LOCALIZATION AND SUMMABILITY OF MULTIPLE FOURIER SERIES

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Introduction

1. Definitions

In this paper we shall deal with the theory of "spherical" summability of multiple Fourier series.

Let $f(x) = f(x_1, x_2 \dots x_k)$ be a Lebesgue integrable function defined on the fundamental cube Q_k, $-\pi < x_i \leqslant \pi$, $i = 1, \dots k$, in Euclidean k-space. We form the Fourier series of $f(x)$

$$f(x) = \sum a_n e^{in\cdot x} = \sum a_{n_1 n_2 \dots n_k} e^{i(n_1 x_1 \cdots + n_k x_k)}, \tag{1.1}$$

where $n = (n_1, \dots, n_k)$ is a vector with integral components, $n \cdot x = n_1 x_1 + n_2 x_2 \cdots + n_k x_k$, with

$$a_n = (2\pi)^{-k} \int_{Q_k} f(x) e^{-in\cdot x} dx,$$

and $dx = dx_1 dx_2 \dots dx_k$.

We next form the spherical Riesz means of order δ of $f(x)$

$$S_R^\delta(x) = S_R^\delta(x, f) = \sum_{|n| < R} \left(1 - \frac{|n|^2}{R^2}\right)^\delta a_n e^{in\cdot x}, \tag{1.2}$$

where $|n| = (n_1^2 + \cdots + n_k^2)^{\frac{1}{2}}$. Unless stated to the contrary, we shall assume that $k \geqslant 2$.

The general problem of the theory concerns itself with the validity (and meaning) of

$$\lim_{R\to\infty} S_R^\delta(x, f) = f(x), \tag{1.3}$$

for some appropriate δ.

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2. Localization

In the theory, the so-called "critical exponent" α ($\alpha = \frac{1}{2}(k-1)$), plays a significant role. If $\delta > \alpha$, the behaviour of the Riesz means $S_R^\delta(x, f)$ is "Fejér-like": the relationship (1.3) holds almost everywhere; the convergence is bounded if $f(x)$ is likewise bounded, and is uniform if $f(x)$ is continuous; finally the validity of the relationship (1.3) depends only on the values of $f(x)$ in any neighborhood of x. When $\delta \leqslant \alpha$, the above is no longer generally true. In the classical case, $k=1$, an important property remains for $\delta = \alpha$. According to the localization theorem of Riemann, the existence of (1.3) (when $k=1$, $\delta=0$) depends only on the values of $f(x)$ in any neighborhood of x. It is natural, therefore, to ask whether the localization property for (1.3) still holds for $\delta=\alpha$ when $k \geqslant 2$. Two results for the critical exponent α, which give a partial answer to the above question, are due to Bochner [3].

First, there exists an $f(x)$ integrable over Q_k, and vanishing in a neighborhood of the origin for which

$$\limsup_{R\to\infty} S_R^\alpha(0, f) = +\infty.$$

Thus the localization principle fails to hold unrestrictedly at the critical exponent, when $k \geqslant 2$. However, by another result of Bochner, the localization principle for (1.3), when $\delta=\alpha$, still holds if we restrict ourselves to functions in $L^2(Q_k)$. Thus the natural question arose whether localization still holds at the critical exponent if we limit ourselves to functions of the class $L^p(Q_k)$, $1<p$.

It will be one of the purposes of this paper to give an *affirmative* answer to the above problem. In fact, we shall show that localization for $\delta=\alpha$ still holds if we restrict ourselves to the class of functions for which

$$\int_{Q_k} |f(x)| \log^+ |f(x)|\, dx < \infty. \tag{2.1}$$

Of course, the class of functions for which (2.1) holds includes every $L^p(Q_k)$ class, $1<p$.

3. Pointwise and dominated summability

If we now consider the relationship (1.3) in the sense of "almost ewerywhere", and not of individual points, we may then obtain results concerning its validity for $\delta<\alpha$, or $\delta=\alpha$, In fact, if $f(x) \in L^p(Q_k)$, $1<p\leqslant 2$, we shall show that (1.3) will hold almost everywhere whenever $\delta > \alpha(2/p-1)$.(1) (The point being that $\alpha(2/p-1)<\alpha$,

(1) For $p=2$, this result is known, see [11]. It is a consequence of the general theory of orthonormal series, as developed in KACZMARZ and STEINHAUS, [9], Chapt. V.

whenever $1 < p \leqslant 2$). Thus whenever $f(x) \in L^p(Q_k)$, $1 < p$, then (1.3) holds almost everywhere for some δ *below* the critical exponent. We shall also show that the relationship (1.3) will hold almost everywhere for $\delta = \alpha$, (α = critical exponent $= \frac{1}{2}(k-1)$), whenever

$$\int_{Q_k} |f(x)| (\log^+ |f(x)|)^2 \, dx < \infty. \tag{3.1}$$

These results will be consequences of results concerning dominated summability —which results seem interesting on their own right. For this purpose we introduce the following definition

$$S_*^\delta(x) = S_*^\delta(x, f) = \sup_{\infty > R > 0} |S_R^\delta(x, f)|. \tag{3.2}$$

We shall prove

$$\Big(\int_{Q_k} (S_*^\delta(x))^p \, dx\Big)^{1/p} \leqslant A_{p,\delta} \Big(\int_{Q_k} |f(x)|^p \, dx\Big)^{1/p}, \tag{3.3}$$

if $\delta > \alpha(2/p - 1)$, and $1 < p \leqslant 2$; $(\alpha = \frac{1}{2}(k-1))$. (1)

We shall also show,

$$\int_{Q_k} S_*^\alpha(x) \, dx \leqslant A \int_{Q_k} |f(x)| (\log^+ |f(x)|)^2 \, dx + B. \tag{3.4}$$

As a further consequence of (3.3) we shall obtain

$$\lim_{R\to\infty} \int_{Q_k} |S_R^\delta(x, f) - f(x)|^p \, dx = 0, \text{ if } f(x) \in L^p(Q_k), \quad 1 < p \leqslant 2, \text{ and } \delta > \alpha(2/p - 1). \tag{3.5}$$

For the analogue of (3.4) in terms of norm convergence, we shall obtain the following improvement:

$$\lim_{R\to\infty} \int_{Q_k} |S_R^\alpha(x, f) - f(x)| \, dx = 0, \quad \text{if } \int_{Q_k} |f(x)| \log^+ |f(x)| \, dx < \infty. \tag{3.6}$$

4. Strong summability

The problem is one of dealing with the validity of the following:

$$\lim_{R\to\infty} \frac{1}{R} \int_0^R |S_u^\delta(x) - f(x)|^2 \, du = 0. \tag{4.1}$$

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(1) When $k = 1$, (3.3) is a known result of Hardy and Littlewood, see [15], Chapt. X.

For $\delta > \alpha$, (4.1) above is an immediate consequence of relation (1.3) (which, of course, holds almost everywhere if $\delta > \alpha$, and $f(x)$ is integrable). Again, only the case $\delta \leqslant \alpha$ will interest us.

Our results are two-fold. First, if $1 < p \leqslant 2$, and $f(x) \in L^p(Q_k)$, then (4.1) holds almost everywhere as long as $\delta > \alpha(2/p - 1) - 1/p'$, where $1/p' + 1/p = 1$, $(\alpha = \frac{1}{2}(k-1))$. Since for $1 < p \leqslant 2$, $\alpha(2/p-1) - 1/p' < \alpha(2/p-1)$, the relation (4.1) is not implied by the results mentioned in § 3.

Secondly, if $f(x) \in L^1(Q_k)$ it is possible to prove a more precise result: The relation (4.1) holds almost everywhere if $\delta = \alpha$. This is the strict analogue of a theorem of Marcinkiewicz on the strong-summability of Fourier series when $k = 1$. We shall however postpone the proof of this to another time, since the method used differs in essence from that of the rest of this paper. It should be pointed out that Bochner and Chandrasekharan [4] had shown that if $f(x) \in L^1(Q_k)$ the relation (4.1) with $\delta = \alpha$ reflects only the local behaviour of $f(x)$.

5. Summary of results

For the sake of convenience we shall briefly summarize our main results. They fall into two classes, and are listed according to self-explanatory notation:

Results for $L^p(Q_k)$, $1 < p \leqslant 2$

Assume that $f(x) \in L^p(Q_k)$, $1 < p \leqslant 2$, then we have:

(L) *If $f(x)$ vanishes in a neighborhood of x_0, then*

$$\lim_{R \to \infty} S_R^{\alpha}(x_0, f) = 0. \quad (\alpha = \tfrac{1}{2}(k-1)).$$

(D) *If $S_*^{\delta}(x, f) = \sup_{R>0} |S_R^{\delta}(x, f)|$, then*

$$\| S_*^{\delta}(x, f) \|_p \leqslant A_{p,\delta} \| f(x) \|_p, \quad \textit{if } \delta > \alpha(2/p - 1).$$

(A.E.) $\lim_{R \to \infty} S_R^{\delta}(x, f) = f(x)$, *for almost every x, if $\delta > \alpha(2/p - 1)$.*

(N) $\lim_{R \to \infty} \| S_R^{\delta}(x, f) - f(x) \|_p = 0$, *if $\delta > \alpha(2/p - 1)$.*

(S) $\lim_{R \to \infty} \dfrac{1}{R} \displaystyle\int_0^R | S_u^{\delta}(x, f) - f(x) |^2 \, du = 0$ *for almost every x, if $\delta > \alpha(2/p - 1) - 1/p'$.*

Results "near" $L^1(Q_k)$

As always, $\alpha = \frac{1}{2}(k-1)$:

(L*) *If $f(x)$ vanishes in a neighborhood of x_0, and $\int_Q |f(x)| \log^+ |f(x)| \, dx$ is finite, then*

$$\lim_{R\to\infty} S_R^\alpha(x_0, f) = 0.$$

(D*) $\int_{Q_k} S_*^\alpha(x, f) \, dx \leqslant A \int_{Q_k} |f(x)| (\log^+ |f(x)|)^2 \, dx + B.$

(A.E.*) *If $\int_{Q_k} |f(x)| (\log^+ |f(x)|)^2 \, dx$ is finite, then $\lim_{R\to\infty} S_R^\alpha(x, f) = f(x)$, for almost every x.*

(N*) $\lim_{R\to\infty} \int_{Q_k} |S_R^\alpha(x, f) - f(x)| \, dx = 0,$ *if,* $\int_{Q_k} |f(x)| \log^+ |f(x)| \, dx < \infty.$

(S*) $\lim_{R\to\infty} \frac{1}{R} \int_0^R |S_u^\alpha(x, f) - f(x)|^2 \, du = 0,$ *for almost any x, if $f(x) \in L^1(Q_k)$.*

6. Methods used

Since our results deal with summability of order δ, $\delta \leqslant \alpha$, we must in each case surmount the same initial difficulty—which we may describe as follows.

Let $K_R^\delta(x)$ denote the function whose Fourier expansion is

$$K_R^\delta(x) = \sum_{|n|<R}' \left(1 - \frac{|n|^2}{R^2}\right)^\delta e^{in\cdot x}. \tag{6.1}$$

Thus we may write

$$S_R^\delta(x, f) = (2\pi)^{-k} \int_{Q_k} K_R^\delta(x-y) f(y) \, dy. \tag{6.2}$$

When $\delta > \alpha$, (or $\delta \geqslant 0$, when $k = 1$), we may obtain estimates for the kernel $K_R^\delta(x)$ which are satisfactory for most purposes. (1)

However, when $\delta \leqslant \alpha$, $k \geqslant 2$, estimates for the kernel $K_R^\delta(x)$ depend heavily on the distribution of lattice points in k-space—and this is a very subtle matter. For this reason no estimates for $K_R^\delta(x)$ when $\delta \leqslant \alpha$, satisfactory for general purposes, have been given.

A novel approach to the problem is therefore needed. The idea of this method was contained in the proof of (N), which appeared earlier (2)—and this result presents the simplest illustration of the method used. The general idea is as follows:

(1) See, for example, [10], formula (7).

(2) See [13].

The expression $S_R^\delta(x, f)$ defined in (1.2) for positive δ, is now extended to *complex* values of δ, thus becoming an analytic function in δ. We then restrict our attention to a suitable strip $a \leqslant \Re(\delta) \leqslant b$. The boundary line $\Re(\delta) = a$ is made to correspond to an L^2 result, and the line $\Re(\delta) = b$ is made to correspond to an L^1 result. The L^2 result on the line $\Re(\delta) = a$ is deduced via Parseval's relation, while for the L^1 result on $\Re(\delta) = b$, rather straightforward estimates are sufficient.

We then use a "Phragmen-Lindelöf" type argument to obtain an L^p estimate on an intermediate line of the strip. This is done via an interpolation theorem for an analytic family of operators—a theorem which generalizes M. Riesz's well-known convexity theorem (Lemma 1).

The above is the general procedure for proving the L^p theorems (AE), (D), (N), and (S). The localization result, (L), is more difficult since the index δ contained in the result is always fixed at α. However, by introducing "fractional integration" into the problem, we may again obtain a situation for which the interpolation method applies. The situation is described more fully in § 12.

Once the L^p results are obtained, the results "near" L^1 (i.e (L*), (D*), (AE*) and (N*)) are obtained by certain limiting arguments from their corresponding L^p results.

A word should be added about a general heuristic principle which makes the convexity property of analytic functions applicable to our situation. It is this: If δ is complex, then the behaviour of $S_R^\delta(x, f)$ is essentially reflected by $S_R^\sigma(x, f)$, where $\sigma = \Re(\delta)$.

7. General remarks; convention

We should point out here that, previously, results concerning summability of order δ, $\delta \leqslant \alpha$, had in general been obtained only at the heavy price of making restrictions on the smoothness of $f(x)$ in the entire cube Q_k. In some circumstances these restrictions were incorporated into restrictions on the order of magnitude of the Fourier coefficients. To be sure, the results thus obtained held at individual points. ([1]) The theorems stated in § 3–§ 5 above show that if we are content with behaviour *almost everywhere*, then we may deal with summability of order δ, $\delta \leqslant \alpha$ by making much milder global restrictions on $f(x)$.

We thus have the interesting phenomenon that a function in L^p, $1 < p$ has a Fourier series which is summable almost everywhere of some order δ, $\delta < \alpha$, while this summability may fail at individual points where the function is very "smooth".

[1] See, for example, [6], Chapter V.

Another phenomenon which seems novel for $k \geqslant 2$, is that if $|f(x)| (\log^+ |f(x)|)^2$ is integrable, the Fourier series of $f(x)$ is summable almost everywhere for the critical exponent. Such a result is unknown for $k = 1$, and its proof (or disproof) would seem to be extremely difficult.

Certain conjectures seem probable, but for which we have no decisive evidence. (1) That the result (L*) cannot be improved. (2) That the result (A.E.*) can be extended to functions for which $|f(x)| \log^+ |f(x)|$ is integrable.

It would be interesting to decide whether the results (D), (A.E.), and (N) are valid for any range of δ for which $\delta \leqslant \alpha(2/p - 1)$. What seems to be needed here most are some good counter-examples.

We wish now to make explicit a convention which we shall use consistently in this paper.

(i) Bounds such as A_ξ, A_p, B_σ, etc. will be used repeatedly to show that the bounds depend on the indicated parameters. These bounds, however, may be different in different contexts.

(ii) When an inequality is given with a bound depending on a parameter (e.g. A_ξ), the range of the parameter will have the following meaning: The function A_ξ is bounded (independently of ξ) for ξ in any *closed* interval of the range of ξ. For example, an inequality with bound A_ξ, for $\xi \geqslant 0$, will mean that A_ξ is bounded in every interval $0 \leqslant \xi \leqslant a < \infty$. However, an inequality with bound A_ξ, for $\xi > 0$, will mean that A_ξ may become infinite as $\xi \to 0$.

CHAPTER I

Basic Lemmas

This chapter contains the basic tools which are needed in the following chapters.

8. Interpolation theorem

Let M and N be two given measure spaces with measures $d\mu$ and $d\nu$ respectively. We shall deal with a family of linear operators T_z (depending on the complex parameter z). We shall assume that the family T_z satisfies the following properties:

(i) for each z, $0 \leqslant \Re(z) \leqslant 1$, T_z is a linear transformation of "simple" functions on M to measurable functions on N.

(ii) If ψ is a simple function on M, ϕ a simple function on N, then

$$\Phi(z) = \int T_z(\psi)\,\phi\,d\nu \quad \text{is analytic in} \quad 0 < \Re(z) < 1$$

and continuous on the closed strip $0 \leqslant \Re(z) \leqslant 1$.

(iii) $\underset{|y| \leqslant r}{\operatorname{Sup}} \underset{0 \leqslant x \leqslant 1}{\sup} \log |\Phi(x+iy)| \leqslant A e^{ar}$, $a < \pi$.

A and a may depend on ψ, and ϕ. We then have the following

LEMMA 1. *Let T_z be a family of operators satisfying conditions* (i), (ii), *and* (iii) *above. Suppose that $1 \leqslant p_1, p_2, q_1, q_2 \leqslant \infty$, and that $1/p = (1-t)/p_1 + t/p_2$, $1/q = (1-t)/q_1 + t/q_2$, where $0 \leqslant t \leqslant 1$.*

Assume that whenever f is simple, the following two inequalities hold:

$$\| T_{iy}(f) \|_{q_1} \leqslant A_0(y) \| f \|_{p_1} \tag{8.1}$$

and

$$\| T_{1+iy}(f) \|_{q_2} \leqslant A_1(y) \| f \|_{p_2}. \tag{8.2}$$

Suppose further that

$$\log A_i(y) \leqslant A\, e^{a|y|}, \quad a < \pi, \text{ for } i = 0, 1. \tag{8.3}$$

Then we may conclude that for any simple f,

$$\| T_t(f) \|_q \leqslant A_t \| f \|_p, \tag{8.4}$$

where

$$\log A_t = \int_{-\infty}^{+\infty} \omega(1-t, y) \log A_0(y)\,dy + \int_{-\infty}^{+\infty} \omega(t, y) \log A_1(y)\,dy$$

with

$$\omega(1-t, y) = \tfrac{1}{2} \cdot \frac{\tan(\tfrac{1}{2}\pi t)}{[\tan^2(\tfrac{1}{2}\pi t) + \tanh^2(\tfrac{1}{2}\pi y)]\cosh^2(\tfrac{1}{2}\pi y)}. \tag{8.5}$$

For a proof, see [13].

9. Class $L\,(\log^+ L)^r$.

Again we shall be given two measure spaces M and N with measures $d\mu$ and $d\nu$ respectively. This time we shall assume that the total measure of M is finite, and we shall denote it by $\mu(M)$.

We recall the standard notation, $\log^+ x = \log x$, if $x \geqslant 1$, otherwise $\log^+ x = 0$. We denote by $L(\log^+ L)^r$ the class of measurable functions for which

$$\int_M |f(x)|\,(\log^+ |f(x)|)^r\,d\mu < \infty. \tag{9.1}$$

We shall say that an operator T defined on simple functions on M to measurable functions on N is *sub-linear* if

(i) $$|T(\psi_1+\psi_2)|\leqslant|T(\psi_1)|+|T(\psi_2)|$$

whenever ψ_1 and ψ_2 are simple, and

(ii) $$|T(k\psi_1)|=|k|\;|T(\psi_1)|, \text{ for every scalar } k.$$

The following lemma has been used in a particular case by Titchmarsh [14].

