Ark. Mat., **32** (1994), 237–243

On a theorem of Korenblum

Kristian Seip

The purpose of this note is to obtain a sharp version of one of the main theorems of Korenblum's paper [3]. This theorem concerns the problem of describing geometrically the zeros of functions in $A^{-\alpha}$, $\alpha>0$, i.e., the set of functions f analytic in the unit disk $U=\{z:|z|<1\}$ with

$$\sup_{z\in U}(1-|z|)^{\alpha}|f(z)|<\infty.$$

We formulate this problem in the following way. For a given sequence $Z=\{z_k\}_{k=1}^{\infty}$ of points from U, we denote by $\varrho(Z)$ the infimum of those α such that some function f in $A^{-\alpha}$ vanishes on Z, i.e., the order of the zero of f at a point z is greater than or equal to the number of occurences of z in Z; we define $\varrho(Z)=\infty$ should no such α exist. The question is, if $\varrho(Z)$ can be expressed in terms of a geometric density of Z.

We solve this problem using the notion of density which was introduced by Korenblum in [3] and in that paper, led to a partial solution. Our contribution is to observe that some fine estimates, due to Specht [7], on the mapping function and its derivative in conformal mapping of so called nearly circular regions, provide the crucial tool for closing the gap in Korenblum's theorem.

We mention that our study also concerns the weighted Bergman spaces [2], since it may be proved that $\varrho(Z)$ does not change if in the definition we replace $A^{-\alpha}$ by those functions f which are analytic in U and satisfy ($z=x+iy$, $0<p<\infty$)

$$\iint_U |f(z)|^p(1-|z|)^{p\alpha-1}\,dx\,dy<\infty.$$

The same is true if we replace $A^{-\alpha}$ by the functions which, according to [6], are of exponential type at most α; this class contains all of the above-mentioned L^p spaces as well as $A^{-\alpha}$. For the definition of this class of functions and some remarks on the analogy to the classical theory of functions of exponential type, we refer to [6]. Here we mention only that the quantity $\varrho(Z)$ appears as the analogue in the unit disk of the closure radius described by Beurling and Malliavin in the context of entire functions of exponential type with a certain growth restriction on the real axis [1].

The density of a sequence Z from U will be measured in the following way. Without loss of generality, assume that $0 \notin Z$. For an arbitrary finite subset F of the unit circle ∂U, let $\{I_k\}$ denote the set of complementary arcs of F. We put

$$\hat{\varkappa}(F) = \sum_k \frac{|I_k|}{2\pi}\left(\log\frac{2\pi}{|I_k|}+1\right),$$

which is called the *Carleson characteristic* of F. The normalized angular distance on ∂U is defined by

$$d(e^{it}, e^{is}) = \frac{|t-s|}{\pi},$$

where it is assumed that $|t-s| \le \pi$. For a finite set $F \subset \partial U$ we define

$$G_F = \left\{ z \in \overline{U} : 1-|z| \ge d\left(\frac{z}{|z|}, F\right)\right\} \cup \{0\}.$$

We put

$$\sigma_F(Z) = \sum_{z_k \in Z \cap G_F} \log\frac{1}{|z_k|},$$

and define the *Korenblum density* of Z to be

$$\delta(Z) = \inf\{\beta : \sup_F(\sigma_F(Z) - \beta\hat{\varkappa}(F)) < \infty\};$$

we define $\delta(Z) = \infty$ if the set on the right-hand side is empty.

Our aim is to show that the following statement is true.

Theorem. *$\varrho(Z) = \delta(Z)$ for every sequence $Z \not\ni 0$.*

Korenblum's original theorem [3,Theorem 3.1] states that

$$\tfrac{1}{2}\varrho(Z) \le \delta(Z) \le 2\varrho(Z).$$

In [5], when solving the interpolation problem for $A^{-\alpha}$, we proved that $\varrho(Z) \le \delta(Z)$ [5, Lemma 4.1]. What remains, therefore, is to prove the inequality

$$\delta(Z) \le \varrho(Z). \tag{1}$$

In order to motivate the proof of this inequality and, indeed, Korenblum's notion of density, we prove first a simple criterion which we obtain directly from Jensen's formula.

