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The complex scaling method for scattering by strictly convex obstacles

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1. Introduction and statement of results

The purpose of this paper is to obtain upper bounds on the number of scattering poles in varying neighbourhoods of the real axis for scattering by strictly convex obstacles with C^∞ boundaries. The new estimates generalize our earlier results on the poles in small conic neighbourhoods of the real axis and include the recent result of Hargé and Lebeau [3] on the pole free region. In fact, one of the new components here is their observation on the choice of the angle of scaling (see Sect. 2).

The starting point of our approach is the same as in [13]: the poles are identified with the square roots of complex eigenvalues of a non-self-adjoint operator obtained by scaling 'all the way to the boundary'. That produces a new elliptic boundary problem for which a semi-classical calculus was developed in [13]. It was then applied to the study of the characteristic values of the scaled operator.

In the present work we adopt a more direct and microlocal approach partly similar to the one used in [9]. By a microlocalization on the boundary we reduce the problem to the study of ordinary differential boundary problem for which a detailed spectral information is available.

We recall that if P is $-\Delta$ on $\mathbf{R}^n\setminus\mathcal{O}$, with the Dirichlet boundary condition, and $\mathcal{O}$ is a bounded subset of $\mathbf{R}^n$ with a connected exterior, then the resolvent

$$(P-\lambda^2)^{-1}\colon L^2(\mathbf{R}^n\setminus\mathcal{O}) \longrightarrow H^2(\mathbf{R}^n\setminus\mathcal{O})\cap H^1_0(\mathbf{R}^n\setminus\mathcal{O}), \quad \operatorname{Im}\lambda>0,$$

extends to a meromorphic operator

$$(P-\lambda^2)^{-1}\colon L^2_{\mathrm{comp}}(\mathbf{R}^n\setminus\mathcal{O}) \longrightarrow H^2_{\mathrm{loc}}(\mathbf{R}^n\setminus\mathcal{O})\cap H^1_{0,\mathrm{loc}}(\mathbf{R}^n\setminus\mathcal{O}),$$

for $\lambda\in\mathbf{C}$ or $\lambda\in\Lambda$, the logarithmic plane, when n is odd or even respectively (see [6], [14], [10]). Here $H^k(\mathbf{R}^n\setminus\mathcal{O})$ is the standard Sobolev space and $H^1_0(\mathbf{R}^n\setminus\mathcal{O})$ is the closure of $C_0^\infty(\mathbf{R}^n\setminus\overline{\mathcal{O}})$ in H^1-norm. Then, L^2_{comp}, H^2_{loc}, $H^1_{0,\mathrm{loc}}$ are defined from these spaces in the usual way. The poles of this continuation are called the scattering poles and can be considered as a replacement of the discrete spectral data for an exterior problem. In our results we count the number of the poles with their multiplicity (see [10]).

We will estimate the poles in the following neighbourhoods of the real axis

$$\{\zeta : 1 \le \operatorname{Re}\zeta \le r,\ -\operatorname{Im}\zeta < \mu(\operatorname{Re}\zeta)\operatorname{Re}\zeta\}, \tag{1.1}$$

where the function μ is assumed to satisfy

$$\begin{gathered} \frac{1}{\mu(x)},\quad \mu(x)^{1/2}x,\quad \frac{\mu(x)^2}{\mu(x/2)}x^{n-1} \text{ non-decreasing,} \\ \frac{1}{C_1}x^{-2/3} \le \mu(x) \le \frac{1}{C_1},\quad x > C_2. \end{gathered} \tag{1.2}$$

We remark that, if $n>4$, the last monotonicity condition is a consequence of the first two and that we could take more general μ's at the expense of some complications in the statements. The natural μ's to take are $\mu(r)=\theta r^{-\alpha}$, $0\le\alpha\le\frac{2}{3}$—see Fig. 1.

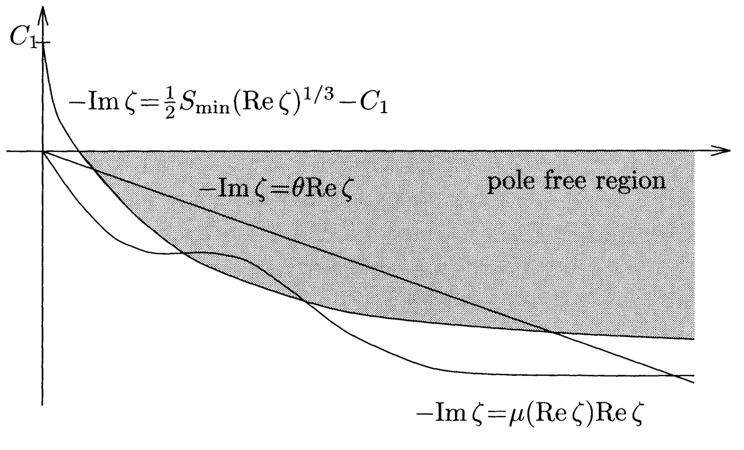

Figure 1. The neighbourhoods of the real axis and the critical curve.

Theorem 1. *If $N(r,\mu)$ is the number of scattering poles in (1.1) with μ satisfying (1.2) then*

$$N(r,\mu) \le C\mu(r)^{3/2}r^n + C,\quad r > C,$$

for some constant C depending only on the $\mathcal{O}$ and the constants in (1.2).

The proof will be given in Sect. 7 as a consequence of a more precise local upper bound in Theorem 4 there. In that bound we also recover the result of Hargé and Lebeau [3] on the pole free region:

$$\mu(r) < \tfrac{1}{2} S_{\min} r^{-2/3} - c_1 r^{-1} \quad \Longrightarrow \quad N(r, \mu) < C, \tag{1.3}$$

where

$$S_{\min} \stackrel{\text{def}}{=} 2^{2/3} \cos\left(\tfrac{1}{6}\pi\right) \zeta_1 \Big(\min_{x' \in \partial\mathcal{O}, i=1,\dots,n-1} K_i(x') \Big)^{2/3},$$

with $K_i(x')$, the principal curvatures of $\partial\mathcal{O}$ at x' and $-\zeta_1$, the first zero of the Airy function. In other words there are only finitely many poles above the critical cubic parabola $-\operatorname{Im}\zeta = \frac{1}{2} S_{\min}(\operatorname{Re}\zeta)^{1/3} - c_1$. Near that curve one expects finer estimates once the geometry is more controlled. To that aim we have

Theorem 2. *If the second fundamental form of $\partial\mathcal{O}$ restricted to the sphere bundle of $\partial\mathcal{O}$ has a non-degenerate minimum on an embedded submanifold of codimension ν (in the sense that the transversal Hessian is nondegenerate), then the number of scattering poles in*

$$\begin{cases} 1 \le \operatorname{Re}\zeta \le r, \\ -\operatorname{Im}\zeta \le \frac{1}{2} S_{\min}(\operatorname{Re}\zeta)^{1/3} + c(\operatorname{Re}\zeta)^{1-\alpha}, \end{cases}$$

$\frac{2}{3} \le \alpha \le 1$, *is bounded by*

$$C r^{n-1} r^{-((\alpha/2)-(1/3))\nu}. \tag{1.4}$$

As in the case of Theorem 1, a more precise local bound is possible, see (7.8).

The special choices of μ in Theorem 1 give the following corollaries. In the first one we take θ large (at least, to get a non-trivial statement, greater than the critical value):

Corollary 1.1. *For $\frac{1}{2} S_{\min} < \theta < \theta_1$ and $r > C(\theta_1)$*

$$\#\{\zeta : \zeta \text{ a scattering pole, } 1 \le \operatorname{Re}\zeta \le r,\ -\operatorname{Im}\zeta \le \theta(\operatorname{Re}\zeta)^{1/3}\} = \mathcal{O}(\theta^{3/2}) r^{n-1}.$$

For θ small and $\alpha = 0$ we recover, in a strengthened form, the result of [14]:

Corollary 1.2. *If $0 < \theta < \theta_0$ then for $r > C$*

$$\#\{\zeta : \zeta \text{ a scattering pole, } 1 \le \operatorname{Re}\zeta \le r,\ -\operatorname{Im}\zeta \le \theta \operatorname{Re}\zeta\} = \mathcal{O}(\theta^{3/2}) r^{n}.$$

The two corollaries and the pole free region estimate are optimal for the sphere. In the non-symmetric case the only lower bound follows from the work of Bardos, Lebeau and Rauch [1]: a non-degenerate, isolated, simple closed geodesic γ of length d_γ on the boundary of a strictly convex analytic obstacle generates infinitely many poles in any region

$$\{\zeta : -\operatorname{Im}\zeta < B(\operatorname{Re}\zeta)^{1/3}\}, \quad B > B_\gamma \stackrel{\text{def}}{=} 2^{-1/3}\zeta_1 \cos\Big(\frac{\pi}{6}\Big)\frac{1}{d_\gamma}\int_0^{d_\gamma} \varrho_\gamma(s)^{2/3}\,ds,$$

where ϱ_γ is the curvature of γ in $\mathbf{R}^n$, n odd and s is the length parameter on γ. From their argument it also seems to follow that there are only finitely many poles in the region with $\frac{1}{2}S_{\min}$ replaced by B, $B<B_{\min}$, where

$$B_{\min} = \sup_{T>0}\ \inf_{\{\gamma \text{ a geodesic}\}} 2^{-1/3}\zeta_1 \cos\Big(\frac{\pi}{6}\Big)\frac{1}{T}\int_0^{T} \varrho_\gamma(s)^{2/3}\,ds.$$

Using a simple Tauberian argument [12] one actually sees that for every $\varepsilon>0$ and $B>B_\gamma$, there exists $r(\varepsilon, B)$ such that

$$\#\{\zeta : \zeta \text{ a scattering pole, } |\operatorname{Re}\zeta| \le r,\ -\operatorname{Im}\zeta < B|\operatorname{Re}\zeta|^{1/3}\} > r^{2/3-\varepsilon}, \quad r > r(\varepsilon, B).$$

Finally, we give an example of an obstacle for which the assumptions of Theorem 2 are nicely satisfied. We let $\partial\mathcal{O}$ be an ellipsoid of revolution. Then the second fundamental form restricted to the sphere bundle takes its minimum on the normal bundle to the shortest geodesic, which is assumed to be the equator. The codimension is 2 and the bound (1.4) becomes $\mathcal{O}(1)r^{(8/3)-\alpha}$ compared to the bound obtained using Theorem 1, $\mathcal{O}(1)r^2$. In the analytic case a better estimate is possible (corresponding to a larger pole free region obtained by using $B_{\min}$ above) but the bound seems new if the boundary is no longer analytic but the geometry is the same.

2. The scaled operator

In this section we will review the complex scaling construction used in the preceding papers [10], [11], [12] stressing the explicit representation of the operator. Thus, let $\mathcal{O}\subset\mathbf{R}^n$ be bounded and open with a smooth boundary. We assume that $\mathcal{O}$ is *strictly convex*. It then follows that $d(x)\stackrel{\text{def}}{=}\operatorname{dist}(x,\mathcal{O})$ is in $C^\infty(\mathbf{R}^n\setminus\mathcal{O})$ and that $d(x)$ is a convex function with $\ker d''_{xx}(x)$ of dimension 1, generated by $x-z(x)$, where $z(x)\in\partial\mathcal{O}$ is the unique point such that $d(x)=|x-z(x)|$. We observe that at $z\in\partial\mathcal{O}$ the exterior unit normal of $\partial\mathcal{O}$ at z is given by

$$n(z) = \nabla d(z). \tag{2.1}$$

If $z_0 \in \partial\mathcal{O}$, we choose some local coordinates $y'=(y_1,\dots,y_{n-1})$ for $\partial\mathcal{O}$ centered at z_0 so that we have a corresponding diffeomorphism

$$s\colon \mathrm{neigh}_{\mathbf{R}^{n-1}}(0) \longrightarrow \mathrm{neigh}_{\partial\mathcal{O}}(z_0). \tag{2.2}$$

We then get the normal geodesic coordinates (y',y_n), $y' \in \mathrm{neigh}_{\mathbf{R}^{n-1}}(0)$, $y_n \geq 0$ for a sector of an extension of $\mathcal{O}$, given by

$$x = s(y') + y_n n(s(y')) = s(y') + y_n \nabla d(s(y')). \tag{2.3}$$

We will also write

$$x = \tilde{s}(y), \quad y = (y', y_n). \tag{2.4}$$

Let $x_0 = s(0) + y_{0,n} n(s(0))$ be some fixed point (we take $y'=0$ for simplicity—any other choice of $y' \in \mathrm{neigh}_{\mathbf{R}^{n-1}}(0)$ would work in the same way). We shall compute the leading contribution to Δ in the y-coordinates at the fixed point $y_0 = (0, y_{0,n})$. After a Euclidean change of the x-coordinates we may assume that $x_0 = (0, y_{0,n})$ lies on the positive x_n-axis. From (2.3) we get

$$\frac{\partial x}{\partial y_n} = \nabla d(s(y')), \quad \frac{\partial x}{\partial y_n}(y_0) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \tag{2.5}$$

We also have,

$$\frac{\partial x}{\partial y'} = \frac{\partial s}{\partial y'} + y_n \nabla^2 d(s(y')) \circ \frac{\partial s}{\partial y'},$$

and in particular at $y'=0$:

$$\frac{\partial x}{\partial y} = \begin{pmatrix} (I + y_n d''_{x'x'}(0)) \partial_{y'} s(0) & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.6}$$

Notice that $d''_{x'x'}(0) > 0$ so that the matrix (2.6) is invertible for $y_n \in \mathbf{C} \setminus (-\infty, 0)$. At $y'=0$ and for y_n small we get from (2.6)

$${}^t\left(\frac{\partial x}{\partial y}\right)^{-1} = \begin{pmatrix} (I + y_n d''_{x'x'}(0))^{-1} ({}^t\partial_{y'} s)^{-1}(0) & 0 \\ 0 & 1 \end{pmatrix} \tag{2.7}$$

$$= \begin{pmatrix} (I - y_n d''_{x'x'}(0)) ({}^t\partial_{y'} s)^{-1}(0) + \mathcal{O}(y_n^2) & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.8}$$

The principal symbol of $-\Delta$ then becomes (still at $y'=0$):

$$\langle\xi,\xi\rangle = \left\langle {}^t\!\left(\frac{\partial x}{\partial y}\right)^{-1}\eta, {}^t\!\left(\frac{\partial x}{\partial y}\right)^{-1}\eta\right\rangle$$

$$= \eta_n^2 + \left(\left((I-y_n d''_{x'x'}(0))\,{}^t\!\left(\frac{\partial s}{\partial y'}\right)^{-1}(0)+\mathcal{O}(y_n^2)\right)\eta'\right)^2 \tag{2.9}$$

$$= \eta_n^2 + \left({}^t\!\left(\frac{\partial s}{\partial y'}(0)\right)^{-1}\eta'\right)^2$$

$$-2y_n\left\langle d''_{x'x'}(0)\,{}^t\!\left(\frac{\partial s}{\partial y'}(0)\right)^{-1}\eta', {}^t\!\left(\frac{\partial s}{\partial y'}(0)\right)^{-1}\eta'\right\rangle + \mathcal{O}(y_n^2\eta'^2). \tag{2.10}$$

Here the second term in the last expression is the principal symbol of $-\Delta_{\partial\mathcal{O}} \stackrel{\text{def}}{=} R(y', D_{y'})$ expressed in the local coordinates y'. We can interpret $d''_{x'x'}(0)$ as the Hessian of d at 0 restricted to $T_0\partial\mathcal{O}$ (viewed as a subspace of $\mathbf{R}^n = T_0\mathbf{R}^n$) and, by the Euclidean duality, as the corresponding Hessian on $T^*\partial\mathcal{O}$. Then

$$\left\langle d''_{x'x'}(0)\,{}^t\!\left(\frac{\partial s}{\partial y'}(0)\right)^{-1}\eta', {}^t\!\left(\frac{\partial s}{\partial y'}(0)\right)^{-1}\eta'\right\rangle$$

is the corresponding quadratic form expressed in the (y', η')-coordinates on $T_0^*\partial\mathcal{O}$. Let $Q(y', D_{y'})$ be the corresponding elliptic differential operator on the boundary (where now we let y' vary). From the discussion above we see that $R(y', \eta')$ is dual to the first fundamental form (the metric on $\partial\mathcal{O}$) and $Q(y', \eta')$, to the second fundamental form (given at $y'=0$ by $\langle d_{y'}n(Y'), (Y', 0)\rangle = \langle d''_{x'x'}(0)Y', Y'\rangle$, $Y' \in T_0\partial\mathcal{O}$). Since the principal curvatures of $\partial\mathcal{O}$ are the eigenvalues of the second fundamental form with respect to the first, we obtain

Lemma 2.1. *The principal curvatures of $\partial\mathcal{O}$ at $x'=s(y')$ are the eigenvalues of the quadratic form $Q(y', \eta')$ with respect to the quadratic form $R(y', \eta')$.*

With the new notation, and for y', y_n small, we now get

$$\begin{aligned} -h^2\Delta &= (hD_{y_n})^2 + R(y', hD_{y'}) - 2y_n Q(y', hD_{y'}) \\ &\quad + \mathcal{O}(y_n^2 (hD_{y'})^2) + \mathcal{O}(h)hD_y + \mathcal{O}(h^2). \end{aligned} \tag{2.11}$$

Here we found it convenient to introduce the semi-classical parameter $h>0$ that we will let tend to 0.