LEMMA 2. *Let T be a sub-linear operator, defined on simple functions of M, $(\mu(M)<\infty)$ as above. Suppose that*

$$\|T(f)\|_1\leqslant A(p-1)^{-r}\|f\|_p \tag{9.2}$$

for every p, $1<p\leqslant 2$, every simple f, and some r, $r\geqslant 0$, with the constant A independent of f and p. Then we may conclude that

$$\|T(f)\|_1\leqslant KA\left[\int_M|f(x)|\,(\log^+|f(x)|)^r\,d\mu+1\right], \tag{9.3}$$

for every simple f; A is the bound of (9.2) and K depends only on the total measure of the space M.

Proof. We write $f(x)=\sum_{n=0}^{\infty}f_n(x)$,

where
$$\begin{aligned}&f_n(x)=f(x), \text{ if } 2^{n-1}\leqslant|f(x)|<2^n,\ n\geqslant 1;\\ &f_n(x)=0, \text{ otherwise, } n\geqslant 1;\\ &f_0(x)=f(x), \text{ if } |f(x)|<1;\\ &f_0(x)=0, \text{ otherwise.}\end{aligned}$$

We let $E_n=$ set where $f_n(x)\neq 0$, $\mu(E_n)$ its measure. Since $f(x)$ is simple, only a finite number of terms appear in the above and following sums.

Now, by properties (i) and (ii) above:

$$|T(f)|\leqslant\sum_{n=0}^{\infty}|T(f_n)|=\sum_{n=0}^{\infty}2^n|T(2^{-n}f_n)|.$$

Therefore,
$$\int_N|T(f)|\,d\mu\leqslant\sum_{n=0}^{\infty}2^n\int_N|T(2^{-n}f_n)|\,d\mu. \tag{9.4}$$

Now, by (9.2),
$$\int_N|T(2^{-n}f_n)|\,d\mu\leqslant A(p_n-1)^{-r}\|2^{-n}f_n\|_{p_n}, \tag{9.5}$$

where the exponents p_n, may be chosen arbitrarily, subject to $1 < p_n \leqslant 2$. We choose p_n as follows: $p_0 = 2$, $p_n = 1 + 1/n$, $n \geqslant 1$. We notice that $|2^{-n} f_n| \leqslant 1$, and f_n vanishes outside E_n.

We therefore have

$$\left.\begin{aligned} (p_n - 1)^{-r} \| 2^{-n} f_n \|_{p_n} &\leqslant n^r \mu(E_n)^{n/(n+1)}, \quad n \geqslant 1. \\ \| f_0 \|_2 &\leqslant (\mu(M))^{\frac{1}{2}}. \end{aligned}\right\} \tag{9.6}$$

Combining (9.6), (9.5), and (9.4) gives

$$\int_N |T(f)| \, d\mu \leqq A (\mu(M))^{\frac{1}{2}} + \sum_{n=1}^{\infty} 2^n n^r (\mu(E_n))^{n/(n+1)}. \tag{9.7}$$

On each term of the infinite series appearing in (9.7) we shall apply the inequality of Young:

$$a b \leqslant a^p/p + b^q/q, \quad 1/p + 1/q = 1.$$

We choose $a = 2^{n+1} n^r \mu(E_n)^{n/(n+1)}$, $b = 2^{-1}$, $p = 1 + 1/n$, $q = n + 1$. Thus

$$\sum_{n=1}^{\infty} 2^n n^r (\mu(E_n))^{n/(n+1)} \leqslant \sum_{n=1}^{\infty} \frac{2^{(n+1)(1+1/n)} n^{r(1+1/n)}}{1 + 1/n} \mu(E_n) + \sum_{n=1}^{\infty} 2^{-n-1}/n + 1.$$

But as is easily verified,

$$\sum_{n=1}^{\infty} \frac{2^{(n+1)(1+1/n)} n^{r(1+1/n)}}{1 + 1/n} \mu(E_n) \leqslant B \sum_{n=1}^{\infty} 2^n n^r \mu(E_n).$$

Combining these estimates with (9.7), we obtain

$$\int_N |T(f)| \, d\nu \leqslant A K \sum_{n=1}^{\infty} 2^n n^r \mu(E_n) + A K. \tag{9.8}$$

However,
$$\sum_{n=1}^{\infty} 2^n n^r \mu(E_n) \leqslant \int_M |f(x)| \, (\log^+ |f(x)|)^r \, d\mu. \tag{9.9}$$

Thus (9.9) and (9.8) together prove Lemma 2.

10. Maximal function

We introduce the spherical means of $f(x)$ and of $|f(x)|$, defined as follows:

$$f(x; t) = \omega_k^{-1} \int f(x_1 + t\xi_1, x_2 + t\xi_2, \ldots, x_k + t\xi_k) \, d\Sigma_\xi, \tag{10.1}$$

$$\bar{f}(x; t) = \omega_k^{-1} \int |f(x_1 + t\xi_1, \ldots, x_k + t\xi_k)| \, d\Sigma_\xi. \tag{10.2}$$

Here $\omega_k = 2(\pi)^{\frac{1}{2}k}/\Gamma(\frac{1}{2}k)$, and Σ is the unit sphere: $\xi_1^2 + \xi_2^2 \cdots + \xi_k^2 = 1$; $d\Sigma_\xi$ its Euclidean measure.

The following lemma is easily deduced from its well-known "non-periodic" analogue. (1)

LEMMA 3. *Let $f(x) = f(x_1 \ldots x_k)$ be of period 2π in each variable x_i, and integrable over the fundamental cube Q_k.*

Let

$$f^*(x) = \sup_{\infty > N > 0} N^{-k} \omega_k^{-1} k \int_{|y| \leq N} |f(x+y)|\, dy$$

$$= \sup_{\infty > N > 0} N^{-k} k \int_0^N f(x;\, t)\, t^{k-1}\, dt, \tag{10.3}$$

Then $f^*(x)$ is finite almost everywhere. Moreover, if $f(x) \in L^p(Q_k)$, $1 < p$, so is $f^*(x)$, and,

$$\Big(\int_{Q_k} (f^*(x))^p\, dx \Big)^{1/p} \leq A\,(p/(p-1)) \cdot \Big(\int_{Q_k} |f(x)|^p\, dx \Big)^{1/p}, \quad 1 < p. \tag{10.4}$$

Proof. Let

$$g(x) = f(x), \quad \text{whenever } -2\pi < x_i \leq 2\pi,\ i = 1, \ldots, k,$$
$$= 0, \quad \text{otherwise.}$$

Form $g^*(x)$, as in (10.3) above. Then by the non-periodic analogue of the lemma, which we take for granted, (1)

$$\Big(\int_{E_k} (g^*(x)^p)\, dx \Big)^{1/p} \leq A\,(p/(p-1)) \Big(\int_{E_k} |g(x)|^p\, dx \Big)^{1/p}, \quad 1 < p. \tag{10.5}$$

Here E_k is the Euclidean k-space.

We next note that

$$N^{-k} \omega_k^{-1} k \int_{|y| \leq N} |g(x+y)|\, dy = N^{-k} \omega_k^{-1} k \int_{|y| \leq N} |f(x+y)|\, dy, \quad \text{if } x \in Q_k,\ 0 < N \leq \pi, \tag{10.6}$$

and

$$N^{-k} \omega_k^{-1} k \int_{|y| \leq N} |f(x+y)|\, dy = N^{-k} k \int_0^N f(x;\, t)\, t^{k-1}\, dt$$

$$\leq \alpha_k \int_{Q_k} |f(x)|\, dx, \quad \text{if } N \geq \pi. \tag{10.7}$$

It follows from (10.6) and (10.7) that

$$f^*(x) \leq g^*(x) + \alpha_k' \Big(\int_{Q_k} |f(x)|^p\, dx \Big)^{1/p}, \quad \text{if } x \in Q_k. \tag{10.8}$$

(1) For a proof of the needed maximal theorem, see [12].

By definition of $g(x)$, however,

$$\int_{E_k} |g(x)|^p\,dx = 2^k \int_{Q_k} |f(x)|^p\,dx. \tag{10.9}$$

A combination of (10.9), (10.8), and (10.5) gives

$$\Big(\int_{Q_k} (f^*(x))^p\,dx\Big)^{1/p} \leqslant A\,(p/(p-1))\,\Big(\int_{Q_k} |f(x)|^p\,dx\Big)^{1/p},\ 1<p, \tag{10.10}$$

which proves the lemma.

Remark. The behaviour for $p\to 1$ of the bound $A\,(p/p-1))$ appearing in (10.10), will be important for later purposes.

11. Riesz means of complex order

Let $\sum_{\nu=0}^{\infty} a_\nu$, be a numerical series. We shall define the Riesz means of complex order δ, $\delta=\sigma+i\tau$, as follows.

Let $\sigma=\Re(\delta)>-1$. Define S_R^δ, by

$$S_R^\delta=\sum_{\nu<R^2}\left(1-\frac{\nu}{R^2}\right)^\delta a_\nu, \tag{11.1}$$

where, of course, the principal value is taken for the complex exponentials appearing in (11.1). Thus

$$S_R^\delta = A_R^\delta/R^{2\delta},\quad \text{where } A_R^\delta=\sum_{\nu<R^2}(R^2-\nu)^\delta a_\nu. \tag{11.2}$$

We note that if $\Re(\delta)>-1/p$, then S_R^δ, as a function of R, is locally in L^p.

The relation between S_R^δ, for different complex δ's, is contained in the following.

LEMMA 4. *Let β, δ, be complex numbers, $\Re(\beta)>0$, $\Re(\delta)>-1$, and $\Re(\beta+\delta)>0$. Then*

$$A_R^{\delta+\beta}=\frac{2\,\Gamma(\delta+\beta+1)}{\Gamma(\delta+1)\,\Gamma(\beta)}\int_0^R (R^2-t^2)^{\beta-1}A_t^\delta\,t\,dt, \tag{11.3}$$

the integral converging absolutely.

Proof. We recall that S_R is locally in L^p, as long as $\Re(\delta)>-1/p$. Now, since $\Re(\delta)>-1$, $\Re(\beta)>0$, and $\Re(\beta+\delta)>0$, we can find exponents p, and q so that $1/p+1/q=1$, and both

$$\int_0^R |A_t^\delta\cdot t|^p\,dt,\qquad \int_0^R |(R^2-t^2)^{\beta-1}|^q\,dt$$

converge. Thus the integral in (11.3) converges absolutely, by Hölder's inequality.

Now, $A_t^\delta = \sum_{\nu < t^2} (t^2-\nu)^\delta a_\nu$. Therefore to verify the identity (11.3) it is sufficient to verify that

$$(R^2-\nu)^{\delta+\beta} = \frac{2\,\Gamma(\delta+\beta+1)}{\Gamma(\delta+1)\,\Gamma(\beta)} \int_\nu^R (R^2-t^2)^{\beta-1}(t^2-\nu)^\delta\, t\, dt,$$

$$\text{for } \Re(\beta)>0,\ \Re(\delta)>-1,\ \Re(\beta+\delta)>0. \tag{11.4}$$

We see first that the integral in (11.4) converges absolutely, by the same argument used to establish the absolute convergence of the integral (11.3). For fixed β, this argument also shows that the convergence is uniform in δ, whenever δ is restricted to a closed bounded set lying in $\Re(\delta)>-1$, and $\Re(\beta+\delta)>0$. Thus for fixed β, the right side of (11.4) is analytic in δ; when $\delta>0$, however, (11.4) is easily verified by the well-known equation of the Beta function. Since the left side of (11.4) is clearly analytic in δ, (11.4) is then demonstrated for all values of β and δ in question. This concludes the proof of the lemma.

Let now $f(x)=f(x_1, x_2, \dots, x_k)$ be of period 2π in each x_i, and let it be integrable over the fundamental cube Q_k. We form the Fourier expansion of $f(x)$

$$f(x) \sim \sum a_n e^{in\cdot x}, \tag{11.5}$$

where
$$a_n = (2\pi)^{-k} \int_{Q_k} f(x)\, e^{-in\cdot x}\, dx.$$

If δ is complex, $\Re(\delta)>-1$, we define $S_R^\delta(x)$ by

$$S_R^\delta(x) = S_R^\delta(x, f) = \sum_{\nu<R^2} \left(1-\frac{\nu}{R^2}\right)^\delta \{\sum_{|n|^2=\nu} a_n e^{inx}\}$$
$$= \sum_{|n|<R} \left(1-\frac{|n|^2}{R^2}\right)^\delta a_n e^{inx}. \tag{11.6}$$

Here $|n|^2 = n_1^2+n_2^2+\dots+n_k^2$. We may now extend Bochner's representation theorem to summability of complex order.

Lemma 5. *Let $f(x)$ be integrable over Q_k, and let $\Re(\delta)>\frac{1}{2}(k-1)$, $\delta=\sigma+i\tau$. Let $S_R^\delta(x)$ be as defined above. Then,*

$$S_R^\delta(x) = c_1 R^{\frac{1}{2}k-\delta} \int_0^\infty f(x;\, t)\, t^{\frac{1}{2}k-\delta-1} J_{\delta+\frac{1}{2}k}(tR)\, dt, \tag{11.7}$$

where $c_1 = 2^{\delta-\frac{1}{2}k+1}\,\Gamma(\delta+1)\,\{\Gamma(\frac{1}{2}k)\}^{-1}$; the integral in (11.7) converges absolutely.

Proof. We assume as known the case where δ is positive and $\delta > \frac{1}{2}(k-1)$.(1) Now for each fixed x and R, the left side of (11.7) is clearly analytic in δ. To prove the identity (11.7) it will therefore be sufficient to show that the right side of (11.7) is analytic in δ, when $\Re(\delta) > \frac{1}{2}(k-1)$. Thus the proof of the lemma will be concluded as soon as we show:

(i) for each fixed x and R, the integral

$$\int_0^\infty f(x;\,t)\, t^{\frac{1}{2}k-\delta-1} J_{\delta+\frac{1}{2}k}(tR)\, dt \tag{11.8}$$

converges absolutely and uniformly in δ, whenever δ lies in a closed bounded set within $\Re(\delta) > \frac{1}{2}(k-1)$;

(ii) the integral in (11.8) above is analytic in δ for each fixed x, $R > 0$ and $t > 0$. For this purpose we recall the following well-known facts in the theory of Bessel function.(2)

$$J_\zeta(t) = \frac{2}{\pi} \frac{t^\zeta}{\Gamma(\zeta+\frac{1}{2})} \int_0^1 (1-u^2)^{\zeta-\frac{1}{2}} \cos ut\, du, \quad \Re(\zeta) > -\tfrac{1}{2}. \tag{11.9}$$

$$|J_{\xi+i\eta}(t)| \leqslant A_\xi e^{\pi|\eta|} \cdot t^{-\frac{1}{2}}, \quad t \geqslant 1,\ \xi \geqslant 0. \tag{11.10}$$

$$|J_{\xi+i\eta}(t)| \leqslant A_\xi e^{\frac{1}{2}\pi|\eta|} \cdot t^{\xi}, \quad t > 0,\ \xi \geqslant 0. \tag{11.11}$$

By (11.9) we see that for each fixed x, $R > 0$, and $t > 0$, the integrand in (11.8) is analytic in δ. We also recall that

$$\int_0^u |f(x;\,t)|\, t^{k-1} dt \leqslant A u^k, \quad \text{if } u \geqslant u_0 > 0. \tag{11.12}$$

We now break up the range of integration for the integral of (11.8) into the intervals $(0, 1/R)$, and $(1/R, \infty)$. We further break up the interval $(1/R, \infty)$ into intervals of the form $(2^n/R, 2^{n+1}/R)$. Thus we write (11.8) as

$$\int_0^{1/R} f(x;\,t)\, t^{\frac{1}{2}k-\delta-1} J_{\delta+\frac{1}{2}k}(tR)\, dt + \sum_{n=0}^{\infty} \int_{2^n/R}^{2^{n+1}/R} f(x;\,t)\, t^{\frac{1}{2}k-\delta-1} J_{\delta+\frac{1}{2}k}(tR)\, dt. \tag{11.3}$$

If we replace each integrand in (11.13) by its absolute value, the resulting sum may be estimated as follows. The first term in (11.13) may be estimated by (11.11),

(1) The proof of this case may be found in [6], Chapter V.

(2) See the references in Lemma 8, below.

and each term in the infinite series may be estimated by (11.10). Combining these estimates, we obtain as an estimate for the absolute convergence of (11.8) the following:

$$A'_\tau e^{\frac{1}{2}\pi|\tau|} R^{\sigma+\frac{1}{2}k} \int_0^{1/R} |f(x;t)|\, t^{k-1}\, dt + A'_\sigma e^{\pi|\tau|} R^{-\frac{1}{2}} \sum_{n=0}^{\infty} 2^{-(\frac{1}{2}k+\frac{1}{2}+\sigma)n} \int_{2^n/R}^{2^{n+1}/R} |f(x;t)|\, t^{k-1}\, dt, \qquad (11.14)$$

where $\delta = \sigma + i\tau$.

By (11.12) the infinite sum appearing in (11.14) may be estimated as follows:

$$A'_\sigma e^{\pi|\tau|} R^{-k+\frac{1}{2}} \sum_{n=0}^{\infty} 2^{-(\sigma-\frac{1}{2}(k-1))n}. \qquad (11.15)$$

This last series converges when $\sigma > \frac{1}{2}(k-1)$; that is, when $\Re(\delta) > \frac{1}{2}(k-1)$. Therefore the integral (11.8) converges absolutely when $\Re(\delta) > \frac{1}{2}(k-1)$, and by the above estimates the convergence is uniform in any closed bounded set within $\Re(\delta) > \frac{1}{2}(k-1)$. This concludes the proof of the lemma.

CHAPTER II

Localization

12. Outline of method

Let $f(x) = f(x_1, \ldots, x_k)$ be a periodic function, integrable over the fundamental cube Q_k. Assume that $f(x)$ vanishes in the ε-sphere,

$$|x| \leqslant \varepsilon,\ \varepsilon > 0,\ (|x|^2 = x_1^2 + x_2^2 + \cdots + x_k^2).$$

We consider the spherical Riesz means of order $\frac{1}{2}(k-1)$ of the Fourier expansion of $f(x)$, evaluated at the origin:

$$S_R^{\frac{1}{2}(k-1)}(0) = S_R^{\frac{1}{2}(k-1)}(0; f) = \sum_{|n|<R} \left(1 - \frac{|n|^2}{R^2}\right)^{\frac{1}{2}(k-1)} a_n, \qquad (12.1)$$

where $a_n = (2\pi)^{-k} \int_{Q_k} f(x)\, e^{-in\cdot x}\, dx$.

