \left(\frac{1}{\prod(f_{I_i}(w) - f_{I_i}(\zeta))} - \frac{1}{\prod(f_{I_i}(w) - f_{I_i}(z))} \right) \\ &\quad + \sum_I \int_{\sigma_I} g(w) \frac{D_I(w,\zeta) - D_I(w,z)}{\prod(f_{I_i}(w) - f_{I_i}(z))} = \mathcal{A} + \mathcal{B}. \end{aligned}$$

To study the integral $\mathcal{A}$, just observe that

$$\frac{1}{\prod_{i=1}^n(f_{I_i}(w)-f_{I_i}(\zeta))}-\frac{1}{\prod_{i=1}^n(f_{I_i}(w)-f_{I_i}(z))}$$
$$=\sum_{k=1}^n \frac{1}{\prod_{i=1}^{k-1}(f_{I_i}(w)-f_{I_i}(z))\prod_{i=k}^n(f_{I_i}(w)-f_{I_i}(\zeta))}$$
$$-\frac{1}{\prod_{i=1}^{k}(f_{I_i}(w)-f_{I_i}(z))\prod_{i=k+1}^n(f_{I_i}(w)-f_{I_i}(\zeta))}$$
$$=\sum_{k=1}^n \frac{f_{I_k}(\zeta)-f_{I_k}(z)}{\prod_{i=1}^{k}(f_{I_i}(w)-f_{I_i}(z))\prod_{i=k}^n(f_{I_i}(w)-f_{I_i}(\zeta))}.$$

By this,

$$\mathcal{A}=\sum_{i=1}^N A^i(\zeta,z)(f_i(\zeta)-f_i(z)),$$

where

$$A^i(\zeta,z)=\sum_{I\ni i,i=I_k}\int_{\sigma_I}\frac{g(w)D_I(w,\zeta)}{\prod_{j=1}^k(f_{I_j}(w)-f_{I_j}(z))\prod_{j=k}^n(f_{I_j}(w)-f_{I_j}(\zeta))}. \tag{5.1}$$

Turn to the integral $\mathcal{B}$. If we do any Hefer decomposition β_I of the determinant D_I; $D_I(w,\zeta)-D_I(w,z)=\sum\beta_I^k(w,\zeta,z)(\zeta_k-z_k)$ such that β_I is holomorphic in all variables (across the boundary!), then by the calculation

$$\int_{\sigma_I}\frac{g(w)(D_I(w,\zeta)-D_I(w,z))}{\prod(f_{I_i}(w)-f_{I_i}(z))}=\sum_{k=1}^n\left(\int_{\sigma_I}\frac{g(w)\beta_I^k(w,\zeta,z)}{\prod(f_{I_i}(w)-f_{I_i}(z))}\right)(\zeta_k-z_k)$$

we have a decomposition $\mathcal{B}=\sum_{k=1}^n B^k(\zeta,z)(\zeta_k-z_k)$, where

$$B^k(\zeta,z)=\sum_I\int_{\sigma_I}\frac{g(w)\beta_I^k(w,\zeta,z)}{\prod(f_{I_i}(w)-f_{I_i}(z))}. \tag{5.2}$$

With A and B as in (5.1) and (5.2), we define the Hefer functions like

$${}_gH(\zeta,z)=A(\zeta,z)\,{}_fH(\zeta,z)+B(\zeta,z). \tag{5.3}$$

To get the right estimates for ${}_gh$ (cf. Section 4), we need the following results for A^i and B^i:

Lemma 5.1. *For each i, the functions $A^i(\zeta,z)(f_i(\zeta)-f_i(z))$ and $B^i(\zeta,z)$ are holomorphic and bounded. (Here $\zeta\in\Omega$ and $z\in\sigma$.) Furthermore, if we perform any local change of variables to $\xi(\zeta)$ and $x(z)$ as in Section 4, each of these functions is a sum of functions*

$$D_{k,\alpha}(\xi,x)\prod_{i\in\alpha}(\xi_i-x_i),$$

where $D_{k,\alpha}$ is holomorphic and bounded and depends on the variables ξ_i and x_i with $i\in\alpha$, x_i with $i\notin\alpha$ and $i<k$ and ξ_i with $i\notin\alpha$ and $i\geq k$.