In [13] we considered exterior complex scaling which near $\partial\mathcal{O}$ was of the form

$$z = x + i\theta f'(x) \tag{2.12}$$

with $f(x)=\frac{1}{2}d(x)^2$ so that $f'(x)=d(x)d'(x)$. Replacing x by the corresponding geodesic coordinates above, we get

$$z=s(y')+y_n\nabla d(s(y'))+i\theta y_n\nabla d(s(y'))=s(y')+(1+i\theta)y_n\nabla d(s(y')). \tag{2.13}$$

Following Hargé and Lebeau [3] near $\partial\mathcal{O}$, we shall scale up to the angle $\frac{1}{3}\pi$, so that $|1+i\theta|^{-1}(1+i\theta)=\exp\bigl(i\frac{1}{3}\pi\bigr)$ near $\partial\mathcal{O}$. Further on we connect the scaling to the one used in [13] (with smaller θ). More precisely, we let $\theta>0$ be small enough and let g be an injective C^∞ map $[0,\infty)\to\mathbf{C}$. We demand that $|g'|=1$, $g(0)=0$, $g(t)=t\exp\bigl(i\frac{1}{3}\pi\bigr)$ for t near 0 and that $g(t)=t|1+i\theta|^{-1}(1+i\theta)$ outside a small neighbourhood of 0,

$$\arg(1+i\theta)\le\arg g(t)\le\tfrac{1}{3}\pi,\quad \tfrac{1}{2}\arg(1+i\theta)\le\arg g'(t)\le\tfrac{1}{3}\pi.$$

Let $\Gamma=\Gamma_g\subset\mathbf{C}^n$ be the image of

$$\partial\mathcal{O}\times[0,\infty)\ni(x',x_n)\longmapsto x'+g(x_n)\nabla d(x'). \tag{2.14}$$

Then, replacing $x'\in\partial\mathcal{O}$ by the corresponding local coordinate considered before and denoted by y', we see that

$$\begin{aligned}P_\Gamma\overset{\text{def}}{=}-h^2\Delta|_\Gamma=\frac{1}{(g'(y_n))^2}(hD_{y_n})^2+R(y',hD_{y'})-2g(y_n)Q(y',hD_{y'})\\+\mathcal{O}(y_n^2(hD_{y'})^2)+\mathcal{O}(h)hD_y+\mathcal{O}(h^2),\end{aligned}$$

so that the operator is elliptic in both the semi-classical and the usual sense.

For y_n so small that $g(y_n)=y_n\exp\bigl(i\frac{1}{3}\pi\bigr)$, we get

$$\begin{aligned}-h^2\Delta|_\Gamma=e^{-2\pi i/3}((hD_{y_n})^2+2y_nQ(y',hD_{y'}))+R(y',hD_{y'})\\+\mathcal{O}(y_n^2(hD_{y'})^2)+\mathcal{O}(h)hD_y+\mathcal{O}(h^2).\end{aligned} \tag{2.15}$$

We finally notice that if p_Γ denotes the principal symbol of $P=P_\Gamma$, then p_Γ takes its values in the closed lower half plane and for every $\delta>0$ there exists $\varepsilon>0$ such that

$$y_n\ge\delta\implies\varepsilon\le-\arg p_\Gamma(y,\eta)\le\pi-\varepsilon.$$

We also recall from Sect. 2 of [13] (partly based on Sect. 3 and Sect. 2 of [10] and [11] respectively) the contents of the following

Lemma 2.2. *The poles of the meromorphic continuation of* $(-\Delta-\lambda^2)^{-1}$ *in* $0<-\arg\lambda<\theta/C$ *are given (with multiplicities) by the square roots of the complex eigenvalues in* $0<-\arg z<2\theta/C$ *of the Dirichlet realization of* $-\Delta|_\Gamma$, *provided* C *is taken large enough.*

3. Some facts about the FBI transform

We will now review some basic facts about the FBI transform or rather its simpler version, the Bargmann transform. Our presentation is motivated by the general theory [7] and the discussion of Bargmann transforms in [8] (see [5, Sect. 6]). Although in the application here we will only use one phase function $\phi(z,x)=\frac{1}{2}i(z-x)^2$, it is instructive to proceed in this greater generality.

Thus, let $\phi(z,x)$ be a quadratic form on $\mathbf{C}^m\times\mathbf{C}^m$ satisfying

$$\operatorname{Im}\frac{\partial^2\phi}{\partial x^2}\gg 0,\quad \det\frac{\partial^2\phi}{\partial z\partial x}\neq 0. \tag{3.1}$$

We define $T=T_{\phi,h}$ on $\mathcal{S}(\mathbf{R}^m)$ by

$$\begin{gathered}Tu(z)=c_\phi h^{-3m/2}\int e^{i\phi(z,x)/h}u(x)\,dx,\quad z\in\mathbf{C}^m,\\ c_\phi=2^{-m/2}\pi^{-3m/4}|\det(\operatorname{Im}\partial^2_{xx}\phi)|^{-1/4}|\det\partial^2_{xz}\phi|.\end{gathered} \tag{3.2}$$

Some basic motivation comes from the standard observation that for $\phi=i\frac{1}{2}(z-x)^2$

$$Tu(z)=\frac{2^{m/4}}{(2\pi h)^{3m/4}}e^{(\operatorname{Im}z)^2/2h-i\operatorname{Im}z\operatorname{Re}z/h}\mathcal{F}(e^{-(\,\cdot\,-\operatorname{Re}z)^2/2h}u)(-\operatorname{Im}z/h), \tag{3.3}$$

where $\mathcal{F}\colon v(x)\mapsto\int v(x)e^{-ix\xi}\,dx$ is the Fourier transform on $\mathbf{R}^n$.

We now define the weight

$$\Phi(z)=\max_{x\in\mathbf{R}^m}-\operatorname{Im}\phi(z,x), \tag{3.4}$$

and the corresponding L^2-space, L^2_Φ, with the measure $e^{-2\Phi(z)/h}\mathcal{L}(dz)$, where $\mathcal{L}(dz)$ is the Euclidean measure on $\mathbf{C}^m$.

In the special case of $\phi=i\frac{1}{2}(z-x)^2$ we have $\Phi(z)=\frac{1}{2}(\operatorname{Im}z)^2$ and (3.3) shows that T extends to an isometry

$$T\colon L^2(\mathbf{R}^m)\longrightarrow L^2_\Phi(\mathbf{C}^m), \tag{3.5}$$

and for the case of any ϕ satisfying (3.1) we refer to Proposition 6.1 of [5]. In what follows $\|\cdot\|$, $\langle\cdot,\cdot\rangle$ and $\|\cdot\|_\Phi$, $\langle\cdot,\cdot\rangle_\Phi$ will denote the norms in the source and target spaces in (3.5) respectively.

The same definitions apply if we consider vector valued functions. Thus for a Hilbert space $\mathcal{H}$ we start with $\mathcal{S}(\mathbf{R}^m,\mathcal{H})$ and obtain an isometry

$$T\colon L^2(\mathbf{R}^m,\mathcal{H})\longrightarrow L^2_\Phi(\mathbf{C}^m,\mathcal{H}),$$

where we have the obvious norms:

$$\|u\|^2_{L^2(\mathbf{R}^m,\mathcal{H})} = \int \|u(x)\|^2_{\mathcal{H}}\,dx, \quad \|w\|^2_{L^2_\Phi(\mathbf{C}^m,\mathcal{H})} = \int \|w(z)\|^2_{\mathcal{H}} e^{-2\Phi/h}\,d\operatorname{Re} z\,d\operatorname{Im} z.$$

In our applications we will take $m=n-1$ and $\mathcal{H}=L^2([0,\infty))$. Thus we will either discuss the scalar case (when the vector valued extension is clear) or that specific case.

For the main part of the proof of Theorems 1 and 4 in Sect. 5 we will need the following proposition (motivated by Theorems 1.2 and 2.2 of [9], see also [2]) which for notational simpliticity we state and prove for the phase $i\frac{1}{2}(x-z)^2$ only. It describes the intertwining properties of T on $L^2(\mathbf{R}^{n-1}, L^2([0,\infty)))$, in a way sufficient for our purposes. Let $C_b^\infty(\mathbf{R}^n)$ be the space of smooth functions on $\mathbf{R}^n$ that are bounded with all derivatives, and define $C_b^\infty(\mathbf{R}^{n-1}\times[0,\infty[\,)$ similarly.

Proposition 3.1. *If $A(x,hD)$ is a second order operator on $\mathbf{R}^{n-1}\times[0,\infty)$ with coefficients in $C_b^\infty(\mathbf{R}^{n-1}\times[0,\infty))$, then for $u\in C_0^\infty(\mathbf{R}^{n-1}\times[0,\infty))$*

$$\begin{aligned}\|A(x,hD)u\|^2 &= \|A(\operatorname{Re} z, x_n; -\operatorname{Im} z, hD_{x_n})Tu\|^2_\Phi \\ &\quad +\mathcal{O}(h)(\|(hD_{x_n})^2Tu\|^2_\Phi \\ &\quad +\|(1+|\operatorname{Im} z|)hD_{x_n}Tu\|^2_\Phi+\|(1+|\operatorname{Im} z|^2)Tu\|^2_\Phi),\end{aligned} \tag{3.6}$$

where T is given by (3.2) with $\phi(z,x)=i\frac{1}{2}(z-x)^2$.

Proof. It will be clear from the discussion below that we can neglect the x_n variable. We will first consider $B(x',hD_{x'})$, a differential operator of order p with coefficients in $C_b^\infty(\mathbf{R}^{n-1})$. We claim that for $u,v\in C_0^\infty(\mathbf{R}^{n-1}\times[0,\infty))$

$$\begin{aligned}\langle B(x',hD_{x'})u,v\rangle &= \langle B(\operatorname{Re} z, -\operatorname{Im} z)Tu, Tv\rangle_\Phi \\ &\quad +\mathcal{O}(h)\|(1+|\operatorname{Im} z|^{p_1})Tu\|_\Phi\|(1+|\operatorname{Im} z|)^{p_2}Tv\|_\Phi, \\ &\quad p_i\in\mathbf{N}_0,\ p_1+p_2=p.\end{aligned} \tag{3.7}$$

Proceeding inductively on $p=|\alpha|$ we only need to consider

$$B(x',hD_{x'}) = (hD_{x'})^{\alpha_2}a(x')(hD_{x'})^{\alpha_1}, \quad |\alpha_i|=p_i,$$

so that

$$\begin{aligned}\langle B(x',hD_{x'})u,v\rangle &= \langle a(x')(hD_{x'})^{\alpha_1}u, (hD_{x'})^{\alpha_2}v\rangle \\ &= \langle T(a(hD_{x'})^{\alpha_1}u), T((hD_{x'})^{\alpha_2}v)\rangle_\Phi.\end{aligned}$$

For $u_1, u_2 \in C_0^\infty(\mathbf{R}^{n-1}\times[0,\infty))$ we have

$$\langle T(au_1), Tu_2\rangle_\Phi = \langle a(\operatorname{Re} z)Tu_1, Tu_2\rangle_\Phi + \mathcal{O}(h)\|Tu_1\|_\Phi\|Tu_2\|_\Phi. \tag{3.8}$$

In fact, since $a\in C_b^\infty(\mathbf{R}^{n-1})$ we can write

$$a(x') = a(\operatorname{Re} z) + \langle x' - \operatorname{Re} z, a_1(x', \operatorname{Re} z)\rangle, \quad a_1 \in C_b^\infty(\mathbf{R}^{n-1}\times\mathbf{R}^{n-1}; \mathbf{R}^{n-1}),$$

and thus we need to estimate

$$\begin{aligned}&\langle T(\langle\,\cdot - \operatorname{Re} z, a_1(\,\cdot\,, \operatorname{Re} z)\rangle u_1), Tu_2\rangle_\Phi \\ &\quad = \frac{1}{(2\pi h)^{-(n-1)/2}} \iiint e^{-(x'-\operatorname{Re} z)^2/h}\langle x' - \operatorname{Re} z, a_1(x', \operatorname{Re} z)\rangle \\ &\qquad \times u_1(x', x_n)\overline{u_2(x', x_n)}\, d\operatorname{Re} z\, dx'\, dx_n,\end{aligned} \tag{3.9}$$

where we used (3.3) and the Plancherel formula. Since

$$(x' - \operatorname{Re} z)\exp(-(x'-\operatorname{Re} z)^2/h) = h\nabla_{\operatorname{Re} z}\tfrac{1}{2}\exp(-(x'-\operatorname{Re} z)^2/h),$$

we can integrate by parts so that the $d\operatorname{Re} z$ integral is $\mathcal{O}(h)$ uniformly in x' and x_n. Hence the left hand side of (3.9) is estimated by $\mathcal{O}(h)\|Tu_1\|_\Phi\|Tu_2\|_\Phi$. To see that (3.7) follows from (3.8) we observe that

$$T((hD_{x'})^\beta v_1) = (hD_z)^\beta Tv_1$$

and that for $|\alpha|=1$

$$\begin{aligned}\langle b(\operatorname{Re} z)(hD_z)^\alpha Tv_1, Tv_2\rangle_\Phi &= \langle [(-hD_z - \operatorname{Im} z)^\alpha (b(\operatorname{Re} z))]Tv_1, Tv_2\rangle_\Phi, \\ &\qquad v_i \in C_0^\infty,\ b \in C_b^\infty,\end{aligned}$$

which follows from integration by parts using

$$(-hD_z)^\alpha \exp(-(\operatorname{Im} z)^2/h) = (-\operatorname{Im} z)^\alpha \exp(-(\operatorname{Im} z)^2/h)$$

and $(-hD_z)^\alpha\overline{Tv_2(z)}\equiv 0$ as Tv_2 is holomorphic.