The crux of the proof of the localization theorem for L^p, $1 < p$, (Theorem (L) in § 5) will consist in the proof of the following inequality:

$$\sup_{R \geqslant 0} |S_R^{\frac{1}{2}(k-1)}(0)| \leqslant A_{\varepsilon,p} \|f\|_p, \quad 1 < p. \qquad (12.2)$$

Here $f(x)$ is any function in $L^p(Q_k)$ which vanishes in the ε-sphere, $|x| \leqslant \varepsilon$; $A_{\varepsilon, p}$ is independent of $f(x)$.

For this purpose, we introduce the operator $U_R^\lambda(0) = U_R^\lambda(0, f)$ defined by

$$U_R^\lambda(0) = \sum_{0<|n|<R} \left(1 - \frac{|n|^2}{R^2}\right)^{\frac{1}{2}(k-1)+\lambda} |n|^\lambda a_n, \quad \text{for } \lambda = \sigma + i\tau, \quad -\tfrac{1}{2} \leqslant \sigma \leqslant \tfrac{1}{2}.\text{[1]} \tag{12.3}$$

We notice that if $\lambda = 0$, then (12.3) reduces (except for the constant term) to (12.1).

We then prove the following two "boundary-line" results

$$\sup_{R \geqslant 0} |U_R^\lambda(0)| \leqslant B_{\varepsilon, \lambda} \|f\|_2, \quad \text{if } \Re(\lambda) = \sigma \leqslant 0, \tag{12.4}$$

$$\sup_{R \geqslant 0} |U_R^\lambda(0)| \leqslant C_{\varepsilon, \lambda} \|f\|_1, \quad \text{if } \Re(\lambda) = \sigma > 0. \tag{12.5}$$

$B_{\varepsilon, \lambda}$ and $C_{\varepsilon, \lambda}$ will be appropriate bounds; their estimates will be of importance later.

Basic to the consideration of the above is the following "kernel":

$$H_\lambda^{(k)}(|x|) = \int_{|y| \leqslant 1} e^{-ix \cdot y} (1 - |y|^2)^{\frac{1}{2}(k-1)+\lambda} |y|^\lambda \, dy_1 \, dy_2 \ldots dy_k. \tag{12.6}$$

Now (12.5) will follow rather easily from the fact that the kernel $H_\lambda^{(k)}(|x|)$ will be integrable at infinity, when $\Re(\lambda) > 0$. The deduction of (12.4) will be more subtle. Since it includes the result for $\lambda = 0$, it may be viewed as a variant of the localization result for L^2. Once (12.4) and (12.5) have been proved, then (12.2) can be deduced by the convexity-interpolation argument mentioned earlier (Lemma 1).

In this chapter we shall adopt the following procedure. In § 13 we shall obtain an asymptotic estimate for the kernel $H_\lambda^{(k)}(|x|)$, $(\lambda = \sigma + i\tau)$, for large values of $|x|$, all values of τ, and $-\tfrac{1}{2} \leqslant \sigma \leqslant \tfrac{1}{2}$. In § 14 we shall derive the L^2 result (12.4). Next, in § 15 we shall prove the L^1 result (12.5). We then obtain the general localization theorems in § 16.

13. Aymptotic formula for $H_\lambda^{(k)}(u)$

We consider the function $H_\lambda^{(k)}(u)$ defined by

$$H_\lambda^{(k)}(|x|) = \int_{|y|<1} (1 - |y|^2)^{\frac{1}{2}(k-1)+\lambda} |y|^\lambda e^{-ix \cdot y} \, dy, \; \lambda = \sigma + i\tau, \tag{13.1}$$

the integral being taken over the solid unit sphere in k dimensional Euclidean space. As is well known, a Fourier transform of a radial function in k dimensions may be

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[1] The limitation $-\tfrac{1}{2} \leqslant \sigma \leqslant \tfrac{1}{2}$ is made for the sake of convenience.

written as an appropriate Fourier-Bessel transform, (see [5], p. 69.) Thus we may write (13.1) as

$$H_\lambda^{(k)}(u) = (2\pi)^{\frac{1}{2}k} u^{-\frac{1}{2}(k-2)} \int_0^1 (1-t^2)^{\frac{1}{2}(k-1)+\lambda} t^{\frac{1}{2}k+\lambda} J_{\frac{1}{2}(k-2)}(ut)\,dt. \tag{13.2}$$

We shall prove the following.

THEOREM 1. *Let* $\lambda = \sigma + i\tau$, $\frac{1}{2} \geqslant \sigma \geqslant -\frac{1}{2}$, $u \geqslant 1$. *Then*

$$H_\lambda^{(k)}(u) = A_\lambda^{(1)} u^{-k-\lambda} + A_\lambda^{(2)} u^{-k-\lambda} \cos u + A_\lambda^{(3)} u^{-k-\lambda} \sin u + R(\lambda, u)\, u^{-k-\lambda-1}, \tag{13.3}$$

with

$$|A_\lambda^{(i)}| \leqslant A\, e^{2\pi|\tau|}, \quad i = 1, 2, 3$$

and also

$$|R(\lambda, u)| \leqslant A\, e^{2\pi|\tau|}.\ ^{(1)}$$

Remarks. (i) The dependence on k (= number of dimensions) will not be exhibited in the above constants, and those entering in the proof.

(ii) The function $H_\lambda^{(k)}(u)$ may be transformed into a generalized hypergeometric function, (see [7] p. 178). For λ fixed, an estimate like (13.3) follows from the theory of asymptotic expansions of these functions, such as in Fox [8]. However, for our purposes it is necessary to exhibit the dependence of the remainder of the asymptotic formula on the imaginary part of λ. An attempt to adopt the treatment in [8] for our case would seem very prohibitive. We shall therefore derive (13.3) from "scratch". The proof will be greatly simplified by making use of certain identities in the theory of Bessel functions.

Proof. We shall first prove a series of lemmas.

LEMMA 6. *Let* $\Phi(u) = u^{-\frac{1}{2}(k-2)} \int_0^\infty \phi(\varrho)\, \varrho^{\frac{1}{2}k} J_{\frac{1}{2}(k-2)}(\varrho u)\, d\varrho,$

and let

$$\Delta(\phi) = \frac{d^2\phi}{d\varrho^2} + \frac{(k-1)}{\varrho}\frac{d\phi}{d\varrho}.$$

Let q be an integer so that $2q \geqslant k+2$. Suppose that ϕ has $2q$ continuous derivatives in $(0, \infty)$, and that

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(1) The estimate $A\,e^{2\pi|\tau|}$ contained in Theorem 1 is not the best possible (in τ). Since Lemma 1 allows for a very large growth in the imaginary parameter, we have not bothered to state more precise estimates, especially since it would make the notation even more unwieldly. As a matter of fact the estimate can be sharpened to $A\,e^{\varepsilon|\tau|}$, for any $\varepsilon > 0$. The same remarks can be made for most similar estimates in this paper.

$$\int_0^\infty |\Delta^q(\phi)|\,\varrho^{k-1}\,d\varrho \leqslant M < \infty.$$

Here, $$\Delta^q(\phi) = \Delta(\Delta^{q-1}(\phi)), \text{ etc.}$$

Conclusion: $$|\Phi(u)| \leqslant \omega_k (2\pi)^{-\frac{1}{2}k} M u^{-k-2}, \quad \text{if } u \geqslant 1.$$

Here, ω_k is the $(k-1)$ dimensional volume of the unit sphere.

Proof of the lemma. We shall make use of the following fact (quoted above): Let

$$f(x_1, \ldots, x_k) = \phi(\varrho), \quad \varrho^2 = x_1^2 + \cdots + x_k^2,$$

and $$F(y_1, \ldots, y_k) = (2\pi)^{-\frac{1}{2}k} \int_{E_k} e^{\,ix\cdot y} f(x_1, \ldots, x_k)\,dx,$$

then $$F(y_1, \ldots, y_k) = \Phi(u), \quad u^2 = y_1^2 + \ldots + y_k^2.$$

It is also well known that $\Delta(\phi)$ is the standard k-dimensional laplacean of $f(x_1, \ldots, x_k)$. Hence

$$\begin{aligned} |u|^{2q}\,\Phi(u)| &= \left|(2\pi)^{-\frac{1}{2}k} \int_{E_k} e^{-ix\cdot y}\,\Delta^q(f)\,dx\right| \\ &\leqslant (2\pi)^{-\frac{1}{2}k} \int_{E_k} |\Delta^q(f)|\,dx \\ &= \omega_k (2\pi)^{-\frac{1}{2}k} \int_0^\infty |\Delta^q(\phi)|\,\varrho^{k-1}\,d\varrho \\ &= \omega_k (2\pi)^{-\frac{1}{2}k} M. \end{aligned}$$

Therefore, $$|\Phi(u)| \leqslant \omega_k (2\pi)^{-\frac{1}{2}k} M u^{-2q} \leqslant \omega_k M (2\pi)^{-\frac{1}{2}k} u^{-k-2},$$

if $u \geqslant 1$. This concludes the proof of the lemma.

LEMMA 7. *Let $\zeta = \xi + i\eta$, $\xi \geqslant -\frac{1}{2}$, and $u \geqslant 1$. Then*

$$\int_0^\infty e^{-t}\,t^{\frac{1}{2}k+\zeta} J_{\frac{1}{2}(k-2)}(tu)\,dt = B_\zeta \cdot u^{-\frac{1}{2}k-\zeta-1} + R^{(1)}(\zeta, u)\,u^{-\frac{1}{2}k-\zeta-2}, \tag{13.4}$$

where $$|B_\zeta| \leqslant A_\xi e^{\frac{1}{2}\pi|\eta|}, \quad \text{and} \quad |R^{(1)}(\zeta, u)| \leqslant A_\xi e^{\pi|\eta|}.$$

Proof of the lemma. We make use of the following known identities:

$$\int_0^\infty e^{-t}\,t^n J_m(tu)\,dt = (1+u^2)^{-\frac{1}{2}(n+1)}\,\Gamma(m+n+1)\,P_n{}^m\left((1+u^2)^{-\frac{1}{2}}\right), \tag{13.5}$$

and

$$P_n^{-m}(\cos\theta) = \frac{2}{\Gamma(m+\frac{1}{2})\,(2\pi)^{\frac{1}{2}}}(\sin\theta)^{-m}\int_0^\theta \cos(n+\tfrac{1}{2})\,\psi\,[\cos\psi-\cos\theta]^{m-\frac{1}{2}}\,d\psi, \tag{13.6}$$

for $0<\theta<\pi$.

The first identity may be found in [2], p. 29, formula 6; the second may be found in [1], p. 159, formula 27.

In the above formula we shall let $n=\frac{1}{2}k+\zeta$, and $m=\frac{1}{2}(k-2)$. Now define B_ζ by

$$B_\zeta=\Gamma(k+\zeta)\,P_{\frac{1}{2}k+\zeta}^{\frac{1}{2}(k-2)}(0). \tag{13.7}$$

By choosing $\theta=\frac{1}{2}\pi$ in (13.6) we easily see that

$$|B_\zeta|\leqslant A_\xi\,e^{\frac{1}{2}\pi|\eta|},\quad \zeta=\xi+i\eta. \tag{13.8}$$

By further inspection in (13.6), we may see that

$$|\Gamma(k+\zeta)\,P_{\frac{1}{2}k+\zeta}^{\frac{1}{2}(2-k)}{}'(x)|\leqslant A\,e^{\frac{1}{2}\pi|\eta|},\ 0\leqslant x\leqslant\tfrac{1}{2}. \tag{13.9}$$

(The dash indicates differentiation with respect to x.)

For the choice of n and m made above, (13.5) becomes

$$\int_0^\infty e^{-t}\,t^{\frac{1}{2}k+\zeta}\,J_{\frac{1}{2}(k-2)}(tu)\,dt=(1+u^2)^{-\frac{1}{2}(\frac{1}{2}k+\zeta+1)}\,\Gamma(k+\zeta)\,P_{\frac{1}{2}k+\zeta}^{\frac{1}{2}(2-k)}((1+u^2)^{-\frac{1}{2}}). \tag{13.10}$$

The asymptotic estimate of the right side can now be made as follows. First observe that

$$|(1+u^2)^{-\frac{1}{2}(\frac{1}{2}k+\zeta+1)}-u^{-(\frac{1}{2}k+\zeta+1)}|\leqslant A_\xi(1+|\eta|)\,u^{-(\frac{1}{2}k+\xi+2)}\quad\text{for } u\geqslant 1,\quad \zeta=\xi+i\eta. \tag{13.11}$$

Next, by the mean-value theorem and (13.9),

$$|\Gamma(k+\zeta)\,P_{\frac{1}{2}k+\zeta}^{\frac{1}{2}(2-k)}((1+u^2)^{-\frac{1}{2}})-\Gamma(k+\zeta)\,P_{\frac{1}{2}k+\zeta}^{\frac{1}{2}(2-k)}(0)|\leqslant A_\xi\,e^{\frac{1}{2}\pi|\eta|}\,u^{-1},\quad u\geqslant 1. \tag{13.12}$$

A combination of (13.12) and (13.11) gives as asymptotic estimate for (13.10)

$$\Gamma(k+\zeta)\,P_{\frac{1}{2}k+\zeta}^{\frac{1}{2}(2-k)}(0)\,u^{-\frac{1}{2}k-\zeta-1}+R^{(1)}(\zeta,u)\,u^{-\frac{1}{2}k-\zeta-2}, \tag{13.13}$$

where $\quad |R^{(1)}(\zeta,u)|\leqslant A_\xi(1+|\eta|)\,e^{\frac{1}{2}\pi|\eta|}\leqslant A_\xi\,e^{\pi|\eta|}.$

With the definition of B_ζ made in (13.7) above, (13.13) is the desired asymptotic formula. The estimate for B_ζ in (13.8) completes the proof of the lemma.

LEMMA 8. *Let* $\zeta=\xi+i\eta$, $\xi\geqslant-\frac{1}{2}$, $u\geqslant 1$. *Then*

$$\int_0^1 (1-t^2)^{\frac{1}{2}(k-1)+\zeta}\, t^{\frac{1}{2}k}\, J_{\frac{1}{2}(k-2)}(tu)\, dt$$
$$= (C_\zeta \sin u + D_\zeta \cos u)\, u^{-\frac{1}{2}k-\zeta-1} + R^{(2)}(\zeta, u)\, u^{-\frac{1}{2}k-\zeta-2}, \qquad (13.14)$$

where
$$|C_\zeta| \leqslant A_\xi\, e^{\frac{1}{2}\pi|\eta|}, \qquad |D_\zeta| \leqslant A_\xi\, e^{\frac{1}{2}\pi|\eta|},$$
and
$$|R^{(2)}(\zeta, u)| \leqslant A_\xi\, e^{\pi|\eta|}.$$

Proof of the lemma. We use the identity

$$u^{n-m} J_m(u)\, 2^{m-n-1}\, \Gamma(m-n) = \int_0^1 (1-\varrho^2)^{m-n-1} J_n(\varrho u)\, \varrho^{n+1}\, d\varrho,$$
$$\Re(m-n) > 0, \quad \Re(n) > 0, \qquad (13.15)$$

which may be found in [2], p. 26. We also use the asymptotic expansion

$$J_{\mu+i\nu}(u) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} u^{-\frac{1}{2}} \cos\left[u - \frac{\pi}{4} - (\mu+i\nu)\frac{\pi}{2}\right] + R^{(3)}(\mu+i\nu, u), \qquad (13.16)$$

where
$$|R^{(3)}(\mu+i\nu, u)| \leqslant A_\mu\, e^{\pi|\nu|}\, u^{-\frac{3}{2}}, \quad \text{if } u \geqslant 1.$$

This asymptotic formula may be found in [1], p. 85. We then take $m=k-\frac{3}{2}+\zeta$, $n=\frac{1}{2}(k-2)$, $\mu=k-\frac{3}{2}+\xi$, and $\nu=\eta$. A straight-forward combination of (13.15) and (13.16) leads directly to (13.14) and the proof of the lemma.

Proof of Theorem 1. Consider the integral

$$\int_0^1 (1-t^2)^{\frac{1}{2}(k-1)+\lambda}\, t^{\frac{1}{2}k+\lambda}\, J_{\frac{1}{2}(k-2)}(tu)\, dt, \quad \lambda=\sigma+i\tau. \qquad (13.17)$$

The main contributions to its asymptotic expansion will be due to the "singularities" of the expression $(1-t^2)^{\frac{1}{2}(k-1)+\lambda}\, t^{\frac{1}{2}k+\lambda}$, at $t=0$, and $t=1$. For this reason we separate the two contributions as follows.

Let $\psi(t)\in C^\infty(0,1)$, with $\psi(t)=1$, if $0\leqslant t\leqslant\frac{1}{3}$, and $\psi(t)=0$, if $\frac{2}{3}\leqslant t\leqslant 1$. Then write

$$\int_0^1 (1-t^2)^{\frac{1}{2}(k-1)+\lambda}\, t^{\frac{1}{2}k+\lambda}\, J_{\frac{1}{2}(k-2)}(tu)\, dt = I_1 + I_2, \qquad (13.18)$$

where
$$I_1 = \int_0^{\frac{2}{3}} (1-t^2)^{\frac{1}{2}(k-1)+\lambda}\, t^{\frac{1}{2}k+\lambda}\, \psi(t)\, J_{\frac{1}{2}(k-2)}(tu)\, dt, \qquad (13.19)$$

and
$$I_2=\int_{\frac{1}{3}}^{1}(1-t^2)^{\frac{1}{2}(k-1)+\lambda}\,t^{\frac{1}{2}k+\lambda}\,[1-\psi(t)]\,J_{\frac{1}{2}(k-2)}(tu)\,dt, \tag{13.20}$$

The integrand in (13.19) has now only one "singularity", at $t=0$. In order to obtain an asymptotic expansion for it we shall compare it with the integral in (13.4) which displays the same singularity. Similarly, (13.20) will be compared with (13.14). Consider (13.19) first.

