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Wiggly sets and limit sets

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Abstract. We show that a compact, connected set which has uniform oscillations at all points and at all scales has dimension strictly larger than 1. We also show that limit sets of certain Kleinian groups have this property. More generally, we show that if G is a non-elementary, analytically finite Kleinian group, and its limit set $\Lambda(G)$ is connected, then $\Lambda(G)$ is either a circle or has dimension strictly bigger than 1.

1. Statement of results

The purpose of this paper is to show that a connected, compact set which "oscillates" around every point and at every scale must have dimension strictly larger than 1. Although this seems like a very intuitive result, we do not know any short, elementary proof and ours is based on the "traveling salesman theorem" of the second author [21] (also see [27]). We will apply the result to limit sets of certain Kleinian groups, and eventually prove a generalization of Bowen's dichotomy: a connected limit set of an analytically finite group is either a circle or has Hausdorff dimension strictly larger than 1.

A dyadic square Q in the plane is one of the form $Q=(2^{-n}j, 2^{-n}(j+1)]\times(2^{-n}k, 2^{-n}(k+1)]$. For a positive number $\lambda>0$, we let λQ denote the square concentric with Q but with diameter $\lambda\,\mathrm{diam}(Q)$, e.g., $2Q$ is the "double" of Q. Given a set E in the plane and a square Q we define $\beta(Q)$ as

$$\beta(Q)=\mathrm{diam}(Q)^{-1}\inf_{L\in\mathcal{L}}\sup_{z\in E\cap 3Q}\mathrm{dist}(z,L),$$

where $\mathcal{L}$ is the set of all lines L intersecting Q. A connected set with the property that there is a $\beta_0>0$ such that $\beta_E(Q)>\beta_0$ for every Q with $\frac{1}{3}Q\cap E\neq\emptyset$ and $\mathrm{diam}(Q)\leq\mathrm{diam}(E)$ is called *uniformly wiggly*. We shall prove the following theorem.

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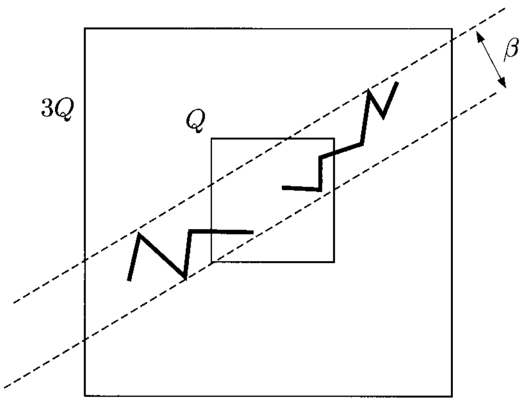

Figure 1.1. Definition of the β's.

Theorem 1.1. *Suppose $E \subset \mathbf{R}^2$ is a closed, connected set and is uniformly wiggly with constant β_0. Then $\dim(E) \geq 1 + C\beta_0^2$, where C is an absolute constant.*

This estimate is sharp (except for the choice of C) as can be seen by simple examples. A stochastic version of this theorem with an application to Brownian motion is given in [10]. The theorem is also true in $\mathbf{R}^n$, but with different constants.

Limit sets of certain Kleinian groups provide a natural class of uniformly wiggly sets. Consider a group G of Möbius transformations acting on the two sphere S^2. We say the group is Kleinian if it is discrete as a subgroup of $\mathrm{PSL}(2, \mathbf{C})$, (i.e., if the identity element is isolated) and we say it is elementary if it has a finite index Abelian subgroup. If G is not elementary, then the limit set, $\Lambda(G) \subset S^2$, is the accumulation set of any orbit. The ordinary set of G, $\Omega(G)$, is the complement of the limit set. The quotient $\Omega(G)/G$ is always a union of (possibly branched) Riemann surfaces.

Theorem 1.2. *If $\Omega(G)/G$ is compact and Λ is connected then Λ is either a Euclidean circle on S^2 or it is uniformly wiggly* (*and hence has dimension* >1).

This result for quasi-Fuchsian groups was obtained by Rohde in [31] who also introduced a version of the uniformly wiggly condition. An alternative version of this condition is used in [36] and [37]. We also show that limit sets of another class of groups are uniformly wiggly, namely the degenerate groups (Theorem 4.2). A Kleinian group action extends naturally from S^2 to a group of isometries of the hyperbolic 3-ball $\mathbf{B}^3$. The *Poincaré exponent* (or *critical exponent*) of the group is

$$\delta(G) = \inf\left\{ s : \sum_G \exp(-s\varrho(0, g(0))) < \infty \right\},$$

where ϱ is the hyperbolic metric in $\mathbf{B}^3$.

The group G is called *analytically finite* if $\Omega(G)/G$ is a finite union of finite type surfaces (i.e., each is compact except for a finite number of punctures). By the Ahlfors finiteness theorem [2], [6], all finitely generated Kleinian groups are analytically finite. If G is analytically finite, but $\Omega(G)/G$ is not compact, then Λ need not be uniformly wiggly, but even in this case we will prove the following results.

Theorem 1.3. *Suppose G is analytically finite and Ω is a simply connected, invariant component of $\Omega(G)$. Then* $\dim(\partial\Omega)=1$ *if and only if* $\delta(G)=1$ *if and only if $\partial\Omega$ is a circle.*

Corollary 1.4. *If G is analytically finite then its limit set is either totally disconnected, a circle or has Hausdorff dimension* >1.

Corollary 1.5. *If G is analytically finite and geometrically infinite then* $\delta(G)>1$.

In [9] we prove that if G is analytically finite and geometrically infinite then $\dim(\Lambda)=2$, but this does not directly imply the corollary (as far as we know).

Theorem 1.3 was first formulated by Bowen [11] for quasi-Fuchsian groups with no parabolics. The geometrically finite, cocompact Kleinian group case was proven by Sullivan [33] and by Braam [12]. See also [34], [26]. The general geometrically finite case was proven by Canary and Taylor in [15]. In this paper we complete the discussion by including the geometrically infinite groups (although our proof covers all cases at once). The fact that the limit set is either a circle or has infinite length (and has tangents almost nowhere) follows from a more general result for divergence type groups by Pommerenke [29]. Pictures of limit sets can be found in several sources such as [13], [25], [28], and [35].

The remaining sections of this paper are organized as follows:

Section 2: We recall some definitions and results.

Section 3: We prove Theorem 1.1.

Section 4: We prove Theorem 1.2.

Section 5: We recall the basic properties of the Schwarzian derivative.

Section 6: We prove Theorem 1.3.

Section 7: We consider the case when Ω is a union of disks.

Section 8: We prove Corollary 1.4 and Corollary 1.5.

The first author thanks Ed Taylor for explaining the results of [15] and for suggesting the problem of proving the Hausdorff dimension of degenerate limit sets is strictly greater than 1. It was by considering this problem that we were led to the other results in this paper. We also thank the referee for a very careful reading of the manuscript and numerous suggestions which improved it.

2. Background

First we recall the definition of Hausdorff dimension. Given an increasing function φ on $[0,\infty)$, we define

$$H_\varphi^\delta(E)=\inf\left\{\sum\varphi(r_j):E\subset\bigcup_j D(x_j,r_j),\ r_j<\delta\right\},$$

and

$$H_\varphi(E)=\lim_{\delta\to 0}H_\varphi^\delta(E).$$

This is the Hausdorff measure associated to φ. H_φ^∞ is called the Hausdorff content. It is not a measure, but has the same null sets as H_φ. When $\varphi(t)=t^\alpha$ we denote the measure H_φ by H_α and we define

$$\dim(E)=\inf\{\alpha:H_\alpha(E)=0\}.$$

For $\alpha=1$ we sometimes denote H_1 by l (for "length"). An upper bound for $\dim(E)$ can be produced by finding appropriate coverings of the set. We will be more interested in finding lower bounds. The usual idea is the *mass distribution principle*: construct a positive measure μ on E which satisfies $\mu(D(x,r))\le Cr^\alpha$. This implies $\dim(E)\ge\alpha$ since for any covering of E we have

$$\sum_j r_j^\alpha\ge C^{-1}\sum_j\mu(D(x_j,r_j))\ge C^{-1}\mu(E)>0.$$

Taking the infimum over all covers gives $H_\alpha^\infty(E)>C^{-1}\mu(E)>0$, which implies that $\dim(E)\ge\alpha$.