We conclude the proof by deriving (3.6) from (3.7). For that let us write

$$A = A_0(hD_{x_n})^2 + A_1(hD_{x_n}) + A_2,$$

where A_i's are of the same form as B above (with some irrelevant dependence on x_n), with $p=i$. Then

$$\begin{aligned}\|Au\|^2 &= \langle A_0^*A_0(hD_{x_n})^2u, (hD_{x_n})^2u\rangle + 2\operatorname{Re}\langle A_1^*A_0(hD_{x_n})^2u, hD_{x_n}u\rangle \\ &\quad + \langle A_1^*A_1hD_{x_n}u, hD_{x_n}u\rangle + 2\operatorname{Re}\langle A_2^*A_0(hD_{x_n})^2u, u\rangle \\ &\quad + 2\operatorname{Re}\langle A_2^*A_1hD_{x_n}u, u\rangle + \langle A_2^*A_2u, u\rangle,\end{aligned}$$

where A_i^*'s are the formal adjoints. We can now apply (3.7) to each individual term, taking $p_i \le 2$. □

As is well known and as is also indicated by the proposition above, the behaviour of the FBI transform (3.2), $\phi = i\frac{1}{2}(x-z)^2$, at z reflects the microlocal behaviour of u at $(\operatorname{Re} z, -\operatorname{Im} z) \in T^*\mathbf{R}^m$. Hence in Sect. 5 we shall use the notation

$$z = x' - i\xi'.$$

In the remainder of the section we shall review some facts needed in Sect. 6 for the proof of Theorem 2. In doing this we will allow any phase ϕ satisfying (3.1). To such ϕ and the corresponding T we associate a linear canonical transformation (with respect to the complex symplectic forms $\sum_{j=1}^m d\xi_j \wedge dx_j$ and $\sum_{j=1} d\zeta_j \wedge dz_j$):

$$\chi_\phi : T^*\mathbf{C}^m \longrightarrow T^*\mathbf{C}^m, \quad (x, -\partial_x \phi(x,z)) \longmapsto (z, \partial_z \phi(x,z)). \tag{3.10}$$

We can then quote from [7], [8] (see also Proposition 6.2 of [5]):

Lemma 3.1. *The quadratic form $\Phi(z)$ given by* (3.4) *is strictly pluri-subharmonic* (*that is, strictly subharmonic on any complex line in $\mathbf{C}^m$*), *and the canonical transformation χ_ϕ is a bijection of $T^*\mathbf{R}^m$ onto*

$$\Lambda_\Phi = \{(z, -2i\partial_z \Phi(z)) : z \in \mathbf{C}^m\},$$

which is a totally real submanifold of $T^*\mathbf{C}^m$, Lagrangian with respect to the symplectic form $\operatorname{Im} \sum_{j=1}^m d\zeta_j \wedge dz_j$ (*that is, I-Lagrangian*).

Since $Tu(z)$ is clearly holomorphic, the closed subspace of the holomorphic elements of $L^2_\Phi(\mathbf{C}^m)$, $H_\Phi(\mathbf{C}^m)$, makes a natural appearance. We will now follow [8] and give a well-known expression for the kernel of the orthogonal projection

$$\Pi : L^2_\Phi(\mathbf{C}^m) \longrightarrow H_\Phi(\mathbf{C}^m).$$

Let $\Psi(x,y)$ be the unique holomorphic quadratic form on $\mathbf{C}^m \times \mathbf{C}^m$ such that $\Phi(x) = \Psi(x, \bar{x})$ (in the special case, $\Phi(x) = \frac{1}{2}(\operatorname{Im} x)^2$, $\Psi(x,y) = -\frac{1}{8}(x-y)^2$). We will use it to deform the contour in the following representation of identity in $H_\Phi(\mathbf{C}^m)$:

$$\begin{aligned} u(x) &= \frac{1}{(2\pi h)^m} \iint_{\Gamma(x)} e^{i(x-y)\xi/h} u(y)\, dy\, d\xi, \\ \Gamma(x) &: y \longmapsto \xi = \frac{2}{i}\partial_x \Phi(x) + iC\overline{(x-y)}, \quad C \gg 1. \end{aligned} \tag{3.11}$$

This can be seen by introducing polar coordinates at x, the mean value theorem for holomorphic functions and an evaluation of a Gaussian integral—the absolute convergence is guaranteed by

$$\operatorname{Re}(2\partial_x\Phi(x)(x-y)+\Phi(y)-\Phi(x)-C|x-y|^2)\leq -|x-y|^2, \tag{3.12}$$

if C is large enough. We want to change (3.11) to obtain a kernel giving a self-adjoint operator on L^2_Φ. For that we use Ψ and make a change of variables $\theta\mapsto\xi$:

$$2(\Psi(x,\theta)-\Psi(y,\theta))=i(x-y)\xi, \quad \frac{\partial\xi}{\partial\theta}=\frac{2}{i}\Psi''_{xy}.$$

Putting $\theta=\bar{y}$ we obtain another 'good contour' (compare (3.12)):

$$\operatorname{Re}(2\Psi(x,\bar{y})-2\Psi(y,\bar{y}))+\Phi(y)-\Phi(x)=-\left\langle\frac{\partial^2\Phi}{\partial z\partial\bar{z}}(x-y),\overline{x-y}\right\rangle\leq\frac{-|x-y|^2}{C}. \tag{3.13}$$

Thus, for $u\in H_\Phi(\mathbf{C}^m)$

$$u(x)=\frac{1}{(2\pi h)^m}\left(\frac{2}{i}\right)^m \det\Psi''_{xy}\iint e^{2\Psi(x,\bar{y})/h}u(y)e^{-2\Phi(y)/h}\,dy\,d\bar{y}\stackrel{\text{def}}{=}\widetilde{\Pi}u. \tag{3.14}$$

The operator $\widetilde{\Pi}$ is defined for any $u\in L^2_\Phi(\mathbf{C}^m)$ and gives an element of $H_\Phi(\mathbf{C}^m)$. Since it is self-adjoint and equal to the identity on H_Φ, it must be equal to Π. The inequality (3.13) shows that the reduced kernel of Π, $e^{-\Phi/h}\Pi e^{\Phi/h}$ is smooth and $\mathcal{O}(h^{-m})e^{-|x-y|^2/Ch}$. Thus for any compact $K\subset\mathbf{C}$, $\mathbf{1}_K\Pi$ and $\Pi\mathbf{1}_K$ are of trace class. From this observation we will pass to traces of Toeplitz operators.

For $q\in L^\infty_{\text{comp}}(\mathbf{C}^m)$ we define the operator

$$\Pi q\Pi^*: H_\Phi(\mathbf{C}^m)\to H_\Phi(\mathbf{C}^m).$$

Here we consider Π^* as an operator $H_\Phi(\mathbf{C}^m)\to L^2_\Phi(\mathbf{C}^m)$. From the comments above it follows that

$$\begin{aligned} &q\Pi\in\mathcal{L}_1(L^2_\Phi(\mathbf{C}^m),L^2_\Phi(\mathbf{C}^m)), \quad \Pi q\in\mathcal{L}_1(L^2_\Phi(\mathbf{C}^m),H_\Phi(\mathbf{C}^m)),\\ &q\Pi^*\in\mathcal{L}_1(H_\Phi(\mathbf{C}^m),L^2_\Phi(\mathbf{C}^m)),\end{aligned}$$

so that $\Pi q\Pi^*\in\mathcal{L}_1(H_\Phi(\mathbf{C}^m),H_\Phi(\mathbf{C}^m))$, and from the cyclicity of the trace we see that

$$\operatorname{tr}_{H_\Phi}\Pi q\Pi^*=\operatorname{tr}_{L^2_\Phi} q\Pi=\frac{1}{(2\pi h)^m}\left(\frac{2}{i}\right)^m\det\Phi''_{x\bar{x}}\iint q(x)\,dx\,d\bar{x}. \tag{3.15}$$

For the future reference we shall restate (3.15) in a more elegant form:

Lemma 3.2. *Let* $q\in L^\infty_{\mathrm{comp}}(\mathbf{C}^m)$ *and let* Π *be the orthogonal projection* $L^2_\Phi(\mathbf{C}^m)\to H_\Phi(\mathbf{C}^m)$. *If* $\Lambda_\Phi=\{(x,-2i\partial_x\Phi(x)):x\in\mathbf{C}^m\}$ *then*

$$\operatorname{tr}_{H_\Phi}\Pi q\Pi^*=\frac{1}{(2\pi h)^m}\iint_{\Lambda_\Phi} q(x)\,dx\,d\xi. \tag{3.16}$$

The method of the proof of (3.14) can also be used to establish the following basic fact, roughly half of which was already seen in (3.5):

Lemma 3.3. *The FBI transform* (3.2) *is unitary as a map*

$$T: L^2(\mathbf{R}^m)\to H_\Phi(\mathbf{C}^m).$$

We will now review briefly the Weyl quantization in the usual and H_Φ settings:

$$S^0_{0,0}(T^*\mathbf{R}^m)\ni b\longmapsto b^w(x,hD_x)\in \mathrm{Op}^w S^0_{0,0}$$
$$S^0_{0,0}(\Lambda_\Phi)\ni a\longmapsto a^w_\Phi(z,hD_z)\in \mathrm{Op}^w_\Phi S^0_{0,0}$$

where $S^0_{0,0}(\mathbf{R}^{2m})$ is the class of symbols satisfying the estimates $|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)|\le C_{\alpha\beta}$. The operators are initially defined for $a\in\mathcal{S}(\Lambda_\Phi)$ and $b\in\mathcal{S}(T^*\mathbf{R}^m)$:

(3.17)
$$b^w(x,hD_x)u=\frac{1}{(2\pi h)^m}\iint e^{i\langle x-y,\xi\rangle/h}b\big(\tfrac12(x+y),\xi\big)u(y)\,dy\,d\xi,$$

(3.18)
$$a^w_\Phi(z,hD_z)v=\frac{1}{(2\pi h)^m}\iint_{\Gamma_0(z)} e^{i\langle z-w,\zeta\rangle/h}a\big(\tfrac12(z+w),\zeta\big)v(w)\,d\zeta\,dw,$$

$u\in L^2(\mathbf{R}^m)$, $v\in H_\Phi(\mathbf{C}^m)$, and where $\Gamma_0(z)$ is an integration contour in $\mathbf{C}^{2m}$: $w\mapsto \zeta=-2i\partial_z\Phi\big(\frac12(z+w)\big)$.

The oscillatory behaviour of the exponential when $z\neq w$, the non-degeneracy of $\partial^2_{z\bar z}\Phi$ (Lemma 3.1), and an integration by parts based on

$$\frac{\partial}{\partial\bar w}\exp\left(2(z-w)\frac{\partial}{\partial z}\Phi\left(\frac{z+w}{2}\right)\right)=\exp\left(2(z-w)\frac{\partial}{\partial z}\Phi\left(\frac{z+w}{2}\right)\right)(z-w)\frac{\partial^2\Phi}{\partial\bar z\partial z},$$

allow a definition of $a^w_\Phi(z,hD_z)$ for any $a\in S^0_{0,0}(\Lambda_\Phi)$ and give

Proposition 3.2. *For* $a\in S^0_{0,0}(\Lambda_\Phi)$ *the Weyl quantization* (3.18) *defines an operator*

$$a^w_\Phi(z,hD_z): H_\Phi(\mathbf{C}^m)\longrightarrow H_\Phi(\mathbf{C}^m).$$

We can now apply the method of Theorem 18.5.9 of [4], first for $a\in\mathcal{S}$ and then by approximation for $a\in S^0_{0,0}$, to obtain

Proposition 3.3. *The FBI transform* (3.2) *gives a one-to-one correspondence between* $\mathrm{Op}^w S^0_{0,0}$ *and* $\mathrm{Op}^w_\Phi S^0_{0,0}$:

$$T^{-1}\circ a^w_\Phi(z,hD_z)\circ T=(a\circ\chi_\phi)^w(x,hD_x),\quad a\in S^0_{0,0}(\Lambda_\Phi).$$

As an immediate corollary of Propositions 3.2 and 3.3 we obtain the well-known boundedness of the elements of $\mathrm{Op}^w S^0_{0,0}$ on $L^2(\mathbf{R}^m)$. Our goal here is the following

Theorem 3. *If* $a\in S^0_{0,0}(\Lambda_\Phi)$ *then*

$$a^w_\Phi(z,hD_z)-a(z,-2i\partial_z\Phi(z))=\mathcal{O}(h^{1/2})\colon H_\Phi(\mathbf{C}^m)\longrightarrow L^2_\Phi(\mathbf{C}^m).$$

Consequently, if $b\in S^0_{0,0}(T^*\mathbf{R}^m)$ *then*

$$T\circ b^w(x,hD_x)-a(z,-2i\partial_z\Phi(z))T=\mathcal{O}(h^{1/2})\colon L^2(\mathbf{R}^m)\longrightarrow L^2_\Phi(\mathbf{C}^m),\quad b=a\circ\chi_\phi.$$

Proof. We start from the expression (3.18) where we want to deform the integration contour $\Gamma_0(z)$ to a 'good contour' in order to obtain an exponentially decaying integrand. To control the error coming from Stokes's formula we introduce an almost analytic extension of $a\in S^0_{0,0}(\Lambda_\Phi)$ (also denoted by a) with the support in $\Lambda_\Phi+B_{\mathbf{C}^{2m}}(0,1)$ and satisfying

$$\bar\partial_{z,\zeta}a(z,\zeta)=\mathcal{O}_N(1)\operatorname{dist}((z,\zeta),\Lambda_\Phi)^N,\quad N\in\mathbf{N}. \tag{3.19}$$

We then define a family of contours $\Gamma_t(z)\colon w\mapsto -2i\partial_z\Phi\big(\tfrac12(z+w)\big)+it(\overline{z-w})$, $0\le t\le 1$, and put

$$A_1u(x)=\frac{1}{(2\pi h)^m}\iint_{\Gamma_1(z)}e^{i\langle z-w,\zeta\rangle/h}a\big(\tfrac12(z+w),\zeta\big)u(w)\,d\zeta\,dw.$$

In this notation $a^w(z,hD_z)u$ becomes $A_0u(z)$ and we claim that

$$\begin{aligned}\|A_1u-a(z,-2i\partial_z\Phi(z))u\|_{L^2_\Phi}&=\mathcal{O}(h^{1/2})\|u\|_{H_\Phi},\\ \|A_1u-A_0u\|_{L^2_\Phi}&=\mathcal{O}(h^\infty)\|u\|_{H_\Phi}.\end{aligned}\tag{3.20}$$

In fact, since on $\Gamma_1(z)$

$$\operatorname{Re}(i\langle z-w,\zeta\rangle)=\operatorname{Re}\Big\langle 2\frac{\partial\Phi}{\partial z}\Big(\frac{z+w}{2}\Big),z-w\Big\rangle-|z-w|^2=\Phi(z)-\Phi(w)-|z-w|^2,$$

the reduced kernel, $\exp(-\Phi/h)A_1\exp(\Phi/h)$, is $\mathcal{O}(h^{-m})\exp(-|z-w|^2/h)$. By expanding $a\big(\tfrac12(z+w),-2i\partial_z\Phi\big(\tfrac12(z+w)\big)\big)$ in Taylor series around $w=z$ we similarly see that the reduced kernel of $A_1-a(z,-2i\partial_z\Phi(z))$ is $\mathcal{O}(h^{-m})|z-w|\exp(-|z-w|^2/h)$ $=\mathcal{O}(h^{1/2})h^{-m}\exp(-|z-w|^2/2h)$, so that the first part of (3.20) follows from Schur's lemma (see for instance Lemma 18.1.12 in [4]).

