A THEOREM OF COMPLETENESS FOR COMPLEX ANALYTIC FIBRE SPACES

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1. Introduction

We begin by recalling several definitions, introduced in the authors' paper [3], concerning complex analytic families of complex manifolds.

By a complex analytic fibre space we mean a triple $(\mathcal{V}, \varpi, M)$ of connected complex manifolds $\mathcal{V}$, M and a holomorphic map ϖ of $\mathcal{V}$ onto M. A fibre $\varpi^{-1}(t)$, $t \in M$, of the fibre space is *singular* if there exists a point $p \in \varpi^{-1}(t)$ such that the rank of the jacobian matrix of the map ϖ at p is less than the dimension of M.

DEFINITION 1. We say that $\mathcal{V} \xrightarrow{\varpi} M$ is a complex analytic family of compact, complex manifolds if $(\mathcal{V}, \varpi, M)$ is a complex analytic fibre space without singular fibres whose fibres are connected, compact manifolds and whose base space M is connected.

With reference to a complex manifold $V_0 = \varpi^{-1}(0)$, $0 \in M$, we call any $V_t = \varpi^{-1}(t)$, $t \in M$, a deformation of V_0 and we call $\mathcal{V} \xrightarrow{\varpi} M$ a complex analytic family of deformations of V_0.

DEFINITION 2. A complex analytic family $\mathcal{V} \xrightarrow{\varpi} M$ of compact, complex manifolds is (complex analytically) complete at the point $t \in M$ if, for any complex analytic family $\mathcal{W} \xrightarrow{\pi} N$ such that $\pi^{-1}(0) = \varpi^{-1}(t)$ for a point $0 \in N$, there exist a holomorphic map $s \to t(s)$, $t(0) = t$, of a neighborhood U of 0 on N and a holomorphic map g of $\pi^{-1}(U)$ into $\mathcal{V}$ which maps each fibre $\pi^{-1}(s)$, $s \in U$ of $\mathcal{W}$ biregularly onto $\varpi^{-1}(t(s))$. The complex analytic family $\mathcal{V} \xrightarrow{\varpi} M$ is called (complex analytically) complete if it is (complex analytically) complete at each point t of M.

Let $\mathcal{V} \xrightarrow{\varpi} M$ be a complex analytic family of compact, complex manifolds and let $V_t = \varpi^{-1}(t)$ be the fibre of $\mathcal{V}$ over $t \in M$. Denote by Θ_t the sheaf over V_t of germs of holomorphic vector fields, and denote by $(T_M)_t$ the (complex) tangent space of M at the point t. Of fundamental importance in the study of the deformation of complex structure is the complex linear map

$$\varrho_t : (T_M)_t \to H^1(V_t, \Theta_t)$$

which measures the magnitude of dependence of the complex structure of the fibre V_t on the parameter t (see [3], Sections 5 and 6). A definition of ϱ_t will be given below (see formula (9)). For a tangent vector $v \in (T_M)_t$ the image $\varrho_t(v) \in H^1(V_t, \Theta_t)$ is called the infinitesimal deformation of V_t along v.

Our purpose is to prove the following theorem:

THEOREM. *Let $\mathcal{V} \xrightarrow{\varpi} M$ be a complex analytic family of compact, complex manifolds and suppose that, for some point $t \in M$, the map $\varrho_t : (T_M)_t \to H^1(V_t, \Theta_t)$ is surjective. Then $\mathcal{V} \xrightarrow{\varpi} M$ is (complex analytically) complete at t.*

The proof of this theorem is elementary, in particular it makes no use of the theory of harmonic differential forms.

We remark that the question remains open whether $\mathcal{V} \xrightarrow{\varpi} M$ is differentiably complete at $t \in M$ (in the sense of [3], Definition 1.7) if the map $\varrho_t : (T_M)_t \to H^1(V_t, \Theta_t)$ is surjective; in particular, Problem 6, Section 22 of [3], remains unsolved. If we assume the additional condition that $H^2(V_t, \Theta_t) = 0$ at this particular point t, then it can be proved, by the method of harmonic differential forms, that $\mathcal{V} \xrightarrow{\varpi} M$ is differentiably complete at t (see Kodaira [2]).

In [3] the authors constructed several simple examples of complex analytic families of compact, complex manifolds, namely:

(1) family of complex tori of arbitrary dimension n;

(2) family $\mathcal{V}_{n,h}$ of all non-singular hypersurfaces of order h on complex projective n-space $(n \geqslant 2, h \geqslant 2)$;

(3) family of non-singular hypersurfaces on abelian varieties of arbitrary dimension $n \geqslant 2$;

(4) family of compact Hopf surfaces.

It was shown in Section 18 of [3], on the basis of special properties of the families, that the families (1) and (2) are complex analytically complete, except for the case $n = 2$, $h = 4$ of (2) in which the map ϱ_t is not surjective. The (complex analytic) completeness of all four families (except the case $n=2$, $h=4$ of (2) in which the family is not complete) now follows at once from the above theorem.

We remark that each of the above families (except $\mathcal{V}_{2,4}$) is differentiably complete (see [3]).