Let q be the smallest integer so that $2q\geqslant k+2$. Define a polynomial $P(t)$ of degree $2q$ by the following properties:

$$P(t)=1+\sum_{j=1}^{2q}a_j t^j;$$

if we set
$$\delta(t)=e^{-t}P(t)-(1-t^2)^{\frac{1}{2}(k-1)+\lambda}\,\psi(t), \tag{13.21}$$

then
$$\delta^{(n)}(0)=0,\quad 0\leqslant n\leqslant 2q. \tag{13.22}$$

It is clear that the conditions (13.22) determine the coefficients a_j completely. Because these conditions involve the derivatives up to order $2q$ of $\psi(t)\,(1-t^2)^{\frac{1}{2}(k-1)+\lambda}$, then

$$|a_j|\leqslant A\,(1+|\lambda|^{2q}). \tag{13.23}$$

By Taylor's theorem with the remainder, (13.22), and (13.23) it also follows that

$$|\delta^{(n)}(t)|\leqslant A'\,(1+|\lambda|^{2q})\,t^{2q-n},\quad 0\leqslant n\leqslant 2q,\quad 0\leqslant t\leqslant \tfrac{2}{3}. \tag{13.24}$$

Moreover, using the fact that $\psi(t)$ vanishes for $t>\frac{2}{3}$, we also get

$$|\delta^{(n)}(t)|\leqslant A'\,(1+|\lambda|^{2q})\,t^{2q}\,e^{-t},\quad 0\leqslant n\leqslant 2q,\ t>\tfrac{1}{3}. \tag{13.25}$$

Now consider $\Delta_1(u)$ defined by

$$\Delta_1(u)=I_1-\int_0^\infty e^{-t}\,P(t)\,t^{\frac{1}{2}k+\lambda}\,J_{\frac{1}{2}(k-2)}(tu)\,dt. \tag{13.26}$$

By (13.21), and (13.19) we have

$$-\Delta_1(u)=\int_0^\infty \delta(t)\,t^{\frac{1}{2}k+\lambda}\,J_{\frac{1}{2}(k-2)}(tu)\,dt.$$

Because of (13.24) and (13.25), it is an easy matter to verify that the function $\phi(t)=\delta(t)\,t^{\lambda}$ satisfies the conditions of Lemma 6, with $M=A\,(1+|\lambda|^{4q})$. Thus

$$|\Delta_1(u)| \leqslant A'(1+|\lambda|^{4q})\, u^{-\frac{1}{2}k-3}, \quad u \geqslant 1. \tag{13.27}$$

Now

$$\int_0^\infty e^{-t} P(t)\, t^{\frac{1}{2}k+\lambda} J_{\frac{1}{2}(k-2)}(tu)\, dt$$
$$= \int_0^\infty e^{-t} t^{\frac{1}{2}k+\lambda} J_{\frac{1}{2}(k-2)}(tu)\, dt + \sum_{j=1}^{2q} a_j \int_0^\infty e^{-t} t^{\frac{1}{2}k+\lambda+j} J_{\frac{1}{2}(k-2)}(tu)\, dt.$$

We may now apply Lemma 7, with $\varrho = \lambda, \lambda+1, \dots, \lambda+2q$. We thus obtain

$$\int_0^\infty e^{-t} P(t)\, t^{\frac{1}{2}k+\lambda} J_{\frac{1}{2}(k-2)}(tu)\, dt = B_\lambda u^{-\frac{1}{2}k-\lambda-1} + R^{(4)}(\lambda, u)\, u^{-\frac{1}{2}k-\lambda-2} \tag{13.28}$$

for $u \geqslant 1$, where

$$|R^{(4)}(\lambda, u)| \leqslant A(1+|\lambda|^{4q})\, e^{\pi|\tau|} \leqslant A\, e^{2\pi|\tau|},$$
$$|B_\lambda| \leqslant A\, e^{\frac{1}{2}\pi|\tau|} \leqslant A\, e^{2\pi|\tau|},$$

with $\lambda = \sigma + i\tau$, and $-\frac{1}{2} \leqslant \sigma \leqslant \frac{1}{2}$.

A combination of (13.27) and (13.28) gives us the asymptotic estimate for I_1 (because of (13.26)). We thus have

$$I_1 = B_\lambda u^{-\frac{1}{2}k-\lambda-1} + R^{(4)}(\lambda, u)\, u^{-\frac{1}{2}k-\lambda-2}, \quad \lambda = \sigma + i\tau, \tag{13.29}$$

with $\quad |B_\lambda| \leqslant A\, e^{2\pi|\tau|}, \ |R^{(5)}(\lambda, u)| \leqslant A\, e^{2\pi|\tau|}, \ -\frac{1}{2} \leqslant \sigma \leqslant \frac{1}{2}$, and $u \geqslant 1$.

We now estimate I_2 in a manner similar to the estimate for I_1.

Let $Q(s) = 1 + \sum_{j=1}^{2q} b_j s^j$ be the polynomial of degree $2q$ determined by the following conditions: If we set

$$\delta_2(t) \equiv Q(1-t^2) - t^\lambda(1-\psi(t)), \tag{13.30}$$

then

$$\delta_2^{(n)}(1) = 0, \quad 0 \leqslant n \leqslant 2q. \tag{13.31}$$

Reasoning as before it follows that

$$|b_j| \leqslant A(1+|\lambda|^{2q}). \tag{13.32}$$

We now redefine $\delta_2(t)$ by setting $\delta_2^*(t) = \delta_2(t)$ if $0 \leqslant t \leqslant 1$, $\delta_2^*(t) = 0$ if $t > 1$. Because of (13.31), this modification does not destroy the continuity of derivatives up to and including order $2q$.

Clearly therefore, (13.33)

$$|\delta_2^{*(n)}(t)| \leqslant A(1+|\lambda|^{2q}), \quad \tfrac{1}{3} \leqslant t, \text{ and } 0 \leqslant n \leqslant 2q.$$

We also recall that $\quad 1-\psi(t)=0, \text{ for } 0\leqslant t\leqslant \tfrac{1}{3}.$

Thus
$$\delta_2^*(t)=\delta_2(t)=Q(1-t^2), \quad \text{for } 0\leqslant t\leqslant \tfrac{1}{3},$$

and therefore
$$\int_0^{\frac{1}{3}} \left|\Delta^q(\delta_2^*(\varrho))\right| \varrho^{k-1}\,d\varrho \leqslant A'(1+|\lambda|^{2q}). \tag{13.34}$$

Now consider $\Delta_2(u)$ defined by

$$\Delta_2(u)=I_2-\int_0^1 Q(1-t^2)(1-t^2)^{\frac{1}{2}(k-1)+\lambda} J_{\frac{1}{2}(k-2)}(tu)\,dt. \tag{13.35}$$

Because of the definitions (13.20) and (13.30) we have

$$-\Delta_2(u)=\int_0^1 \delta_2(t)\,t^{\frac{1}{2}k} J_{\frac{1}{2}(k-2)}(tu)\,dt=\int_0^\infty \delta_2^*(t)\,t^{\frac{1}{2}k} J_{\frac{1}{2}(k-2)}(tu)\,dt. \tag{13.36}$$

We see by (13.33) and (13.34) (and the fact that $\delta_2^*(t)=0$ for $t\geqslant 1$) that $\delta_2^*(t)$ satisfies the conditions of Lemma 6, with $M=A(1+|\lambda|^{2q})$. Thus,

$$|\Delta_2(u)|\leqslant A'(1+|\lambda|^{2q})\,u^{-\frac{1}{2}k-3}, \quad u\geqslant 1. \tag{13.37}$$

Now

$$\int_0^1 Q(1-t^2)(1-t^2)^{\frac{1}{2}(k-1)+\lambda}\,t^{\frac{1}{2}k} J_{\frac{1}{2}(k-2)}(tu)\,dt$$
$$=\int_0^1 (1-t^2)^{\frac{1}{2}(k-1)+\lambda}\,t^{\frac{1}{2}k} J_{\frac{1}{2}(k-2)}(tu)\,dt+\sum_{j=1}^{2q} b_j \int_0^1 (1-t^2)^{\frac{1}{2}(k-1)+\lambda+j}\,t^{\frac{1}{2}k} J_{\frac{1}{2}(k-2)}(tu)\,dt.$$

We may now apply Lemma 8, with $\zeta=\lambda,\ \lambda+1,\ \dots,\ \lambda+2q$ to the above. A combination of this and (13.37) gives us an estimate for (13.35). It is

$$I_2=(C_\lambda \sin u+D_\lambda \cos u)\,u^{-\frac{1}{2}k-\lambda-1}+R^{(6)}(\lambda,u)\,u^{-\frac{1}{2}k-\lambda-2}, \tag{13.38}$$

where
$$|C_\lambda|\leqslant A\,e^{2\pi|\tau|}, \quad |D_\lambda|\leqslant A\,e^{2\pi|\tau|}, \quad |R^{(6)}(\lambda,u)|\leqslant A\,e^{2\pi|\tau|},$$
$$\lambda=\sigma+i\tau, \quad -\tfrac{1}{2}\leqslant\sigma\leqslant\tfrac{1}{2}, \text{ and } u\geqslant 1.$$

If we combine (13.38) with the asymptotic expansion for I_1 in (13.29) we obtain the asymptotic expansion for (13.17). We also notice that the function $H_\lambda^{(k)}(u)$ (defined in (13.2)) differs from (13.17) only by a factor $(2\pi)^{\frac{1}{2}k}u^{-\frac{1}{2}k+1}$; thus the proof of Theorem 1 is complete.

14. The L^2 estimate

Let $f(x)$ be integrable over the fundamental cube $Q = Q_k$, and let it be periodic. Define $U_R^\lambda(x; f)$ by

$$U_R^\lambda(x; f) = \sum_{0<|n|<R} a_n \left(1 - \frac{|n|^2}{R^2}\right)^{\frac{1}{2}(k-1)+\lambda} |n|^\lambda e^{in\cdot x}, \quad \lambda = \sigma + i\tau, \quad -\tfrac{1}{2} \leqslant \sigma \leqslant \tfrac{1}{2}. \tag{14.1}$$

The a_n are the Fourier coefficients; $a_n = (2\pi)^{-k} \int_Q f(x) e^{-in\cdot x} dx$. The main result of this section will be the following:

THEOREM 2. *Let $1 \geqslant \varepsilon \geqslant 0$, ε fixed. Let $f(x) \in L^2(Q)$, and assume that $f(x)$ vanishes in the sphere $|x| \leqslant \varepsilon$. Then*

$$\sup_{\infty > R > 0} |U_R^\lambda(0, f)| \leqslant B_{\varepsilon,\lambda} \|f\|_2, \quad -\tfrac{1}{2} \leqslant \Re(\lambda) \leqslant 0. \tag{14.2}$$

We also have the estimate

$$|B_{\varepsilon,\lambda}| \leqslant B_\varepsilon \cdot e^{3\pi|\tau|}, \quad \lambda = \sigma + i\tau, \tag{14.3}$$

The proof of the theorem will be the consequence of several lemmas. The following lemma may be considered as a justification of the formal relation:

$$U_R^\lambda(x, f) = \omega_k (2\pi)^{-k} R^{k+\lambda} \int_0^\infty H_\lambda^k(Rt) f(x; t) t^{k-1} dt. \text{ (1)}$$

LEMMA 9. *Let $f(x)$ be a trigonometric polynomial, and suppose*

$$\int_Q f(x_1, x_2, \ldots, x_k) dx = 0.$$

Suppose that $\psi(s)$ is continuous in $0 < a \leqslant s \leqslant b < \infty$; then

$$\int_a^b U_s^\lambda(x, f) \psi(s) ds = \lim_{\eta \to 0} \omega_k (2\pi)^{-k} \int_0^\infty e^{-\eta t} H_\lambda^*(t) f(x; t) t^{k-1} dt, \tag{14.4}$$

where

$$H_\lambda^*(t) = \int_a^b s^{k+\lambda} H_\lambda^k(st) \psi(s) ds.$$

Proof of the lemma. We have

$$H_\lambda^k(|x|) = \int_{|y|<1} (1 - |y|^2)^{\frac{1}{2}(k-1)+\lambda} |y|^\lambda e^{-ix\cdot y} dy.$$

(1) See also Lemma 11, below.

Therefore by the Abel-summability of the Fourier inversion we have that the following

$$\lim_{\eta\to 0} (2\pi)^{-k} \int_{E_k} H_\lambda^k(|x|)\, e^{ix\cdot y}\, e^{-\eta|x|}\, dx \tag{14.5}$$

converges uniformly in y, if $|y|\geqslant\delta>0$; moreover, this limit is $(1-|y|^2)^{\frac{1}{2}(k-1)+\lambda}|y|^\lambda$, if $|y|\leqslant 1$, zero otherwise. By a change of variable we then have, for each fixed y, $|y|>0$,

$$\lim_{\eta\to 0} (2\pi)^{-k}\cdot s^{k+\lambda} \int_{E_k} H_\lambda^k(s|x|)\, e^{ix\cdot y}\, e^{-\eta|x|}\, dx \tag{14.6}$$

converging uniformly in s, $0<a\leqslant s\leqslant b<\infty$, the limit being $(1-|y|^2/s^2)^{\frac{1}{2}(k-1)+\lambda}|y|^\lambda$, if $|y|\leqslant s$, zero otherwise. Now in the above, let $|x|=t$, and $y=n$, (where n is a vector with integral components, $|n|\neq 0$). Because of the uniform convergence in (14.6) we may integrate the expression in $a\leqslant s\leqslant b$, after multiplying by $\psi(s)$, and interchange limits. Thus we obtain (14.4) in the case $f(x)=e^{in\cdot x}$, $|n|\neq 0$. A finite linear combination of such monomials will complete the proof of the lemma.

LEMMA 10. *Given a fixed ε, $1\geqslant\varepsilon>0$; assume that $f(x)\in L^1(Q)$ and $f(x)=0$ if $|x|\leqslant\varepsilon$. Let $\varphi(s)$ be of class C^1 in the interval $0\leqslant s\leqslant 1$. Define $U_R^{*\lambda}(x,f)$ by*

$$U_R^{*\lambda}(x,f)=\int_0^1 \left(1+\frac{s}{R}\right)^{k-1+2\lambda} \varphi(s)\, U_{R+s}^\lambda(x,f)\, ds. \tag{14.7}$$

Assume further that $\int_Q f(x)\,dx=0$. Then

$$\sup_{R\geqslant 1} |U_R^{*\lambda}(0,f)| \leqslant A_\varepsilon e^{3\pi|\tau|}\|f\|_1, \quad \lambda=\sigma+i\tau, \quad -\tfrac{1}{2}\leqslant\sigma\leqslant 0. \tag{14.8}$$

Proof of the lemma. Define $H_\lambda^*(R,t)$ by

$$H_\lambda^*(R,t)=\int_0^1 \left(1+\frac{s}{R}\right)^{k-1+2\lambda} (R+s)^{k+\lambda}\varphi(s)\, H_\lambda^k((R+s)t)\, ds. \tag{14.9}$$

According to Lemma 9, then

$$U_R^{*\lambda}(0,f)=\lim_{\eta\to 0}\omega_k(2\pi)^{-k}\int_0^\infty e^{-\eta t} H_\lambda^*(R,t)\, f(0;t)\, t^{k-1}\, dt, \tag{14.10}$$

where $$f(0;t)=\frac{1}{\omega_k}\int_\Sigma f(t\xi_1, t\xi_2, \ldots t\xi_k)\, d\Sigma.$$

We shall first show that if $f(x)$ is an exponential polynomial and $\int_Q f(x)\,dx = 0$, then if $R \geqslant 1$,

$$|U_R^{*\lambda}(0, f)| \leqslant \omega_k (2\pi)^{-k} \left| \int_0^\varepsilon H_\lambda^*(R, t) f(0, t) t^{k-1}\,dt \right| + A_\varepsilon e^{3\pi|\tau|} \|f\|_1, \tag{14.11}$$

with A_ε, independent of R. For this purpose, we break up the range of integration in (14.10) into the intervals $(0, \varepsilon)$ and (ε, ∞). It is therefore sufficient to show that

$$\left| \int_\varepsilon^\infty e^{-\eta t} H_\lambda^*(R, t) f(0, t)\,dt \right| \leqslant A_\varepsilon e^{3\pi|\tau|} \|f\|_1, \tag{14.12}$$

where A_ε is independent of R and η. We first claim that $H_\lambda^*(R, t)$ has the following asymptotic expansion

$$|H_\lambda^*(R, t) - A_\lambda^{(4)} t^{-k-\lambda}| \leqslant A_\varepsilon e^{2\pi|\tau|} t^{-k-\sigma-1}, \tag{14.13}$$

if $R \geqslant 1$, $t \geqslant \varepsilon$, $\lambda = \sigma + i\tau$, and with $A_\lambda^{(4)}$ depending on R, but $|A_\lambda^{(4)}| \leqslant A e^{2\pi|\tau|}$ with A independent of R.

In fact, according to Theorem 1, the first three terms of the asymptotic expansion of $H_\lambda^{(k)}(u)$ are

$$A_\lambda^{(1)} u^{-k-\lambda} + A_\lambda^{(2)} u^{-k-\lambda} \cos u + A_\lambda^{(3)} u^{-k-\lambda} \sin u.$$

Applying formula (14.9) to the first term above, we obtain

$$A_\lambda^{(1)} \int_0^1 \left(1 + \frac{s}{R}\right)^{k-1+2\lambda} (R+s)^{k+\lambda} \phi(s) t^{-k-\lambda} (R+s)^{-k-\lambda}\,ds$$
$$= A_\lambda^{(1)} t^{-k-\lambda} \int_0^1 \left(1 + \frac{s}{R}\right)^{k-1+2\lambda} \phi(s)\,ds.$$

This last is the term $A_\lambda^{(4)} t^{-k-\lambda}$ which appears in (14.13), with

$$A_\lambda^{(4)} = A_\lambda^{(1)} \int_0^1 \left(1 + \frac{s}{R}\right)^{k-1+2\lambda} \phi(s)\,ds.$$

Going over to the second term of the asymptotic expansion, $A_\lambda^{(2)} u^{-k-\lambda} \cos u$, then the contribution in (14.9) is

$$A_\lambda^{(2)} t^{-k-\lambda} \int_0^1 \left(1 + \frac{s}{R}\right)^{k-1+2\lambda} (R+s)^{k+\lambda} \phi(s) (R+s)^{-k-\lambda} \cos[t(R+s)]\,ds.$$

Making use of the fact that $\phi(s) \in C^1$, we integrate the above integral by parts and obtain that it is

$$O(t^{-1}) \text{ uniformly in } R, \quad R \geqslant 1.$$

Thus the entire contribution of $A_\lambda^{(2)} u^{-k-\lambda} \cos u$ is incorporated in the right-hand side of (14.13).

A similar argument is applied to the term $A_\lambda^{(3)} u^{-k-\lambda} \sin u$. Finally, the remainder term, $R(\lambda, u) u^{-k-\lambda-1}$, of the asymptotic expansion is also directly incorporated in the right-hand side of (14.13).

Hence (14.13) is demonstrated.

Now it is easy to see that

$$\int_\varepsilon^\infty t^{-k-\sigma-1} |f(0, t)| t^{k-1} dt \leqslant A_\varepsilon \|f\|_1 \quad \text{if} \quad -\tfrac{1}{2} \leqslant \sigma \leqslant 0, \text{ say.}$$

Thus in order to conclude the estimate (14.12), we must estimate the quantity

$$A_\lambda^{(4)} \int_\varepsilon^\infty e^{-\eta t} t^{-k-\lambda} f(0, t) t^{k-1} dt. \tag{14.14}$$

By changing back to the Cartesian coordinates $x = (x_1, \dots, x_k)$ in Euclidean space k-space, we may write (14.14) as

$$A_\lambda^{(4)} (\omega_k)^{-1} \int_{|x| \geqslant \varepsilon} e^{-\eta|x|} |x|^{-k-\lambda} f(x) dx. \tag{14.15}$$

We recall that $f(x)$ is an exponential polynomial, periodic over the fundamental cube Q, $-\pi \leqslant x_i \leqslant \pi$, $i = 1, \dots, k$; and that $\int_Q f(x) dx = 0$.