A point $x\in\Lambda(G)$ is called a *conical limit point* if there is a sequence of orbit points which converges to x inside a (Euclidean) non-tangential cone with vertex at x (such points are sometimes called radial limit points or points of approximation). The set of such points is denoted $\Lambda_c(G)$. The following is Theorem 1.1 of [9].

Theorem 2.1. *Suppose G is a discrete group of Möbius transformations with more than one limit point. Then $\delta(G)=\dim(\Lambda_c)\le\dim(\Lambda)$.*

Next we want to show that it is enough to estimate the Poincaré series along an orbit in $\Omega(G)$ (rather than along the orbit of $0\in\mathbf{B}^3$). For $z\in\Omega(G)$, let $d(z)=\operatorname{dist}(z,\partial\Omega)$ denote spherical distance.

Lemma 2.2. *If G is a Kleinian group, 0 is the center of $\mathbf{B}^3$ and $z_0 \in \Omega(G) \subset \partial\mathbf{B}^3$, then $d(g(z_0)) \simeq 1-|g(0)|$ for all $g \in G$, with constants that depend on z_0 and G, but not on g.*

Proof. It clearly suffices to prove this with the origin replaced by any other fixed point $x \in \mathbf{B}$. We may also assume that the group is normalized so that $\operatorname{diam}(\Lambda) = \operatorname{diam}(S^2) = 1$. Let D be a disk around z_0 with radius $\frac{1}{2}\operatorname{dist}(z_0, \Lambda)$, let H be the hyperbolic half-plane which meets S^2 in ∂D and let $x \in H$ be the point closest to 0. Then z_0 is one of the endpoints of the geodesic through x which is perpendicular to H. Thus $g(z_0)$ is an endpoint of the geodesic through $g(x)$ perpendicular to $g(H)$. This implies

$$\operatorname{dist}(g(z_0), \partial g(D)) \geq C^{-1}(1-|g(x)|),$$

since both endpoints of the geodesic are at least this far away from $\partial g(D)$. Since $g(z_0) \in g(D)$ and $g(D)$ does not hit Λ, we deduce $\operatorname{dist}(g(z_0), \Lambda) \geq C(1-|g(x)|)$.

Since Λ is separated from $g(z_0)$ by $g(H)$ and Λ is assumed to have large diameter, $g(z_0)$ must be on the side of $g(H)$ with smaller diameter. Thus

$$\operatorname{dist}(g(z_0), g(x)^*) \leq C(1-|g(x)|),$$

where $g(x)^*$ denotes the radial projection onto S^2. Next, note that since x is a fixed hyperbolic distance from the hyperbolic convex hull of Λ, there is a fixed M so that the spherical ball of radius $M(1-|g(x)|)$ around $g(x)^*$ must hit Λ. Thus $\operatorname{dist}(g(z_0), \Lambda) \leq C(1-|g(x)|)$, as desired. □

3. Large β's imply dimension >1

The main tool in the proof of Theorem 1.1 is the second author's "traveling salesman theorem" from [21]. It states that if E is a set in the plane then the shortest curve Γ which contains E has length comparable (with universal constants) to

$$\operatorname{diam}(E) + \sum_Q \beta(Q)^2 \operatorname{diam}(Q),$$

where the sum is over all dyadic squares in the plane. Similarly, the length of the shortest curve which passes within ε of every point of E has length comparable to

$$\operatorname{diam}(E) + \sum_{\operatorname{diam}(Q) \geq \varepsilon} \beta(Q)^2 \operatorname{diam}(Q),$$

where the sum is over all dyadic squares larger than size ε.

Proof of Theorem 1.1. Suppose E_0 is compact, connected and uniformly wiggly. Suppose Q is a square of side length $\operatorname{diam}(Q)=r=2^{-N}\leq\operatorname{diam}(E_0)$ with $\frac{1}{3}Q\cap E_0\neq\emptyset$. Our first objective is to show that for small enough $\varepsilon>0$ (depending only on the β_0 in the definition of uniformly wiggly) we can find more than $1000\varepsilon^{-1}$ subsquares of Q of sidelength $\varepsilon\operatorname{diam}(Q)$ with disjoint doubles and such that each contains a point of E_0 in its middle third. We will then apply the same argument to each subsquare and use the resulting nested collection of squares to build a measure μ on E_0.

Define $E=(E_0\cap Q)\cup\partial Q$. Note that E is connected since $\operatorname{diam}(Q)\leq\operatorname{diam}(E_0)$. Fix some integer n so that $2^{-n}<r$ (possibly much smaller). Because E is connected, there are more than $\frac{1}{3}r2^n$ dyadic subsquares $\{Q_j\}$ of $\frac{2}{3}Q$ of size 2^{-n} such that $Q_j\cap E\neq\emptyset$. To see this, consider concentric "annuli" between $\frac{1}{3}Q$ and $\frac{2}{3}Q$ made of squares of size 2^{-n}; there are $\frac{1}{3}r2^n$ such and each must intersect E. See Figure 3.1.

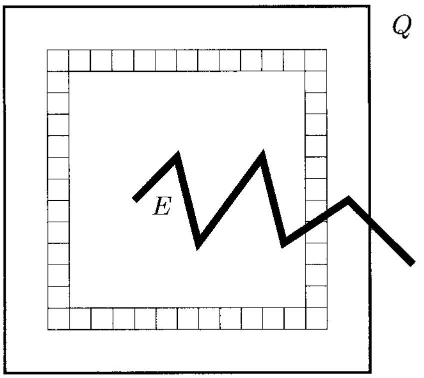

Figure 3.1. Many squares must hit E.

From this we deduce

$$\sum_{Q:\operatorname{diam}(Q)=2^{-n}}\beta_E^2(Q)\operatorname{diam}(Q)\geq\tfrac{1}{3}\beta_0^2r.$$

Thus for any integer $k\geq1$ (recall $r=2^{-N}$),

$$\sum_{n=N+1}^{N+k}\sum_{Q:\operatorname{diam}(Q)=2^{-n}}\beta_E^2(Q)\operatorname{diam}(Q)\geq\tfrac{1}{3}k\beta_0^2r.$$

Suppose Γ_k is the shortest curve in the plane with the property that for each $z\in E$ we have $\operatorname{dist}(z,\Gamma_k)\leq2^{-N-k}$. It is fairly easy to check that for squares with $\operatorname{diam}(Q)\geq10\cdot2^{-N-k}\beta_0^{-1}$, we have

$$\beta_{\Gamma_k}(Q)\geq\tfrac{1}{2}\beta_E(Q)\geq\tfrac{1}{2}\beta_0.$$

Therefore

$$\sum_{n=N+1}^{N+k} \sum_{Q:\operatorname{diam}(Q)=2^{-n}} \beta_{\Gamma_k}^2(Q)\operatorname{diam}(Q) \geq \tfrac{1}{3}(k-10\beta^{-1})\beta_0^2 r.$$

Choose $k>20\beta_0^{-1}$. Then the term on the right is $\geq\frac{1}{6}k\beta_0^2 r$.

By the second author's characterization of rectifiable curves in [21], the length of Γ_k is at least $C_0(\operatorname{diam}(\Gamma_k)+\beta_0^2 kr)$, for some absolute constant C_0. We claim that this implies that there are more than $(C_0\beta_0^2 kr-4r)C2^{N+k}$ boxes $\{Q_j\}$ of side length 2^{-N-k} such that $\frac{1}{3}Q_j\cap E\neq\emptyset$.