2. Complex analytic completeness (proof of the theorem)

Let $\mathcal{V} \xrightarrow{\varpi} M$ be a complex analytic family which satisfies the hypothesis of our theorem, namely that, for some point $0 \in M$, the map

$$\varrho_0 : (T_M)_0 \to H^1(V_0, \Theta_0)$$

is surjective, where $V_0 = \varpi^{-1}(0)$ is the fibre over the point $0 \in M$. Given an arbitrary complex analytic family $\mathcal{W} \xrightarrow{\pi} N$ such that $\pi^{-1}(0) = V_0$ for a point $0 \in N$, we must show that there exist a holomorphic map $s \to t(s)$, $t(0)=0$, of a neighborhood U of 0 on N into M and a holomorphic map g of $\mathcal{W} | U = \pi^{-1}(U)$ into $\mathcal{V}$ which maps each fibre $\pi^{-1}(s)$, $s \in U$, of $\mathcal{W}$ biregularly onto $\varpi^{-1}(t(s))$.

First we fix our notations. We denote by t a point $(t_1, t_2, \ldots, t_m)$ on the space $\mathbf{C}^m$ of m complex variables and by s a point $(s_1, s_2, \ldots, s_l)$ on $\mathbf{C}^l$. We define

$$|t| = \max_r |t_r|,$$

$$|s| = \max_r |s_r|.$$

Similarly we denote by z_i a point $(z_i^1, z_i^2, \ldots, z_i^n)$, by ζ_i a point $(\zeta_i^1, \ldots, \zeta_i^n)$, and let

$$|z_i| = \max_\alpha |z_i^\alpha|,$$

$$|\zeta_i| = \max_\alpha |\zeta_i^\alpha|.$$

If
$$f : s \to f(s) = (f^1(s), \ldots, f^\alpha(s), \ldots, f^n(s))$$

is a holomorphic map of a domain $\{s \,|\, |s| < \varepsilon\}$ into $\mathbf{C}^n$, we write the power series expansion of $f^\alpha(s)$ in the form

$$f^\alpha(s) = f_0^\alpha + f_1^\alpha(s) + \cdots + f_\mu^\alpha(s) + \cdots,$$

where $f_\mu^\alpha(s)$ is a homogeneous polynomial in $(s_1, s_2, \ldots, s_l)$ of degree μ. Moreover, letting

$$f_\mu(s) = (f_\mu^1(s), \ldots, f_\mu^\alpha(s), \ldots, f_\mu^n(s)),$$

we write
$$f(s) = f_0 + f_1(s) + \cdots + f_\mu(s) + \cdots$$

and call this the power series expansion of the vector-valued holomorphic function $f(s)$.

We may assume the following:

i) M is a polycylinder: $M=\{t \mid |t|<1\}$ and $V_0=\varpi^{-1}(0)$.

ii) $\mathcal{V}$ is covered by a finite number of coordinate neighborhoods $\mathcal{U}_i$. Each $\mathcal{U}_i$ is covered by a system of holomorphic coordinates (ζ_i, t) such that $\varpi(\zeta_i, t)=t$ and

$$\mathcal{U}_i=\{(\zeta_i, t) \mid |\zeta_i|<1,\ |t|<1\}.$$

(We indicate by (ζ_i, t) a set of $n+m$ complex numbers $\zeta_i^1, \ldots, \zeta_i^n, t_1, \ldots, t_m$ *and* the point on $\mathcal{U}_i$ with the coordinates $(\zeta_i^1, \ldots, \zeta_i^n, t_1, \ldots, t_m)$.)

iii) (ζ_i, t) coincides with (ζ_k, t) if and only if

$$\zeta_i=g_{ik}(\zeta_k, t),$$

where $g_{ik}(\zeta_k, t)$ is a vector-valued holomorphic function of (ζ_k, t) defined on $\mathcal{U}_k \cap \mathcal{U}_i$.

iv) N is a polycylinder: $N=\{s \mid |s|<1\}$ and $V_0=\pi^{-1}(0)$.

v) $\mathcal{W}$ is covered by a finite number of coordinate neighborhoods $\mathcal{W}_i$ such that

$$V_0 \cap \mathcal{W}_i=V_0 \cap \mathcal{U}_i.$$

Each $\mathcal{W}_i$ is covered by a system of holomorphic coordinates (z_i, s) such that $\pi(z_i, s)=s$ and

$$\mathcal{W}_i=\{(z_i, s) \mid |z_i|<1,\ |s|<1\}.$$

Moreover, on $V_0 \cap \mathcal{W}_i=V_0 \cap \mathcal{U}_i$, the system of coordinates (z_i) coincides with (ζ_i), i.e., $(z_i, 0)$ and $(\zeta_i, 0)$ are the same point on $V_0 \cap \mathcal{W}_i=V_0 \cap \mathcal{U}_i$ if and only if $z_i^1=\zeta_i^1, \ldots, z_i^n=\zeta_i^n$.

vi) (z_i, s) coincides with (z_k, s) if and only if

$$z_i=h_{ik}(z_k, s),$$

where $h_{ik}(z_k, s)$ is a vector-valued holomorphic function of (z_k, s) defined on $\mathcal{W}_k \cap \mathcal{W}_i$.