Let now Q^n denote the translation of the cube Q by the vector $2\pi n$, where $n = (n_1, \dots, n_k)$, n_i are integers. Thus $Q^n = Q + 2\pi n$, and $E_k = U_n Q^n$, where the union ranges over all integral component vectors. Thus except for the constant $A_\lambda^{(4)} (\omega_k)^{-1}$, we may rewrite (14.15) as

$$\int_{\{|x| \geqslant \varepsilon\} \cap Q} e^{-\eta|x|} |x|^{-k-\lambda} f(x) dx + \Sigma_n' \int_{Q^n} e^{-\eta|x|} |x|^{-k-\lambda} f(x) dx, \tag{14.16}$$

where Σ' indicates that we sum over all n, with $|n| \neq 0$. Since $\int_{Q^n} f(x) dx = 0$, then

$$\int_{Q^n} e^{-\eta|x|} |x|^{-k-\lambda} f(x) dx = \int_{Q^n} [e^{-\eta|x|} |x|^{-k-\lambda} - e^{-\eta|n|} |n|^{-k-\lambda}] f(x) dx. \tag{14.17}$$

Now it is an easy matter to verify that if $x \in Q^n$ then

$$|e^{-\eta|x|}|x|^{-k-\lambda} - e^{-\eta|n|}|n|^{-k-\lambda}| \leqslant A\,[1+|\tau|]\,|n|^{-k-1-\sigma},$$

$$0 < \eta \leqslant 1, \quad \lambda = \sigma + i\tau, \quad -\tfrac{1}{2} \leqslant \sigma \leqslant 0.$$

Therefore by (14.17)

$$\left|\int_{Q^n} e^{-\eta|x|}|x|^{-k-\lambda} f(x)\,dx\right| \leqslant A\,|\tau|\cdot|n|^{-k-1-\sigma}\int_Q |f(x)|\,dx, \quad \text{if } |n| \neq 0,$$

since $\int_{Q^n} |f(x)|\,dx = \int_Q |f(x)|\,dx$. Thus the infinite sum appearing in (14.16) is estimated by

$$A\,|\tau|\,(\Sigma'\,|n|^{-k-1-\sigma})\int_Q |f|\,dx.$$

Since $-\tfrac{1}{2} \leqslant \sigma \leqslant 0$, then certainly $\Sigma'\,|n|^{-k-1-\sigma} \leqslant A$. The first member of (14.16) is clearly estimated by $A_\varepsilon \int_Q |f(x)|\,dx$. Combining these two estimates, we obtain as an estimate for (14.16)

$$A_\varepsilon (1+|\tau|)\int_Q |f(x)|\,dx.$$

We thus obtain the estimate for (14.15), and then via (14.14) we arrive at the estimate (14.12). (Here we used $(1+|\tau|)\,e^{2\pi|\tau|} \leqslant e^{3\pi|\tau|}$.) Hence the proof for (14.11) is completed, when $f(x)$ is an exponential polynomial, and $\int_Q f(x)\,dx = 0$. A simple limiting argument (keeping R fixed) shows that (14.11) still holds if $f(x) \in L^1(Q)$, and $\int_Q f(x)\,dx = 0$. If we now assume that $f(x)$ vanishes if $|x| \leqslant \varepsilon$, then $f(0,t) = 0$ if $0 \leqslant t \leqslant \varepsilon$. Therefore (14.11) becomes

$$|U_R^{*\lambda}(0,f)| \leqslant A_\varepsilon\, e^{3\pi|\tau|}\,\|f\|_1 \tag{14.18}$$

with A_ε independent of R, and $f(x)$ assumed to vanish for $|x| \leqslant \varepsilon$. The above completes the proof of the lemma.

COROLLARY. *The conclusion of Lemma* 10 *still holds if we drop the assumption that* $\int_Q f(x)\,dx = 0$.

Proof. Choose $g(x)$ as a fixed periodic function of class C^∞ with properties $g(x) = 0$, for $|x| \leqslant 1$, and $\int_Q g(x)\,dx = 1$. Apply Lemma 10 to the function

$$f_1(x) = f(x) - g(x)\int_Q f(x)\,dx.$$

Then clearly $\int_Q f_1(x)\,dx = 0, \quad \text{and} \quad f_1(x) = 0 \text{ if } |x| \leqslant \varepsilon.$

Moreover, $U_R^{*\lambda}(0, f_1) = U_R^{*\lambda}(0, f) - \left(\int_Q f(x)\,dx\right) U_R^{*\lambda}(0, g).$

It is easy to verify that $|U_R^{*\lambda}(0, g)| \leqslant A$, by the absolute convergence of the Fourier expansion of $g(x)$. Thus

$$|U_R^{*}(0, f)| \leqslant A_\varepsilon\, e^{3\pi|\tau|}\, \|f\|_1 + A\, \|f\|_1,$$

and the corollary is proved.

Proof of Theorem 2. We fix the function $\varphi(s)$ appearing in (14.7) once and for all, as follows: Let $\varphi(s)$ be the polynomial of degree $2k-1$ which satisfies:

$$\int_0^1 \varphi(s)\,ds = 1, \quad \int_0^1 \varphi(s)\, s^j\,ds = 0, \quad 1 \leqslant j \leqslant 2k-1. \tag{14.19}$$

With $\varphi(s)$ so defined, the proof of Theorem 2 will be concluded as soon as we show that

$$|U_R^{\lambda}(0, f) - U_R^{*\lambda}(0, f)| \leqslant A\, e^{3\pi|\tau|}\, \|f\|_2, \; R \geqslant 2, \tag{14.20}$$

where f is any function in $L^2(Q)$, and A is independent of R.

Write $f(x) \sim \Sigma\, a_n e^{in\cdot x}$. Then

$$\Sigma\, |a_n|^2 = (2\pi)^{-k} \int_Q |f(x)|^2\,dx.$$

We recall that the number of lattice points in the spherical shell contained between spheres of radius $R-1$, and $R+1$, is $O(R^{k-1})$. Thus an application of Schwarz's inequality yields:

$$\sum_{R-1 \leqslant |n| < R+1} |n|^\lambda\, |a_n| \leqslant A\, R^{\frac{1}{2}(k-1)+\sigma}\, \|f\|_2, \; \lambda = \sigma + i\tau. \tag{14.21}$$

For the proof of (14.20) it is sufficient to show that

$$R^{k-1+2\lambda}\, U_R^{\lambda}(0, f) - R^{k-1+2\lambda}\, U_R^{*\lambda}(0, f) \tag{14.22}$$

is bounded in absolute value by $R^{k-1+2\sigma} \cdot A \cdot e^{3\pi|\tau|}\, \|f\|_2$. The quantity in (14.22) can be written as

$$\sum_{1 \leqslant |n| < R} (R^2 - n^2)^{\frac{1}{2}(k-1)+\lambda}\, |n|^\lambda a_n - \int_0^1 \Big\{ \sum_{1 \leqslant |n| < R+s} ((R+s)^2 - n^2)^{\frac{1}{2}(k-1)+\lambda}\, |n|^\lambda a_n \Big\}\, \varphi(s)\,ds. \tag{14.23}$$

We shall write (14.22) (or 14.23) in the form S_1+S_2, where S_2 involves all terms with $R-1\leqslant|n|<R+1$, and S_1 the remaining terms of the sums. Now if $R-1\leqslant|n|$, then clearly $(R^2-n^2)\leqslant 2R$; and similarly $(R+s)^2-n^2\leqslant 4R$. Thus for S_2 we have the following estimate

$$|S_2|\leqslant A\,R^{\frac{1}{2}(k-1)+\sigma}\sum_{R-1\leqslant|n|<R+1}|n|^{\lambda}|a_n|. \tag{14.24}$$

Because of (14.21), this becomes

$$|S_2|\leqslant A\,R^{k-1+2\sigma}\|f\|_2. \tag{14.25}$$

Now

$$S_1=\sum_{1\leqslant|n|<R-1}(R^2-n^2)^{\frac{1}{2}(k-1)+\lambda}|n|^{\lambda}a_n-\int_0^1\Big\{\sum_{1\leqslant|n|<R-1}((R+s)^2-n^2)^{\frac{1}{2}(k-1)+\lambda}|n|^{\lambda}a_n\Big\}\,\varphi(s)\,ds.$$

Remembering that $\int_0^1\varphi(s)\,ds=1$, we may rewrite the above as

$$S_1=\int_0^1\sum_{1\leqslant|n|<R-1}\big[(R^2-n^2)^{\frac{1}{2}(k-1)+\lambda}-((R+s)^2-n^2)^{\frac{1}{2}(k-1)+\lambda}\big]\,\varphi(s)\,|n|^{\lambda}a_n\,ds. \tag{14.26}$$

Now by Taylor's expansion, if $0\leqslant s\leqslant 1$,

$$(R^2-n^2)^{\frac{1}{2}(k-1)+\lambda}-((R+s)^2-n^2)^{\frac{1}{2}(k-1)+\lambda}=c_1 s+c_2 s^2\cdots+c_{2k-1}s^{2k-1}+O_{2k}, \tag{14.27}$$

where $$|O_{2k}|\leqslant\frac{1}{2k!}\,\Big|\sup_{0\leqslant s\leqslant 1}\big[((R+s)^2-n^2)^{\frac{1}{2}(k-1)+\lambda}\big]^{(2k)}\Big|.$$

Then it is an easy matter to verify, if $|n|<R-1$, that

$$|O_{2k}|\leqslant A\,[1+|\tau|^{2k}]\,n^{-\frac{1}{2}k-\frac{1}{2}+2\sigma},\quad \lambda=\sigma+i\tau,\quad -\tfrac{1}{2}\leqslant\sigma\leqslant 0.$$

Substituting (14.27) in (14.26), and using the orthogonality relation (14.19) we obtain

$$|S_1|\leqslant A\,[1+|\tau|^{2k}]\sum_{1\leqslant|n|<R-1}|n|^{-\frac{1}{2}k-\frac{1}{2}+2\sigma}|a_n|.$$

If we now use Schwarz's inequality, and the fact that $-\frac{1}{2}\leqslant\sigma\leqslant 0$, we obtain

$$|S_1|\leqslant A\,[1+|\tau|^{2k}]\,(\Sigma|a_n|^2)^{\frac{1}{2}}\leqslant A\,e^{3\pi|\tau|}\|f\|_2\leqslant A\,e^{3\pi|\tau|}R^{k-1+2\sigma}\|f\|_2,\quad R\geqslant 2.$$

Combining this with (14.25) we obtain

$$|S_1+S_2|\leqslant A\,e^{3\pi|\tau|}R^{k-1+2\sigma}\|f\|_2.$$

Since S_1+S_2 equals the quantity in (14.22), then the above proves (14.20). If we combine (14.20) and Lemma 10, corollary, then we obtain. whenever $f(x)=0$ for $|x|\leqslant\varepsilon$,

$$\sup_{R\geqslant 2} |U_R^\lambda(0, f)| \leqslant A_\varepsilon e^{3\pi|\tau|} \|f\|_1 + A e^{3\pi|\tau|} \|f\|_2 \leqslant A'_\varepsilon e^{3\pi|\tau|} \|f\|_2.$$

Since the above inequality is trivial if $0 < R < 2$, we have thus concluded the proof of Theorem 2.

15. The L^1 estimate

LEMMA 11. *Let $f(x) \in L^1(Q_k)$, and assume that $\frac{1}{2} \geqslant \sigma \equiv \Re(\lambda) > 0$, then*

$$U_R^\lambda(x, f) = \omega_k (2\pi)^{-k} R^{k+\lambda} \int_0^\infty H_\lambda^k(Rt) f(x; t) t^{k-1} dt, \tag{15.1}$$

the integral converging absolutely. The quantity $U_R^\lambda(x, f)$ is defined in (14.1), *and the kernel $H_\lambda^k(u)$ is defined in* (13.1).

Proof. Using the asymptotic estimate of Theorem 1, we see that for fixed R, the kernel $R^{k+\lambda} H_\lambda^k(R|x|)$ is absolutely integrable over E_k, whenever $R(\lambda) > 0$. Thus we may convolve $R^{k+\lambda} H_\lambda^k(R|x|)$ with an arbitrary periodic integrable function $f(x)$, and the usual multiplication formula for the Fourier coefficients holds. The Fourier transform of $R^{k+\lambda} H(R|x|)$ is immediately deduced from (13.1), and from this the proof of Lemma 11 is concluded.

THEOREM 3. *Let ε be fixed, $\varepsilon > 0$. Assume that $f(x) \in L^1(Q)$ and that $f(x) = 0$, if $|x| \leqslant \varepsilon$. Assume further that $\frac{1}{2} \geqslant \Re(\lambda) > 0$, $\lambda = \sigma + i\tau$. Then*

$$\sup_{R\geqslant 0} |U_R^\lambda(0, f)| \leqslant A_\varepsilon \cdot e^{2\pi|\tau|} \cdot \frac{1}{\sigma} \cdot \|f\|_1. \tag{15.2}$$

Proof. If we make use of (15.1) and the fact $f(0; t) = 0$, if $0 \leqslant t \leqslant \varepsilon$, then we have

$$U_R^\lambda(0, f) = \omega_k (2\pi)^{-k} R^{k+\lambda} \int_\varepsilon^\infty H_\lambda^k(Rt) f(0, t) t^{k-1} dt. \tag{15.3}$$

If we use the asymptotic formula for $H_\lambda^k(u)$ of Theorem 1 we then see that

$$|U_R^\lambda(0, f)| \leqslant A_\varepsilon e^{2\pi|\tau|} \int_\varepsilon^\infty t^{-k-\sigma} |f(0; t)| t^{k-1} dt, \tag{15.4}$$

where A_ε is independent of R. Now we use the fact that

$$\int_0^1 |f(0; t)| t^{k-1} dt \leqslant A \|f\|_1$$

and $$\int_n^{n+1} |f(0;t)|\,dt \leqslant A\,\|f\|_1, \quad n=1,2,\ldots .$$

Thus (15.4) becomes $$|U_R^\lambda(0,f| \leqslant A'_\varepsilon e^{2\pi|\tau|}\cdot\left(\sum_{n=1}^{\infty} n^{-1-\sigma}\right)\|f\|_1$$

and therefore $$|U_R^\lambda(0,f)| \leqslant A_\varepsilon e^{2\pi|\tau|}\cdot\frac{1}{\sigma}\cdot\|f\|_1,$$

with A_ε independent of R. This concludes the proof of Theorem 3.

16. The localization theorems

The main result is easily derivable from the following.

THEOREM 4. *Let $f(x)\in L^p(Q)$, $1<p\leqslant 2$. Assume that an ε is given, $1\geqslant\varepsilon>0$, and that $f(x)=0$, if $|x|\leqslant\varepsilon$. Let*

$$S_R^{\frac12(k-1)}(0)=\sum_{|n|<R}\left(1-\frac{|n|^2}{R^2}\right)^{\frac12(k-1)} a_n$$

be the Riesz means of order $\frac12(k-1)$ evaluated at the origin. Then

$$\sup_{R\geqslant 0}|S_R^{\frac12(k-1)}(0)| \leqslant (A_\varepsilon/(p-1))\,\|f\|_p, \quad 1<p\leqslant 2. \tag{16.1}$$

Proof. It is clearly sufficient to prove (16.1) for $S_R'^{\frac12(k-1)}(0)$ in place of $S_R^{\frac12(k-1)}(0)$, where

$$S_R'^{\frac12(k-1)}(0)=\sum_{0<|n|<R} a_n e^{inx}\left(1-\frac{|n|^2}{R^2}\right)^{\frac12(k-1)}.$$

This follows from the observation that

$$|S_R^{\frac12(k-1)}(0)-S_R'^{\frac12(k-1)}(0)|=|a_0|\leqslant(2\pi)^{-k}\int_Q|f(x)|\,dx\leqslant(2\pi)^{-k/p}\|f\|_p.$$

We shall prove the result for $S_R'^{\frac12(k-1)}(0)$ by applying Lemma 1 to the operator $U_R^\lambda(0,f)$, as follows. Assume first that $1<p<2$, since the case $p=2$ is contained in Theorem 2, when $\lambda=0$.

We now define an analytic family of operators T_z mapping simple functions of M to measurable functions of N as follows. For our fixed ε, let M be set of points in the fundamental cube Q complementary to the sphere $|x|\leqslant\varepsilon$; define $d\mu$ to be the induced Lebesgue measure on it.

To define N, we pick an arbitrary positive R_0, and we identify N with the interval $[0, R_0]$, giving the space N the standard Lebesgue measure. The family $T_z(\cdot)$ is now defined by

$$T_z(f)(R) = U_R^\lambda(0, f), \tag{16.2}$$

where
$$\lambda = \lambda(z) = (p-1)/2 - z \cdot p/4. \tag{16.3}$$

In defining $U_R^\lambda(0, f)$ we have set $f = 0$, for $|x| \leqslant \varepsilon$. Following the notation of Lemma 1, we shall let $p_1 = 1$, $p_2 = 2$, $q_1 = q_2 = q = \infty$. t will be the parameter so that $0 < t < 1$, and $1/p = 1 - t + t/2$.

(We should point out here that it will be important that estimates that follow are made independently of R_0. At the conclusion of the proof we shall let $R_0 \to \infty$.)

It is an easy matter to verify that the family of operators T_z is an analytic family in the sense of (i), (ii), and (iii) of § 8.

We shall also use the following notation, which should not lead to confusion:

$$\|f\|_p = \Big(\int_M |f(x)|^p \, d\mu\Big)^{1/p} = \Big(\int_{Q \cap \{|x| > \varepsilon\}} |f(x)|^p \, dx\Big)^{1/p}.$$

We then claim the following bounds on $T_z(f)$:

$$\|T_{iy}(f)\|_\infty \leqslant A_0(y) \|f\|_1, \tag{16.4}$$

$$\|T_{1+iy}(f)\|_\infty \leqslant A_1(y) \|f\|_2, \tag{16.5}$$

where
$$A_0(y) \leqslant (A_\varepsilon/(p-1)) \, e^{\pi|y|}, \tag{16.6}$$

and
$$A_1(y) \leqslant A_\varepsilon \, e^{3\pi|y|/2}. \tag{16.7}$$

(16.4) and (16.6) follow from Theorem 3, since in this case $\lambda = \lambda(iy) = \frac{1}{2}(p-1) - iyp/4$ and $\frac{1}{2} > \Re(\lambda) = \frac{1}{2}(p-1) > 0$, while $I(\lambda) = -yp/4$.

(16.5) and (16.7) follow similarly from Theorem 2. It should be noted that the bounds above do not depend on R_0.