Fix some number ε, $0<\varepsilon<1$, and let $0<r<1$. We assume that $f\in A^{-\alpha}$, $f(0)=1$, $|f(z)|\le C(1-|z|)^{-\alpha}$, and let $z_1, z_2, z_3, \ldots$ denote the zeros of f in U, counting multiplicities. Then Jensen's formula applied to the circle $|z|=R=1-\varepsilon(1-r)$ yields

$$\sum_{|z_k|\le r} \log\frac{R}{|z_k|} \le \alpha\log\frac{1}{1-r} + \log C + \alpha\log\frac{1}{\varepsilon},$$

since $r<R$. For $|z_k|\le r$ we have

$$R = 1-\varepsilon(1-r) > r^{\varepsilon} \ge |z_k|^{\varepsilon}.$$

Thus the above inequality implies

$$\sum_{|z_k|<r} \log\frac{1}{|z_k|} \le \frac{\alpha}{1-\varepsilon}\log\frac{1}{1-r} + C',$$

where we have put $C' = (\log C + \alpha\log(1/\varepsilon))/(1-\varepsilon)$. This permits us to conclude that

$$\sup_{r<1}\left(\sum_{|z_k|<r} \log\frac{1}{|z_k|} - \beta\log\frac{1}{1-r}\right) < \infty$$

holds for all $\beta>\alpha$. The corresponding argument with $f\in A^{-\alpha}$ replaced by some function f in the Nevanlinna class leads to the Blaschke condition which, rather remarkably, in that case is sufficient as well. In our situation, however, it is easy to see that the condition just proved is far from being sufficient, since, e.g., a radial sequence $\{z_k\}$ must satisfy the Blaschke condition.

Korenblum's idea is to replace the disk $|z|<r$ in the above argument by the star-like domain G_F so that we focus on the density of $\{z_k\}$ along different rays from the origin; we replace the disk $|z|<R$ by a larger star-like domain containing G_F, map the larger star by a Riemann mapping to U, and apply Jensen's formula as above. For this approach to be successful, on the one hand, the larger star must be so large and smoothly bounded that the Riemann mapping is close to the identity, while on the other hand, the larger star should be sufficiently close to G_F to make the integral appearing in Jensen's formula not too large. These are seemingly contradictory requirements, and the problem is to find an optimal trade-off between them.

We now turn to the proof of (1). We need to show that given some function $f\in A^{-\alpha}$ vanishing on the sequence $Z=\{z_k\}$, we have for every $\beta>\alpha$ that

$$\sup_F(\sigma_F(Z) - \beta\widehat{\varkappa}(F)) < \infty. \tag{2}$$

We fix a finite set $F=\{e^{it_1},e^{it_2},\dots,e^{it_n}\}$, where $0\le t_1<t_2<\dots<t_n<2\pi$; for convenience we define $t_{n+1}=t_1+2\pi$ and $\Delta_k=t_{k+1}-t_k$, $1\le k\le n$. We introduce parameters $1<q\le 2$, $0<\varepsilon\le q/2^{q-1}$ and associate with F a closed Jordan curve $\Gamma_{F;q,\varepsilon}$; $\Gamma_{F;q,\varepsilon}$ is star-shaped with respect to the origin and given in polar coordinates by $z(t)=\varrho(t)e^{it}$, where $\varrho(t)$ is 2π-periodic and defined in the following way. We let

$$\lambda_{q,\varepsilon}(t)=\frac{\varepsilon}{\pi}\left(\frac{t}{\pi}\right)^{q-1},$$