To prove this claim, let $\{z_j\}$ be a collection of points on Γ_k so that $j\neq j'$ implies $|z_j-z_{j'}|\geq 2^{-N-k}$, but so that $\bigcup_j B(z_j, 2^{-N-k+2})$ covers E. Let $\mathcal{C}$ be the collection $\{\partial Q_k\}$ of all dyadic squares of size 2^{-N-k} contained in Q which contain some z_j. Let $\Gamma=\bigcup_{Q_j\in\mathcal{C}}\partial Q_j\cup\bigcup_j S_j$ be the union of the boundaries of these squares, together with segments S_j which connect ∂Q_j with the point z_j. Then obviously

$$l(\Gamma)\leq 6\cdot 2^{-N-k}\#(\mathcal{C}).$$

Since Γ has the property that it passes within 2^{-N-k} of every point of E and since Γ_k was defined to be the shortest such curve, we must have

$$6\cdot 2^{-N-k}\#(\mathcal{C})\geq l(\Gamma)\geq l(\Gamma_k)\geq C_0\beta_0^2 k 2^{-N}$$

and hence

$$\#(\mathcal{C})\geq\tfrac{1}{6}C_0\beta_0^2 k2^k.$$

Now set $\varepsilon=2^{-k}$. If $k>600000\beta_0^{-2}C_0^{-1}$, then $|\mathcal{C}|\geq 100000\varepsilon^{-1}$, and consists of disjoint dyadic squares of size $\varepsilon\operatorname{diam}(Q)$ each of which contains a point of E. Replace each square by its triple. By throwing away $\frac{80}{81}$'s of the squares we can assume the remaining ones have disjoint triples, and each hits E in its middle third, as desired.

To finish the proof of the theorem, we build nested generations of squares using the construction above. The initial square Q_0 forms the first generation. The squares of size $\varepsilon\operatorname{diam}(Q_0)$ constructed above form the first generation. In general, given an nth generation square containing a point of E_0 in its middle third, we construct $1000\varepsilon^{-1}$ subsquares as above (with disjoint triples and containing a point of E_0 in their middle thirds), and put these into the $(n+1)$st generation.

We then define a measure μ by assigning each nth generation square equal mass (namely $(\varepsilon/1000)^n$)). Since an nth generation square Q has size $\varepsilon^n\operatorname{diam}(Q_0)$, this measure satisfies

$$\mu(Q)\leq C\operatorname{diam}(Q)^\alpha,$$

where

$$\alpha = \frac{\log\varepsilon - \log 1000}{\log\varepsilon} = 1 + \frac{\log 1000}{\log(1/\varepsilon)} > 1.$$

Since $\varepsilon = 2^{-k}$ and $k \simeq \beta_0^{-2}$ we get $\log\varepsilon^{-1} \simeq \beta_0^{-2}$. This gives the estimate in the theorem. It only remains to check that this inequality holds for all squares in the plane, but this is a standard argument. Thus by the mass distribution principle $\dim(E_0) \geq \alpha > 1$. $\square$

Given a compact set K in the plane let $\Omega = \mathbf{R}^2 \setminus K$ be its complement. A Whitney decomposition of Ω is a collection of squares $\{Q_j\}$ which are disjoint, except along their boundaries, and such that there is a $C < \infty$ such that

$$\frac{1}{C}\operatorname{dist}(Q_j, \partial\Omega) \leq \operatorname{diam}(Q_j) \leq C \operatorname{dist}(Q_j, \partial\Omega).$$

The existence of a Whitney decomposition for any open set is a standard fact in real analysis (e.g., [32]). One can define $\{Q_j\}$ to be the maximal collection of dyadic squares in Ω such that $\operatorname{diam}(Q) \leq \operatorname{dist}(Q, \partial\Omega)$. This gives

$$\tfrac{1}{4}\operatorname{dist}(Q, \partial\Omega) \leq \operatorname{diam}(Q) \leq \operatorname{dist}(Q, \partial\Omega).$$

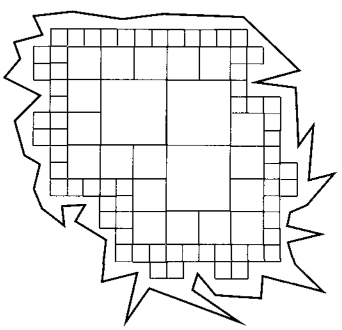

Figure 3.2. Whitney squares.

Corollary 3.1. *Suppose K is connected and $\beta_K(Q) > \beta_0 > 0$ for every square of the form $Q = 10Q'$ with Q' a Whitney square for Ω (the complement of K) and* $\operatorname{diam}(Q') \leq \operatorname{diam}(K)$. *Then* $\dim(K) > 1$.

Proof. To prove this we simply note that the hypotheses imply that $\beta_K(Q) \geq \beta_0/C$ for every square intersecting K and some absolute $C < \infty$. If $\beta_K(Q) > 1/100$ there is nothing to do (take $C = 100$). If $\beta_K(Q) \leq 1/100$ then Q contains a Whitney square Q' for Ω of comparable size with $10Q' \subset Q$ and so $\beta_K(Q) \geq \beta_K(10Q')/C \geq \beta_0/C$. $\square$

We should point out that in the previous result, one needs the estimate for Whitney squares in all the complementary components of K. One can construct a Jordan domain so that the β's for Whitney squares of the bounded component are all bounded away from zero, but so that the boundary has dimension 1.

4. Limit sets which are uniformly wiggly

Proof of Theorem 1.2. It suffices to show that if Λ is not uniformly wiggly, then Λ is a circle. Consider a square Q with $\operatorname{diam}(Q) \leq \operatorname{diam}(\Lambda)$, $\frac{1}{3}Q \cap \Lambda \neq \emptyset$ and $\beta = \beta_\Lambda(Q)$ very small. Since Λ is connected we may pass to a subsquare if necessary and assume that $\Lambda \cap \beta^{-1/2}Q$ is contained in a strip S of width $\leq C\beta^{1/2} \operatorname{diam}(Q)$ and that it "crosses" Q. Choose a point $z_1 \in \Omega \cap 3Q$ with distance $\operatorname{diam}(Q)$ from Λ. Since $\Omega(G)/G$ is compact, we can find a group element $g \in G$ so that $\operatorname{dist}(g(z_1), \Lambda) > \delta$ (in Euclidean metric) for some δ depending on G, but not on z_1. (This would be false if Ω/G was not compact.) Since Λ is G-invariant, $g(\Lambda) = \Lambda$. On the other hand, $g(\Lambda)$ is contained in $g(S) \cup g(S^2 \setminus \beta^{-1/2}Q)$. See Figure 4.1.

Since lines map to circles under Möbius transformations, $g(S)$ is the region between two circles and the distance between these circles looks like at most $C\beta^{1/2}$. To see this let L be the line through z_1 which is perpendicular to the strip S and let z_2 be the point on L midway between z_1 and S. By Koebe's distortion theorem $\operatorname{dist}(g(z_2), \Lambda)$ is also bounded uniformly away from 0. Let z_3, z_4 be the points where L intersects the two sides of the strip S. The cross ratios of (z_1, z_2, ∞, z_3) and (z_1, z_2, ∞, z_4) clearly differ by only a factor bounded by $C\beta^{1/2}$. Thus the same is true after we map by g. Since $g(L)$ is perpendicular to the two circles bounding $g(S)$ and $g(z_1)$, $g(z_2)$ are bounded away from these circles, we see that $g(z_3)$ and $g(z_4)$ are bounded away from $g(\infty)$. Thus the maximum width of $g(S)$ is bounded by $C\beta^{1/2}$, as desired.

Similarly, $g(S^2 \setminus \beta^{-1/2}Q)$ is contained in a small disk. Thus Λ is contained in a neighborhood of a circle on the sphere where the width of the region can be estimated in terms of $\beta_\Lambda(Q)$. Thus if $\beta_\Lambda(Q)$ is not bounded below, Λ must be a circle. $\square$

Its easy to see that if Ω/G is compact then Λ is uniformly perfect (e.g., [14], [30]). Using this, the proof above gives the following corollary.

Corollary 4.1. *Suppose Ω/G is compact. Then Λ is either contained in a circle or $\beta_\Lambda(Q) \geq \beta_0$ for every square Q such that $\frac{1}{3}Q \cap \Lambda \neq \emptyset$ and* $\operatorname{diam}(Q) \leq \operatorname{diam}(\Lambda)$.