Let
$$b_{ik}(z_k)=h_{ik}(z_k, 0).$$

By v) we have
$$b_{ik}(\zeta_k)=g_{ik}(\zeta_k, 0).$$

Let
$$U_i=V_0 \cap \mathcal{W}_i=V_0 \cap \mathcal{U}_i$$

and let
$$N_\varepsilon=\{s \mid |s|<\varepsilon\},$$

where $0<\varepsilon<1$. In view of ii) and v) we may write

$$\mathcal{U}_i = U_i \times M,$$
$$\mathcal{W}_i = U_i \times N.$$

We may suppose therefore that

$$U_i \times N_\varepsilon \subset U_i \times N = \mathcal{W}_i,$$
$$\mathcal{U}_i = U_i \times M \subset \mathbf{C}^n \times M.$$

In order to prove our theorem it suffices to construct a holomorphic map $s \to t = t(s)$ of N_ε into M such that $t(0)=0$ and holomorphic maps

$$g_i : (z_i, s) \to (\zeta_i, t) = (g_i(z_i, s),\ t(s))$$

of $U_i \times N_\varepsilon$ into $\mathbf{C}^n \times M$ such that $g_i(z_i, 0) = z_i$ which satisfy the equations

$$g_i(h_{ik}(z_k, s), s) = g_{ik}(g_k(z_k, s), t(s)) \tag{1}$$

whenever $z_k \varepsilon U_k \cap U_i$ and $|s|$ is sufficiently small (or, more precisely, $|s| < \varepsilon(z_k)$, $\varepsilon(z_k)$ being a *continuous* function of z_k defined on $U_k \cap U_i$ such that $0<\varepsilon(z_k)<\varepsilon$). In fact, let $\{U_i^*\}$ be a covering of V_0 such that the closure of each U_i^* is a compact subset of U_i and such that $\{U_i^* \times N_\varepsilon\}$ covers $\mathcal{W} | N_\varepsilon = \pi^{-1}(N_\varepsilon)$. Moreover, let $\delta < \varepsilon$ be a sufficiently small positive number and let g_i^* be the restriction of g_i to $U_i^* \times N_\delta$. Since $g_i(z_i, 0) = z_i$ and $t(0)=0$, we infer that g_i^* maps $U_i^* \times N_\delta$ into $U_i \times M = \mathcal{U}_i$. Thus g_i^* is a holomorphic map of $U_i^* \times N_\delta$ into $\mathcal{V}$. Moreover, (1) implies that g_i^* and g_k^* coincide on the intersection $U_i^* \times N_\delta \cap U_k^* \times N_\delta$. Consequently the collection $\{g_i^*\}$ determines a holomorphic map g^* of $\mathcal{W} | N_\delta = \pi^{-1}(N_\delta)$ into $\mathcal{V}$ which clearly maps each fibre $\pi^{-1}(s)$ of $\mathcal{W} | N_\delta$ biregularly onto the fibre $\varpi^{-1}(t(s))$ of $\mathcal{V}$. This proves our theorem.

Let
$$t(s) = t_1(s) + t_2(s) + \cdots + t_\mu(s) + \cdots \tag{2}$$

be the power series expansion of $t(s)$ and let

$$t^\mu(s) = t_1(s) + t_2(s) + \cdots + t_\mu(s). \tag{3}$$

Moreover, let
$$g_i(z_i, s) = z_i + g_{i|1}(z_i, s) + \cdots + g_{i|\mu}(z_i, s) + \cdots \tag{4}$$

be the power series expansion of $g_i(z_i, s)$ and let

$$g_i^\mu(z_i, s) = z_i + g_{i|1}(z_i, s) + \cdots + g_{i|\mu}(z_i, s). \tag{5}$$

We remark that $g_{i|\mu}(z_i, s)$ is a homogeneous polynomial in $(s_1, s_2, \cdots, s_l)$ whose coefficients are vector-valued holomorphic functions of z_i defined on $\{z_i \mid |z_i| < 1\}$. For any vector-valued holomorphic functions $P(s)$, $Q(s)$ in $(s_1, s_2, \ldots, s_l)$, we indicate by writing $P(s) \underset{\mu}{\equiv} Q(s)$ that the power series expansion of $P(s)-Q(s)$ in $(s_1, s_2, \ldots, s_l)$ contains no terms of degree $\leqslant \mu$. Clearly (1) is equivalent to the system of congruences

$$g_i^\mu(h_{ik}(z_k, s), s) \underset{\mu}{\equiv} g_{ik}(g_k^\mu(z_k, s), t^\mu(s)), \qquad (\mu = 0, 1, 2, \ldots). \tag{6$_\mu$}$$

Note that the power series expansions of both sides of $(6)_\mu$ are well-defined at each point $z_k \in U_k \cap U_i$.