The holomorphicity is in no doubt, the thing to prove is the boundedness and the decomposition after changes of variables.

To prove this lemma, start with the part including A and rewrite the expression like

$$\begin{aligned}A^i(\zeta,z)(f_i(\zeta)-f_i(z)) &= \sum_{I\ni i, i=I_k}\int_{\sigma_I}\frac{g(w)D_I(w,\zeta)(f_{I_k}(\zeta)-f_{I_k}(z))}{\prod_{j=1}^{k}(f_{I_j}(w)-f_{I_j}(z))\prod_{j=k}^{n}(f_{I_j}(w)-f_{I_j}(\zeta))}\\ &= \sum_{I\ni i, i=I_k}\int_{\sigma_I}\frac{g(w)D_I(w,\zeta)}{\prod_{j=1}^{k-1}(f_{I_j}(w)-f_{I_j}(z))\prod_{j=k}^{n}(f_{I_j}(w)-f_{I_j}(\zeta))}\\ &\quad-\int_{\sigma_I}\frac{g(w)D_I(w,\zeta)}{\prod_{j=1}^{k}(f_{I_j}(w)-f_{I_j}(z))\prod_{j=k+1}^{n}(f_{I_j}(w)-f_{I_j}(\zeta))}\end{aligned}$$

and look at one of the integrals, say one of the first kind.

Take any $r\in\sigma_I$. Let $f_I=(f_{I_1},\dots,f_{I_n})$ be a local change of variables in some neighbourhood of r. Cover σ_I with a finite number of such neighbourhoods U_i. (We shall let $t=f_I(w)$ (that is, $t_j=f_{I_j}(w)$) be new variables instead of w there.) Let $\{\chi_i\}$ be a partition of unity subordinate to $\{U_i\}$.

Consider the integrals J_i given by

$$\begin{aligned}&\int_{\sigma_I}\frac{g(w)D_I(w,\zeta)}{\prod_{j=1}^{k-1}(f_{I_j}(w)-f_{I_j}(z))\prod_{j=k}^{n}(f_{I_j}(w)-f_{I_j}(\zeta))}\\ &\qquad=\sum_i\int_{\sigma_I}\frac{g(w)D_I(w,\zeta)\chi_i(w)}{\prod_{j=1}^{k-1}(f_{I_j}(w)-f_{I_j}(z))\prod_{j=k}^{n}(f_{I_j}(w)-f_{I_j}(\zeta))}=\sum_i J_i.\end{aligned}$$

We want to prove the decomposition for each such J_i.

Assume that $I=(1,\dots,n)$ and let us suppress the index i. This means that we want to show the decomposition for the integral

$$J=\int_{\sigma_I}\frac{g(w)\chi(w)D(w,\zeta)}{\prod_{j=1}^{k-1}(f_j(w)-f_j(z))\prod_{j=k}^{n}(f_j(w)-f_j(\zeta))}.$$

To achieve this, we will perform changes of variables to convert J into a sum of Cauchy integrals. The changes of variables will be according to the situation in Section 4; z is to live near some point $p\in\sigma_{\tilde{I}}$ and ζ is to live near some $q\in\partial\Omega$. Let us assume that $|f_i(p)|=1$ for $i=1,\dots,\mu$, $|f_i(p)|<1$ for $i=\mu+1,\dots,k-1$, $|f_i(q)|<1$ for $i=k,\dots,\nu-1$ and $|f_i(q)|=1$ for $i=\nu,\dots,n$. Then we will have new variables $t(w)$ near r, $x(z)$ near p and $\xi(\zeta)$ near q. In particular $t_i=f_i(w)$ for $i=1,\dots,n$, $x_i=f_i(z)$ for $i=1,\dots,\mu$ and $\xi_i=f_i(\zeta)$ for $i=\nu,\dots,n$.