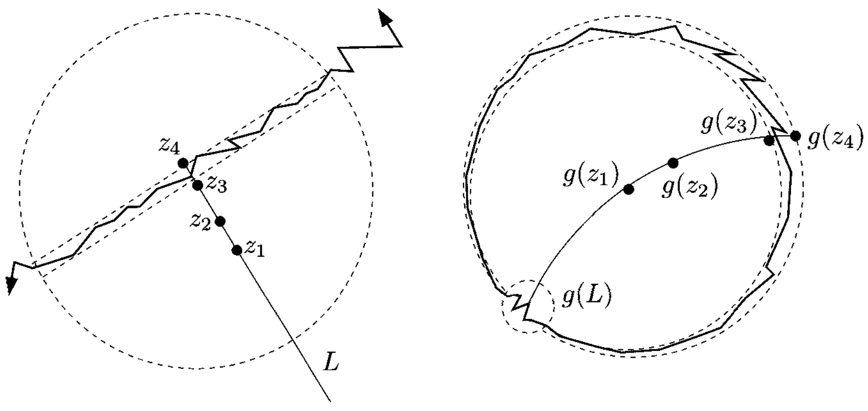

Figure 4.1. Co-compact groups have big β's.

Next we will consider another class of groups which have uniformly wiggly limit sets. Suppose G is a finitely generated degenerate group, i.e., $\Omega(G)=\Omega$ has a unique component and this component is simply connected. Such a group must be geometrically infinite, so results from [9] imply we actually have $\dim(\Lambda)=2$, but this is much harder, and does not directly imply the set is uniformly wiggly.

Theorem 4.2. *If G is a finitely generated degenerate group then $\Lambda(G)$ is uniformly wiggly.*

Proof. If the Fuchsian equivalent $\widehat{G}=\Phi\circ G\circ\Phi^{-1}$ has no parabolics, then we already know this (e.g., Theorem 1.2). Therefore we may assume $\widehat{G}$ does have parabolics. Fix a small number δ and for each parabolic fixed point of $\widehat{G}$, choose a generator $\hat{g}_j$ of the parabolic group fixing that point and let B_j^1 be the set of points in $\mathbf{D}$ which are moved less than hyperbolic distance δ by $\hat{g}_j$. Similarly, let B_j^2 be the points moved by less than $\frac{1}{2}\delta$ by $\hat{g}_j$. Let $X=\mathbf{D}\setminus\bigcup_j B_j^2$. Then $X/\widehat{G}$ is compact.

The domain $\Phi(B_j^1)$ must be a quasidisk with constant depending only on δ. To prove this, suppose $g\in G$ corresponds to $\hat{g}$ and conjugate g so its fixed point is ∞, i.e., g is a translation. Let γ be an arc of length δ on ∂B_j^1. By the Koebe distortion theorem its image is a smooth arc and $\Phi(\partial B_j^1)$ consists of translations of this arc joined end to end, and hence is a quasicircle since it satisfies the three point condition: there is an $M<\infty$ so that if z_1, z_2, z_3 are any three points on the curve with z_1 on the shorter arc between z_2 and z_3, then

$$\frac{|z_3-z_1|}{|z_2-z_1|}\le M,$$

(e.g., Theorem IV.4, [3]). Composing with Möbius transformations preserves quasicircles, so any image of a horoball conjugate to B_j^1 is a quasidisk with the same constant.

Now suppose Λ is not uniformly wiggly, i.e., suppose we have a square Q with $\frac{1}{3}Q\cap\Lambda\neq\emptyset$, $\operatorname{diam}(Q)\leq\operatorname{diam}(\Lambda)$ and $\beta=\beta_\Lambda(Q)$ is small. Let S be a strip of width $\beta\operatorname{diam}(Q)$ centered on a line L containing $\Lambda\cap Q$. Choose a point $z\in\frac{1}{3}Q$ which is about distance $\beta^{1/2}$ from L. If we can choose $z\in\Phi(X)$ then the proof of Theorem 1.2 gives a contradiction, if β is small enough.

So suppose $z\notin\Phi(X)$. Then z is in the image of some horoball B_j^2 and is moved less than hyperbolic distance $\frac{1}{2}\delta$ by some element g of G. Let z' be the reflection of z across L. By normal families, it is easy to see that if β is small enough then g moves z' by less than δ in the hyperbolic metric on Ω (because in the limit as $\beta\to 0$, the transformation tends to one preserving L and is symmetric with respect to L). This means that $z'\in\Phi(B_j^1)$.

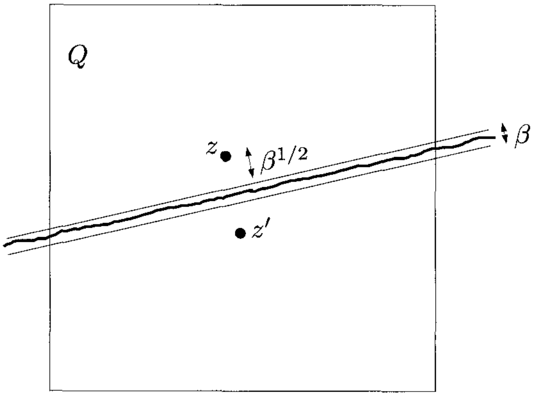

Figure 4.2. Degenerate limit sets cannot have small β's.

Thus z and z' are both in $\Phi(B_j^1)\subset\Omega$ which is a quasidisk with constant depending on G but not on z, Q or β. The boundary of this quasidisk must intersect the segment connecting z to z' in at least two points and must leave Q between these two points. Thus if β is small enough the three point condition must be violated, giving a contradiction. This proves the result. □

5. The Schwarzian derivative

The Schwarzian derivative of a locally univalent function f is defined by

$$S(f)(z)=\left[\frac{f''(z)}{f'(z)}\right]'-\frac{1}{2}\left[\frac{f''(z)}{f'(z)}\right]^2.$$

The following facts are standard (e.g., see [17]). First, $S(f)\equiv 0$ if and only if f is a Möbius transformation. Furthermore, S satisfies the composition law

$$S(f\circ g)=S(f)(g')^2+S(g).$$

In particular, if g is Möbius then

$$S(f\circ g)=S(f)(g')^2 \quad\text{and}\quad S(g\circ f)=S(f).$$

If G is a Kleinian group and Ω is a simply connected invariant component of $\Omega(G)$ and $\Phi\colon \mathbf{D}\to\Omega$ is a Riemann mapping then $|S(\Phi)|(1-|z|^2)^2$ is constant along orbits of the Fuchsian equivalent $\widehat{G}=\Phi^{-1}\circ G\circ\Phi$. Thus either Φ is Möbius or $|S(\Phi)|(1-|z|^2)^2$ bounded away from zero along some orbit of $\widehat{G}$. Suppose f is analytic on $\mathbf{D}$. If f is univalent and $\varphi=\log f'$ then

$$|S(f)(z)|\le\frac{6}{(1-|z|^2)^2} \quad\text{and}\quad |\varphi'(z)|\le\frac{6}{1-|z|^2}.$$

The following is from [8] but we give a proof for completeness.

Lemma 5.1. *There is a $C<\infty$ such that if $\Phi\colon\mathbf{D}\to\Omega$ is univalent and $\partial\Omega$ is rectifiable (i.e., has finite 1-dimensional measure) then*

$$\iint_{\mathbf{D}}|\Phi'(z)|\,|S(\Phi)(z)|^2(1-|z|^2)^3\,dx\,dy<Cl(\partial\Omega).$$

The same holds if Φ is only defined on a Lipschitz subdomain $\mathcal{D}\subset\mathbf{D}$.