We insert here a remark on the first cohomology group $H^1(V_0, \Theta_0)$ of V_0 with coefficients in the sheaf Θ_0 of germs of holomorphic vector fields on V_0. Denote the covering $\{U_i\}$ by $\mathfrak{U}$. Since each U_i is a Stein manifold we have the canonical isomorphism (see Cartan [1], Leray [4])

$$H^1(V_0, \Theta_0) \cong H^1(\mathfrak{U}, \Theta_0). \tag{7}$$

Let $\{\theta_{ik}\}$ be a 1-cocycle on $\mathfrak{U} = \{U_i\}$ with coefficients in Θ_0, i.e., a system of holomorphic vector fields θ_{ik} defined respectively on $U_i \cap U_k$ such that

$$\theta_{ik} = \theta_{ij} + \theta_{jk}, \qquad \text{on } U_i \cap U_j \cap U_k. \tag{8}$$

We write θ_{ik} explicitly in the form

$$\theta_{ik}(z_i) = (\theta_{ik}^1(z_i), \ldots, \theta_{ik}^\alpha(z_i), \ldots, \theta_{ik}^n(z_i))$$

with reference to the system of coordinates $(z_i) = (z_i^1, \ldots, z_i^\alpha, \ldots, z_i^n)$. The explicit form of the cocycle condition (8) is:

$$\theta_{ik}^\alpha(z_i) = \theta_{ij}^\alpha(z_i) + \sum_{\beta=1}^n \frac{\partial b_{ij}^\alpha(z_j)}{\partial z_j^\beta} \cdot \theta_{jk}^\beta(z_j),$$

where $z_i = b_{ij}(z_j)$. Using matrix notation we write this in the form

$$\theta_{ik}(z_i) = \theta_{ij}(z_i) + B_{ij}(z_j) \cdot \theta_{jk}(z_j), \qquad (z_i = b_{ij}(z_j)),$$

where $B_{ij}(z_j)$ denotes the $n \times n$ matrix

$$B_{ij}(z_j) = \left(\frac{\partial b_{ij}^\alpha(z_j)}{\partial z_j^\beta}\right)_{\substack{\alpha=1,2,\cdots,n \\ \beta=1,2,\cdots,n}}$$

Letting
$$\beta_{ikr}(z_i) = \left.\frac{\partial g_{ik}(z_k, t)}{\partial t_r}\right|_{t=0}, \qquad \text{where } z_i = b_{ik}(z_k),$$

we obtain a 1-cocycle $\{\beta_{ikr}(z_i)\}$ on $\mathfrak{U} = \{U_i\}$ with coefficients in Θ_0. For any tangent vector

$$v = \sum_{r=1}^{m} v_r \frac{\partial}{\partial t_r}$$

of M at 0, the infinitesimal deformation $\varrho_0(v) \in H^1(V_0, \Theta_0)$ is, by definition, the cohomology class of the 1-cocycle

$$\left\{ \sum_{r=1}^{m} v_r \beta_{ikr}(z_i) \right\}. \tag{9}$$

By hypothesis, $\varrho_0 : (T_M)_0 \to H^1(V_0, \Theta_0)$ is surjective. In view of the canonical isomorphism (7), we infer therefore that any 1-cocycle $\{\theta_{ik}(z_i)\}$ is cohomologous to a linear combination of $\{\beta_{ikr}(z_i)\}$, $r = 1, 2, \ldots, m$. In other words, for any 1-cocycle $\{\theta_{ik}(z_i)\}$, we can find constants $\gamma_1, \ldots, \gamma_r, \ldots, \gamma_m$ and holomorphic vector fields

$$\theta_i(z_i) = (\theta_i^1(z_i), \ldots, \theta_i^\alpha(z_i), \ldots, \theta_i^n(z_i))$$

defined respectively on U_i such that

$$\sum_{r=1}^{m} \gamma_r \beta_{ikr}(z_i) + B_{ik}(z_k) \cdot \theta_k(z_k) - \theta_i(z_i) = \theta_{ik}(z_i), \tag{10}$$

where $z_i = b_{ik}(z_k)$.

We may assume that $\beta_{ikr}(z_i)$ and $B_{ik}(z_k)$ are uniformly bounded:

$$|\beta_{ikr}(z_i)| < K_1, \quad |B_{ik}(z_k)| < K_1, \tag{11}$$

where $|B_{ik}(z_k)|$ denotes the usual norm of the matrix $B_{ik}(z_k)$. For any 1-cocycle $\sigma = \{\theta_{ik}(z_i)\}$, we define the norm $\|\sigma\|$ of σ by

$$\|\sigma\| = \max_{i,k} \sup_{z_i} |\theta_{ik}(z_i)|.$$

LEMMA 1. *For any 1-cocycle $\sigma = \{\theta_{ik}(z_i)\}$, we can find γ_r and $\theta_i(z_i)$ satisfying* (10) *such that*

$$|\gamma_r| < K_2 \cdot \|\sigma\|, \quad |\theta_i(z_i)| < K_2 \cdot \|\sigma\|, \tag{12}$$

where K_2 is a positive constant which is independent of σ.

Proof. We define

$$\iota(\sigma) = \inf \max \{|\gamma_r|, \sup_{z_i} |\theta_i(z_i)|\},$$

where inf is taken with respect to all solutions $\{\gamma_r, \theta_i(z_i)\}$ of the equations (10). It suffices to prove the existence of a constant K_2 such that

$$\iota(\sigma) < K_2 \cdot \|\sigma\|.$$

Suppose that such a constant K_2 does not exist. Then we can find a sequence $\sigma^{(1)}, \sigma^{(2)}, \ldots, \sigma^{(\nu)}, \ldots$ of 1-cocycles $\sigma^{(\nu)} = \{\theta_{ik}^{(\nu)}(z_i)\}$ such that