We may then apply Lemma 1 to (16.4) and (16.5). The conclusion is

$$\|T_t(f)\|_\infty \leqslant A_t \|f\|_p, \tag{16.8}$$

where
$$\log A_t = \int_{-\infty}^{+\infty} \log A_0(y) \, \omega(1-t, y) \, dy + \int_{-\infty}^{+\infty} \log A_1(y) \, \omega(t, y) \, dy. \tag{16.9}$$

Now observe that

$$T_t(f)(R) = U_R^{\lambda(t)}(0, f), \quad \text{and} \quad \lambda(t) = \tfrac{1}{2}(p-1) - t \cdot p/4.$$

Since $1/p = 1 - t/2$, then $\lambda(t) = 0$. Thus

$$T_t(f)(R) = U_R^0(0, f) = S_R'^{\frac{1}{2}(k-1)}(0).$$

Therefore (16.8) becomes

$$\sup_{0 < R \leqslant R_0} |S_R'^{\frac{1}{2}(k-1)}(0)| \leqslant A_t \|f\|_p. \tag{16.10}$$

To calculate the bound A_t we proceed as follows. We note first that

$$\int_{-\infty}^{+\infty} \omega(1-t, y)|y| \, dy \leqslant A; \qquad \int_{-\infty}^{+\infty} \omega(t, y)|y| \, dy \leqslant A,$$

where A is independent of t.

We also note that $\omega(t, y) \geqslant 0$ and

$$\int_{-\infty}^{+\infty} \omega(1-t, y) \, dy \leqslant 1; \qquad \int_{-\infty}^{+\infty} \omega(t, y) \, dy \leqslant 1.$$

This last fact follows from the fact that $\omega(t, y)$ is the "Poisson kernel" for the strip $0 \leqslant t \leqslant 1$, $-\infty < y < \infty$.

Thus using (16.6) we obtain

$$\log A_t \leqslant \log \frac{1}{p-1} + \log A_\varepsilon + \pi A + \log A_\varepsilon + \frac{3\pi}{2} A.$$

Therefore,
$$A_t \leqslant A_\varepsilon'/(p-1). \tag{16.11}$$

Combining (16.11) and (16.10) we get

$$\sup_{0 < R < R_0} |S_R'^{\frac{1}{2}(k-1)}(0)| \leqslant (A_\varepsilon'/(p-1)) \|f\|_p, \quad 1 < p < 2. \tag{16.12}$$

We note that the calculations made above for the bound A_t where independent of R_0, and therefore A_ε' in the above is independent of R_0. Letting $R_0 \to \infty$ we obtain

$$\sup_{R \geqslant 0} |S_R'^{\frac{1}{2}(k-1)}(0)| \leqslant (A_\varepsilon'/(p-1)) \|f\|_p, \quad 1 < p < 2,$$

whenever f is simple. A standard limit argument proves the above for general $f \in L^p(Q)$. By the remark made at the beginning of the proof, this suffices for the proof of the theorem.

As a consequence of the above theorem we obtain:

THEOREM 5. *Let* $1 \geqslant \varepsilon > 0$; *then there exists a constant* B_ε, *so that whenever* $f(x) \in L \log^+ L(Q)$, *and* $f(x) = 0$ *for* $|x| \leqslant \varepsilon$, *then*

$$\sup_{R\geqslant 0} |S_R^{\frac{1}{2}(k-1)}(0)| \geqslant B_\varepsilon \int_Q |f(x)| \log^+ |f(x)|\, dx + B_\varepsilon. \tag{16.13}$$

Proof. If we fix R, then Theorem 4 implies that for every simple f, vanishing in $|x|\leqslant\varepsilon$,

$$|S_R^{\frac{1}{2}(k-1)}(0)| \leqslant (A_\varepsilon/(p-1))\cdot\|f\|_p, \quad 1<p<2. \tag{16.14}$$

A_ε is independent of R, of course.

Now apply Lemma 2 to the above, where $T(f)=S_R^{\frac{1}{2}(k-1)}(0)$; the space N consists of the single point with measure 1; and $r=1$. Thus we obtain

$$|S_R^{\frac{1}{2}(k-1)}(0)| \leqslant B_\varepsilon \Big(\int_Q |f(x)| \log^+ |f(x)|\, dx + 1\Big), \tag{16.15}$$

where B_ε is independent of R. This concludes the proof of the theorem in the case when $f(x)$ is simple. The general case then follows by a standard limiting argument.

We are now in a position to prove our main result.

THEOREM 6. *Let $f(x)\in L\log^+ L(Q)$, and assume that $f(x)$ vanishes in a neighborhood of the point x_0. Then $\lim_{R\to\infty} S_R^{\frac{1}{2}(k-1)}(x_0, f)$ exists and is zero.*

Proof. We may, after translation, assume that $x_0=0$, and that our neighborhood contains the sphere $|x|\leqslant\varepsilon$, for some ε, $0<\varepsilon\leqslant 1$.

It will be sufficient to show that given any $\eta>0$, there exists an $R_0=R_0(\eta)$, so that

$$|S_R^{\frac{1}{2}(k-1)}(0,f)|<\eta, \quad \text{whenever } R>R_0(\eta).$$

Now for any ξ, $0<\xi$, (16.3) may be rewritten as

$$\sup_{R\geqslant 0} |S_R^{\frac{1}{2}(k-1)}(0)| \leqslant B_\varepsilon \int_Q |f(x)| \log^+ \left|\frac{f(x)}{\xi}\right| dx + \xi B_\varepsilon. \tag{16.16}$$

This follows by writing $f(x)/\xi$ instead of $f(x)$ in (16.3). Now choose ξ no small that $\xi B_\varepsilon<\eta/3$, and keep ξ fixed.

Next write $f(x)=f_1(x)+f_2(x)$ where $f_1(x)\in C^\infty$ and $f_1(x)$ and $f_2(x)$ vanishes in the ε-neighborhood of 0; and

$$B_\varepsilon \int_Q |f_2(x)| \log^+ \left(\frac{|f_2(x)|}{\xi}\right) dx < \eta/3.$$

Now $$S_R^{\frac{1}{2}(k-1)}(0,f) = S_R^{\frac{1}{2}(k-1)}(0,f_1) + S_R^{\frac{1}{2}(k-1)}(0,f_2).$$

Because f_1 is sufficiently smooth and $f(0)=0$, then $S_R^{\frac{1}{2}(k-1)}(0, f_1) \to 0$, as $R \to \infty$. Thus

$$|S_R^{\frac{1}{2}(k-1)}(0, f)| < \eta/3, \text{ if } R > R_0(\eta).$$

However, by applying (16.16) to $f_2(x)$ we obtain $|S_R^{\frac{1}{2}(k-1)}(0, f_2)| < 2\eta/3$. Combining these two, we get: $|S_R^{\frac{1}{2}(k-1)}(0, f)| < \eta$, whenever $R > R_0(\eta)$. This concludes the proof of Theorem 6.

CHAPTER III

Dominated Summability

17. An L^1 estimate for dominated summability

With
$$S_R^\delta(x, f) = \sum_{|n|<R} \left(1 - \frac{|n|^2}{R^2}\right)^\delta a_n e^{in\cdot x},$$

where
$$a_n = (2\pi)^{-k} \int_{Q_k} f(x) e^{-in\cdot x} dx,$$

we define $S_*^\delta(x, f)$ by
$$S_*^\delta(x, f) = \sup_{R>0} |S_R^\delta(x, f)|. \tag{17.1}$$

The result of this section is contained in the following:

LEMMA 12. *Let $f(x) \in L^1(Q)$, and let $\Re(\delta) > \frac{1}{2}(k-1)$. Also let $f^*(x)$ be the maximal function defined in Lemma 3. Then*

(a) $S_*^\delta(x, f) \leqslant A_\sigma e^{\pi|\tau|} (\sigma - \frac{1}{2}(k-1))^{-1} f^*(x)$,

(b) $\int_{Q_k} |S_R^\delta(x, f)| dx \leqslant A_\sigma e^{\pi|\tau|} (\sigma - \frac{1}{2}(k-1))^{-1} \int_{Q_k} |f(x)| dx \quad (\delta = \sigma + i\tau)$,

where A_σ is independent of R and f, and A_σ remains bounded as $\sigma \to \frac{1}{2}(k-1)$. (1)

Proof. According to Lemma 5

$$S_R^\delta(x, f) = c_1 R^{\frac{1}{2}k-\delta} \int_0^\infty f(x; t) t^{\frac{1}{2}k-\delta-1} J_{\delta+\frac{1}{2}k}(tR) dt$$

with
$$c_1 = 2^{\delta-\frac{1}{2}k+1} \Gamma(\delta+1)/\Gamma(\tfrac{1}{2}k).$$

We break up the above integral into two integrals corresponding respectively to the intervals $(0, 1/R)$, and $(1/R, \infty)$. Using the estimate (11.11) for the first integral, and (11.10) for the second integral, we then easily obtain

(1) See footnote p. 109.

$$|S_R^\delta(x, f)| \leqslant A_\sigma e^{\pi|\tau|} \{R^k \int_0^{1/R} |f(x; t)| t^{k-1} dt + R^{\frac{1}{2}(k-1)-\sigma} \int_{1/R}^{\infty} |f(x; t)| t^{\frac{1}{2}(k-1)-\sigma-1} dt\}. \quad (17.2)$$

(We note here that A_σ remains bounded as long as $\sigma \geqslant 0$, and σ is restricted to some finite interval.)

Now by definition of $f^*(x)$ given in (10.3), we have

$$R^k \int_0^{1/R} |f(x; t)| t^{k-1} dt \leqslant f^*(x) k^{-1}.$$

Moreover, a simple argument of integration by parts gives

$$R^{\frac{1}{2}(k-1)-\sigma} \int_{1/R}^{\infty} |f(x; t)| t^{\frac{1}{2}(k-1)-\sigma-1} dt \leqslant A [\sigma - \tfrac{1}{2}(k-1)]^{-1} f^*(x) \quad \text{for } \sigma > \tfrac{1}{2}(k-1).$$

If we combine these two estimates in (17.2) we obtain

$$S_*^\delta(x, f) = \sup_R |S_R^\delta(x, f)| \leqslant A_\sigma e^{\pi|\tau|} [\sigma - \tfrac{1}{2}(k-1)]^{-1} f^*(x),$$

with A_σ bounded as $\sigma \to \frac{1}{2}(k-1)$. This proves part (a).

In order to prove part (b), we may rewrite (17.2) in terms of a convolution with an integrable kernel.

In effect we have

$$|S_R^\delta(x, f)| \leqslant A_\sigma e^{\pi|\tau|} R^k \int_{E_k} |f(x-y)| \phi(R|y|) dy, \quad (17.3)$$

where $\phi(u) = 1$, if $0 \leqslant u < 1$, and $\phi(u) = u^{-\sigma-\frac{1}{2}k+\frac{1}{2}}$, if $1 \leqslant u$.

Integrating (17.3) with respect to x and inverting the order of integration, we obtain

$$\int_{Q_k} |S_R^\delta(x, f)| dx \leqslant A_\sigma e^{|\pi|\tau} \left\{ \int_{Q_k} |f(x)| dx \right\} R^k \int_{E_k} \phi(R|y|) dy.$$

However, $$R^k \int_{E_k} \phi(R|y|) dy = \omega_k \int_0^\infty \phi(t) t^{k-1} dt \leqslant A [\sigma - \tfrac{1}{2}(k-1)]^{-1}.$$

Thus part (b) is also proved, and the proof of the lemma is complete.

18. An L^2 estimate

The result of this section is contained in the following theorem.

THEOREM 7: *Let $f(x) \in L^2(Q)$; and $\delta = \sigma + i\tau$. Then*

(a) $\left(\int_Q |S_*^\delta(x, f)|^2 dx\right)^{\frac{1}{2}} \leqslant A_\sigma e^{\pi|\tau|} \left(\int_Q |f(x)|^2 dx\right)^{\frac{1}{2}}$ *for* $\sigma > 0$,

(b) $\left(\int_Q |S_R^\delta(x, f)|^2 dx\right)^{\frac{1}{2}} \leqslant \left(\int_Q |f(x)|^2 dx\right)^{\frac{1}{2}}$, *for* $\sigma \geqslant 0$.

Proof. We introduce the following two auxiliary functions:

$$\Omega_\delta(x, f) = \left(\int_0^\infty \frac{|S_R^\delta(x, f) - S_R^{\delta-1}(x, f)|^2}{R} dR\right)^{\frac{1}{2}}, \tag{18.1}$$

and

$$\Lambda_\delta(x, f) = \sup_{R\geq 0} \left\{\frac{1}{R}\int_0^R |S_u^{\delta-1}(x, f)|^2 du\right\}^{\frac{1}{2}}. \tag{18.2}$$

The proof of the theorem will be a consequence of the following lemma.

LEMMA 13. *Let $f(x) \in L^2(Q)$, $\Re(\delta) > \frac{1}{2}$, $(\delta = \sigma + i\tau)$. Then $\Omega_\delta(x, f)$ and $\Lambda_\delta(x, f)$ are finite almost everywhere; moreover*

$$\left(\int_Q [\Omega_\delta(x, f)]^2 dx\right)^{\frac{1}{2}} \leq A_\sigma \left(\int_Q |f(x)|^2 dx\right)^{\frac{1}{2}} \tag{18.3}$$

and

$$\left(\int_Q [\Lambda_\delta(x, f)]^2 dx\right)^{\frac{1}{2}} \leq A_\sigma e^{\pi|\tau|} \left(\int_Q |f(x)|^2 dx\right)^{\frac{1}{2}}. \tag{18.4}$$

Proof of the lemma: We consider Ω_δ first. Since

$$[\Omega_\delta(x, f)]^2 = \int_0^\infty \frac{|S_R^\delta(x, f) - S_R^{\delta-1}(x, f)|^2}{R} dR,$$

We integrate with respect to x and interchange the order of integration. Thus

$$\int_Q [\Omega_\delta(x, f)]^2 dx = \int_0^\infty \left\{\int_Q |S_R^\delta(x, f) - S_R^{\delta-1}(x, f)|^2 dx\right\} \frac{dR}{R}. \tag{18.5}$$

We now evaluate the inner integral by Parseval's formula.

$$\begin{aligned} S_R^\delta(x, f) - S_R^{\delta-1}(x, f) &= \sum_{|n|<R} \left[\left(1 - \frac{|n|^2}{R^2}\right)^\delta - \left(1 - \frac{|n|^2}{R^2}\right)^{\delta-1}\right] a_n e^{in\cdot x} \\ &= \sum_{|n|<R} \left(1 - \frac{|n|^2}{R^2}\right)^{\delta-1} \frac{|n|^2}{R^2} a_n e^{in\cdot x}. \end{aligned}$$

Therefore,

$$(2\pi)^{-k} \int_Q |S_R^\delta(x, f) - S_R^{\delta-1}(x, f)|^2 dx = \sum_{|n|<R} \left(1 - \frac{|n|^2}{R^2}\right)^{2\sigma-2} \frac{|n|^4}{R^4} |a_n|^2 \tag{18.6}$$

Assume first that $\sigma \geq 1$. Then by the above

$$(2\pi)^{-k} \int_Q |S_R^\delta(x, f) - S_R^{\delta-1}(x, f)|^2 dx \leq R^{-4} \sum_{|n|<R} |n|^4 |a_n|^2.$$

Substituting this in (18.5) we then have

$$\int_a [\Omega_\delta(x, f)]^2\,dx \leqslant (2\pi)^k \int_0^\infty \left\{\sum_{|n|<R} |n|^4 |a_n|^2\right\} \frac{dR}{R^5}$$

$$= (2\pi)^k \sum |a_n|^2 |n|^4 \cdot \left(\int_{|n|}^\infty \frac{dR}{R^5}\right)$$

$$\leqslant A \sum |a_n|^2.$$

This gives $$\int_Q [\Omega_\delta(x, f)]^2\,dx \leqslant A \int_Q |f(x)|^2\,dx, \quad \text{if } \sigma \geqslant 1,$$

and therefore (18.3) is proved in this case.

Assume now $\sigma < 1$. Then

$$\left(1 - \frac{|n|^2}{R^2}\right)^{2\sigma-2} \leqslant A, \quad \text{if } |n| \leqslant \frac{R}{2}.$$

Using (18.6) we obtain

$$(2\pi)^{-k} \int_Q |S_R^\delta(x, f) - S_R^{\delta-1}(x, f)|^2\,dx$$

$$\leqslant A \sum_{|n|\leqslant \frac{1}{2}R} \frac{|n|^4}{R^4} |a_n|^2 + \sum_{\frac{1}{2}R<|n|<R} \left(1 - \frac{|n|^2}{R^2}\right)^{2\sigma-2} \frac{|n|^4}{R^4} |a_n|^2$$

$$= \sum\nolimits_1 + \sum\nolimits_2, \text{ say.}$$

By the argument of interchange of order of integration used above, we may see that

$$\int_0^\infty \left\{\sum\nolimits_1\right\} \frac{dR}{R} \leqslant A \sum |a_n|^2 = A \int_Q |f(x)|^2\,dx. \tag{18.7}$$

Now $$\sum\nolimits_2 = \sum_{\frac{1}{2}R<|n|<R} \left(1 - \frac{|n|^2}{R^2}\right)^{2\sigma-2} \frac{|n|^4}{R^4} |a_n|^2$$

$$\leqslant R^{4-4\sigma} \sum_{\frac{1}{2}R<|n|<R} (R^2 - n^2)^{2\sigma-2} |a_n|^2$$

$$\leqslant R^{2-2\sigma} \sum_{\frac{1}{2}R<|n|<R} (R - |n|)^{2\sigma-2} |a_n|^2.$$

Therefore, $$\int_0^\infty \left\{\sum\nolimits_2\right\} \frac{dR}{R} \leqslant \int_0^\infty R^{2\sigma-2} \left\{\sum_{\frac{1}{2}R<|n|<R} (R - |n|)^{2\sigma-2} |a_n|^2\right\} \frac{dR}{R}$$

$$= \sum |a_n|^2 \cdot \left\{\int_{|n|}^{2|n|} R^{1-2\sigma} (R - |n|)^{2\sigma-2}\,dR\right\}.$$

But it is easy to check that

$$\int_{|n|}^{2|n|} R^{1-2\sigma}(R-|n|)^{2\sigma-2}\,dR \leqslant A(\sigma-\tfrac{1}{2})^{-1} = A_\sigma, \quad \text{if } \sigma > \tfrac{1}{2}.$$

Hence
$$\int_0^\infty \{\textstyle\sum_2\} \frac{dR}{R} \leqslant A_\sigma \sum |a_n|^2 = A_\sigma \int_Q |f(x)|^2\,dx, \quad \sigma > \tfrac{1}{2}. \tag{18.8}$$

A combination of (18.7) and (18.8) proves (18.3) when $\frac{1}{2} < \sigma \leqslant 1$. Thus (18.3) is completely proved. The finiteness almost everywhere of $\Omega_\delta(x, f)$ follows from (18.3), of course.