Proof. The lemma is an application of the following version of Green's formula. Suppose that F is holomorphic on $\mathbf{D}$ and satisfies

$$\iint_{\mathbf{D}}|F'(z)|^2(1-|z|)\,dx\,dy<\infty.$$

Then $F\in H^2(\mathbf{D})$ (the Hardy space) and

$$\int_{\partial\mathbf{D}}|F(z)|^2\,ds(z)\simeq|F(0)|^2+\iint_{\mathbf{D}}|F'(z)|^2(1-|z|)\,dx\,dy.$$

Setting $F(z)=\sum_n a_n z^n$, then the proof is an easy calculation using the fact that the right hand side is $\sum_n |a_n|^2$ and that the left hand side is

$$|a_0|^2+\sum_{n=1}^{\infty} n^2|a_n|^2 \int_0^1 r^{2(n-1)}(1-r)r\,dr = |a_0|^2+\sum_{n=1}^{\infty} n^2|a_n|^2\left(\frac{1}{2n}-\frac{1}{2n+1}\right)$$
$$\simeq \sum_{n=0}^{\infty} |a_n|^2.$$

A similar calculation shows

$$\int_{\partial\mathbf{D}} |F(z)|^2\,ds(z) \simeq |F(0)|^2+|F'(0)|^2+\iint_{\mathbf{D}} |F''(z)|^2(1-|z|)^3\,dx\,dy.$$

We apply this with $F=(\Phi')^{1/2}$. Then if $\varphi=\log(\Phi')$, $F'=\frac{1}{2}(\Phi')^{1/2}(\varphi')$, and

$$F''=\tfrac{1}{2}(\Phi')^{1/2}\big(\varphi''+\tfrac{1}{2}(\varphi')^2\big)=\tfrac{1}{2}(\Phi')^{1/2}(S(\Phi)+(\varphi')^2).$$

Thus

$$\begin{aligned}\iint_{\mathbf{D}} |\Phi'(z)|\,|S(\Phi)|^2(1-|z|)^3\,dx\,dy &\le C\iint_{\mathbf{D}} |\Phi'(z)|\,|S(\Phi)+(\varphi'(z))^2|^2(1-|z|)^3\,dx\,dy\\ &\quad +C\iint_{\mathbf{D}} |\Phi'(z)|\,|\varphi'(z)|^4(1-|z|)^3\,dx\,dy\\ &\le C\iint_{\mathbf{D}} |F''(z)|^2(1-|z|)^3\,dx\,dy\\ &\quad +C\sup_z |\varphi'(z)|^2(1-|z|^2)\\ &\quad \times\iint_{\mathbf{D}} |F'(z)|^2(1-|z|)\,dx\,dy.\end{aligned}$$

Since Φ is univalent, $\sup_z |\varphi'(z)|^2(1-|z|^2)$ is uniformly bounded and so by the formulas above, each of the two terms on the right is bounded by $C\int|\Phi'|\,d\theta\le Cl(\partial\Omega)$.

The same proof works for Lipschitz domains, using the following version of Green's theorem for such domains ([16], [20]). Suppose $0\in\mathcal{D}$, $\mathcal{D}$ is Lipschitz with constant M, and that $\operatorname{dist}(0,\partial\mathcal{D})\sim\operatorname{diam}(\mathcal{D})=1$. Then if $d(z)=\operatorname{dist}(z,\partial\mathcal{D})$,

$$\int_{\partial\mathcal{D}} |F(z)|^2\,ds(z)\simeq |F(0)|^2+\iint_{\mathcal{D}} |F'(z)|^2(1-d(z))\,dx\,dy,$$
$$\int_{\partial\mathcal{D}} |F(z)|^2\,ds(z)\simeq |F(0)|^2+|F'(0)|^2+\iint_{\mathcal{D}} |F''(z)|^2(1-d(z))^3\,dx\,dy. \quad \square$$

Although we will not need it in this paper, we should mention that there is a close relationship between the Schwarzian of a Riemann mapping Φ and the β's of the corresponding $\partial\Omega$. In [7] the following is proven: suppose E is compact and connected, $\Omega=\overline{\mathbf{C}}\backslash E$ and $\Phi\colon \mathbf{D}\to\Omega$ the Riemann map. Then

$$|S(\Phi)(w)|(1-|w|^2)^2 \le C\sum_{n=0}^{\infty}\beta_E(2^nQ)2^{-\mu n}$$

where $r=\operatorname{dist}(\Phi(w),\partial\Omega)$. The number μ satisfies $0<\mu<1$ but can be taken as close to 1 as we wish. The constant C depends only on the choice of μ. There is also a version of this for disconnected E using the universal covering map on the complement. Thus large Schwarzian implies large β's, although the converse need not be true.

A set $\{z_n\}\subset\mathbf{D}$ is called *non-tangentially dense* if almost every point of $\partial\mathbf{D}$ is a non-tangential limit point of the set. Although in this definition the angle associated to each boundary point may be different, this is not really a restriction; if we fix any positive angle and only consider convergence within cones of that angle then almost every boundary point is still a limit point (e.g., see the proof of Theorem IX.5.1 of [19]). Thus if we fix any $0<C<\infty$ and let I_n be the interval on $\partial\mathbf{D}$ with center $z_n/|z_n|$ of length $C(1-|z_n|)$ we see that $\{z_n\}$ is non-tangentially dense if and only if almost every point of the circle is in arbitrarily small I_n's.

The orbit of a Fuchsian group is non-tangentially dense if and only if the group is divergence type, i.e.,

$$\sum_{g\in G}(1-|g(0)|)=\infty.$$

See, for example, Theorem 6.3.3 of [26]. If G is analytically finite with an invariant simply connected component Ω, then the Fuchsian equivalent $\widehat{G}$ is divergence type. The following is a quantitative version of a result of Pommerenke (Theorem 4 in [29]).

Lemma 5.2. *Suppose G has an invariant simply connected component Ω which is not a disk and that the Fuchsian equivalent $\widehat{G}$ of G is divergence type. Let z_0 be in the $\widehat{G}$ orbit of 0 and let I_0 be the interval centered at $z_0/|z_0|$ of length $10(1-|z_0|)$. Let Φ be a Riemann mapping onto Ω such that $\varepsilon=|S(\Phi)(0)|>0$ and for $I\subset I_0$ let $tI=\{tz:z\in I\}$. Then*

$$\frac{l(\Phi((1-r|z_0|)I))}{\operatorname{dist}(\Phi(z_0),\partial\Omega)}\to\infty,$$

as $r\to 0$ with estimates that only depend on $\widehat{G}$, $|I|/|I_0|$ and $|S(\Phi)(0)|$.

Proof. By rescaling we may assume that $\operatorname{dist}(\Phi(z_0), \partial\Omega)=1$. Suppose that $\Phi((1-r|z_0|)I)$ has length $\leq L$ for all $0<r<1$. Then Φ' is in the Hardy space H^1 for the Carleson square with base I, so by the non-tangential maximal theorem there is a compact set $E\subset I$ with $|E|\geq\frac{1}{2}|I|$ so that for each $x\in E$ there is a cone (say of angle $\frac{1}{2}\pi$) in which

$$\frac{1}{M}\leq|\Phi'|\leq M,$$

where M depends only on L. By taking the union of these cones we construct a "sawtooth" domain $W\subset\mathbf{D}$ such that $M^{-1}\leq|\Phi'|\leq M$ on W.

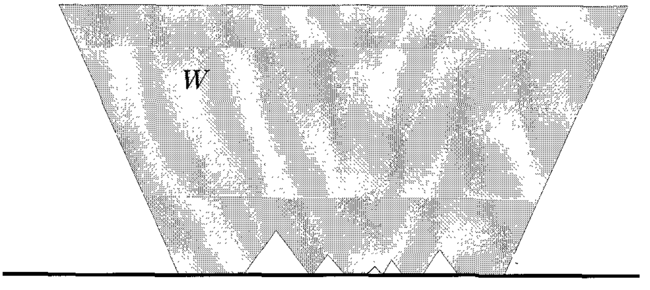

Figure 5.1. The sawtooth domain.

For each orbit point $z_n\in W$, let D_n denote the intersection of W with the disk centered at z_n of radius $\frac{1}{2}(1-|z_n|)$. By the Koebe $\frac{1}{4}$ theorem $|\Phi'|\geq C|\Phi'(z_n)|$ on all of D_n and by subharmonicity,

$$\int_{D_n}|S(\Phi)|^2\,dx\,dy\geq C\operatorname{area}(D_n)|S(\Phi)(z_n)|^2.$$

Thus

$$\begin{aligned}\iint_{D_n}|\Phi'(z)|\,|S(\Phi)(z)|^2(1-|z|^2)^3\,dx\,dy&\geq C\varepsilon^2\iint_{D_n}|\Phi'(z)|(1-|z|)^{-1}\,dx\,dy\\&\geq CM^{-1}\varepsilon^2(1-|z_n|).\end{aligned}$$

Therefore by the Lipschitz case of Lemma 5.1,

$$L\geq C\iint_{W}|\Phi'(z)|\,|S(\Phi)(z)|^2(1-|z|^2)^3\,dx\,dy\geq CM^{-1}\varepsilon^2\sum_{n:z_n\in W}1-|z_n|.$$

We want to see that the last term grows to infinity as $r\to1$ with an estimate which is independent of W, giving a contradiction. Suppose this was false, i.e., there are sequences of sawtooth domains $\{W_n\}$ and radii $\{r_n\}$ so that $|E_n|=|\partial W_n \cap r_n I| \geq \frac{1}{2}|I|$ and

$$\sum_{j:z_j\in W_n} 1-|z_j| \leq C < \infty,$$

for all n. Passing to the limit (in the Hausdorff metric) of some subsequence we get a sawtooth domain W so that $|E|=|\partial W \cap I| \geq \frac{1}{2}|I|$ and

$$\sum_{n:z_n\in W} 1-|z_n| \leq C < \infty.$$

(We leave it as an easy exercise to verify that there is a limiting domain W with these properties.)