To obtain the second part we apply Stokes's formula:

$$\begin{aligned}A_1u(z)-A_0u(z)&=\frac{1}{(2\pi h)^m}\iint_\Omega d_{w,\zeta}\big(e^{i\langle z-w,\zeta\rangle/h}a\big(\tfrac{1}{2}(z+w),\zeta\big)u(w)\,dw\wedge d\zeta\big)\\&=\frac{1}{(2\pi h)^m}\iint_\Omega e^{i\langle z-w,\zeta\rangle/h}u(w)\bigg(\sum_{j=1}^m\frac{1}{2}\frac{\partial a}{\partial \bar z_j}\Big(\frac{z+w}{2},\zeta\Big)\,d\bar w_j\\&\qquad+\sum_{j=1}^m\frac{1}{2}\frac{\partial a}{\partial\bar\zeta_j}\Big(\frac{z+w}{2},\zeta\Big)d\bar\zeta_j\bigg)\wedge dw\wedge d\zeta,\end{aligned}$$

where $\Omega=\bigcup_{t=0}^1\Gamma_t$ is parametrized by $w\in\mathbf{C}^m$ and $t\in[0,1]$. Thus,

$$d\bar w_j\wedge dw\wedge d\zeta|_\Omega,\quad d\bar\zeta_j\wedge dw\wedge d\zeta|_\Omega=\mathcal{O}(|z-w|)\mathcal{L}(dw)\,dt.$$

The almost analyticity of a guarantees that on Γ_t,

$$\bar\partial_{w,\zeta}a\big(\tfrac{1}{2}(z+w),\zeta\big)=\mathcal{O}_N(t^N|z-w|^N),\quad\text{for any } N\in\mathbf{N}.$$

Hence we can write $A_1-A_0=\int_0^1 B_t\,dt$, where the reduced kernel of B_t is

$$\mathcal{O}(h^{-m})e^{-t|z-w|^2/h}t^N|z-w|^{N+1}=\mathcal{O}(h^{-m+(N+1)/2}t^{(N-1)/2})e^{-t|z-w|^2/2h}.$$

Schur's lemma shows that the $L^2_\Phi\to L^2_\Phi$ norm of B_t is $\mathcal{O}(h^{(N+1)/2}t^{(N-1)/2-m})$, from which the second part of (3.20) follows. This completes the proof of the theorem as the first part is immediate from (3.20) while the second one follows from Proposition 3.2. □

4. Estimates for localized ordinary differential operators

The purpose of this section is to provide lower bounds for ordinary differential operators arising by freezing $(y',\eta')\in T^*\partial\mathcal{O}$ in (2.15) and considering it as an operator on $[0,\infty)$. We start by discussing the Dirichlet realization of $(hD_t)^2+t$ on $[0,\infty)$. Since $\tau^2+t\to\infty$ as $|\tau|,t\to\infty$, its resolvent is compact and the spectrum discrete. By a simple scaling argument (putting $t=h^{2/3}s$) we see that the eigenvalues are of the form $\zeta_jh^{2/3}$, $0<\zeta_1<\zeta_2<...$, where ζ_j's are the eigenvalues of the Dirichlet realization of D_s^2+s on $[0,\infty)$:

$$\begin{aligned}&(D_s^2+s)Ai(s-\zeta_j)=\zeta_jAi(s-\zeta_j),\quad Ai(-\zeta_j)=0,\\&Ai(s)=\frac{1}{2\pi}\int_{\operatorname{Im}\sigma=\delta>0}e^{i(\sigma^3/3)+i\sigma s}\,d\sigma.\end{aligned}\tag{4.1}$$

If $N(\mu,h)=N(\mu h^{-2/3},1)$ is the number of eigenvalues less than μ, then the semi-classical Weyl law or the well-known asymptotics of the zeros of Airy functions show that $N(\mu,h)=(2/3\pi)h^{-1}\mu^{3/2}(1+o(1))$.

The spectral theorem gives the following trivial lower bound

$$\langle((hD_t)^2+t)u,u\rangle \geq \mu\|u\|^2-(\mu-\zeta_1 h^{2/3})_+\langle\Pi_\mu u,\Pi_\mu u\rangle, \tag{4.2}$$

$$u\in C_0^\infty([0,\infty)),\quad u(0)=0,$$

where $\Pi_\mu\colon L^2([0,\infty))\to L^2([0,\infty))$ denotes the orthogonal projection onto the space spanned by the first $N=N(\mu,h)$ eigenvalues of $(hD_t)^2+t$.

The motivating operator (2.15) contains additional terms—to control them we start by studying the stability of (4.2) with the potential t in the left hand side replaced by a potential $\min(t,2R\mu)$ with $R\geq 2$. To do that we shall first review exponentially weighted estimates on the eigenfunctions. That can be done using asymptotic expansions of the Airy functions (4.1) but we prefer a direct approach in the spirit of Lithner–Agmon estimates.

Recall that if $P=-h^2\Delta+V(x)$ on Ω and $(P-\lambda)u=0$, then under reasonable assumptions (which will be satisfied below), we have

$$h^2\|\nabla(e^{\phi/h}u)\|^2_{L^2(\Omega)}+\int_\Omega (V(x)-\lambda-|\nabla\phi(x)|^2)e^{2\phi(x)/h}|u(x)|^2\,dx=0. \tag{4.3}$$

In our case $\Omega=[0,\infty)$, $P=hD_t^2+t$ and $\lambda\in(0,\mu]$ is an eigenvalue with u the corresponding normalized eigenfunction. We then define $\phi(t)$ depending on μ but not on λ by

$$\phi(t)=\begin{cases} 0, & 0\leq t\leq\mu,\\ \displaystyle\int_\mu^t \sqrt{s-\mu}\,ds, & \mu\leq t\leq R\mu,\\ \phi(R\mu), & R\mu\leq t.\end{cases} \tag{4.4}$$

In (4.3) this gives

$$h^2\|D_t(e^{\phi/h}u)\|^2+\int_0^\mu (t-\lambda)|u(t)|^2\,dt+\int_\mu^{R\mu}(\mu-\lambda)e^{2\phi(t)/h}|u(t)|^2\,dt \\ +e^{2\phi(R\mu)/h}\int_{R\mu}^\infty (t-\lambda)|u(t)|^2\,dt=0,$$

which implies

$$e^{2\phi(R\mu)/h}\int_{R\mu}^\infty (t-\lambda)|u(t)|^2\,dt\leq\int_0^\mu(\lambda-t)|u(t)|^2\,dt\leq\mu\int_0^\mu|u(t)|^2\,dt\leq\mu.$$

Since $t-\lambda\geq(R-1)\mu$ on $[R\mu,\infty)$ and $\phi(R\mu)=\frac{2}{3}(R-1)^{3/2}\mu^{3/2}$ we obtain from this

$$\|u\|_{L^2([R\mu,\infty))}\leq\frac{1}{\sqrt{R-1}}e^{-(2/3)(R-1)^{3/2}\mu^{3/2}/h} \tag{4.5}$$

for any eigenfunction u of $(hD_t)^2+t$ with an eigenvalue $\lambda\in(0,\mu]$. This is crucial for

Lemma 4.1. *If $\mu\leq1$ and $R\geq2$ then for $u\in C_0^\infty([0,\infty))$, $u(0)=0$*

$$\begin{aligned}(4.6)\quad &\langle((hD_t)^2+\min(t,2R\mu))u,u\rangle\\ &\geq\mu\left(1-\mathcal{O}\left(\frac{h^2}{R^2\mu^3}\right)\right)\|u\|^2+(R-1)\mu\|\chi_1u\|^2-(\mu-\zeta_1h^{2/3})_+\|\Pi_\mu u\|^2,\end{aligned}$$

where Π_μ, ζ_1 are as in (4.2) *and $\chi_1\in C^\infty((R\mu,\infty);[0,1])$, $\chi_1\equiv1$ for $t>2R\mu$, $\chi_1=0$ for t close to $R\mu$.*

Proof. We take χ_1 with the properties in the statement of the lemma and in addition such that

$$1-\chi_1^2=\chi_0^2,\quad \chi_0\in C^\infty((-\infty,2R\mu);[0,1]),\quad \partial^\alpha\chi_j=\mathcal{O}_\alpha((R\mu)^{-\alpha}).$$

It then follows that

$$\chi_0[\chi_0,(hD_t)^2]+\chi_1[\chi_1,(hD_t)^2]=-(\chi_0(hD_t)^2(\chi_0)+\chi_1(hD_t)^2(\chi_1))=\mathcal{O}\left(\left(\frac{h}{R\mu}\right)^2\right),$$

from which we get

$$\begin{aligned}&\langle((hD_t)^2+\min(t,2R\mu))u,u\rangle\\ &\qquad\geq\langle((hD_t)^2+t)\chi_0u,\chi_0u\rangle+R\mu\|\chi_1u\|^2-\mathcal{O}\left(\left(\frac{h}{R\mu}\right)^2\right)\|u\|^2.\end{aligned}$$

Combining this with (4.2) we obtain

$$\begin{aligned}(4.7)\quad &\langle((hD_t)^2+\min(t,2R\mu))u,u\rangle\\ &\geq\mu\left(1-\mathcal{O}\left(\frac{h^2}{R^2\mu^3}\right)\right)\|u\|^2+(R-1)\mu\|\chi_1u\|^2-(\mu-\zeta_1h^{2/3})_+\|\Pi_\mu\chi_0u\|^2.\end{aligned}$$

This is almost (4.6)—we only need to see that the last term can be replaced by $-(\mu-\zeta_1h^{2/3})_+\|\Pi_\mu u\|^2$, that is we have to estimate $\|\Pi_\mu(1-\chi_0)u\|^2$. Let us write

$$\Pi_\mu u=\sum_{j=1}^{N(\mu,h)}\langle u,e_j\rangle e_j,\quad ((hD_t)^2+t)e_j=\lambda_je_j,\ e_j(0)=0.$$

Then

$$\|\Pi_\mu(1-\chi_0)u\|^2 \le \sum_{j=1}^{N(\mu,h)} |\langle u,(1-\chi_0)e_j\rangle|^2 \le \|u\|^2 \sum_{j=1}^{N(\mu,h)} \|(1-\chi_0)e_j\|^2,$$

so that, using (4.5)

$$\begin{aligned}\|\Pi_\mu(1-\chi_0)\| &\le \Big(\sum_{j=1}^{N(\mu,h)} \|e_j\|^2_{[R\mu,\infty)}\Big)^{1/2} \le \sqrt{\frac{N(\mu,h)}{R-1}}\, e^{-(2/3)(R-1)^{3/2}\mu^{3/2}/h} \\ &\le C\Big(\frac{\mu^{3/2}}{h(R-1)}\Big)^{1/2} e^{-(3/2)(R-1)^{3/2}\mu^{3/2}/h} \\ &= \frac{1}{(R-1)^{5/4}}\mathcal{O}\Big(\Big(\frac{h}{(R-1)^{3/2}\mu^{3/2}}\Big)^M\Big),\end{aligned}$$

for any M. Taking $M=2$ we obtain (4.6). □

When we restrict the support of u to a fixed interval and optimize the parameters we get

Lemma 4.2. *For $L>0$, $0<h<h_0(L)$ and $0\le\mu\le\mu_0(L)$ the following estimate holds uniformly for $u\in C_0^\infty([0,(2L)^{-1}])$, $u(0)=0$:*

$$\begin{aligned}\langle((hD_t)^2+t)u,u\rangle \ge \mu\Big(1-\mathcal{O}(1)\max\Big(\mu L,\frac{h\sqrt{L}}{\mu}\Big)\Big)\|u\|^2 \\ -(\mu-\zeta_1 h^{2/3})_+\|\Pi_\mu u\|^2+L\|tu\|^2.\end{aligned} \tag{4.8}$$

Proof. Writing $T=2R\mu$, we observe that (4.6) implies that for $T\ge4\mu$ (this reflects the condition that $R\ge2$), some $C>0$ and any $u\in C_0^\infty([0,a[)$, $u(0)=0$,

$$\begin{aligned}\langle((hD_t)^2+t)u,u\rangle \ge \mu(1-C\mu^{-1}h^2T^{-2}-\delta)\|u\|^2 \\ -(\mu-\zeta_1 h^{2/3})_+\|\Pi_\mu u\|^2+L\|tu\|^2,\end{aligned} \tag{4.9}$$

provided that

$$\begin{aligned} Lt^2-\mu\delta\le0 \quad &\text{if } 0\le t\le T, \\ Lt^2-\mu\delta+T\le t \quad &\text{if } T\le t\le a.\end{aligned} \tag{4.10}$$

We choose $a=(2L)^{-1}$ and require that $T\le(2L)^{-1}$. Then (4.10) follows if we have the last inequality of (4.10) satisfied at the end points $t=T$, $t=(2L)^{-1}$: $LT^2\le\mu\delta$, $T-(4L)^{-1}\le\mu\delta$, we choose $\delta=\mu^{-1}\max(LT^2,T-(4L)^{-1})$. Remembering also that $T=2R\mu$, $R\ge2$, it is enough to restrict T to the interval $4\mu\le T\le(2L)^{-1}$. Assuming that μ and h are sufficiently small depending on L, we can then take $T=\max(C^{1/4}L^{-1/4}h^{1/2},4\mu)$ and get $\delta=\mu^{-1}LT^2$, and

$$\delta+C\mu^{-1}T^{-2}h^{-2}=\mathcal{O}(1)\max(\mu^{-1}L^{1/2}h,L\mu). \quad \square$$

We still need to control more terms and for that we have

Lemma 4.3. *For* $u\in C_0^\infty([0,\infty))$ *with* $u(0)=0$

$$\sqrt{\zeta_1}h^{1/3}\|hD_tu\|\le\|((hD_t)^2+t)u\|, \tag{4.11}$$

$$\|(hD_t)^2u\|\le\|((hD_t)^2+t)u\|. \tag{4.12}$$

Proof. For u in the statement of the lemma we have

$$\|((hD_t)^2+t)u\|^2=\|(hD_t)^2u\|^2+\|tu\|^2+2\operatorname{Re}\langle tu,(hD_t)^2u\rangle,$$

where the last term is equal to

$$2\operatorname{Re}\langle hD_t(tu),hD_tu\rangle=2\operatorname{Re}\langle thD_tu,hD_tu\rangle+h\operatorname{Re}\frac{2}{i}\langle u,hD_tu\rangle=2\|t^{1/2}hD_tu\|^2.$$

Hence (4.12) follows. To get (4.11) we observe that

$$\|u\|\le(\zeta_1h^{2/3})^{-1}\|((hD_t)^2+t)u\|,$$

so that, by (4.12)

$$\|hD_tu\|^2=\langle(hD_t)^2u,u\rangle\le(\zeta_1h^{2/3})^{-1}\|((hD_t)^2+t)u\|^2,$$

which is (4.11). □

To motivate the proof of the main result of this section let us now consider the model scaled operator $e^{-2\pi i/3}((hD_t)^2+t)$. Let $\omega_0=\operatorname{Re}\omega_0+ir_0$ satisfy $0<\arg\omega_0<\frac{2}{3}\pi$ and let $\mu\ge h^{2/3}/C$ be close to 0. Then for $u\in C_0^\infty([0,(CL)^{-1}])$, L, $C\gg1$, $u(0)=0$, we have

$$\begin{aligned}\|(e^{-2\pi i/3}((hD_t)^2+t)-\omega_0)u\|^2&=|\omega_0|^2\|u\|^2+\|((hD_t)^2+t)u\|^2\\&\quad+2\operatorname{Re}(-e^{-2\pi i/3}\bar\omega_0)\langle((hD_t)^2+t)u,u\rangle.\end{aligned} \tag{4.13}$$

Since $\operatorname{Re}(-e^{-2\pi i/3}\bar\omega_0)=|\omega_0|\cos\big(\frac{1}{3}\pi-\arg\omega_0\big)>0$, the combination of (4.13) and (4.8) gives for $h<h_0(L)$ and u same as above

(4.14)
$$\begin{aligned}\|(e^{-2\pi i/3}((hD_t)^2+t)-\omega_0)u\|^2&\ge\big(|\omega_0-e^{-2\pi i/3}\mu|^2-\mathcal{O}(1)\max\big(\sqrt{L}\,h,L\mu^2\big)\big)\|u\|^2\\&\quad-2\operatorname{Re}(-e^{-2\pi i/3}\bar\omega_0)(\mu-\zeta_1h^{2/3})_+\|\Pi_\mu u\|^2\\&\quad+L\|tu\|^2+\|((hD_t)^2+t)u\|^2,\end{aligned}$$

where we also used $|\omega_0-e^{-2\pi i/3}\mu|^2=|\omega_0|^2+2\operatorname{Re}(-\mu e^{-2\pi i/3}\bar\omega_0)+\mathcal{O}(\mu^2)$.