$$\iota(\sigma^{(\nu)}) = 1, \quad \|\sigma^{(\nu)}\| < \frac{1}{\nu}.$$

$\iota(\sigma^{(\nu)}) = 1$ implies that there exist $\gamma_r^{(\nu)}$, $\theta_i^{(\nu)}(z_i)$ satisfying

$$\sum_{r=1}^{m} \gamma_r^{(\nu)} \beta_{ikr}(z_i) + B_{ik}(z_k)\, \theta_k^{(\nu)}(z_k) - \theta_i^{(\nu)}(z_i) = \theta_{ik}^{(\nu)}(z_i), \tag{13}$$

$$|\gamma_r^{(\nu)}| < 2, \quad |\theta_i^{(\nu)}(z_i)| < 2, \tag{14}$$

where $z_i = b_{ik}(z_k)$. Hence, replacing $\sigma^{(1)}, \sigma^{(2)}, \ldots$ by a suitable subsequence if necessary, we may suppose that

$$\gamma_r = \lim_{\nu\to\infty} \gamma_r^{(\nu)},$$

$$\theta_i(z_i) = \lim_{\nu\to\infty} \theta_i^{(\nu)}(z_i)$$

exist, where the convergence $\theta_i^{(\nu)}(z_i) \to \theta_i(z_i)$ is uniform on each compact subset of U_i and $\theta_i(z_i)$ is holomorphic on U_i. Since

$$|\theta_{ik}^{(\nu)}(z_i)| \leqslant \|\sigma^{(\nu)}\| \to 0 \ (\nu \to \infty), \tag{15}$$

we obtain from (13) the equality

$$\sum_{r=1}^{m} \gamma_r \beta_{ikr}(z_i) + B_{ik}(z_k)\, \theta_k(z_k) - \theta_i(z_i) = 0. \tag{16}$$

Let $\{U_i^*\}$ be a covering of V_0 such that the closure of each U_i^* is a compact subset of U_i. For each point $z_i \in U_i$ there exists at least one U_k^* which contains $z_k = b_{ki}(z_i)$. Hence we infer from (13) and (15) that $\theta_i^{(\nu)}(z_i)$ converges to $\theta_i(z_i)$ uniformly on the whole of U_i.

Letting

$$\gamma_r' = \gamma_r^{(\nu)} - \gamma_r, \quad \theta_i'(z_i) = \theta_i^{(\nu)}(z_i) - \theta_i(z_i)$$

for a sufficiently larger integer ν, we have therefore

$$|\gamma_r'| < \tfrac{1}{2}, \quad |\theta_i'(z_i)| < \tfrac{1}{2}$$

while we infer from (13) and (16) that

$$\sum_{r=1}^{m} \gamma_r' \beta_{ikr}(z_i) + B_{ik}(z_k)\theta_k'(z_k) - \theta_i'(z_i) = \theta_{ik}^{(\nu)}(z_i).$$

This contradicts with $\iota(\sigma^{(\nu)}) = 1$, q.e.d.

Now we construct $t^\mu(s)$ and $g_i^\mu(z_i, s)$ satisfying $(6)_\mu$ by induction on μ. It follows from the identity

$$h_{ik}(z_k, 0) = b_{ik}(z) = g_{ik}(z_k, 0)$$

that $t^0(s) = 0$ and $g_i^0(z_i, s) = z_i$ satisfy $(6)_0$. Suppose therefore that $t^{\mu-1}(s)$ and $g_i^{\mu-1}(z_i, s)$ satisfying $(6)_{\mu-1}$ are already determined. We define a homogeneous polynomial $\Gamma_{ik|\mu}(z_i, s)$ of degree μ in $(s_1, s_2, \dots, s_l)$, whose coefficients are vector-valued holomorphic functions of z_i defined on $U_i \cap U_k$, by the congruence

$$\Gamma_{ik|\mu}(z_i, s) \underset{\mu}{\equiv} g_i^{\mu-1}(h_{ik}(z_k, s), s) - g_{ik}(g_k^{\mu-1}(z_k, s), t^{\mu-1}(s)),$$

where $z_i = b_{ik}(z_k)$.

LEMMA 2. *We have the identity*

$$\Gamma_{ik|\mu}(z_i, s) = \Gamma_{ij|\mu}(z_i, s) + B_{ij}(z_j) \cdot \Gamma_{jk|\mu}(z_j, s), \tag{17}$$

where $z_i = b_{ij}(z_j)$.

Proof. For simplicity let $\Gamma_{ik|\mu} = \Gamma_{ik|\mu}(z_i, s)$, $\Gamma_{ij|\mu} = \Gamma_{ij|\mu}(z_i, s)$ and $\Gamma_{jk|\mu} = \Gamma_{jk|\mu}(z_j, s)$, where $z_i = b_{ij}(z_j) = b_{ik}(z_k)$, $z_j = b_{jk}(z_k)$. Since

$$g_{ik}(z_k, t) = g_{ij}(g_{jk}(z_k, t), t),$$

we have

$$\Gamma_{ik|\mu} \underset{\mu}{\equiv} g_i^{\mu-1}(h_{ik}(z_k, s), s) - g_{ij}(g_{jk}(g_k^{\mu-1}(z_k, s), t^{\mu-1}(s)), t^{\mu-1}(s)).$$