To show this is impossible we use the Vitali covering lemma: if E is a set and $\{I_j\}$ is a collection of intervals such that each point of E is contained in intervals in $\{I_j\}$ of arbitrarily small length, then there is a disjoint subcollection which covers almost every point of E (e.g., see [38]).

By assumption, $\widehat{G}$ has non-tangentially dense orbits, so if we associate to each orbit point z_n the interval I_n of length $1-|z_n|$ centered at $z_n/|z_n|$, then almost every point of the circle is in infinitely many of the intervals $\mathcal{F}=\{I_n\}$. Let F be the subset of $\partial W \cap I$ which is Vitali covered by $\{I_n\}$. Then $|F| \geq \frac{1}{2}|I|$ and we can use the Vitali covering lemma to obtain a disjoint covering of almost every point of F by intervals in $\{I_n\}$. In fact, by repeated applications of the covering lemma we can find infinitely many collections $\mathcal{F}_k=\{I_j^k\}\subset\mathcal{F}$ each of which is a disjoint covering of almost all of F, and so that no interval belongs to more than one collection. Thus

$$\sum_{n:z_n\in W} 1-|z_n| \geq \sum_k \sum_{I_n\in\mathcal{F}_k} |I_n| = \sum_k \tfrac{1}{2}|I| = \infty.$$

Therefore $\Phi(rI)$ must have length tending to ∞ as $r\to 1$ with estimates depending only on I, the group $\widehat{G}$ and $|S(\Phi)(0)|$.

If we divide I_0 into N equal intervals, we can apply the proof to each interval and then by taking the minimum growth rate for $l(\Phi(rI))$, get an estimate which is valid for all N intervals. Since any interval of length $\geq 4\pi/N$ contains at least one of the above intervals, the growth rate also holds for such an interval. This finishes the proof. □

The hypothesis in this result that the Fuchsian equivalent be divergence type is necessary. Astala and Zinsmeister [5] have shown that any convergence group (i.e., $\sum_{\gamma\in G}(1-|\gamma(0|)<\infty)$ has a quasiconformal deformation to a Kleinian group whose limit set is a rectifiable curve but not a circle.

6. Proof of Theorem 1.3

Proof of Theorem 1.3. If the limit set is a circle then clearly $\dim(\partial\Omega)=1$ and hence $\delta\leq 1$ by Theorem 2.1. Since it is easy to see that $\delta\geq 1$ under the hypotheses of the theorem, we get $\delta=1$. Thus to complete the proof we need only show that if Ω is not a disk then $\delta>1$.

So assume Ω is not a disk. Let $\Phi\colon \mathbf{D}\to\Omega$ be a Riemann mapping and normalize so that $|S(\Phi)(0)|\neq 0$. Let $\widehat{G}=\Phi^{-1}\circ G\circ\Phi$ denote the Fuchsian equivalent of G. Let $\{z_j\}$ be the orbit of 0 under $\widehat{G}$ and for each j, let $I_j\subset\partial\mathbf{D}$ be the interval on the boundary centered at $z_j/|z_j|$ with length $1-|z_j|$. We let S_j denote the Carleson square with base I_j,

$$S_j=\{z:z/|z|\in I_j,\ 1-|I_j|\leq|z|<1\}.$$

Let

$$d_j=d(\Phi(z_j))=\operatorname{dist}(\Phi(z_j),\partial\Omega)\simeq(1-|z_j|)|\Phi'(z_j)|.$$

We will show the following lemma.

Lemma 6.1. *There is a $C<\infty$ (depending only on G) such that if $g\in\widehat{G}\setminus\{\mathrm{Id}\}$, and $z_0=g(0)$ then there is a collection of orbit points $\{z_k\}=\{g_k(0)\}\subset S_0$ such that*

(1) $\sum_k d_k\geq 2d_0$,
(2) $\frac{1}{2}(1-|z_0|)\geq(1-|z_k|)\geq(1-|z_0|)/C$,
(3) *the intervals $\{I_k\}$ are disjoint and $I_k\subset I_0$ for all k.*

Note that conditions (1) and (2) imply $\sum_k d_k^{1+\varepsilon}\geq d_0^{1+\varepsilon}$, if ε is small enough (depending on C). Using condition (3), we can break the orbit of $0\in\mathbf{D}$ into generations $\mathcal{G}_n$ so that

$$\sum_{z_k\in\mathcal{G}_n} d_k^{1+\varepsilon}\geq\sum_{z_k\in\mathcal{G}_{n-1}} d_k^{1+\varepsilon}\geq 1,$$

which implies

$$\sum_{z_k\in\widehat{G}(0)} d_k^{1+\varepsilon}=\infty.$$

By Lemma 2.2, this proves $\delta(G)>1$.

Thus it suffices to prove Lemma 6.1. Let $\Phi\colon\mathbf{D}\to\Omega$ be a Riemann map, normalized so that $S(\Phi)(0)\neq 0$. Conjugate G so that $\Phi(0)=\infty$ and $\operatorname{diam}(\partial\Omega)=1$. If Ω/G is a surface with punctures, then we can find a G invariant collection of disjoint balls $\mathcal{B}_1=\{B_j^1\}$ in Ω, each invariant under a parabolic element of G and so that $\bigl(\Omega\setminus\bigcup_j B_j^1\bigr)/G$ is compact (i.e., we are taking a neighborhood of each puncture on Ω/G and lifting it to Ω). This is the only place in the proof where we use analytic finiteness.

Each B_j^1 is thus conjugate to one of a finite subcollection and each ball has a parabolic fixed point of G on its boundary. To each ball in $\mathcal{B}_1$ we associate smaller invariant balls $B_j^2 \subset B_j^1$ so that the hyperbolic distances between ∂B_j^1 and ∂B_2^j is 1. Also any point of Ω which is outside $\bigcup_j B_j^2$ is within a bounded hyperbolic distance of the orbit of ∞. Let C_1 denote this bound.

If Ω/G has no punctures, just replace the collections $\mathcal{B}_j$ by the empty set in the proof that follows. Note that in this case every point of Ω is within a bounded hyperbolic distance of the orbit of ∞.

For each $t>0$ consider the level line of Green's function in the disk

$$\Gamma_t = \{z : |z| = 1-t\},$$

and for each orbit point $z_j \in \widehat{G}(0)$ let

$$\Gamma_t^j = \Gamma_{t(1-|z_j|)} \cap S_j.$$

The first thing we want to see is that the image of Γ_t^j is very long if t is small enough (independent of j).

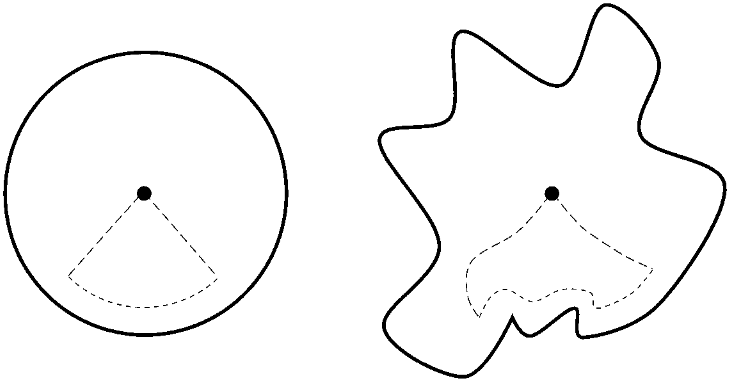

Figure 6.1. Level lines for Green's function.