We will now proceed to the main result of this section:

Proposition 4.1. *Suppose that a second order ordinary differential operator P on $[0,\infty)$ satisfies*

$$P = e^{-2\pi i/3}((hD_t)^2+t)+\mathcal{O}(h)hD_t+\mathcal{O}(h+h^{1/2}t+t^2). \tag{4.15}$$

If $L>0$ is sufficiently large and $h>0$, $\mu\geq 0$ are sufficiently small depending on L, then for $\omega_0\in\mathbf{C}$, $0<\arg\omega_0<\frac{2}{3}\pi$ and $u\in C_0^\infty([0,(CL)^{-1}))$, $u(0)=0$:

$$\begin{aligned}\|(P-\omega_0)u\|^2 \geq (|\omega_0-e^{-2\pi i/3}\mu|^2-\mathcal{O}(1)\max(\sqrt{L}\,h,\mu^2L))\|u\|^2+\tfrac{1}{2}L\|tu\|^2 \\ -2\operatorname{Re}(-e^{-2\pi i/3}\bar{\omega}_0)(\mu-\zeta_1h^{2/3})_+\|\Pi_\mu u\|^2+\tfrac{1}{2}\|((hD_t)^2+t)u\|^2.\end{aligned} \tag{4.16}$$

Here Π_μ is the orthogonal projection onto the eigenspaces corresponding to the intersection $\sigma((hD_t)^2+t)\cap(-\infty,\mu)$.

Proof. Since $h^{1/2}t=\mathcal{O}(t^2+h)$ we can neglect that term in (4.15). Thus, to apply (4.14) and Lemma 4.3 we first estimate the left hand side of (4.16) from below by

$$\begin{aligned}&\|(e^{-2\pi i/3}((hD_t)^2+t)-\omega_0)u\|^2-\langle(\mathcal{O}(h)hD_t+\mathcal{O}(h+t^2))u,(hD_t)^2u\rangle \\ &\quad-\langle(\mathcal{O}(h)hD_t+\mathcal{O}(h+t^2)t)u,u\rangle-\langle\mathcal{O}(h)hD_tu,\mathcal{O}(h)hD_tu\rangle.\end{aligned}$$

The last three terms are bounded from below by

$$\begin{aligned}&-\mathcal{O}(1)\big[h\|hD_tu\|\|(hD_t)^2u\|+h\|u\|\|(hD_t)^2u\| \\ &+\|t^2u\|\|(hD_t)^2u\|+h\|hD_tu\|\|u\|+h\|u\|^2+\|tu\|^2+h^2\|hD_tu\|^2\big] \\ &\quad\geq-\mathcal{O}(1)\Big[h^{2/3}(h^{1/3}\|hD_tu\|)^2+h^{2/3}\|(hD_t)^2u\|^2+h\|u\|^2+h\|(hD_t)^2u\|^2 \\ &\quad+\frac{M}{2}\|tu\|^2+\frac{1}{2M}\|(hD_t)^2u\|^2+h^{1/3}(h^{1/3}\|hD_tu\|)^2+h\|u\|^2+\|tu\|^2\Big],\end{aligned}$$

which by Lemma 4.3 is bounded from below by

$$-\mathcal{O}(1)\left[\left(\frac{1}{2M}+h^{1/3}+2h^{2/3}+h\right)\|((hD_t)^2+t)u\|^2+2h\|u\|^2+\left(\frac{M}{2}+1\right)\|tu\|^2\right].$$

By taking M sufficiently large and then $L\gg M$ this estimate combined with (4.14) gives (4.16) provided h is sufficiently small depending on L. □

It is clear that $(hD_t)^2+t$ in (4.15) can be replaced by $(hD_t)^2+Qt$ where $Q\in I$ with I a compact subset of $(0,\infty)$. The projection Π_μ is then to be replaced by the spectral projections associated to $\sigma((hD_t)^2+Qt)\cap(-\infty,\mu)$. All the estimates remain uniform for $Q\in I$.

5. Lower bounds for $P-\omega_0$

As already indicated in Sect. 3 we want to freeze $(y',\eta')\in T^*\partial\mathcal{O}$ in (2.15) and to apply Proposition 4.1 to the resulting ordinary differential operator.

Let $\Omega\subset\partial\mathcal{O}$ be a neighbourhood of a fixed point $y_0'\in\partial\mathcal{O}$ near which we consider coordinates such that (2.15) holds. Consequently for $P=-h^2\Delta|_\Gamma$

$$\begin{aligned}P(x',x_n;\xi',hD_{x_n}) &= e^{-2i\pi/3}((hD_{x_n})^2+2x_nQ(x',\xi')) \\ &\quad +R(x',\xi')+\mathcal{O}(x_n^2+h)\langle\xi'\rangle^2+\mathcal{O}(h)hD_{x_n},\end{aligned} \tag{5.1}$$

where we identify Ω with a subset of $\mathbf{R}^{n-1}$ so that $(x',\xi')\in T^*\mathbf{R}^{n-1}$, and x_n is small enough so that $g(x_n)=x_ne^{i\pi/3}$ (see (2.14)). For a fixed ω_0 in the first quadrant, we want to obtain lower bounds (positive except for a finite rank contribution) on $(P-\omega_0)^*(P-\omega_0)$. In view of Lemma 2.2 we can start with the corresponding differential operator obtained from (5.1). The estimate in the critical region $1/C<|\xi'|<C$ is provided by

Lemma 5.1. *Suppose that* $\omega_0\in\mathbf{C}$, $\operatorname{Re}\omega_0$, $\operatorname{Im}\omega_0>0$ *and* $|\operatorname{Re}\omega_0-R(x',\xi')|$ *is sufficiently small, L is large enough and* $0<h<h_0$. *Then for $\tilde\mu$ close to* 0 *and any* $v\in C_0^\infty([0,(CL)^{-1}))$, $v(0)=0$,

$$\begin{aligned}&\|(P(x',t,\xi',hD_t)-\omega_0)v\|^2 \\ &\quad\geq \left(|\omega_0-R(x',\xi')-e^{-2i\pi/3}\tilde\mu|^2-\mathcal{O}(1)\max\{\sqrt{L}\,h,\tilde\mu^2\}\right)\|v\|^2 \\ &\quad\quad -2\operatorname{Re}(-e^{-2i\pi/3}(\bar\omega_0-R(x',\xi')))(\tilde\mu-\zeta_1(x',\xi')h^{2/3})_+\|\Pi_{(x',\xi',\tilde\mu)}v\|^2 \\ &\quad\quad +\tfrac12\|((hD_t)^2+2tQ(x',\xi'))v\|^2+\tfrac12L\|tv\|^2,\end{aligned} \tag{5.2}$$

where $\|\cdot\|$ *is the* L^2*-norm on* $[0,\infty)$, $\Pi_{(x',\xi',\tilde\mu)}$ *is the orthogonal projection associated to* $\sigma((hD_t)^2+2tQ(x',\xi'))\cap(-\infty,\tilde\mu)$ *and* $\zeta_1(x',\xi')=\zeta_1(2Q(x',\xi'))^{2/3}$ *is the first eigenvalue of* $D_t^2+2tQ(x',\xi')$.

Proof. The bound (5.2) follows immediately from Proposition 4.1 and (5.1) once ω_0 in (4.16) is replaced by $\omega_0-R(x',\xi')$. The size condition on $|\operatorname{Re}\omega_0-R(x',\xi')|$ guarantees that $0<\arg(\omega_0-R)<\frac23\pi$, so that $2\operatorname{Re}(-e^{-2i\pi/3}(\bar\omega_0-R))>0$. The uniformity of the constants follows from the ellipticity of Q and R and the bound $\operatorname{Re}\omega_0-c\leq R(x',\xi')\leq c+\operatorname{Re}\omega_0$. □

In the 'easy' region, the estimate is immediate:

Lemma 5.2. *Suppose that* $\omega_0\in\mathbf{C}$, $\operatorname{Re}\omega_0$, $r_0=\operatorname{Im}\omega_0>0$ *and that*

$$|R(x',\xi')-\omega_0|>c+r_0,\quad c>0. \tag{5.3}$$

Then for L large enough, h sufficiently small, and $v\in C_0^\infty([0,(CL)^{-1}))$, $v(0)=0$,

$$\|(P(x',t,\xi',hD_t)-\omega_0)v\|^2 \geq \left(r_0+\frac{1}{C}\right)^2\|v\|^2+\frac{1}{C}(\|(hD_t)^2v\|^2+\langle\xi'\rangle^4\|v\|^2). \tag{5.4}$$

Proof. In place of Proposition 4.1 we use the following elementary inequality ($a\in\mathbf{R}$):

$$\begin{aligned}\|(e^{-2i\pi/3}(hD_t)^2+a-ir_0)v\|^2 &= \|(hD_t)^2v\|^2+(r_0^2+a^2)\|v\|^2\\ &\quad +2\operatorname{Re}(e^{-2i\pi/3}(a+ir_0))\langle(hD_t)^2v,v\rangle\\ &\geq \left(1-\sin\left(\tfrac{1}{6}\pi\right)\right)\|(hD_t)^2v\|^2\\ &\quad +\left(r_0^2+a^2\left(1-\sin\left(\tfrac{1}{6}\pi\right)\right)\right)\|v\|^2,\end{aligned}$$

which holds since

$$2\operatorname{Re}(e^{-2i\pi/3}(a+ir_0))\langle(hD_t)^2v,v\rangle \geq -a\sin\left(\frac{\pi}{6}\right)\left(\frac{1}{a}\|(hD_t)^2v\|^2+a\|v\|^2\right).$$

We then put $a=R(x',\xi')-\operatorname{Re}\omega_0$ so that in view of (5.3) $|a|\geq C^{-1}\langle\xi'\rangle^2$. Hence for L sufficiently large and h sufficiently small

$$\|(\mathcal{O}(t+t^2+h)\langle\xi'\rangle^2+\mathcal{O}(h)hD_t)v\|^2 \leq \frac{1}{C}(a^2\|v\|^2+\|(hD_t)^2v\|^2),$$

so that (5.4) with yet another constant C follows from (5.1). □

Using the term $O(th^{1/2})$ in (4.15), the lower bound given in Proposition 4.1 allows us to vary (x',ξ') in the left hand side of (5.2) within the distance $h^{1/2}$. More precisely, let us fix (x',ξ'), $x'\in\Omega$, $1/C<|\xi'|<C$. For $\mu>0$ and $\varepsilon>0$ small, we define $\hat\mu(x',\xi')\geq 0$ by

$$\inf_{|(y',\eta')-(x',\xi')|\leq\varepsilon h^{1/2}} |\omega_0-R(y',\eta')-e^{-2i\pi/3}\hat\mu(x',\xi')| = r_0+\mu, \quad r_0=\operatorname{Im}\omega_0, \tag{5.5}$$

with the convention that $\hat\mu=0$ if

$$\inf_{|(y',\eta')-(x',\xi')|\leq\varepsilon h^{1/2}} |\omega_0-R(y',\eta')|\geq r_0+\mu.$$

We notice that $\partial_{\hat\mu}(|\omega_0-R(y',\eta')-e^{-2i\pi/3}\hat\mu|^2)=2\hat\mu+2\operatorname{Re}(-e^{-2i\pi/3}(\bar\omega_0-R))>0$ if C is chosen as in Lemma 5.1. Hence, $\hat\mu\mapsto|\omega_0-R(y',\eta')-e^{-2i\pi/3}\hat\mu|$ is an increasing function of $\hat\mu\geq 0$ and the definition makes sense, $\hat\mu(x',\xi')=\mathcal{O}(\max(h^{2/3},\mu))$.

We also observe that for $|(x',\xi')-(y',\eta')|<\varepsilon h^{1/2}$

$$\begin{aligned}P(y',t,\eta',hD_t) &= (P(x',t,\xi',hD_t)-R(x',\xi'))+R(y',\eta')\\ &\quad +\mathcal{O}(h^{1/2})(t+h+\mathcal{O}(h)hD_t).\end{aligned}$$

Hence, proceeding as in the proof of Lemma 5.1 we obtain

Lemma 5.3. *Under the assumptions of Lemma* 5.1 *and for*

$$\tilde{\mu} = \max(\zeta_1(x', \xi')h^{2/3}, \hat{\mu}(x', \xi'))$$

with $\hat{\mu}$ *given by* (5.5) *we have for* $|(y', \eta') - (x', \xi')| < \varepsilon h^{1/2}$ *and* $v \in C_0^\infty([0, (CL)^{-1}))$, $v(0) = 0$,

(5.6)

$$\|(P(y', t; \eta', hD_t) - \omega_0)v\|^2 \geq \left((r_0 + \mu)^2 - \mathcal{O}(1)\max(\sqrt{L}\,h, \mu^2 L)\right)\|v\|^2 \\ - \mathcal{O}(\mu)\|\Pi_{x', \xi', \tilde{\mu}} v\|^2 + \tfrac{1}{2}\|((hD_t)^2 + 2tQ(x', \xi'))v\|^2. \quad \square$$

The advantage we gained is in having a fixed projection $\Pi_{x', \xi', \tilde{\mu}}$ for varying (y', η') in P as long as they remain in an $\varepsilon h^{1/2}$ neighbourhood of (x', ξ'). We will now follow Sect. 3 of [9] and introduce finite rank operators associated to partitions of unity.

Let $K_\Omega \subset \overline{\Omega} \times \mathbf{R}^{n-1}$ be the compact set

$$K_\Omega = \{(x', \xi') \in \overline{\Omega} \times \mathbf{R}^{n-1} : \hat{\mu}(x', \xi') \geq \zeta_1(x', \xi')h^{2/3}\} \tag{5.7}$$

where $\hat{\mu}$ is defined by (5.5). We observe that $\hat{\mu} > 0$ implies $|\omega_0 - R(x', \xi')| \leq r_0 + \mu$ and hence $|\operatorname{Re}\omega_0 - R(x', \xi')| < \mu^{1/2}(2r_0 + \mu)^{1/2}$, and finally

$$\operatorname{vol}_{\mathbf{R}^{2(n-1)}}(K_\Omega) = \mathcal{O}(\mu^{1/2}). \tag{5.8}$$

Let us assume that K_Ω can be covered by $\widetilde{M} = \widetilde{M}(K_\Omega, \varepsilon h^{1/2})$ balls $B((x'_j, \xi'_j); \varepsilon h^{1/2})$. We then choose a partition of K:

$$K_\Omega = \bigcup_{j=1}^{\widetilde{M}} K_j, \quad K_j \cap K_k = \emptyset,\ j \neq k, \quad K_j \subset B((x'_j, \xi'_j), \varepsilon h^{1/2}),\ (x'_j, \xi'_j) \in K_j.$$

We note that (5.6) holds precisely for $(y', \eta') \in K_j$ and $(x', \xi') = (x'_j, \xi'_j)$ in the right hand side. Motivated by this we define a modified projection operator for $(x', \xi') \in T^*\overline{\Omega}$:

$$\widetilde{\Pi}_{(x', \xi')} = \begin{cases} 0 & \text{if } (x', \xi') \notin K_\Omega \\ \Pi_{(x'_j, \xi'_j, \tilde{\mu})} & \text{if } (x', \xi') \in K_j. \end{cases} \tag{5.9}$$

With this notation we can state:

Proposition 5.1. *Suppose that* $u\in C^\infty(\mathbf{R}^n\setminus\mathcal{O})$ *satisfies*

$$\operatorname{supp} u\subset\Omega\times[0,(CL)^{-1}),\quad u|_{\partial\mathcal{O}}=0,$$

where L *is sufficiently large,* Ω *is a sufficiently small open subset of* $\partial\mathcal{O}$ *and we use the coordinates* (2.3) *near* Ω. *Then for* $h^{2/3}/C<\mu<1/C$, $\omega_0\in\mathbf{C}$, $0<\operatorname{Im}\omega_0<1/C$, *and* $0<h<h_0(L)$:

$$\|(P-\omega_0)u\|^2\geq\big((r_0+\mu)^2-\mathcal{O}(1)\max\big(\mu^2L,\sqrt{L}\,h\big)\big)\|u\|^2-\mathcal{O}(\mu)\|\widetilde{\Pi}Tu\|_\Phi, \tag{5.10}$$

where T *is given by* (3.2), $z=x'-i\xi'$ *and*

$$\widetilde{\Pi}u(x',\xi';x_n)\stackrel{\text{def}}{=}(\widetilde{\Pi}_{(x',\xi')}(u(x',\xi',\cdot)))(x_n).$$

Proof. We apply Proposition 3.1 with $A=P-\omega_0$ (since $\operatorname{supp} u$ is compact, the coefficients of $P-\omega_0$ are effectively C_b^∞):

$$\begin{aligned}\|(P-\omega_0)u\|^2&=\|(P(x',x_n;\xi',hD_{x_n})-\omega_0)Tu\|_\Phi^2\\&\quad+\mathcal{O}(h)(\|(hD_{x_n})^2Tu\|_\Phi^2+\|(1+|\xi'|^2)Tu\|_\Phi^2),\end{aligned}\tag{5.11}$$

where, as $Tu|_{x_n=0}=0$, we simplified (3.6) by interpolation.