Using
$$g_{jk}(g_k^{\mu-1}(z_k, s), t^{\mu-1}(s)) \underset{\mu}{\equiv} g_j^{\mu-1}(h_{jk}(z_k, s), s) - \Gamma_{jk|\mu},$$

we get

$$g_{ij}(g_{jk}(g_k^{\mu-1}(z_k, s), t^{\mu-1}(s)), t^{\mu-1}(s)) \underset{\mu}{\equiv} g_{ij}(g^{\mu-1}(h_{jk}(z_k, s), s), t^{\mu-1}(s)) - B_{ij}(z_j) \cdot \Gamma_{jk|\mu},$$

since
$$g_j^{\mu-1}(h_{jk}(z_k, 0), 0) = b_{jk}(z_k) = z_j.$$

Hence we obtain

$$\Gamma_{ik|\mu} \underset{\mu}{\equiv} g_i^{\mu-1}(h_{ik}(z_k, s), s) - g_{ij}(g_j^{\mu-1}(h_{jk}(z_k, s), s), t^{\mu-1}(s)) + B_{ij}(z_j) \cdot \Gamma_{jk|\mu}.$$

Now, using $h_{ik}(z_k, s) = h_{ij}(h_{jk}(z_k, s), s)$, we get

$$g_i^{\mu-1}(h_{ik}(z_k, s), s) - g_{ij}(g_j^{\mu-1}(h_{jk}(z_k, s), s), t^{\mu-1}(s)) = g_i^{\mu-1}(h_{ij}(h_{jk}(z_k, s), s), s)$$
$$- g_{ij}(g_j^{\mu-1}(h_{jk}(z_k, s), s), t^{\mu-1}(s)) \underset{\mu}{\equiv} \Gamma_{ij|\mu}(b_{ij}(h_{jk}(z_k, s)), s) \underset{\mu}{\equiv} \Gamma_{ij|\mu}(b_{ik}(z_k), s).$$

Consequently we obtain

$$\Gamma_{ik|\mu} \underset{\mu}{\equiv} \Gamma_{ij|\mu} + B_{ij}(z_j) \cdot \Gamma_{jk|\mu}, \quad \text{q.e.d.}$$

Our purpose is to determine

$$t^\mu(s) = t^{\mu-1}(s) + t_\mu(s), \quad g_i^\mu(z_i, s) = g_i^{\mu-1}(z_i, s) + g_{i|\mu}(z, s)$$

which satisfy $(6)_\mu$. Letting

$$t_\mu(s) = (t_{1|\mu}(s), \ldots, t_{r|\mu}(s), \ldots, t_{m|\mu}(s)),$$

we have

$$g_{ik}(g_k^{\mu-1}(z_k, s) + g_{k|\mu}(z_k, s), t^{\mu-1}(s) + t_\mu(s))$$
$$\underset{\mu}{\equiv} g_{jk}(g_k^{\mu-1}(z_k, s), t^{\mu-1}(s)) + B_{ik}(z_k) \cdot g_{k|\mu}(z_k, s) + \sum_{r=1}^{m} t_{r|\mu}(s)\, \beta_{ik|r}(z_j),$$

where $z_i = b_{ik}(z_k)$, while

$$g_{i|\mu}(h_{ij}(z_k, s), s) \underset{\mu}{\equiv} g_{i|\mu}(z_i, s).$$

Therefore, $(6)_\mu$ is equivalent to the equalities

$$\sum_{r=1}^{m} t_{r|\mu}(s)\, \beta_{ik|r}(z_i) + B_{ik}(z_k) \cdot g_{k|\mu}(z_k, s) - g_{i|\mu}(z_i, s) = \Gamma_{ik|\mu}(z_i, s). \tag{18}$$

Now the formula (17) shows that $\{\Gamma_{ik|\mu}(z_i, s)\}$ is a homogeneous polynomial in s of degree μ whose coefficients form a 1-cocycle on $\mathfrak{U} = \{U_i\}$ with coefficients in Θ_0. Consequently, by the above remark (see (10)), we can find homogeneous polynomials $t_{r|\mu}(s)$ with constant coefficients and homogeneous polynomials $g_{i|\mu}(z_i, s)$ whose coefficients are vector-valued holomorphic functions on U_i which satisfy (18). This completes our inductive construction of $t^\mu(s)$ and $g_i^\mu(z_i, s)$.

Now we prove that, if we choose proper solutions $t_{r|\mu}(s)$, $g_{i|\mu}(z_i, s)$ of the equation (18) in each step of the above construction, the power series

$$t(s) = t_1(s) + t_2(s) + \cdots + t_\mu(s) + \cdots, \quad g_i(z_i, s) = z_i + g_{i|1}(z_i, s) + \cdots + g_{i|\mu}(z_i, s) + \cdots$$

converge absolutely and uniformly for $|s| < \varepsilon$ provided that $\varepsilon > 0$ is sufficiently small.