Lemma 6.2. *For any $M>0$ there is a t_0 such that if $t \le t_0$ then*

$$l(\Phi(\Gamma_t^j)) \ge M d_j.$$

Proof. This is just Lemma 5.2. □

We would like to take the orbits for Lemma 6.1 to be near the curve Γ_t^j. More precisely, we will break Γ_t^j into unit hyperbolic segments $\{\gamma_k\}$ and to each segment associate the closest orbit point z_k. If there were no parabolic points then each orbit point would be associated to a bounded number of segments, say N (depending only on G) and by the standard distortion theorems for conformal maps,

$$l(\Phi(\gamma_k)) \simeq d(\Phi(z_k)).$$

Thus by throwing out repeated segments we would have a collection of points $z_k \in S_j$ with

$$\sum_k d_k \geq C \frac{M}{N} d_j,$$

and

$$d_k \geq C d_j$$

(where C depends on t and constants in certain distortion theorems for conformal maps). This proves the Lemma 6.1 when there are no parabolics.

If the surface Ω/G has punctures, then there may be points of Γ_t^j which are very far from the closest orbit of 0, and we need to replace such pieces by new curves which are closer to orbit points. The idea is that if Γ_t^j passes through the "bottom half" of a horoball B in the unit disk with tangent point p on the unit circle, then we replace $\Gamma_t^j \cap B$ by the arcs

$$\Gamma_B = (\partial B \cap \{z : 1-|z| \leq t(1-|z_j|)\}) \setminus B(p, t(1-|z_j|)).$$

See Figure 6.2. Then the Euclidean distance of Γ_B from the unit circle is greater than $\varepsilon_B t(1-|z_j|)$ where ε_B is a positive constant depending only on the conjugacy class of B.

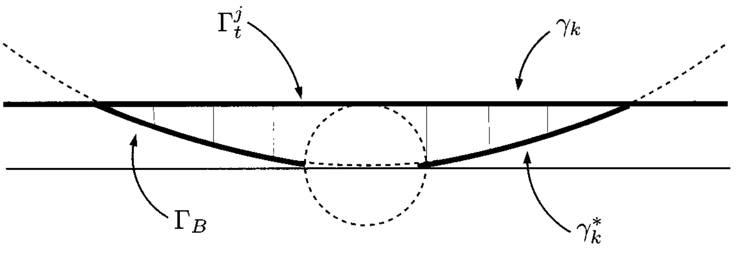

Figure 6.2. Changing Γ inside horoballs.

The arcs Γ_B remain within a bounded hyperbolic distance of the orbit of 0. Most importantly, Beurling's projection theorem implies that the length of $\Phi(\Gamma_B)$ is bounded below by a universal constant times the length of $\Phi(\Gamma_t^j \cap B)$. To see this, recall that Beurling's projection theorem (e.g., [4]) says that if Ω is simply connected and $z_0 \in \Omega$ then

$$\omega(z_0, \partial\Omega \cap D(x,r), \Omega) \leq C\left(\frac{r}{\operatorname{dist}(z_0, \partial\Omega)}\right)^{1/2},$$

for any disk $D(x,r)$. In particular, if a subset of $\partial\Omega$ has large harmonic measure with respect to z_0, it must have diameter bounded away from zero by a multiple of $\operatorname{dist}(z_0, \partial\Omega)$.

Cut $\Gamma_t^j \cap B$ into unit hyperbolic segments $\{\gamma_j\}$ and project each, except the center one, vertically onto an arc γ_j^* of Γ_B. Then γ_j^* has harmonic measure bounded uniformly away from zero with respect to w_j, the center of γ_j. Thus by Beurling's theorem, $\Phi(\gamma_j^*)$ must have length bounded below by a universal constant times $\operatorname{dist}(\Phi(w_j), \partial\Omega)$, which by Koebe's distortion theorem, is comparable to $l(\Phi(\gamma_j))$. The center piece of $\Gamma_t^j \cap B$ is handled by using Koebe's theorem to compare it to adjacent arcs.

Thus we have proved the following lemma.

Lemma 6.3. *For each z_j there is an arc $\widetilde{\Gamma}_t^j$ consisting of pieces of Γ_t^j and arcs of horoballs such that*

(1) $\widetilde{\Gamma}_t^j \subset S_j \cap \{|z| \leq 1 - \varepsilon_t(1-|z_j|)\}$,
(2) $l(\Phi(\widetilde{\Gamma}_t^j)) \geq M d_j$,
(3) *every component of $\widetilde{\Gamma}_t^j$ has hyperbolic length at least* 1,
(4) *each point of $\widetilde{\Gamma}_t^j$ has at most hyperbolic distance C_1 from $G(0)$.*

We can now finish the proof of Lemma 6.1 just as in the case without parabolics described above. This finishes the lemma and hence completes the proof of Theorem 1.3. □

7. Groups with round components

Suppose G is a finitely generated Kleinian group and $\Omega(G)$ contains only components which are round disks. Then either $\Omega(G)$ consists of exactly two components or has infinitely many components which are disks. In the first case, G is an extended Fuchsian group and the limit set is a circle. In the second case Λ must have dimension >1 by a more general result of Larman [23]: there is an $\varepsilon > 0$ so that if $\{D_j\}$ is any collection of three or more disjoint open disks in the plane, then $\dim\left(\mathbf{R}^2 \setminus \bigcup_j D_j\right) \geq 1+\varepsilon$.

Theorem 7.1. *Suppose G is an analytically finite Kleinian group and $\Omega(G)$ contains infinitely many components which are disks. Then* $\dim(\Lambda(G)) \geq \delta(G) > 1$.

This is slightly different than Larman's result because we do not insist that every component be a disk. This is a known result; it is a special case of Theorem 1 of [15] which Canary and Taylor prove using a result of Furusawa [18]. It is also contained in results of Sullivan and of Patterson. Some very interesting pictures of this type of limit set appear in [13]. Further results on such limit sets and the corresponding groups are given in [22].

We include a short proof of Theorem 7.1 for completeness. It is a fairly standard computation involving Hausdorff measures. Let D_1, D_2 be distinct round components with disjoint closures. Denote the stabilizers by G_1 and G_2. Since the orbit of D_1 under G_2 accumulates densely on ∂D_2 and vice versa, we may assume (by choosing new disks if necessary) that

$$\begin{aligned} \operatorname{dist}(D_1, D_2) &\geq 1, \\ \operatorname{diam}(D_1) &\leq \tfrac{1}{1000}, \end{aligned}$$

and the double of each disk is contained in a fundamental polygon of the other group (so the translates of the doubles are disjoint).

Fix values of $\delta > 0$ and $N < \infty$. Suppose we construct a Cantor set E by an iterative construction in which a disk D is replaced by at most N disks $\{D_j\}$ such that

(1) $D_j \subset 2D$, and $\{2D_j\}$ are disjoint,
(2) $\delta \operatorname{diam}(D) \leq \operatorname{diam}(D_j) \leq \operatorname{diam}(D)/100$,
(3) $\sum_j \operatorname{diam}(D_j) \geq 2 \operatorname{diam}(D)$.

Then it is easy to see that $\dim(E) \geq \alpha(\delta, N) > 1$.

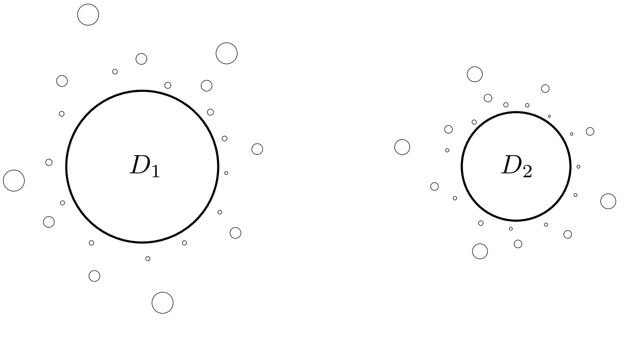

Figure 7.1. Constructing a Cantor set.