Lemmas 5.2, 5.3 and 4.3 show that

$$\int_0^\infty|(P(x',x_n;\xi',hD_{x_n})-\omega_0)Tu(x',\xi',x_n)|^2\,dx_n$$

is bounded from below by

$$\begin{gathered}\big((r_0+\mu)^2-\mathcal{O}(1)\max\big(\mu^2L,\sqrt{L}\,h\big)\big)\int_0^\infty|Tu(x',\xi',x_n)|^2\,dx_n\\-\mathcal{O}(\mu)\int_0^\infty|\widetilde{\Pi}_{(x',\xi')}Tu(x',\xi',x_n)|^2\,dx_n+\frac{1}{2}\int_0^\infty|(hD_{x_n})^2Tu(x',\xi',x_n)|^2\,dx_n,\end{gathered}$$

if $|R(x',\xi')-\operatorname{Re}\omega_0|<c$, and by

$$\frac{1}{C_1}\bigg(\langle\xi'\rangle^4\int_0^\infty|Tu(x',\xi',x_n)|^2\,dx_n+\frac{1}{2}\int_0^\infty|(hD_{x_n})^2Tu(x',\xi',x_n)|^2\,dx_n\bigg),$$

otherwise. If $0<r_0+\mu<2/C$ for $C\gg C_1$, integration in (x',ξ') (with the weight function $\exp(-|\xi'|^2/2h)$), gives (5.10), as the remainder terms in (5.11) can be absorbed into the lower bound. □

Since the second term on the right hand side of (5.10) is obtained from integration over the finite volume subset of the phase space (of $\partial\mathcal{O}$), K, and $\widetilde{\Pi}_{(x',\xi')}$ projects on a finite number of eigenspaces, we would like to replace that term by the square of the norm of a finite rank operator acting on u.

We recall from Sect. 3 of [9] that for any $\varepsilon>0$, which is the same as the ε in the construction of the partition $\{K_j\}$ of K above, there exists an operator $\Xi\colon L^2_\Phi(\mathbf{C}^{n-1})\to L^2_\Phi(\mathbf{C}^{n-1})$ of finite rank less than or equal to $\widetilde{M}=\widetilde{M}(K,\varepsilon h^{1/2})$, satisfying

$$\|\mathbf{1}_K(Tu-\Xi(Tu(\,\cdot\,,x_n)))\|_\Phi\le C\varepsilon\|Tu\|_\Phi. \tag{5.12}$$

In fact, by the mean value theorem for holomorphic functions we have for $v\in H_\Phi(\mathbf{C}^{n-1})$

$$v(z)=\iint e^{i(z-w)\operatorname{Im} z/h}\chi_0((z-w)h^{-1/2})h^{-n}v(w)\,\mathcal{L}(dw),$$

where $\chi_0\in C_0^\infty(B(0,1))$, $\int\chi_0(w)\,\mathcal{L}(dw)=1$, $\chi_0(z)=\widetilde{\chi}(|z|)$ (compare (3.11) where $(2\pi)^{-n/2}\exp(-|x|^2)$ is used in place of χ_0). We then define an operator of rank less than or equal to $\widetilde{M}$

(5.13)

$$\Xi v(z)=\begin{cases}\displaystyle\int e^{i(z-w)\operatorname{Im} z/h}\chi_0((z_j-w)h^{-1/2})h^{-n}v(w)\,\mathcal{L}(dw), & z\in K_j,\ z_j=x_j-i\xi_j\\ 0, & z\notin K.\end{cases}$$

In our case the relevant operator $v(z,x_n)\mapsto(\Xi(v(\,\cdot\,,x_n)))(z)$ is not of finite rank. However, it becomes one when composed with the projection $\widetilde{\Pi}$:

$$v(z,x_n)\longmapsto(\widetilde{\Pi}_{(\operatorname{Re} z,-\operatorname{Im} z)}((\Xi v)(z,\,\cdot\,)))(x_n)=(\widetilde{\Pi}\Xi v)(z,x_n).$$

The rank of $\widetilde{\Pi}\Xi$ is less than or equal to $\sum_1^{\widetilde{M}}N_j$, where N_j is the rank of the projection $\Pi_{(\operatorname{Re} z_j,-\operatorname{Im} z_j,\tilde{\mu})}$, $z_j\in K_j$. On the other hand by (5.12)

$$\|\widetilde{\Pi}Tu-\widetilde{\Pi}\Xi Tu\|_\Phi\le\|\mathbf{1}_K(Tu-\Xi Tu)\|_\Phi\le C\varepsilon\|Tu\|_\Phi,$$

and consequently

$$\|\widetilde{\Pi}Tu\|_\Phi^2\le(\|\widetilde{\Pi}\Xi Tu\|_\Phi+\|\widetilde{\Pi}Tu-\widetilde{\Pi}\Xi Tu\|_\Phi)^2\le\|\widetilde{\Pi}\Xi Tu\|_\Phi^2+\mathcal{O}(\varepsilon)\|u\|^2. \tag{5.14}$$

We have thus proved the local version of the main result of this section:

Proposition 5.2. *Suppose that $h,\mu>0$ and $\omega_0\in\mathbf{C}$ satisfy*

$$0<h<h_0,\quad h^{2/3}/C<\mu<1/C,\quad r_0=\operatorname{Im}\omega_0>0,\quad \operatorname{Re}\omega_0>2(\operatorname{Im}\omega_0+\mu).$$

Then for every $\varepsilon>0$ there exist finite rank operators

$$\widetilde{\Xi}_\varepsilon^p: L^2(\mathbf{R}^n\setminus\mathcal{O})\longrightarrow L^2_\Phi(\mathbf{C}^{n-1}\times[0,\infty))$$

such that for $u\in C_0^\infty(\mathbf{R}^n\setminus\mathcal{O})$, $u|_{\partial\mathcal{O}}=0$ we have

$$\|(P-\omega_0)u\|^2\geq((r_0+\mu)^2-\mathcal{O}(h+\mu^2+\varepsilon\mu))\|u\|^2-\mathcal{O}(\mu)\left(\sum_{p=1}^{Q}\|\widetilde{\Xi}_\varepsilon^p u\|_\Phi^2\right) \tag{5.15}$$

$$\operatorname{rank}\widetilde{\Xi}_\varepsilon^p\leq\sum_{j=1}^{\widetilde{M}(K_p,\varepsilon h^{1/2})}\operatorname{rank}(\widetilde{\Pi}_{(x_j^p,\xi_j^p)}),$$

where $K_p=K_{\Omega_p}$ given by (5.7), $\partial\mathcal{O}=\bigcup_{p=1}^{Q}\Omega_p$, Ω_p open, and $\widetilde{\Pi}_{(x_j^p,\xi_j^p)}$ is defined by (5.9).

Proof. We start by proving (5.15) for u with the support sufficiently close the boundary: $\operatorname{supp}u\subset\{x:d(x)<(CL)^{-1}\}$. If $\partial\Omega=\bigcup_{p=1}^{Q}\Omega_p$ where Ω_p are open sets which are images of coordinate maps $\tilde{s}_p=\tilde{s}$ (see (2.2)), then we choose $\chi_p\in C^\infty(\partial\mathcal{O};[0,1])$, $\operatorname{supp}\chi_p\subset\Omega_p$, $\sum_{p=1}^{Q}\chi_p^2=1$. We have already seen that (5.15) holds for u replaced by $\chi_p u$:

$$\|(P-\omega_0)\chi_p u\|^2\geq((r_0+\mu)^2-\mathcal{O}(1)(h+\mu^2+\varepsilon\mu))\|\chi_p u\|^2-\mathcal{O}(\mu)\|\widetilde{\Pi}\Xi_\varepsilon T\tilde{s}_p^*\chi_p u\|_\Phi^2,$$

with $\Xi_\varepsilon=\Xi$ given by (5.13) and using (5.10) and (5.14). We also used the fact that since a small neighbourhood of Ω_p in $\mathbf{R}^n\setminus\mathcal{O}$ is identified with $\Omega_p\times[0,\delta)$, the χ_p's can be considered as functions on $\mathbf{R}^n\setminus\mathcal{O}$.

We now write

$$\begin{aligned}\|(P-\omega_0)u\|^2&=\sum_{p=1}^{Q}\|\chi_p(P-\omega_0)u\|^2\\&\geq\sum_{p=1}^{Q}\|(P-\omega_0)\chi_p u\|^2+\sum_{p=1}^{Q}\|[\chi_p,P]u\|^2-2\sum_{p=1}^{Q}\|(P-\omega_0)\chi_p u\|\|[\chi_p,P]u\|\\&\geq\sum_{p=1}^{Q}\|(P-\omega_0)\chi_p u\|^2-\sum_{p=1}^{Q}\|[\chi_p,P]u\|^2-2\sum_{p=1}^{Q}\|\chi_p(P-\omega_0)u\|\|[\chi_p,P]u\|\\&\geq\sum_{p=1}^{Q}\|(P-\omega_0)\chi_p u\|^2-\sum_{p=1}^{Q}\|[\chi_p,P]u\|^2\\&\quad-2\|(P-\omega_0)u\|\left(\sum_{p=1}^{Q}\|[\chi_p,P]u\|^2\right)^{1/2}.\end{aligned}$$

We also have $\|u\|^2=\sum_{p=1}^{Q}\|\chi_p u\|^2$ so that if we put

$$\widetilde{\Xi}_\varepsilon^p \stackrel{\text{def}}{=} \widetilde{\Pi}\Xi_\varepsilon T\tilde{s}_p^*\chi_p\colon L^2(\mathbf{R}^n\setminus\mathcal{O}) \longrightarrow L^2_\Phi(\mathbf{C}^{n-1}\times[0,\infty)),$$

then (5.15) follows, once we show that

$$\|[\chi_p,P]u\| = \mathcal{O}(h)(\|(P-\omega_0)u\|+\|u\|).$$

Since $\|[\chi_p,P]u\|=\mathcal{O}(h)(\|hD_{x_n}u\|+\|hD_{x'}u\|+\|u\|)$, this is immediate from the ellipticity of the Dirichlet problem for $P-\omega_0$ (see the proof of Lemma 5.2).

It remains to remove the restriction that the support of u is close to $\partial\mathcal{O}$. For that let $\phi_0,\phi_1\in C^\infty(\mathbf{R}^n\setminus\mathcal{O};[0,1])$ satisfy $\phi_0^2+\phi_1^2=1$, $\operatorname{supp}\phi_0\subset\{x:d(x)<(CL)^{-1}\}$, $\phi_0\equiv1$ on $\{x:d(x)<(2CL)^{-1}\}$. Let μ be small enough so that $\operatorname{Im}((1+i\theta)\omega_0)>r_0+\mu$, where θ is the same as in the definition of Γ (2.14), $P=-\Delta|_\Gamma$. We claim that then

$$\|(P-\omega_0)\phi_1 u\|^2 > (r_0+\mu)^2\|\phi_1 u\|^2. \tag{5.16}$$

In fact, we can replace P on Γ by $-\Delta|_{\widetilde{\Gamma}}$ where $\widetilde{\Gamma}$ extends the totally real submanifold $\Gamma\subset\mathbf{C}^n\setminus\mathcal{O}$ to a smooth totally real submanifold in $\mathbf{C}^n$. By the construction of Γ and by choosing the extension suitably, we see that the symbol of $-\Delta|_{\widetilde{\Gamma}}$ takes its values in $\arg(1+i\theta)<-\arg z<\frac{1}{3}\pi$. Hence, $\inf|\sigma(-\Delta|_{\widetilde{\Gamma}})-\omega_0|>\operatorname{Im}((1+i\theta)\omega_0)$ and (5.16) holds if h is small enough. We conclude the argument by writing:

$$\begin{aligned}\|(P-\omega_0)u\|^2 &\geq \sum_{i=1,2}\|(P-\omega_0)\phi_i u\|^2-\sum_{i=1,2}\|[\phi_i,P]u\|^2\\ &\quad -2\sum_{i=1,2}\|\phi_i(P-\omega_0)u\|\|[\phi_i,P]u\|\\ &\geq ((r_0+\mu)^2-\mathcal{O}(h+\mu^2+\varepsilon\mu))\|\phi_0 u\|^2\\ &\quad +(r_0+\mu)^2\|\phi_1 u\|^2-\mathcal{O}(\mu)\sum_{p=1}^{P}\|\widetilde{\Xi}_\varepsilon^p u\|^2_\Phi\\ &\quad -\sum_{i=1,2}\|[\phi_i,P]u\|^2-2\sum_{i=1,2}\|\phi_i(P-\omega_0)u\|\|[\phi_i,P]u\|.\end{aligned}$$

By estimating the commutator terms by $\mathcal{O}(h)(\|(P-\omega_0)u\|+\|u\|)$ as before we obtain (5.15). □

It is clear that for μ smaller than $C^{-1}h^{2/3}$ a better estimate is possible if C is large enough. Thus we want to find the largest μ for which one gets a positive lower bound.

Proposition 5.3. *Suppose that* $\omega_0 \in \mathbf{C}$ *satisfies* $0<r_0=\operatorname{Im}\omega_0<1/C$, $\operatorname{Re}\omega_0>0$ *and that* $0<h<h_0$ *for some sufficiently small* $h_0>0$. *Then*

$$\|(P-\omega_0)u\|^2 \geq |r_0+S_{\min}(\operatorname{Re}\omega_0)^{2/3}h^{2/3}-\mathcal{O}(h)|^2\|u\|^2, \tag{5.17}$$

$$S_{\min} \stackrel{\text{def}}{=} 2^{2/3}\cos\left(\tfrac{1}{6}\pi\right)\zeta_1\Big(\min_{x'\in\partial\mathcal{O}, i=1,\dots,n-1} K_i(x')\Big)^{2/3}, \tag{5.18}$$

where $K_i(x')$ *are the principal curvatures of* $\partial\mathcal{O}$ *at* x' *and* $-\zeta_1$ *is the first zero of the Airy function* (4.1).