Consider a power series

$$f(s) = \sum f_{h_1 h_2 \ldots h_l} s_1^{h_1} s_2^{h_2} \ldots s_l^{h_l}$$

whose coefficients $f_{h_1 h_2 \ldots h_l}$ are vectors and a power series

$$a(s) = \sum a_{h_1 h_2 \ldots h_l} s_1^{h_1} s_2^{h_2} \ldots s_l^{h_l}$$

with non-negative coefficients $a_{h_1 h_2 \ldots h_l}$. We indicate by writing $f(s) \ll a(s)$ that

$$|f_{h_1 h_2 \ldots h_l}| < a_{h_1 h_2 \ldots h_l}.$$

Let
$$A(s) = \frac{b}{64\, c} \sum_{\mu=1}^{\infty} \frac{1}{\mu^2} c^{\mu} (s_1 + s_2 + \cdots + s_l)^{\mu}.$$

We remark that

$$A(s)^{\nu} \ll \left(\frac{b}{c}\right)^{\nu-1} A(s), \quad \nu = 2, 3, 4, \ldots. \tag{19}$$

Let
$$z_k + y = (z_k^1 + y_1, \ldots, z_k^{\alpha} + y_{\alpha}, \ldots, z_k^n + y_n).$$

We may assume that the power series expansion of $g_{ik}(z_k + y, t)$ in $n+m$ variables $y_1, \ldots, y_n, t_1, \ldots, t_m$ satisfies

$$g_{ik}(z_k + y, t) - b_{ik}(z_k) \ll A_0(y, t), \quad z_k \in U_k \cap U_i, \tag{20}$$

where
$$A_0(y, t) = \frac{b_0}{c_0} \sum_{\mu=1}^{\infty} c_0^{\mu} (y_1 + \cdots + y_n + t_1 + \cdots + t_m)^{\mu}.$$

Moreover, we may assume that

$$h_{ik}(z_k, s) - b_{ik}(z_k) \ll A_0(s), \quad z_k \in U_k \cap U_i, \tag{21}$$

where $A_0(s)$ is the function $A(s)$ in which the constants b, c are replaced by b_0, c_0.

For our purpose it suffices to derive the estimates

$$t^{\mu}(s) \ll A(s), \quad g_i^{\mu}(z_i, s) - z_i \ll A(s) \tag{22$_\mu$}$$

by induction on μ provided that the constants b, c are chosen properly. For $\mu = 1$ the estimates $(22)_1$ are obvious if b is sufficiently large. Assume therefore that estimates $(22)_{\mu-1}$ are established for some μ. We have

$$\Gamma_{ik|\mu}(z_i, s) \underset{\mu}{\equiv} g_i^{\mu-1}(h_{ik}(z_k, s), s) - g_{ik}(g_k^{\mu-1}(z_k, s), t^{\mu-1}(s)),$$

where $z_i = b_{ik}(z_k)$. Letting $$U_i^\delta = \{z_i \mid |z_i| < 1-\delta\},$$

we first estimate $\Gamma_{ik|\mu}(z_i, s)$ for $z_i \in U_i^\delta \cap U_k$, where δ is a sufficiently small positive number such that $\{U_i^\delta\}$ forms a covering of V_0. Set

$$G_i(z_i, s) = g_i^{\mu-1}(z_i, s) - z_i$$

for simplicity and expand $G_i(z_i + y, s)$ into power series in $y_1, \ldots, y_n, s_1, \ldots, s_l$. Since by our hypothesis,

$$G_i(z_i, s) \ll A(s), \quad \text{for } |z_i| < 1,$$

we get

$$G_i(z_i + y, s) - G_i(z_i, s) \ll A(s) \sum \frac{y_1^{\nu_1} y_2^{\nu_2} \cdots y_n^{\nu_n}}{\delta^{\nu_1+\nu_2+\cdots+\nu_n}}, \quad \text{for } |z_i| < 1-\delta,$$

where $\sum$ is extended over all non-negative integers $\nu_1, \nu_2, \ldots, \nu_n$ with $\nu_1 + \nu_2 + \cdots + \nu_n \geq 1$. Letting $y = h_{ik}(z_k, s) - b_{ik}(z_k)$, $z_i = b_{ik}(z_k)$ and using (21), we obtain from this

$$G_i(h_{ik}(z_k, s), s) - G_i(z_i, s) \ll A(s) \left\{ \left(\sum_{\nu=0}^{\infty} \delta^{-\nu} A_0(s)^\nu \right)^n - 1 \right\}, \quad \text{for } z_i \in U_i^\delta \cap U_k.$$

Since $A_0(s)^\nu \ll (b_0/c_0)^{\nu-1} A_0(s)$ for $\nu \geq 2$, we have

$$\frac{A_0(s)^\nu}{\delta^\nu} \ll \left(\frac{b_0}{c_0 \delta}\right)^{\nu-1} \cdot \frac{A_0(s)}{\delta}, \quad \text{for } \nu \geq 2.$$

We may assume that
$$\frac{b_0}{c_0 \delta} < \frac{1}{2}, \tag{23}$$

since (20) and (21) remain valid if we replace c_0 by a larger constant. Hence we have

$$\frac{A_0(s)^\nu}{\delta^\nu} \ll \frac{A_0(s)}{2^{\nu-1}\delta}, \quad \text{for } \nu \geq 2.$$