We now consider Λ_δ. Let $\nu = [\frac{1}{2}(k-1)] =$ greatest integer in $\frac{1}{2}(k-1)$. Then by Minkowski's inequality,

$$\begin{aligned}
&\left\{\frac{1}{R}\int_0^R |S_u^{\delta-1}(x, f)|^2\,du\right\}^{\frac{1}{2}} \\
&\quad \leqslant \left\{\frac{1}{R}\int_0^R |S_u^{\delta-1}(x, f) - S_u^{\delta+\nu}(x, f)|^2\,du\right\}^{\frac{1}{2}} + \left\{\frac{1}{R}\int_0^R |S_u^{\delta+\nu}(x, f)|^2\,du\right\}^{\frac{1}{2}} \\
&\quad = I_1 + I_2.
\end{aligned} \tag{18.9}$$

Since $\sigma = R(\delta) > \frac{1}{2}$, then $R(\delta+\nu) > \frac{1}{2}(k-1)$, and we may apply Lemma 12, part (a) to $S_u^{\delta+\nu}(x, f)$. This gives

$$|S_R^{\delta+\nu}(x, f)| \leqslant A_\sigma e^{\pi|\tau|} f^*(x).$$

Substituting this estimate in the second term of (18.9) gives

$$I_2 \leqslant A_\sigma e^{\pi|\tau|} f^*(x). \tag{18.10}$$

Moreover,
$$\begin{aligned}
I_1 &= \left\{\frac{1}{R}\int_0^R (S_u^{\delta-1}(x, f) - S_u^{\delta+\nu}(x, f)|^2\,du\right\}^{\frac{1}{2}} \\
&\leqslant \left(\int_0^\infty \frac{|S_R^{\delta-1}(x, f) - S_R^{\delta+\nu}(x, f)|^2}{R}\,dR\right)^{\frac{1}{2}} \\
&\leqslant \Omega_\delta(x, f) + \Omega_{\delta+1}(x, f) + \cdots + \Omega_{\delta+\nu}(x, f).
\end{aligned} \tag{18.11}$$

When we combine (18.9), (18.10) and (18.11) we obtain

$$\Lambda_\delta(x, f) \leqslant \Omega_\delta(x, f) + \cdots + \Omega_{\delta+\nu}(x, f) + A_\sigma e^{\pi|\tau|} f^*(x). \tag{18.12}$$

To the above we apply (18.3) which we have already proved) successively to $\delta, \delta+1, \ldots, \delta+\nu$. To the term containing $f^*(x)$ we apply Lemma 3. We therefore obtain

$$\Big(\int_Q \Lambda_\delta(x, f)]^2 dx\Big)^{\frac{1}{2}} \leqslant A_\sigma e^{\pi|\tau|}\Big(\int_Q |f(x)|^2 dx\Big)^{\frac{1}{2}}.$$

This completes the proof of Lemma 13.

Proof of theorem 7. We consider (a) first. We shall make use of Lemma 4 (of §11), recalling that $R^{2\delta} S_R^\delta = A_R^\delta$. For δ, β appearing in the statement of Lemma 4, we substitute $\frac{1}{2}(\sigma-1)$, $\frac{1}{2}(\sigma+1)+i\tau$, respectively. We thus have:

$$R^{2\delta} S_R^\delta = \frac{2\Gamma(\sigma+i\tau+1)}{\Gamma(\frac{1}{2}(\sigma+1))\Gamma(\frac{1}{2}(\sigma+1)+i\tau)} \cdot \int_0^R (R^2-t^2)^{\frac{1}{2}(\sigma-1)+i\beta} t^{2\sigma} S_t^{\frac{1}{2}(\sigma-1)} dt. \tag{18.13}$$

If $\sigma>0$, the factor involving the Γ functions is certainly bounded by $A_\sigma e^{\pi|\tau|}$. Thus from (18.13) we obtain

$$|S_R^\delta(x, f)| \leqslant A_\sigma e^{\pi|\tau|} R^{-2\sigma}\int_0^R (R^2-t^2)^{\frac{1}{2}(\sigma-1)} t^{2\sigma} |S_t^{\frac{1}{2}(\sigma-1)}(x, f)|\, dt.$$

Applying now Schwarz's inequality tn the above we obtain

$$|S_R^\delta(x, f)| \leqslant A_\tau e^{\pi|\tau|} R^{-2\sigma}\Big\{\int_0^R (R^2-t^2)^{\sigma-1} t^{4\sigma} dt\Big\}^{\frac{1}{2}} \Big\{\int_0^R |S_t^{\frac{1}{2}(\sigma-1)}(x, f)|^2 dt\Big\}^{\frac{1}{2}}.$$

Therefore, $$|S_R^\delta(x, f)| \leqslant A_\sigma e^{\pi|\tau|} \cdot B(R) \cdot \Lambda_{\frac{1}{2}(\sigma+1)}(x, f),$$

where
$$\begin{aligned} B(R) &= R^{-2\sigma+\frac{1}{2}} \Big\{\int_0^R (R^2-t^2)^{\sigma-1} t^{4\sigma} dt\Big\}^{\frac{1}{2}} \\ &= \Big\{\int_0^1 (1-t^2)^{\sigma-1} t^{4\sigma} dt\Big\}^{\frac{1}{2}} \\ &\leqslant A/\sigma = A_\sigma, \quad \text{if } \sigma>0. \end{aligned}$$

We thus have

$$S_*^\delta(x, f) = \sup_R |S_R^\delta(x, f)| \leqslant A_\sigma e^{\pi|\tau|} \Lambda_{\frac{1}{2}(\sigma+1)}(x, f). \tag{18.14}$$

Finally then

$$\int_Q [S_*(x, f)]^2 dx \leqslant A_\sigma^2 e^{2\pi|\tau|}\int_Q [\Lambda_{\frac{1}{2}(\sigma+1)}(x, f)]^2 dx \leqslant A_\sigma^2 e^{2\pi|\tau|}\int_Q |f(x)|^2 dx,$$

by applying (18.4) of Lemma 13 (to the case $\frac{1}{2}(\sigma+1)$, since $\frac{1}{2}(\sigma+1)>\frac{1}{2}$, when $\sigma>0$).

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This proves part (a) of Theorem 7.

Part (b) is proved by observing when $\mathfrak{R}(\delta) \geqslant 0$, that $S_R^\delta(x, f)$ multiplies the Fourier coefficients of $f(x)$ by constants of absolute value not exceeding one.

19. Proof of dominated, pointwise, and norm summability

(a) Poof of Theorem (D), (see § 5, for statement).

The idea of the proof is as follows. We notice that the case which corresponds to $p=2$ has already been disposed of in Theorem 7, part (a), of § 18. We should like to have an analogous result for $p=1$, and then interpolate between indices $p=2$, and $p=1$. However, Theorem (D) fails when $p=1$, so that we must content ourselves with a weaker substitute. Such a substitute result, satisfactory for our purposes, is contained in Lemma 12, part (a), of § 17.

Now to the proof. Let p and δ be the indices given in Theorem (D). Assume that $1 < p < 2$, since the case $p=2$ is contained in Theorem 7. Let p_1 be an exponent (to be determined later) which satisfies $1 < p_1 < p$. Thus we may write

$$p_1 = \frac{1}{1-\eta}, \quad 0 < \eta < 1 - \frac{1}{p} \cdot \tag{19.1}$$

Choose two further parameters, ε_0, and ε_1 (to be determined later) subject presently only to the conditions

$$0 < \varepsilon_0 < \infty, \quad \text{and} \quad 0 < \varepsilon_1 < \infty.$$

Define p_0, by $p_0 = 2$. Thus $p_1 < p < p_0$, and we may therefore write

$$1/p = (1-t)/p_0 + t/p_1, \quad 0 < t < 1.$$

Using (19.1) we obtain after a simple calculation

$$t = \left(\frac{2}{p} - 1\right)(1 - 2\eta)^{-1}. \tag{19.2}$$

Now define $\delta(z)$ by

$$\delta(z) = \varepsilon_0(1-z) + (\tfrac{1}{2}(k-1) + \varepsilon_1)z. \tag{19.3}$$

We show now that our arbitrary parameters η, ε_0, ε_1 can be chosen so that

$$\delta(t) = \delta, \tag{19.4}$$

(where δ is the index given in the statement of the theorem). In fact, using (19.2), we may write

$$\delta(t)=\left(\frac{k-1}{2}\right)\left(\frac{2}{p}-1\right)+\left(\frac{k-1}{2}\right)\left(\frac{2}{p}-1\right)\frac{2\eta}{1-2\eta}+\varepsilon_1 t+\varepsilon_0(1-t)$$
$$=\left(\frac{k-1}{2}\right)\left(\frac{2}{p}-1\right)+E(\eta)+E'(\varepsilon_1)+E''(\varepsilon_0).$$

However, by assumption, $\delta>(\frac{1}{2}(k-1))(2/p-1)$. Thus we may find an η small enough so that still $\delta>(\frac{1}{2}(k-1))(2/p-1)+E(\eta)$. We fix such an η. This determines p_1 (by (19.1)), and t, (by (19.2)). However, $0<t<1$, thus we may find ε_0, and ε_1 so that

$$\delta(t)=\frac{k-1}{2}\left(\frac{2}{p}-1\right)+E(\eta)+E'(\varepsilon_1)+E''(\varepsilon_0).$$

This proves (19.4), and we proceed with the parameters η, ε_0, ε_1 fixed in this manner.

Let $R(x)$ be a measurable function defined on Q_k subject only to the conditions that:

$$0\leqslant R(x)\leqslant R_0<\infty. \tag{19.5}$$

With the aid of $\delta(t)$ and $R(x)$ we now define an analytic family of transformations, $T_z(\cdot)$, as follows:

$$T_z(f)(x)=S^{\delta(z)}_{R(x)}(x, f), \tag{19.6}$$

and we verify that the family satisfies the conditions of Lemma 1 of § 8. That the conditions (i), (ii) and (iii) are satisfied follows easily when one makes use of the restriction (19.5). We next claim that $T_z(\cdot)$ satisfies the following bounds

$$\left.\begin{aligned}\|T_{iy}(f)\|_{p_0}&\leqslant A_0(y)\|f\|_{p_0},\\ \|T_{1+iy}(f)\|_{p_1}&\leqslant A_1(y)\|f\|_{p_1},\end{aligned}\right\} \tag{19.7}$$

where

$$A_i(y)\leqslant A_i\, e^{a_i|y|},\quad i=0,\,1. \tag{19.8}$$

and where the A_i and a_i do not depend on the function $R(x)$ and R_0.

In fact, by (19.6) it follows immediately that:

$$|T_z(f)(x)|\leqslant S^{\delta(z)}_*(x, f). \tag{19.9}$$

Now

$$\delta(iy)=\varepsilon_0(1-iy)+(\tfrac{1}{2}(k-1)+\varepsilon_1)\,iy.$$

Thus $\Re(\delta(iy))=\varepsilon_0>0$. We may therefore apply Theorem 7, part (a) and obtain

$$\|T_{iy}(f)\|_{p_0}=\|T_{iy}(f)\|_2\leqslant\|S^{\delta(iy)}_*(x, f)\|_2\leqslant A_0\, e^{a_0|y|}\,\|f\|_2=A_0\, e^{a_0|y|}\,\|f\|_{p_0}, \tag{19.10}$$

where $$a_0 = \pi \left| \tfrac{1}{2}(k-1) + \varepsilon_1 - \varepsilon_0 \right|.$$

(The constant A_0 clearly does not depend on $R(x)$ or R_0.) Next,

$$\Re(\delta(1+iy)) = -\varepsilon_0(iy) + (\tfrac{1}{2}(k-1) + \varepsilon_1)(1+iy).$$

Hence, $$\Re(\delta(1+iy)) = \tfrac{1}{2}(k-1) + \varepsilon_1 > \tfrac{1}{2}(k-1).$$

We may thus use Lemma 12, part (a), and obtain

$$\| T_{1+iy}(f) \|_{p_1} \leqslant \| S_*^{(1+iy)}(f) \|_{p_1} \leqslant A\, e^{a_1|y|} \| f^* \|_{p_1},$$

where $$a_1 = \pi \left| \tfrac{1}{2}(k-1) + \varepsilon_1 - \varepsilon_0 \right|.$$

Since $1 < p_1$, we may apply Lemma 3 of § 10. Therefore,

$$\| T_{1+iy}(f) \|_{p_1} \leqslant A_1\, e^{a_1|y|} \| f \|_{p_1}. \tag{19.11}$$

(The constant A_1 again clearly does not depend on $R(x)$ and R_0.) This establishes (19.7) and (19.8).

Using the interpolation lemma, we thus obtain

$$\| T_t(f) \|_p \leqslant A_t \| f \|_p. \tag{19.12}$$

Now by (8.5) of Lemma 1 the constant A_1 appearing in (19.12) depends only on the $A_t(y)$ of (19.7). Since these latter are independent of $R(x)$ and R_0, the same holds for A_t. However, (19.12) may be rewritten as

$$\left(\int_{Q_k} | S_{R(x)}^{\delta(t)}(x, f) |^p \, dx \right)^{1/p} \leqslant A_t \left(\int_{Q_k} | f(x) |^p \, dx \right)^{1/p} \tag{19.13}$$

and by (19.4) $$\delta(t) = \delta.$$

Thus we have $$\left(\int_{Q_k} | S_{R(x)}^{\delta}(x, f) |^p \, dx \right)^{1/p} \leqslant A_t \left(\int_{Q_k} | f(x) |^p \, dx \right)^{1/p}$$

with $R(x)$ subject only to the condition $0 \leqslant R(x) \leqslant R_0 < \infty$, and with A_t independent of $R(x)$ and R_0. By an appropriate choice of $R(x)$ we deduce

$$\left(\int_{Q_k} \left(\sup_{0 \leqslant R \leqslant R_0} | S_R^{\delta}(x, f) | \right)^p dx \right)^{1/p} \leqslant A_t \left(\int_{Q_k} | f(x) |^p \right)^{1/p}.$$

Now since the integrand of the left-hand side increases with R_0, we obtain

$$\left(\int_{Q_k} \left(\sup_{0 \leqslant R < \infty} | S_R^{\delta}(x, f) | \right)^p dx \right)^{1/p} \leqslant A_t \left(\int_{Q_k} | f(x) |^p \, dx \right)^{1/p}.$$

This concludes the proof of Theorem (D).

(b) Proof of Theorem (D*) (see § 5 for statement).

Theorem (D*) will be a consequence of the following lemma.

LEMMA 14. *Let* $f(x) \in L^p(Q_k)$, $1 < p \leqslant 2$. *Then,*

$$\left(\int_{Q_k} [S_*^{\frac{1}{2}(k-1)}(x, f)]^p \, dx\right)^{1/p} \leqslant A\,(p-1)^{-2} \left(\int_{Q_k} |f(x)|^p \, dx\right)^{1/p}, \tag{19.14}$$

where A *does not depend on* p *or* f.

Proof. This lemma is already contained implicitly in the proof of Theorem (D) above. In fact, for our given p, $1 < p \leqslant 2$, fix the index p_1, $1 < p_1 < p$, by

$$1 - \frac{1}{p_1} = \frac{1}{2}\left(1 - \frac{1}{p}\right). \tag{19.15}$$

Thus if $1/p = (1-t)/p_0 + t/p_0$, $(p_0 = 2)$, then

$$t = 2 - p, \quad \text{and} \quad 1 - t = p - 1. \tag{19.16}$$

Now define $\delta(z)$ by

$$\delta(z) = \left(\frac{k-1}{2} - \left(\frac{2-p}{4}\right)\right)(1-z) + \left(\frac{k-1}{2} + \left(\frac{p-1}{4}\right)\right) z. \tag{19.17}$$

As is easily verified,

$$\delta(t) = \frac{k-1}{2}. \tag{19.18}$$

Next define the family $T_z(\cdot)$, as before, by

$$T_z(f)(x) = S_{R(x)}^{\delta(z)}(x, f). \tag{19.20}$$

$R(x)$ is again an arbitrary measurable function limited only by

$$0 \leqslant R(x) \leqslant R_0 < \infty.$$

We then show that T_z obey bounds as in (19.7) with

$$A_0(y) \leqslant A\, e^{a|y|} \tag{19.21}$$

and

$$A_1(y) \leqslant A\, e^{a|y|} \cdot (p-1)^{-2}. \tag{19.22}$$

A and a do not depend on $R(x)$, R_0, or p. First,

$$\delta(iy) = \left(\frac{k-1}{2} - \frac{2-p}{4}\right)(1 - iy) + \left(\frac{k-1}{2} + \frac{p-1}{4}\right) iy.$$

Thus
$$\Re(\delta(iy)) = \frac{k-1}{2} - \frac{2-p}{4} \geqslant \frac{k-1}{2} - \frac{1}{4} \geqslant \frac{1}{4},$$

since $k \geqslant 2$. Therefore,

$$\| T_{iy}(f) \|_2 \leqslant \| S_*^{\delta(iy)}(x, f) \|_2 \leqslant A\, e^{a|y|} \| f \|_2, \tag{19.23}$$

with $a = \frac{1}{2}\pi$, by Theorem 7, part (a).

Secondly,
$$\delta(1+ip) = \left(\frac{k-1}{2} - \frac{2-p}{4}\right)(-iy) + \left(\frac{k-1}{2} + \frac{p-1}{4}\right)(1+iy).$$

Hence,
$$\Re(\delta(1+iy)) = \frac{k-1}{2} + \frac{p-1}{4} > \frac{k-1}{2}.$$

Then by Lemma 12, part (a),

$$\| T_{1+iy}(f) \|_{p_1} \leqslant \| S_*^{\delta(1+iy)}(x, f) \|_{p_1} \leqslant A\, e^{a|y|} (p-1)^{-1} \| f^* \|_p,$$

with $a = \frac{1}{2}\pi$. While by Lemma 3 of § 10,

$$\| f^* \|_{p_1} \leqslant A (p_1 - 1)^{-1} \| f \|_{p_1} \leqslant A' (p-1)^{-1} \| f \|_{p_1},$$

(since $(p_1 - 1)^{-1} \leqslant 2(p-1)^{-1}$ by (19.15)).

Combining the estimate we obtain:

$$\| T_{1+iy}(f) \|_{p_1} \leqslant A\, e^{a|y|} (p-1)^{-2} \| f \|_{p_1}.$$

Therefore the estimates (19.21) and (19.22) are established. Again, as in the proof of Theorem D above, we now apply the interpolation argument (Lemma 1). Following a similar argument, the result is

$$\left(\int_{Q_k} [S_*^{\frac{1}{2}(k-1)}(x, f)]^p \, dx \right)^{1/p} \leqslant A_t \left(\int_{Q_k} |f(x)|^p \, dx \right)^{1/p}, \tag{19.24}$$

where the A_t is given by

$$\log A_t = \int_{-\infty}^{+\infty} \omega(1-t, y) \log A_0(y)\, dy + \int_{-\infty}^{+\infty} \omega(t, y) \log A_1(y)\, dy.$$

We then use the fact (see § 16) that $\omega(t, y) \geqslant 0$,

$$\int_{-\infty}^{+\infty} \omega(t, y)\, dy \leqslant 1, \quad \text{and} \quad \int_{-\infty}^{+\infty} \omega(t, y) |y| \, dy \leqslant A.$$

We therefore have (using (19.21) and (19.22))

$$\log A_t \leqslant B + 2 \log \left(\frac{1}{p-1}\right);$$

therefore, $$A_t \leqslant e^B (p-1)^{-2}.$$

Applying this estimate in (19.24), completes the proof of Lemma 14.