For D_1 we can choose δ, N and such disks $\{D_j^1\}$ by taking part of the orbit of D_2 under G_1 (using the fact that G_1 is divergence type). Similarly for D_2 and $\{D_j^2\}$. At a general step in the construction suppose we have a disk D' which is a "child" of D'' (i.e, $D' \subset 2D''$ and $\operatorname{diam}(D') \le \operatorname{diam}(D'')/100$). Then D' corresponds to either D_1 or D_2 under the action of $G_1 * G_2$, so assume it is D_1. Then there is an element $g \in G_1 * G_2$ so that $g(D_1) = D'$ and $g(D_2) = D''$. It is easy to check that g has bounded distortion on $2D_1$ (since it corresponds to the much smaller disk D'). Thus $\{g(D_j^1)\}$ satisfy the desired conditions with respect to D' and the construction may be continued (the constants may be different, but we have uniform bounds). The bounded distortion also shows that the orbit of a single point in the first generation disk stays near the center of each higher generation disk and this shows $\delta(G) > 1$ as well.

8. Proof of Corollary 1.4 and Corollary 1.5

Proof of Corollary 1.4. Suppose G has a connected limit set. Then any component Ω of $\Omega(G)$ is simply connected. The subgroup fixing any component Ω of $\Omega(G)$ is a finitely generated Kleinian group G_Ω and Ω is an invariant component of its ordinary set. Thus by Theorem 1.3 either $\dim(\Lambda(G)) > 1$ or every component of $\Omega(G)$ is a disk. If the latter case holds then either $\Omega(G)$ has two components or infinitely many. If it has two then $\Lambda(G)$ is a circle. Otherwise $\Lambda(G) = \overline{\mathbf{C}} \setminus \bigcup_j D_j$ for some infinite collection of disjoint open disks. Thus $\dim(\Lambda(G)) > 1$ by Larman's theorem [23] or Theorem 7.1.

It follows from the Klein–Maskit combination theorems that either $\Lambda(G)$ is totally disconnected or $\Lambda(G)$ contains a connected component which is itself the limit set of a finitely generated subgroup (see e.g. [1], [24]). If this component is not a circle then we are done by Theorem 1.3. If the component is a circle but $\Lambda(G)$ is not then there are infinitely many circular components and the argument of Section 7 shows $\dim(\Lambda) > 1$. $\square$

To deduce Corollary 1.5, we need to show that the first two cases of Corollary 1.4 do not occur if G is geometrically infinite. First, if G is analytically finite and Λ is a circle then G is Fuchsian (or has an index 2 Fuchsian subgroup) and must be geometrically finite. Secondly, it follows from the Klein–Maskit combination theorems that if G is analytically finite and $\Lambda(G)$ is totally disconnected then G is geometrically finite. Thus the third case of Corollary 1.4 must hold and this implies Corollary 1.5

References

1. ABIKOFF, W. and MASKIT, B., Geometric decompositions of Kleinian groups, *Amer. J. Math.* **99** (1977), 687–697.
2. AHLFORS, L. V., Finitely generated Kleinian groups, *Amer. J. Math.* **86** (1964), 413–429.
3. AHLFORS, L. V., *Lectures on Quasiconformal Mappings,* Math. Studies **10**, Van Nostrand, Toronto–New York–London, 1966.
4. AHLFORS, L. V., *Conformal Invariants: Topics in Geometric Function Theory*, McGraw-Hill, New York–Düsseldorf–Johannesburg, 1973.
5. ASTALA, K. and ZINSMEISTER, M., Mostow rigidity and Fuchsian groups, *C. R. Acad. Sci. Paris Sér. I Math.* **311** (1990), 301–306.
6. BERS, L., Inequalities for finitely generated Kleinian groups, *J. Analyse Math.* **18** (1967), 23–41.
7. BISHOP, C. J. and JONES, P. W., Harmonic measure and arclength, *Ann. of Math.* **132** (1990), 511–547.
8. BISHOP, C. J. and JONES, P. W., Harmonic measure, L^2 estimates and the Schwarzian derivative, *J. Analyse Math.* **62** (1994), 77–113.
9. BISHOP, C. J. and JONES, P. W., Hausdorff dimension and Kleinian groups, *Acta Math.* **179** (1997), 1–39.
10. BISHOP, C. J., JONES, P. W., PEMANTLE, R. and PERES, Y., The dimension of the Brownian frontier is greater than 1, *J. Funct. Anal.* **143** (1997), 309–336.
11. BOWEN, R., Hausdorff dimension of quasicircles, *Inst. Hautes Études Sci. Publ. Math.* **50** (1979), 11–25.
12. BRAAM, P., A Kaluza–Klein approach to hyperbolic three-manifolds, *Enseign. Math.* **34** (1988), 275–311.
13. BULLETT, S. and MANTICA, G., Group theory of hyperbolic circle packings, *Nonlinearity* **5** (1992), 1085–1109.
14. CANARY, R. D., The Poincaré metric and a conformal version of a theorem of Thurston, *Duke Math. J.* **64** (1991), 349–359.
15. CANARY, R. D. and TAYLOR, E., Kleinian groups with small limit sets, *Duke Math. J.* **73** (1994), 371–381.
16. COIFMAN, R., JONES, P. W. and SEMMES, S., Two elementary proofs of the L^2 boundedness of Cauchy integrals on Lipschitz graphs, *J. Amer. Math. Soc.* **2** (1989), 553–564.
17. DUREN, P., *Univalent Functions*, Springer-Verlag, Berlin–Heidelberg, 1983.
18. FURUSAWA, H., The exponent of convergence of Poincaré series of combination groups, *Tôhoku Math. J.* **43** (1991), 1–7.
19. GARNETT, J. B., *Bounded Analytic Functions*, Academic Press, Orlando, Fla., 1981.
20. JERISON, D. S. and KENIG, C. E., Hardy spaces, A_∞ and singular integrals on chord-arc domains, *Math. Scand.* **50** (1982), 221–248.
21. JONES, P. W., Rectifiable sets and the traveling salesman problem, *Invent. Math.* **102** (1990), 1–15.
22. KEEN, L., MASKIT, B. and SERIES, C., Geometric finiteness and uniqueness for Kleinian groups with circle packing limit sets, *J. Reine Angew. Math.* **436** (1993), 209–219.

23. Larman, D. H., On the Besicovitch dimension of the residual set of arbitrary packed disks in the plane, *J. London Math. Soc.* **42** (1967), 292–302.
24. Maskit, B., *Kleinian Groups*, Springer-Verlag, Berlin–Heidelberg, 1988.
25. McShane, G., Parker, J. R. and Redfern, I., Drawing limit sets of Kleinian groups using finite state automata, *Experiment. Math.* **3** (1994), 153–170.
26. Nicholls, P. J., *The Ergodic Theory of Discrete Groups,* London Math. Soc. Lecture Note Ser. **143**, Cambridge Univ. Press, Cambridge, 1989.
27. Okikiolu, K., Characterizations of subsets of rectifiable curves in $\mathbf{R}^n$, *J. London Math. Soc.* **46** (1992), 336–348.
28. Parker, J. R., Kleinian circle packings, *Topology* **34** (1995), 489–496.
29. Pommerenke, C., Polymorphic functions for groups of divergence type, *Math. Ann.* **258** (1982), 353–366.
30. Pommerenke, C., On uniformly perfect sets and Fuchsian groups, *Analysis* **4** (1984), 299–321.
31. Rohde, S., On conformal welding and quasicircles, *Michigan Math. J.* **38** (1991), 111–116.
32. Stein, E., *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, N. J., 1970.
33. Sullivan, D., The density at infinity of a discrete group of hyperbolic motions, *Inst. Hautes Études Sci. Publ. Math.* **50** (1979), 172–202.
34. Sullivan, D., Discrete conformal groups and measureable dynamics, *Bull. Amer. Math. Soc.* **6** (1982), 57–73.
35. Tomaschitz, R., Quantum chaos on hyperbolic manifolds: a new approach to cosmology, *Complex Systems* **6** (1992), 137–161.
36. Väisälä, J., Bilipschitz and quasisymmetric extension properties, *Ann. Acad. Sci. Fenn. Ser. A I Math.* **11** (1986), 239–274.
37. Väisälä, J., Vuorinen, M. and Wallin, H., Thick sets and quasisymmetric maps, *Nagoya Math. J.* **135** (1994), 121–148.
38. Wheeden, R. and Zygmund, A., *Measure and Integral*, Marcel Dekker, New York, 1977.

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