Using this we obtain

$$G_i(h_{ik}(z_k, s), s) - G_i(z_i, s) \ll A(s) \left\{ \left(1 + \frac{2A_0(s)}{\delta}\right)^n - 1 \right\} \ll \frac{K_0}{\delta} A(s) A_0(s),$$

or

$$g_i^{\mu-1}(h_{ik}(z_k, s), s) - h_{ik}(z_k, s) - g_i^{\mu-1}(z_i, s) + z_i \ll \frac{K_0}{\delta} A(s) A_0(s), \quad \text{for } z_i \in U_i^\delta \cap U_k, \tag{24}$$

where $z_i = b_{ik}(z_k)$ and where K_0 is a constant depending only on n. Assuming that

$$b > b_0, \quad c > c_0, \tag{25}$$

we have $$A_0(s) \ll \frac{b_0}{b} \cdot A_0(s),$$

and therefore $$A(s)\, A_0(s) \ll \frac{b_0}{b} A(s)^2 \ll \frac{b_0}{c} A(s).$$

Consequently we infer from (24) and (21) that

$$g_i^{\mu-1}(h_{ik}(z_k, s), s) - g_i^{\mu-1}(z_i, s) \ll \left(\frac{K_0 b_0}{\delta c} + \frac{b_0}{b}\right) A(s), \quad \text{for } z_i \in U_i^\delta \cap U_k. \tag{26}$$

For any power series

$$f(s) = f_0 + f_1(s) + \cdots + f_\mu(s) + \cdots$$

we denote by $[f(s)]_\mu$ the term $f_\mu(s)$ of degree μ. Then we get from (26)

$$[g_i^{\mu-1}(h_{ik}(z_k, s), s)]_\mu \ll \left(\frac{K_0 b_0}{\delta c} + \frac{b_0}{b}\right) A(s) \quad \text{for } z_k \in U_k \cap U_i^\delta. \tag{27}$$

Next we estimate $g_{ik}(g_k^{\mu-1}(z_k, s), t^{\mu-1}(s))$. We expand $g_{ik}(z_k + y, t)$ into power series in $y_1, \ldots, y_n, t_1, \ldots, t_m$ and let

$$L_{ik}(z_k, y, t) = [g_{ik}(z_k + y, t)]_1$$

be the linear term of the power series. Then we have, by (20),

$$g_{ik}(z_k + y, t) - b_{ik}(z_k) - L_{ik}(z_k, y, t) \ll \frac{b_0}{c_0} \sum_{\mu=2}^{\infty} c_0^\mu (y_1 + \cdots + y_n + t_1 + \cdots + t_m)^\mu.$$

Letting $y = g_k^{\mu-1}(z_k, s) - z_k$, $t = t^{\mu-1}(s)$ and using our inductive hypothesis $(22)_{\mu-1}$, we obtain from this the estimate

$$[g_{ik}(g_k^{\mu-1}(z_k, s), t^{\mu-1}(s))]_\mu \ll \frac{b_0}{c_0} \sum_{\mu=2}^{\infty} c_0^\mu (m+n)^\mu A(s)^\mu.$$

Assume that $$\frac{(m+n)\, b\, c_0}{c} < \frac{1}{2}. \tag{28}$$

Then we have

$$\sum_{\mu=2}^{\infty} c_0^\mu (m+n)^\mu A(s)^\mu \ll \sum_{\mu=2}^{\infty} c_0^\mu (m+n)^\nu \left(\frac{b}{c}\right)^{\mu-1} A(s) \ll \frac{2(m+n)^2 b\, c_0^2}{c} \cdot A(s),$$

and therefore $$[g_{ik}(g_k^{\mu-1}(z_k, s), t^{\mu-1}(s))]_\mu \ll \frac{2(m+n)^2 b\, b_0\, c_0}{c} \cdot A(s).$$

Combining this with (27) we obtain

$$\Gamma_{ik|\mu}(z_i, s) \ll c^* \cdot A(s), \quad \text{for } z_i \in U_i^\delta \cap U_k, \tag{29}$$

where
$$c^* = \frac{K_0 b_0}{\delta c} + \frac{b_0}{b} + \frac{2(m+n)^2 b b_0 c_0}{c}.$$

Now we recall that the $\Gamma_{ik|\mu}(z_i, s)$ satisfy the cocycle condition (17). In particular we have

$$\Gamma_{ik|\mu}(z_i, s) = B_{ik}(z_k) \cdot \Gamma_{ki|\mu}(z_k, s).$$

Combining this with (11) and (29) we get

$$\Gamma_{ik|\mu}(z_i, s) \ll c^* K_1 A(s), \quad \text{for } z_i \in U_i \cap U_k^\delta. \tag{30}$$

For an arbitrary point $z_i \in U_i \cap U_k$ there exists one U_j^δ which contains z_i. Therefore we infer from (17), (29) and (30) that

$$\Gamma_{ik|\mu}(z_i, s) \ll 2 c^* K_1 A(s),$$

and consequently, by Lemma 1, we can choose solutions $t_{r|\mu}(s)$ and $g_{i|\mu}(z_i, s)$ of the equations (18) such that

$$t_\mu(s) \ll 2 c^* K_1 K_2 A(s), \quad g_{i|\mu}(z_i, s) \ll 2 c^* K_1 K_2 A(s).$$

On the other hand, it is clear that, by a proper choice of the constants b and c satisfying our requirements (25), (28), we obtain

$$2 c^* K_1 K_2 < 1.$$

Consequently we obtain

$$t_\mu(s) \ll A(s), \quad g_{i|\mu}(z_i, s) \ll A(s).$$

This proves $(22)_\mu$, q.e.d.

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