Rewrite J, changing the bounded, holomorphic function D as we go along:

$$\begin{aligned} J&=\int_{\sigma_I}\frac{g(w)\chi(w)D(w,\zeta)}{\prod_{j=1}^{k-1}(f_j(w)-f_j(z))\prod_{j=k}^{n}(f_j(w)-f_j(\zeta))}\\ &=\int_{\sigma_I}\frac{g(w)\chi(w)D(w,\zeta,z)}{\prod_{j=1}^{\mu}(f_j(w)-f_j(z))\prod_{j=\nu}^{n}(f_j(w)-f_j(\zeta))}\\ &=\int_{T^n}\frac{g(t)\chi(t)D(t,\xi,x)}{\prod_{j=1}^{\mu}(t_j-x_j)\prod_{j=\nu}^{n}(t_j-\xi_j)}=\int_{T^n}\frac{g(t)\chi(t)D(t,\xi,x)}{\prod_{j=1}^{k-1}(t_j-x_j)\prod_{j=k}^{n}(t_j-\xi_j)}. \end{aligned}\tag{5.4}$$

The next step is to rewrite D in order to get the properties that we want. We want to see that J has the properties of the bounded functions of Lemma 3.2; that it is holomorphic and bounded and only depends on the variables x_i and ξ_i in a certain order. We will not be able to achieve exactly that, but will see that J in fact can be split into a sum of functions with that ordering property for some of the variables, while the dependence on the remaining variables will be harmless. Study the first variable number. We obtain a decomposition

$$D(t;\xi;x)=(D(t;\xi;x)-D(t;x_1,\xi_2,\dots,\xi_n;x))+D(t;x_1,\xi_2,\dots,\xi_n;x)$$

of D into two functions, where the first function depends on both ξ_1 and x_1 but remains bounded even after division by ξ_1-x_1, and the second function depends on x_1 but not on ξ_1. Repeat this decomposition for each of these two functions (and all their descendants) for variables with numbers $2,\dots,n$. This yields the decomposition

$$D(t;\xi;x)=\sum_{|\alpha|\le n}D_{k,\alpha}(t;\xi;x)\prod_{i\in\alpha}(\xi_i-x_i),$$

where the function $D_{k,\alpha}$ depends on all t_i, on all ξ_i and x_i with $i\in\alpha$, on all x_i with $i\notin\alpha$ and $i<k$ and finally on all ξ_i with $i\notin\alpha$ and $i\ge k$. This induces the corresponding decomposition of J as a sum of integrals

$$J_{k,\alpha}=\prod_{i\in\alpha}(\xi_i-x_i)\int_{T^n}\frac{g(t)\chi(t)D_{k,\alpha}(t,\xi,x)}{\prod_{j=1}^{k-1}(t_j-x_j)\prod_{j=k}^{n}(t_j-\xi_j)}=\prod_{i\in\alpha}(\xi_i-x_i)D(\xi,x),$$

where the function D is defined by the weighted Cauchy integral; D is holomorphic and (by Lemma 5.2 below) bounded and depends on all ξ_i and x_i with $i\in\alpha$, all x_i with $i\notin\alpha$ and $i<k$ and all ξ_i with $i\notin\alpha$ and $i\ge k$. This proves the assertion.

Lemma 5.2. *If* $b(w,\zeta)\in H^\infty(D^n\times D^n)$ *and* $\chi(w,\zeta)\in C^\infty(\bar{D}^n\times\bar{D}^n)$, *then the weighted Cauchy integral*

$$J(\zeta)=\int_{w\in T^n}\frac{b(w,\zeta)\chi(w,\zeta)}{\prod_{i=1}^n(w_i-\zeta_i)}$$

is bounded in D^n.

That was the part of the lemma involving A^i. When we study B^i, considerations similar to those above show that the integral

$$\int_{\sigma_I}\frac{g(w)\beta(w,\zeta,z)\chi(w)}{\prod(f_{I_j}(w)-f_{I_j}(z))}=\int_{T^n}\frac{g(t)\beta(t,\xi,x)B(t,x)\chi(t)}{\prod_{j=1}^{\mu}(t_j-x_j)}$$

is bounded. Also, when we perform changes of variables, it will have a decomposition as above; this case corresponds to $k=n+1$.

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Jörgen Boo
Department of Mathematics
Chalmers University of Technology and Göteborg University
S-412 96 Göteborg
Sweden
email: jboo@math.chalmers.se