Proof. Following the arguments in the proof of Proposition 5.1 we only need to consider the critical region $|R(x',\xi')-\operatorname{Re}\omega_0|<c$ and prove the bound on the FBI-transform side:

$$\|(P(x',t,\xi',hD_t)-\omega_0)v\|^2 \geq (r_0+S_{\min}(\operatorname{Re}\omega_0)^{2/3}-\mathcal{O}(h))^2\|v\|^2,$$

where v satisfies the assumptions of Lemma 5.1 (from which we take the notation). If we put $\tilde{\mu}=\zeta_1(2Q(x',\xi'))^{2/3}h^{2/3}$ in (5.2) then we see that the minimum of

$$|\omega_0-R(x',\xi')-e^{-2i\pi/3}\zeta_1(x',\xi')h^{2/3}|^2 = \left|\operatorname{Re}\omega_0-R(x',\xi')+\cos\left(\tfrac{1}{3}\pi\right)\zeta_1(x',\xi')h^{2/3}\right|^2 + \left|r_0+\cos\left(\tfrac{1}{6}\pi\right)\zeta_1(x',\xi')h^{2/3}\right|^2$$

is obtained by taking $R(x',\xi')=\operatorname{Re}\omega_0+\mathcal{O}(h^{2/3})$ and the minimum of $\zeta_1(x',\xi')$ with that constraint. Since $\zeta_1(x',\xi')=\zeta_1(2Q(x',\xi'))^{2/3}$, Lemma 2.1 gives the minimal value in terms of $S_{\min}$:

$$|r_0+S_{\min}(\operatorname{Re}\omega_0)^{2/3}h^{2/3}|^2+\mathcal{O}(h^{4/3}). \quad \square$$

6. Refined estimates near the critical curve

We will now investigate the lower bounds with the parameter μ close to the critical value $S_{\min}(\operatorname{Re}\omega_0)^{2/3}h^{2/3}$. In the case of the model ordinary differential operator—the Airy operator—this corresponds to taking $\zeta_1 h^{2/3}<\hat{\mu}<\zeta_2 h^{2/3}$ in (4.2). In the analysis in Sect. 5, $\hat{\mu}$ was replaced by $\hat{\mu}(x',\xi')$ given by (5.5). Here we will consider a fixed $(x',\xi')\in T^*\partial\mathcal{O}$ and determine $\hat{\mu}(x',\xi')$ by the equation

$$|\omega_0-R(x',\xi')-e^{-2i\pi/3}\hat{\mu}(x',\xi')|=r_0+\mu, \tag{6.1}$$

with the convention that $\hat{\mu}=0$ if $|R-\omega_0|\geq r_0+\mu$.

Let $e_{x',\xi'}(t)$ be the first normalized positive eigenfuction of $(hD_t)^2+2tQ(x',\xi')$ corresponding to the eigenvalue $\zeta_1(x',\xi')h^{2/3}$. We also define

$$\gamma_1(x',\xi')v=\int_0^\infty v(x_n)\overline{e_{x',\xi'}(x_n)}\,dx_n,$$

so that for $\zeta_1(x',\xi')h^{2/3}<\hat{\mu}<\zeta_2(x',\xi')h^{2/3}$, we have

$$\|\Pi_{x',\xi',\hat{\mu}}u\|^2=|\gamma_1(x',\xi')u|^2.$$

Let Ω and u be as in the statement of Proposition 5.1. Then, using (5.2) and the proof of Proposition 5.1 we get with the same notation and for $\hat{\mu}(x',\xi')<\zeta_2(x',\xi')h^{2/3}$

$$\begin{aligned}\|(P-\omega_0)u\|^2&\geq((r_0+\mu)^2-\mathcal{O}(h)-\mathcal{O}(\mu^2))\|u\|^2\\ &\quad-\iint_{\Omega\times\mathbf{R}^{n-1}}\chi_1(\xi')q(x',\xi')|\gamma_1(x',\xi')Tu(x',\xi')|^2e^{-|\xi'|^2/h}\,dx'\,d\xi',\\ q(x',\xi')&=2\chi_2(\xi')^2\operatorname{Re}((R(x',\xi')-\bar{\omega}_0)e^{-2i\pi/3})(\hat{\mu}(x',\xi')-\zeta_1(x',\xi')h^{2/3})_+,\end{aligned}\tag{6.2}$$

with $\chi_i\in C_0^\infty(\mathbf{R}^{n-1},[0,1])$, $\chi_i(\xi')=0$ for $|\xi'|>2C$ or $|\xi'|<(2C)^{-1}$ and $\chi_i(\xi')=1$ for $C^{-1}<|\xi'|<C$, $\chi_2\chi_1=\chi_1$.

We shall reexamine, in a more microlocal way, the approximation of the negative term in the lower bound (6.2) by $-(Qu,u)$, where Q is a finite rank operator. For that we investigate the symbol properties of $\gamma_1(x',\xi')$. If $e_0(t)$ is the first normalized positive eigenfunction of D_t^2+t then

$$e_{x',\xi'}(x_n)=\left(\frac{2Q(x',\xi')}{h^2}\right)^{1/6}e_0\left(\left(\frac{2Q(x',\xi')}{h^2}\right)^{1/3}x_n\right).$$

The function e_0 and all its derivatives belong to $\mathcal{S}([0,\infty))$, so that for $C^{-1}<|\xi'|<C$

$$\partial_{x'}^{\alpha'}\partial_{\xi'}^{\beta'}e_{x',\xi'}(x_n)=h^{-1/3}f_{x',\xi'}^{\alpha',\beta'}\left(\left(\frac{2Q(x',\xi')}{h^2}\right)^{1/3}x_n\right),$$

where $f_{x',\xi'}^{\alpha',\beta'}(t)$ is smooth in x',ξ' with values in $\mathcal{S}([0,\infty))$.

Hence, $\gamma(x',\xi')=\chi(\xi')\gamma_1(x',\xi')$ satisfies $\partial_{x'}^{\alpha'}\partial_{\xi'}^{\beta'}\gamma(x',\xi')=\mathcal{O}(1)\colon L^2([0,\infty))\to\mathbf{C}$, and consequently gives a pseudodifferential operator

$$\gamma(x',hD_{x'})\colon L^2(\Omega\times[0,\infty))\longrightarrow L^2(\mathbf{R}^{n-1}).$$

Theorem 3 now shows that

$$\|(T\gamma(x',hD_{x'})-\gamma(x',\xi')T)u\|_K=\mathcal{O}(h^{1/2})\|u\|,$$

for any compact $K\subset\mathbf{C}^{n-1}$, $z=x'-i\xi'$. If $\chi_2\in C_0^\infty(\mathbf{R}^{n-1},[0,1])$ has the property $\chi_2\chi_1=\chi_1$ we define

$$r: L^2(\Omega\times[0,\infty))\longrightarrow L^2(\Omega\times\mathbf{R}^{n-1}),\quad r=\mathbf{1}_\Omega\chi_2(\xi')(T\gamma(x',hD_{x'})-\gamma(x',\xi')T).$$

Using r we rewrite the last term in (6.2) as

$$-\iint_{\Omega\times\mathbf{R}^{n-1}} q(x',\xi')|T\gamma(x',hD_{x'})u|^2\chi_2(\xi')^2e^{-|\xi'|^2/h}\,dx'\,d\xi'$$
$$-\iint \mathcal{O}(\mu)|ru(x',\xi')|^2\,dx'\,d\xi'-\iint \mathcal{O}(\mu)|ru(x',\xi')|\,|T\gamma(x',hD_{x'})u|\,dx'\,d\xi',$$

and here the last two terms can be estimated by $\mathcal{O}(\mu h^{1/2})\|u\|^2=\mathcal{O}(\mu^2+h)\|u\|^2$. Thus, (6.2) can be rewritten as

$$\begin{aligned}\|(P-\omega_0)u\|^2\geq((r_0+\mu)^2-\mathcal{O}(h)-\mathcal{O}(\mu^2))\|u\|^2\\ -\iint_{\Omega\times\mathbf{R}^{n-1}} q(x',\xi')|T\gamma(x',hD_{x'})u|^2e^{-|\xi'|^2/h}\,dx'\,d\xi'.\end{aligned}\tag{6.3}$$

We now gained an advantage of having $T\gamma(x',hD_{x'})u\in H_\Phi(\mathbf{C}^{n-1})$ so that the last term above can be written using a Toeplitz operator:

$$-\langle\Pi q\Pi^*T\gamma(x',hD_{x'})u,T\gamma(x',hD_{x'})u\rangle_\Phi.$$

We now have a simple

Lemma 6.1. *If $q\in L^\infty_{\mathrm{comp}}(\mathbf{C}^n)$ and $q\geq0$, then for every $\varepsilon>0$ there exists a finite rank operator Q_ε such that*

$$\|\Pi q\Pi^*-Q_\varepsilon\|_{\mathcal{L}(H_\Phi,H_\Phi)}\leq\varepsilon,\quad \operatorname{rank}Q_\varepsilon\leq\frac{1}{\varepsilon}\operatorname{tr}\Pi q\Pi^*.\tag{6.4}$$

Proof. Put $Q_\varepsilon=\mathbf{1}_{[\varepsilon,\infty)}(\Pi q\Pi^*)\Pi q\Pi^*$. The first part of (6.4) clearly holds while for the second part we observe that the rank of Q_ε is equal to the number, N_ε, of eigenvalues of $\Pi q\Pi^*$ larger than or equal to ε. Since $\Pi q\Pi^*$ is self-adjoint and positive, $\varepsilon N_\varepsilon\leq\operatorname{tr}\Pi q\Pi^*$. □

From Lemmas 3.2 and 6.1 we now see that the last term in (6.3) can be replaced by

$$
\begin{aligned}
&-\langle Q_\varepsilon u,u\rangle-\varepsilon\|u\|^2,\\
(6.5)\qquad &\operatorname{rank} Q_\varepsilon \le \frac{1}{\varepsilon(2\pi h)^{n-1}}\iint 2\chi_2(\xi')^2\operatorname{Re}((R(x',\xi')-\bar\omega_0)e^{-2i\pi/3})\\
&\qquad\times(\hat\mu(x',\xi)-\zeta_1(x',\xi')h^{2/3})_+\,dx'\,d\xi'.
\end{aligned}
$$

We will use this to prove the next lemma which will then be used in Sect. 7 to estimate the number of poles near the critical line. Let us first recall that $R(x',\xi')$ is the symbol of the tangential Laplacian so that $S^*\partial\mathcal{O}=\{m\in T^*\partial\mathcal{O}:R(m)=1\}$.

Lemma 6.2. *Let us define* $\mu_j=\zeta_j S_{\min}(\operatorname{Re}\omega_0)^{2/3}h^{2/3}$, $j=1,2$, *where* $S_{\min}$ *is given by* (5.18) *and assume*

$$h\ll\mu-\mu_1,\quad \mu\le\mu_2-h^{2/3}/C.$$

Let us also assume that $Q|_{S^*\partial\mathcal{O}}$ *attains its minimum on a submanifold* $\Gamma_0\subset T^*\partial\mathcal{O}$ *of codimension* ν *and that the transversal Hessian of* $Q|_{S^*\partial\mathcal{O}}$ *is non-degenerate. Then for any fixed* $0<\delta\ll1$ *there exists a finite rank operator* Q_δ *such that*

$$
\begin{aligned}
&\|(P-\omega_0)u\|^2\ge((r_0+\mu)^2-\delta(\mu-\mu_1)-\mathcal{O}(h))\|u\|^2-\langle Q_\delta u,u\rangle,\\
(6.6)\qquad &\operatorname{rank} Q_\delta\le\frac{C}{\delta}(\mu-\mu_1)^{(\nu/2)+(1/2)}h^{-(\nu/3)-(n-1)}.
\end{aligned}
$$

Proof. We only need to consider a local version of (6.6)—the global one follows as in the proof of Proposition 5.2. We start by observing that $\hat\mu(x',\xi')>0$ implies

$$\hat\mu(x',\xi')=\frac{2r_0\mu-(R(x',\xi')-\operatorname{Re}\omega_0)^2}{2r_0\cos\left(\frac16\pi\right)}+\mathcal{O}(h). \tag{6.7}$$

In fact, (6.1) is equivalent to

$$\left(r_0+\cos\left(\tfrac16\pi\right)\hat\mu(x',\xi')\right)^2+\left(\operatorname{Re}\omega_0-R(x',\xi')+\cos\left(\tfrac13\pi\right)\hat\mu(x',\xi')\right)^2=(r_0+\mu)^2, \tag{6.8}$$

and $\hat\mu(x',\xi')>0$ implies $\operatorname{Re}\omega_0-R=\mathcal{O}(\mu^{1/2})=\mathcal{O}(h^{1/3})$. Expanding (6.8) gives (6.7).

To simplify the notation let us now assume that $\operatorname{Re}\omega_0=1$ (a scaling argument then treats the general case). Denoting by $\mathbf{A}$ the annulus $\{C^{-1}<|\xi'|<C\}$, we introduce new coordinates on $T^*\partial\mathcal{O}\cap(\Omega\times\mathbf{A})$, $z_1,\dots,z_{2n}$, so that

$$S^*\partial\mathcal{O}\cap(\Omega\times\mathbf{A})=\{z_1=0\},\quad \Gamma_0\cap(\Omega\times\mathbf{A})=\{z_1=\dots=z_{\nu+1}=0\}.$$

Motivated by this we write $z=(z_1,z',z'')$, where $z'=(z_2,...,z_{\nu+1})$. The nondegeneracy assumption allows a more particular choice of coordinates in which

$$Q(z)|_{z_1=0}=Q(0,0,z'')+|z'|^2.$$

Since the minimal value of $\zeta_1(2Q(z))^{2/3}$ is $\zeta_1 S_{\min}/\cos(\frac{1}{6}\pi)$ we get using (6.7) and the fact that $z_1=\mathcal{O}(h^{1/3})$ if $\hat{\mu}(z)>0$

$$\begin{aligned}\hat{\mu}(z)-\zeta_1(z)h^{2/3}&=\hat{\mu}(z)-\zeta_1[Q(0,0,z'')+|z'|^2+\mathcal{O}(z_1)]^{2/3}h^{2/3}\\&=\frac{\mu-\mu_1}{\cos(\frac{1}{6}\pi)}-\frac{z_1^2}{2r_0\cos(\frac{1}{6}\pi)}\\&\quad-\frac{4}{3}\left(\frac{\cos(\frac{1}{6}\pi)}{S_{\min}}\right)^{1/2}\zeta_1 h^{2/3}|z'|^2(1+\mathcal{O}(|z'|^2))+\mathcal{O}(h).\end{aligned}$$

We now insert this into (6.5):

$$\begin{aligned}\operatorname{rank} Q_\varepsilon&\le\frac{C_1}{\varepsilon h^{n-1}}\iint_{\Omega\times\mathbf{R}^{n-1}}\chi_2(\xi')^2(r_0+\mu)(\hat{\mu}(x',\xi')-\zeta_1(x',\xi')h^{2/3})_+\,dx'\,d\xi'\\&\le\frac{C_2}{\varepsilon h^{n-1}}\iint_{\Omega\times\mathbf{R}^{n-1}}\chi_2(\xi'(z))^2\left(\frac{\mu-\mu_1}{\cos(\frac{1}{6}\pi)}+\mathcal{O}(h)-c_1z_1^2-c_2h^{2/3}|z'|^2\right)_+dz_1\,dz'dz'',\end{aligned}$$

where $c_1,c_2>0$. This is bounded by

$$\frac{C}{\varepsilon h^{n-1}}(\mu-\mu_1)\int_{c_1z_1^2+c_2h^{2/3}|z'|^2<C(\mu-\mu_1)}dz_1\,dz'<\frac{C}{\varepsilon}h^{-n+1}(\mu-\mu_1)^{(\nu/2)+(3/2)}h^{-\nu/3}.$$

Putting $\varepsilon=\delta(\mu-\mu_1)$, we get (6.6) from (6.5). □

7. Distribution of scattering poles

We will now use the lower bounds of Sect. 5 and 6 to prove Theorems 1 and 2. This will be done by the method originating from Sect. 3 and 4 of [9] (see also [10], [11], [13]) but for the convenience of the reader we will try to make the presentation self-contained.

We start with the results of Sect. 5 which give:

Theorem 4. *If $0<h<h_0$ and $h^{2/3}/C<\mu<\mu_0\ll 1$, then the number of eigenvalues of $P=-h^2\Delta|_\Gamma$ in*

(7.1) $$|\operatorname{Re} z-1|<\mu^{1/2}/C,\quad -\operatorname{Im} z<2\mu,$$

$n(h,\mu)$, satisfies

$$n(h,\mu)=\mathcal{O}(1)(2\mu-S_{\min}h^{2/3}+c_0h)_+^0\mu^2h^{-n}, \tag{7.2}$$

for some c_0 and $S_{\min}$ is given by (5.18).