Now Hölder's inequality applied to (19.14) gives

$$\int_{Q_k} |S_*^{\frac{1}{2}(k-1)}(x, f)|\, dx \leqslant A\,(2\pi)^{k(1-1/p)}(p-1)^{-2}\Big(\int_{Q_k} |f(x)|^p\, dx\Big)^{1/p}, \quad 1 < p \leqslant 2.$$

We then apply the case $r=2$ of Lemma 2 of § 10 to the above inequality. The result is

$$\int_{Q_k} S_*^{\frac{1}{2}(k-1)}(x, f)\, dx \leqslant B \int_{Q_k} |f(x)|\,(\log^+ |f(x)|)^2\, dx + B,$$

which proves Theorem (D*).

(c) **Proof of Theorem (AE.)** (see § 5 for statement).

Let $f(x) \in L^p(Q_k)$. Then we must show that given any $\varepsilon > 0$, there exists a positive number R_ε, and a set $E_\varepsilon \subset Q_k$ so that $m(E_\varepsilon) < \varepsilon$, and so that

$$|S_R^\delta(x, f) - f(x)| < \varepsilon, \text{ if } R > R_\varepsilon, \text{ and } x \notin E_\varepsilon.$$

Let us first fix the constant $A_{p,\delta}$ which occurs in the statement of Theorem (D) Write

$$f(x) = f_1(x) + f_2(x),$$

where $f_1(x)$ is periodic and in C^∞, and $\|f_2(x)\|_p < \eta$, η to be determined momentarily. Now by the inequality of Theorem (D) we obtain

$$\|S_*^\delta(x, f_2)\|_p \leqslant A_{p,\delta}\cdot\eta.$$

Hence we may choose η so small so that the set where either $|f_2(x)| \geqslant \frac{1}{3}\varepsilon$, or $S_*(x, f_2) \geqslant \frac{1}{3}\varepsilon$ is of measure less than ε. Write

$$S_R^\delta(x, f) - f(x) = \{S_R^\delta(x, f_1) - f_1(x)\} + \{S_R^\delta(x, f_2) - f_2(x)\}.$$

Thus, $$|S_R^\delta(x, f) - f(x)| \leqslant |S_R^\delta(x, f_1) - f_1(x)| + |S_R^\delta(x, f_2)| + |f_2(x)|. \tag{19.25}$$

Since $f_1(x)$ is C^∞ and periodic, then $S_R(x, f_1) \to f_1(x)$ uniformly. Therefore $|S_R^\delta(x, f_1) - f_1(x)| < \frac{1}{3}\varepsilon$ if $R \geqslant R_\varepsilon$. Since

$$|S_R^\delta(x, f_2)| + |f_2^\delta(x)| \leqslant S_*^\delta(x, f_2) + |f_2(x)|,$$

then
$$|S_R^\delta(x, f_2)| + |f_2(x)| \leqslant \frac{2\varepsilon}{3},$$

except in a set E_ε with $m(E_\varepsilon) < \varepsilon$.

Combining this with (19.25) concludes the proof of the theorem.

(d) Proof of Theorem (AE*) (see § 5 for statement).

If $f \in L(\log L)^2$, then according to Theorem (D*) we have

$$\int_{Q_k} S_*^{\frac{1}{2}(k-1)}(x, f)\, dx \leqslant A \int_{Q_k} |f(x)|\, (\log^+ |f(x)|)^2\, dx + B.$$

Using the above for $f(x)/\xi$, instead of $f(x)$, we obtain

$$\int_{Q_k} S_*^{\frac{1}{2}(k-1)}(x, f)\, dx \leqslant A \int_{Q_k} f(x) \left(\log^+ \left|\frac{f(x)}{\xi}\right|\right)^2 dx + \xi B. \tag{19.26}$$

Now choose ξ so small so that $\xi B < \varepsilon^2/12$, and keep ξ fixed. Since $f(x) \in L(\log L)^2$, we may write $f(x) = f_1(x) + f_2(x)$, where $f_1(x)$ is periodic and C^∞,

$$\int_{Q_k} f_2(x) \left(\log^+ \frac{|f_2(x)|}{\xi}\right)^2 dx < \eta,$$

and if E_1 is the set where $|f_2(x)| > \frac{1}{3}\varepsilon$, then $m(E_1) > \frac{1}{2}\varepsilon$. Choose now η, so that $A < \varepsilon^2/12$. Then by (19.26) applied to $f_2(x)$ instead of $f(x)$,

$$\int_{Q_k} S_*^{\frac{1}{2}(k-1)}(x, f_2)\, dx \leqslant A \cdot \frac{\varepsilon^2}{12} \cdot A^{-1} + B \cdot \frac{\varepsilon^2}{12} B^{-1} \leqslant \frac{\varepsilon^2}{6}.$$

Thus if E_2 is the set where $S_*^{\frac{1}{2}(k-1)}(x, f_2) > \frac{1}{3}\varepsilon$, then $m(E_2) < \frac{1}{2}\varepsilon$. Now

$$\begin{aligned}
&|S_R^{\frac{1}{2}(k-1)}(x, f) - f(x)| \\
&\quad \leqslant |S_R^{\frac{1}{2}(k-1)}(x, f_1) - f_1(x)| + |S_R^{\frac{1}{2}(k-1)}(x, f_2)| + |f_2(x)| \\
&\quad \leqslant |S_R^{\frac{1}{2}(k-1)}(x, f_1) - f_1(x)| + |S_*^{\frac{1}{2}(k-1)}(x, f_2)| + |f_2(x)| \\
&\quad \leqslant |S_R^{\frac{1}{2}(k-1)}(x, f_1) - f_1(x)| + \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon, \quad \text{if } x \notin E_1 \cup E_2.
\end{aligned}$$

However, $S_R^{\frac{1}{2}(k-1)}(x, f_1)$ converges uniformly to $f_1(x)$ as $R \to \infty$. Thus there exists an R_ε so that $|S_R^{\frac{1}{2}(k-1)}(x, f) - f_1(x)| < \frac{1}{3}\varepsilon$, whenever $R \geqslant R_\varepsilon$. Hence,

$$|S_R^{\frac{1}{2}(k-1)}(x, f) - f(x)| < \varepsilon, \quad \text{if } R \geqslant R_\varepsilon, \quad \text{and } x \notin E_1 \cup E_2.$$

However, $m(E_1 \cup E_2) < \varepsilon$, and this concludes the proof of Theorem (AE*).

(e) Proof of Theorem (N) (see § 5).

Theorem (N) is an immediate consequence of Theorems (D), (AE), and the Lebesgue-dominated convergence theorem.

(f) Proof of Theorem (N*) (see § 5).

We prove first

LEMMA 15. *Let $f(x) \in L^p(Q_k)$, $1 < p \leqslant 2$, then*

$$\| S_R^{\frac{1}{2}(k-1)}(x, f) \|_p \leqslant A\,(p-1)^{-1} \| f(x) \|_p, \tag{19.27}$$

where A does not depend on R, p, or f. If $f(x) \in L \log^+ L$, then

$$\int_{Q_k} | S_R^{\frac{1}{2}(k-1)}(x, f) | \, dx \leqslant A \int_{Q_k} | f(x) | \log^+ | f(x) | \, dx + B, \tag{19.28}$$

where A and B do not depend on R or f.

Proof: We prove (19.27) first. Let p be a fixed index, $1 < p \leqslant 2$. Thus we may write

$$\frac{1}{p} = \frac{1-t}{2} + \frac{t}{1}, \quad 0 \leqslant t < 1.$$

A simple calculation yields

$$t = \frac{2}{p} - 1, \quad 1 - t = 2 - \frac{2}{p}. \tag{19.29}$$

Now define $\delta(z)$, by

$$\delta(z) = \left(\frac{k}{2} - \frac{1}{p}\right)(1-z) + \left(\frac{k+1}{2} - \frac{1}{p}\right) z. \tag{19.30}$$

Then one may verify that

$$\delta(t) = \frac{k-1}{2}. \tag{19.31}$$

Fix R. Define the family of operators $\{T_z(\cdot)\}$, by

$$T_z(f) = S_R^{\delta(z)}(x, f). \tag{19.32}$$

We will verify that

$$\| T_{iy}(f) \|_2 \leqslant \| f \|_2 \tag{19.33}$$

and

$$\| T_{1+iy}(f) \|_1 \leqslant A\,(p-1)^{-1} e^{\frac{1}{2}\pi |y|} \| f \|_1. \tag{19.34}$$

Since $\Re(\delta(iy)) = k + \frac{1}{2} - 2/p \geqslant 0$, $(k \geqslant 2)$, we apply Theorem 7, part (b), of § 18, and we obtain (12.33).

We next observe that

$$\Re(\delta(1+iy)) = \frac{k+1}{2} - \frac{1}{p} = \frac{k-1}{2} + 1 - \frac{1}{p};$$

hence
$$\Re(\delta(1+iy)) - \frac{k-1}{2} = 1 - \frac{1}{p} \geqslant \frac{p-1}{2}.$$

Moreover,
$$\Im(\delta(1+iy)) = \frac{y}{2}.$$

We then apply part (b) of Lemma 12, § 17, and obtain

$$\| T_{1+iy}(f) \|_1 = \| S_R^{\delta(1+iy)}(x, f) \|_1 \leqslant A_{y,p} \| f \|_1,$$

where
$$A_{y,p} \leqslant A \left[\Re(\delta(1+iy)) - \frac{k-1}{2} \right]^{-1} e^{\pi |\Im(\delta(1+iy))|}$$
$$\leqslant A(p-1)^{-1} e^{\frac{1}{2}\pi|y|}.$$

Thus we have established (12.34). We now apply again Lemma 1 of § 8, and we obtain

$$\| T_t(f) \|_p = \| S_R^{\delta(t)}(x, f) \|_p \leqslant A_t \| f \|_p.$$

We also have the estimate (see Lemma 1)

$$\log A_t \leqslant \int_{-\infty}^{+\infty} \omega(t, y) \log [A(p-1)^{-1} e^{\frac{1}{2}\pi|y|}] \, dy$$
$$\leqslant \log \frac{1}{p-1} + B.$$

The last estimate is obtained by recalling that

$$\omega(t, y) \geqslant 0, \quad \int_{-\infty}^{+\infty} \omega(t, y) \, dy \leqslant 1, \quad \text{and} \quad \int_{-\infty}^{+\infty} \omega(t, y) |y| \, dy \leqslant A.$$

Hence
$$A_t \leqslant e^B [p-1]^{-1}.$$

If we recall (12.31), we see that we have therefore established

$$\| S_R^{\frac{1}{2}(k-1)}(x, f) \|_p = \| S_R^{\delta(t)}(x, f) \|_p \leqslant A [p-1]^{-1} \| f \|, \quad 1 < p \leqslant 2.$$

We have therefore proved (19.27) of Lemma 15. We then apply the case $r=1$ of Lemma 2 of § 10 to the inequality in (19.27). The result is (19.28), and thus the proof of Lemma 15 is complete.

Theorem (N^*) now follows from (19.28) by standard arguments. (We have already given a very similar argument in (d), above.)

CHAPTER IV

Strong Summability

20. The interpolation argument

Recalling the definition

$$\Lambda_\delta(x, f) = \sup_{R>0} \frac{1}{R}\left\{\int_0^R |S_u^{\delta-1}(x, f)|^2 du\right\}^{\frac{1}{2}},$$

we notice that in Lemma 13 of § 18 we have already shown that

$$\|\Lambda_\delta(x, f)\|_2 \leqslant A_\delta \|f\|_2, \quad \text{if } \delta > \tfrac{1}{2}. \tag{20.1}$$

As we shall presently see, Lemma 3 of § 10, and 12 of § 17 allow us to prove rather easily that

$$\|\Lambda_\delta(x, f)\|_p \leqslant A_{\delta, p} \|f\|_p, \quad 1 < p \leqslant 2, \ \delta > \tfrac{1}{2}(k+1). \tag{20.2}$$

We then take p arbitrarily close to 1 in the above and interpolate between (20.1) and (20.2). The results are contained in the following theorem.

THEOREM 8. *Let $f(x) \in L^p(Q)$, $1 < p \leqslant 2$, and let $\delta > k/p + \frac{1}{2}(1-k)$. Then*

$$\|\Lambda_\delta(x, f)\|_p \leqslant A_{\delta, p} \|f\|_p. \tag{20.3}$$

Proof. We make some remarks first. Most of the proof of this theorem follows the pattern of the proofs given in § 19 above, with one difference. The operator Λ_δ is not linear as it stands, and we thus need to introduce a "linearization" of the operator in order to apply the interpolation argument of Lemma 1 of § 8.

We therefore proceed as follows. Let $R(x)$ be a strictly positive, bounded, and measurable function on Q. Except for these restrictions $R(x)$ is arbitrary. Next let $\psi(x, u)$ be a measurable function which satisfies the condition:

$$\frac{1}{R(x)} \int_0^{R(x)} |\psi(x, u)|^2 du \leqslant 1, \quad \text{all} \quad x \in Q. \tag{20.4}$$

Keeping the functions $R(x)$ and $\psi(x, u)$ momentarily fixed, we define the family of operators V_δ, by

$$V_\delta(f) = \frac{1}{R(x)} \int_0^{R(x)} S_u^{\delta-1}(x, f)\, \psi(x, u)\, du. \tag{20.5}$$

We notice now that V_δ are linear operators. However, Schwarz's inequality (and (20.4)) show that

$$|V_\delta(f)(x)| \leqslant \Lambda_\delta(x, f). \tag{20.6}$$

Moreover, by using the converse of Schwarz's inequality, it is not difficult to verify that for any p

$$\|\Lambda_\delta(x, f)\|_p = \sup \|V_\delta(f)(x)\|_p. \tag{20.7}$$

Here, the supremum is taken over all functions $R(x)$ and $\psi(x, u)$ of the type described above.

We next recall that by Lemma 13

$$\|\Lambda_\delta(x, f)\|_2 \leqslant A_\sigma e^{\pi|\tau|} \|f\|_2, \quad \delta = \sigma + i\tau, \ \sigma > \tfrac{1}{2}. \tag{20.8}$$

Now by Lemma 12 of § 17 it follows that

$$\Lambda_\delta(x, f) \leqslant S_*^{\delta-1}(x, f) \leqslant B_\sigma e^{\pi|\tau|} f^*(x),$$

where $\delta = \sigma + i\tau$, $\sigma > \frac{1}{2}(k+1)$. However, by Lemma 12 of § 7, we have

$$\|f^*(x)\|_p \leqslant A_p \|f\|_p, \text{ if } 1 < p.$$

Combining the above yields

$$\|\Lambda_\delta(x, f)\|_p \leqslant A_{p,\sigma} e^{\pi|\tau|} \|f\|_p, \quad \sigma > \tfrac{1}{2}(k-1), \ p > 1. \tag{20.9}$$

Let p and δ be the indices given in the statement of the theorem. Since $1 < p$, we may find a p_1, so that $1 < p_1 < p$. We let $p_0 = 2$. We then can determine a t, $0 < t < 1$, so that

$$\frac{1}{p} = \frac{1-t}{p_0} + \frac{t}{p_1}.$$

We define $\delta(z)$, by

$$\delta(z) = (\varepsilon_0 + \tfrac{1}{2})(1-z) + (\tfrac{1}{2}(R+1) + \varepsilon_1) z, \tag{20.10}$$

where $\varepsilon_0 > 0$, and $\varepsilon_1 > 0$. Proceeding as in the proof of Theorem (D), we may choose p_1 suitably close to 1, also ε_0, and ε_1 appropriately so that

$$\delta(t) = \delta\left(> \frac{k}{p} + \frac{1-k}{2}\right). \tag{20.11}$$

(We omit the elementary but painful calculation.)

We now define our analytic family of operators by

$$T_z(f) = V_{\delta(z)}(f), \tag{20.12}$$

where V_δ is defined in (20.5).

Using (20.6), (20.8), and (20.9) we see easily that:

$$\| T_{iy}(f) \|_2 \leqslant A\, e^{a|y|} \| f \|_2,$$

and

$$\| T_{1+iy}(f) \|_{p_1} \leqslant A\, e^{a|y|} \| f \|_{p_1}. \tag{20.13}$$

It is important to notice that the bounds A and a, which appear in the above, do not depend on $R(x)$ or $\psi(x, u)$, since the right side of (20.6) does not depend on $R(x)$ and $\psi(x, u)$. Applying the interpolation lemma of § 8 to the above we obtain

$$\| T_t(f) \|_p \leqslant A_t \| f \|_p. \tag{20.14}$$

Because of (20.11) and (20.12) this is

$$\| V_\delta(x, f) \|_p \leqslant A_t \| f \|_p. \tag{20.15}$$

We now use (20.7) and the fact that A_t did not depend on $R(x)$ or $\psi(x, u)$. We therefore obtain:

$$\| \Lambda_\delta(x, f) \|_p \leqslant A_t \| f \|_p,$$

which concludes the proof of the theorem.

21. Proof of theorem (S) (see § 5).

We first observe that whenever $P(x)$ is a trigonometric polynomial

$$P(x) = \sum_{|n| \leqslant N} b_n e^{in \cdot x},$$

then

$$\lim_{R\to\infty} \frac{1}{R} \int_0^R | S_u^{\delta-1}(x, P) - P(x) |^2 du \to 0, \tag{21.1}$$

uniformly in x, when $\delta > \frac{1}{2}$.

Now if $f(x) \in L^p(Q)$, then $f(x) - P(x)$ can be made arbitrarily small in L^p norm by an appropriate choice of $P(x)$. Theorem 8 tells us, however, that

$$\left\| \left\{ \sup_{R>0} \frac{1}{R} \int_0^R | S_u^{\delta-1}(x, f) - S_u^{\delta-1}(x, P) |^2 du \right\}^{\frac{1}{2}} \right\|_p \leqslant A_{\delta, p} \| f(x) - P(x) \|_p, \quad \text{if } \delta > \frac{k}{p} + \frac{1-k}{2}. \tag{21.2}$$

We can now combine (21.1) and (21.2) by the use of a standard argument—a very similar argument was used in the proof of Theorem (AE) in § 19—and obtain

$$\lim_{R\to\infty} \frac{1}{R} \int_0^R | S_R^{\delta-1}(x, f) - f(x) |^2 du = 0, \quad \text{almost every } x, \quad \text{if } \delta > \frac{k}{p} + \frac{1-k}{2}.$$

This may be written as

$$\lim_{R\to\infty} \frac{1}{R} \int_0^R | S_u^{\delta}(x, f) - f(x) |^2 du \to 0, \quad \text{almost every } x, \quad \text{if } \delta > \frac{k}{p} + \frac{1-k}{2} - 1.$$

The last condition is clearly

$$\delta > \left(\frac{k-1}{2}\right)\left(\frac{2}{p} - 1\right) - \frac{1}{p'},$$

where

$$\frac{1}{p'} + \frac{1}{p} = 1.$$

This concludes the proof of Theorem (S).

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