Proof. We start by showing that if $2\mu<S_{\min}h^{2/3}-c_0h$, for some c_0, then P has no eigenvalues in the rectangle (7.1). In fact, if z were an eigenvalue in (7.1), then by Proposition 5.3 applied with $\omega_0=\operatorname{Re} z+ir_0$, $\mu<r_0<2\operatorname{Re} z$,

$$r_0-\operatorname{Im} z\geq r_0+S_{\min}(\operatorname{Re} z)^{2/3}h^{2/3}-\mathcal{O}(h).$$

Since $(\operatorname{Re} z)^{2/3}=1+\mathcal{O}(\mu^{1/2})=1+\mathcal{O}(h^{1/3})$ we get a contradiction once c_0 is large enough.

Let us now assume that $2\mu>S_{\min}h^{2/3}-c_0h$ and take $\omega_0=1+ir_0$ with r_0 sufficiently small. We observe that for C large enough the rectangle (7.1) is contained in the disc

$$D=D(\omega_0,r_0+4\mu).$$

If $z_1,...,z_N$ are the eigenvalues of P in D then $N\geq n(h,\mu)$, so we will estimate N. Let us introduce the characteristic values of $P-\omega_0$, $\mu_1\leq...\leq\mu_N\leq...$ as the eigenvalues of $[(P-\omega_0)^*(P-\omega_0)]^{1/2}$ (with the convention that in case there are only finitely many such eigenvalues we repeat the infimum of the essential spectrum infinitely many times). We then use the Weyl inequality (see Apppendix A of [9]):

$$\mu_1 ... \mu_N\leq|z_1-\omega_0| ... |z_N-\omega_0|. \tag{7.3}$$

We start by estimating

$$N^\#=\#\{\mu_j:\mu_j\leq r_0+6\mu\},$$

with $\mu<\mu_0$ so that

$$\inf\sigma_{\mathrm{ess}}([(P-\omega_0)^*(P-\omega_0)]^{1/2})\geq\operatorname{Im}((1+i\theta)\omega_0)>r_0+8\mu,$$

see the proof of Proposition 5.2. The max-min principle shows that

$$N^\#<M \iff \begin{cases} \forall\,\delta>0\,\exists \text{ a closed subspace } E\subset\mathcal{D}(P-\omega_0)\subset L^2(\Gamma) \\ \text{of codimension less than or equal to } M-1 \text{ such that} \\ \|(P-\omega_0)u\|^2\geq((r_0+6\mu)^2-\delta)\|u\|^2,\ u\in E. \end{cases} \tag{7.4}$$

Let us now apply Proposition 5.2 with μ replaced by 8μ to see that

$$\|(P-\omega_0)u\|^2\geq((r_0+8\mu)^2-\mathcal{O}(1)(h+\mu^2+\varepsilon\mu))\|u\|^2\geq(r_0+6\mu)^2\|u\|^2,$$

if $\widetilde{\Xi}^p_\varepsilon u=0$, for $p=1,...,Q$ and h,ε are small enough. Thus (7.4) implies that

$$N^{\#}<\sum_{p=1}^{Q}\operatorname{rank}\widetilde{\Xi}^p_\varepsilon\le\sum_{p=1}^{Q}\sum_{j=1}^{\widetilde{M}(K_p,\varepsilon h^{1/2})}\operatorname{rank}\widetilde{\Pi}_{x^p_j,\xi^p_j}.$$

We now recall (5.8) and (5.9) to see that $\widetilde{M}(K_p,\varepsilon h^{1/2})=\mathcal{O}_\varepsilon(\mu^{1/2}h^{-(n-1)})$ and that $\operatorname{rank}\widetilde{\Pi}_{x^p_j,\xi^p_j}=\mathcal{O}(\mu^{3/2}h^{-1})$. Hence $N^{\#}=\mathcal{O}(\mu^2h^{-n})$ and the proof is completed by showing that $N\le CN^{\#}$ and that is done exactly as in [9], [10], [11], [13]: if $N>N^{\#}$ then

$$\mu_1^{N^{\#}}(r_0+6\mu)^{N-N^{\#}}\le(r_0+4\mu)^N.$$

Since by Proposition 5.3, $\mu_1>r_0$ if $h<h_0$, we get

$$N\le\left(\log\left(\frac{r_0+6\mu}{r_0+4\mu}\right)\right)^{-1}\log\left(\frac{r_0+6\mu}{r_0}\right)N^{\#}=\mathcal{O}(1)N^{\#}=\mathcal{O}(\mu^2h^{-n}).\quad\square$$

Writing $\lambda=h^{-1}$, $\zeta^2=h^{-2}z$ the semi-classical statement about resonances translates immediately into a statement about the scattering poles: for $\lambda\gg1$ and $\lambda^{-2/3}/C\le\mu\le1/C$, the number of scattering poles in a rectangle

$$|\operatorname{Re}\zeta-\lambda|\le\mu^{1/2}\lambda/\widetilde{C},\quad -\operatorname{Im}\zeta\le\mu\lambda,$$

is bounded by

$$\mathcal{O}(1)(\mu-S_{\min}\lambda^{-2/3}+c_1\lambda^{-1})_+^0\mu^2\lambda^n,\tag{7.5}$$

for some c_1. Theorem 1 is a somewhat weaker global version of (7.5) and to obtain it we need

Lemma 7.1. *Suppose that m,f and g are measurable functions on $[1,\infty)$, $f(x),f\left(\frac12x\right)^{-1}g(x)$ are non-decreasing, $C^{-1}\le f(x)\le C^{-1}x$ and*

$$m(x)-m(x-f(x))\le g(x),\tag{7.6}$$

then

$$m(r)\le Crf\left(\tfrac12r\right)^{-1}g(r)+C.$$

Proof. We first obtain a bound on $m(\lambda)-m\left(\frac12\lambda\right)$, and for that we define a sequence $\lambda_0=\lambda$, $\lambda_{k+1}=\lambda_k-f(\lambda_k)$. Then

$$m(\lambda)-m\left(\tfrac12\lambda\right)\le\sum_{k=0}^{K}g(\lambda_k)\le Kg(\lambda),$$

where K is the smallest integer for which $\lambda_{K+1}<\frac{1}{2}\lambda$. Hence $\frac{1}{2}\lambda+Kf\left(\frac{1}{2}\lambda\right)\leq\lambda$ and $K\leq\frac{1}{2}\lambda f\left(\frac{1}{2}\lambda\right)^{-1}$. Consequently, using the monotonicity of $f\left(\frac{1}{2}\lambda\right)^{-1}g(\lambda)$

$$m(r)\leq\frac{1}{2}\sum_{k=0}^{M}\frac{2^{-k}r}{f(2^{-k-1}r)}g(2^{-k}r)+C\leq C\frac{rg(r)}{f\left(\frac{1}{2}r\right)}+C,$$

where M is the largest integer for which $2^{-M}r>2$. □

Proof of Theorem 1. We put $m(\lambda)=N(\lambda,\mu)$, $f(\lambda)=\mu(\lambda)^{1/2}\lambda/\widetilde{C}$ and $g(\lambda)=C\mu(\lambda)^2\lambda^n$. The estimate (7.5) implies that (7.6) and the assumptions of Lemma 7.1 are satisfied in view of (1.2). Thus the bound

$$m(r)\leq Crf\left(\tfrac{1}{2}r\right)^{-1}g(r)+C\leq C\mu(r)^{3/2}r^n$$

follows from the monotonicity of μ. □

Proof of Theorem 2. The proof is based on Lemma 6.2 (from which we borrow the notation) in the same way as that of Theorem 4 was based on Proposition 5.2. Thus we start by estimating the number of eigenvalues of $P=-h^2\Delta|_\Gamma$ in the disc

$$\begin{aligned}&|z-\omega_0|<r_0+\mu_1+\tfrac{1}{2}(\mu-\mu_1), && r_0=\operatorname{Im}\omega_0<\tfrac{1}{2}\operatorname{Re}\omega_0-3\mu_1,\\ &\mu-\mu_1\gg h, && \mu\leq\mu_2-h^{2/3}/C,\end{aligned}\tag{7.7}$$

and claim that it is bounded by

$$\mathcal{O}(1)(\mu-\mu_1)^{(1/2)+(\nu/2)}h^{-\nu/3}h^{-(n-1)}.$$

To see that, we observe that for $\delta\ll r_0$

$$\begin{aligned}(r_0+\mu)^2-\mathcal{O}(h)-\delta(\mu-\mu_1)&\geq(r_0+\mu_1+(\mu-\mu_1))^2-\left(\delta-\frac{\mathcal{O}(h)}{\mu-\mu_1}\right)(\mu-\mu_1)\\ &\geq\left(r_0+\mu_1+\left(1-\frac{\delta}{3r_0}\right)(\mu-\mu_1)\right)^2.\end{aligned}$$

Hence, (see Lemma 6.2) except on a space of codimension less than or equal to $\operatorname{rank}Q_\delta$

$$\|(P-\omega_0)u\|^2\geq\left(r_0+\mu_1+\left(1-\frac{\delta}{3r_0}\right)(\mu-\mu_1)\right)^2\|u\|^2.$$

Since $1-\delta/3r_0>\frac{2}{3}$ for small δ we conclude that the number of characteristic values of $P-\omega_0$ in $\left[0,r_0+\mu+\frac{2}{3}(\mu-\mu_1)\right]$ is $\mathcal{O}(1)\delta^{-1}(\mu-\mu_1)^{(1/2)+(\nu/2)}h^{-\nu/3}h^{-(n-1)}$. The

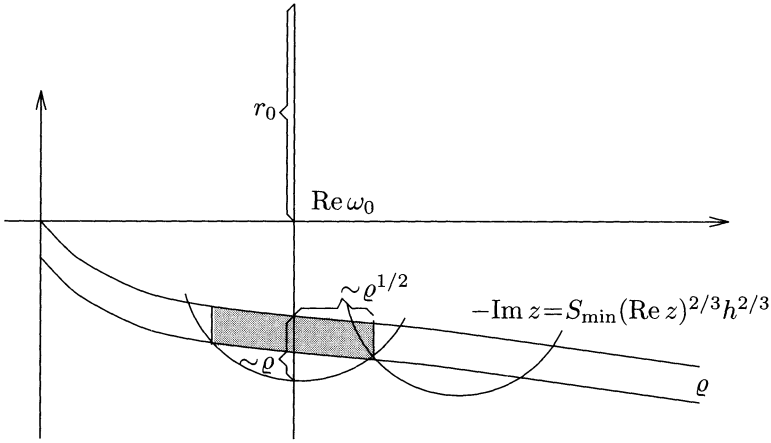

Figure 2. The covering of a neighbourhood of the critical curve.

Weyl inequality applied as in the proof of Theorem 4 gives the claimed bound on the number of eigenvalues of P in the disc (7.7).

A covering argument analogous to the one used in [13] (see Fig. 2) shows that the number of eigenvalues of P in

$$\begin{cases} \frac{1}{2} \le \operatorname{Re} z \le \frac{3}{2}, \\ -\operatorname{Im} z \le S_{\min}(\operatorname{Re} z)^{2/3}h^{2/3} + \varrho, \end{cases}$$

is bounded by

$$\mathcal{O}(1)\varrho^{\nu/2}h^{-\nu/3}h^{-(n-1)}. \tag{7.8}$$

We now need to translate this bound to the ζ-plane with $\zeta = h^{-1}\sqrt{z}$, $\varrho = \operatorname{Re} f(z)h^{\alpha}$, $\frac{2}{3} \le \alpha \le 1$, where f is holomorphic near the positive real axis and positive on it. From (7.8) we immediately get a bound in the region

$$\begin{cases} \frac{1}{2}h^{-2} \le \operatorname{Re}\zeta^2 \le \frac{3}{2}h^{-2}, \\ -\operatorname{Im}\zeta^2 \le S_{\min}(\operatorname{Re}\zeta^2)^{2/3} + \operatorname{Re}(f(h^2\zeta^2)h^{\alpha-2}), \end{cases}$$

which by choosing $f(z) = c_0 z^{1-(\alpha/2)}$, $\frac{2}{3} \le \alpha \le 1$, becomes

$$\begin{cases} \frac{1}{2}h^{-2} \le \operatorname{Re}\zeta^2 \le \frac{3}{2}h^{-2}, \\ -\operatorname{Im}\zeta \le \frac{1}{2}S_{\min}(1-(\operatorname{Re}\zeta)^{-2}(\operatorname{Im}\zeta)^2)^{2/3}(\operatorname{Re}\zeta)^{1/3} + \frac{1}{2}c_0\operatorname{Re}\zeta^{2-\alpha}(\operatorname{Re}\zeta)^{-1}. \end{cases}$$

Using $\operatorname{Im}\zeta = \mathcal{O}(\operatorname{Re}\zeta)^{1/3}$, we get this to be the same as

$$\begin{cases} \sqrt{\frac{1}{2}}\,h^{-1} \le \operatorname{Re}\zeta(1+\mathcal{O}(\operatorname{Re}\zeta)^{-2/3}) \le \sqrt{\frac{3}{2}}\,h^{-1}, \\ -\operatorname{Im}\zeta \le \frac{1}{2}S_{\min}(\operatorname{Re}\zeta)^{1/3} + \frac{1}{2}c_0(\operatorname{Re}\zeta)^{1-\alpha} + \mathcal{O}((\operatorname{Re}\zeta)^{-1}). \end{cases}$$

Thus the number of the scattering poles in this region is bounded by (7.8) with $\varrho = h^{\alpha}$. Another standard scaling argument completes the proof. □

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References

1. BARDOS, C., LEBEAU, G. and RAUCH, J., Scattering frequencies and Gevrey 3 singularities, *Invent. Math.* **90** (1987), 77–114.
2. CORDOBA, A. and FEFFERMAN, C., Wave packets and Fourier integral operators, *Comm. Partial Differential Equations* **3** (1978), 979–1006.
3. HARGÉ, T. and LEBEAU, G., Diffraction par un convexe, *Invent. Math.* **118** (1994), 161–196.
4. HÖRMANDER, L., *The Analysis of Linear Partial Differential Operators* **I–IV**, Springer-Verlag, Berlin–Heidelberg, 1983–1985.
5. HÖRMANDER, L., Quadratic hyperbolic operators, in *Microlocal Analysis and Applications* (Cattabriga, L. and Rodino, L., eds.), pp. 118–160, Lecture Notes in Math. **1495**, Springer-Verlag, Berlin–Heidelberg, 1991.
6. LAX, P. and PHILLIPS, R., *Scattering Theory*, Academic Press, New York–London, 1967.
7. SJÖSTRAND, J., Singularités analytiques microlocales, *Astérisque* **95** (1982).
8. SJÖSTRAND, J., Lecture Notes, *Lund University* (1985–86).
9. SJÖSTRAND, J., Geometric bounds on the density of resonances for semiclassical problems, *Duke Math. J.* **60** (1990), 1–57.
10. SJÖSTRAND, J. and ZWORSKI, M., Complex scaling and the distribution of scattering poles, *J. Amer. Math. Soc.* **4** (1991), 729–769.
11. SJÖSTRAND, J. and ZWORSKI, M., Distribution of scattering poles near the real axis, *Comm. Partial Differential Equations* **17** (1992), 1021–1035.
12. SJÖSTRAND, J. and ZWORSKI, M., Lower bounds on the number of scattering poles, *Comm. Partial Differential Equations* **18** (1993), 847–858.
13. SJÖSTRAND, J. and ZWORSKI, M., Estimates on the number of scattering poles for strictly convex obstacles near the real axis, *Ann. Inst. Fourier* (*Grenoble*) **43** (1993), 769–790.
14. VAINBERG, B., *Asymptotic Methods in Equations of Mathematical Physics*, Moscow Univ. Press, Moscow, 1982 (Russian). English transl: Gordon and Breach, New York, 1989.

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