$t\ge 0$; then we put $\varrho(t_1)=1$ and require

$$\varrho'(t)=\begin{cases}-\lambda_{q,\varepsilon}(t-t_k), & t_k\le t<t_k+\frac{1}{4}\Delta_k\\ -\lambda_{q,\varepsilon}(t_k+\frac{1}{2}\Delta_k-t), & t_k+\frac{1}{4}\Delta_k\le t<t_k+\frac{1}{2}\Delta_k\\ \lambda_{q,\varepsilon}(t-t_k-\frac{1}{2}\Delta_k), & t_k+\frac{1}{2}\Delta_k\le t<t_k+\frac{3}{4}\Delta_k\\ \lambda_{q,\varepsilon}(t_k+\Delta_k-t), & t_k+\frac{3}{4}\Delta_k\le t<t_k+\Delta_k.\end{cases}$$

Note that $\varrho(t_k)=1$, $1\le k\le n$, while otherwise $\varrho(t)<1$.

Let $\Omega_{F;q,\varepsilon}$ denote the domain enclosed by the curve $\Gamma_{F;q,\varepsilon}$ and observe that

$$G_F\subset\overline{\Omega_{F;q,\varepsilon}}.$$

We denote by $w=w(z)$ the Riemann mapping function which maps $\Omega_{F;q,\varepsilon}$ conformally onto U so that $w(0)=0$ and $w'(0)>0$; we denote by $z=z(w)$ the inverse function and define $\phi(\theta)=\arg z(e^{i\theta})$.

In order to state the crucial estimates on w and w', we perform a few simple calculations. It follows from the definition of ϱ that

$$\varrho(t)\ge 1-\varepsilon=m,$$
$$|\varrho'(t)|\le\frac{\varepsilon}{\pi},$$

and also that

$$|\varrho'(t)-\varrho'(t_0)|\le\varepsilon\left|\frac{t-t_0}{2\pi}\right|^{q-1}$$

for arbitrary real numbers t,t_0, because of the concavity of $\lambda_{q,\varepsilon}(t)$. Using these facts, we find that

$$\sup_{t_0}\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\left(\frac{\varrho'(t)}{\varrho(t)}-\frac{\varrho'(t_0)}{\varrho(t_0)}\right)\Big/\sin\left(\frac{t-t_0}{2}\right)\right|dt\le\frac{\varepsilon}{m(q-1)}+\frac{\varepsilon^2}{m^2}=\delta.$$

We put

$$\eta=\eta(q,\varepsilon)=6\,\frac{6\varepsilon+\delta}{1-\delta}$$

and claim that we have[1]

$$|z(w)-w|\le\eta, \tag{3}$$

$$|\phi'(\theta)-1|\le\eta, \tag{4}$$

$$|z'(w)-1|\le\eta \tag{5}$$

for $|w|\le 1$, $0\le\theta\le 2\pi$, provided that $m>\frac{1}{2}$ and $\delta<1$. Indeed, note that $z(w)/m$ maps U onto a nearly circular region in the sense of Specht [7]. Then we see that (3) follows from Theorem I of [7] (or trivially, in this case, from (5)), (4) follows from Theorem II of [7], and (5) follows from Theorem III of [7]. We see from Specht's work that the above estimates are not optimal, but they are convenient for our purpose. The main point is that η can be made arbitrarily small for any fixed $q>1$ by choosing a sufficiently small ε.

We consider now a function $f\in A^{-\alpha}$ vanishing on $Z=\{z_k\}$, with $f(0)=1$ and $|f(z)|\le C(1-|z|)^{-\alpha}$. Jensen's formula applied to the function $w\mapsto f(z(w))$ yields

$$\sum_{z_k\in G_F}\log\frac{1}{|w(z_k)|}\le\log C+\frac{\alpha}{2\pi}\int_0^{2\pi}\log\frac{1}{1-|z(e^{i\theta})|}\,d\theta. \tag{6}$$

To estimate the right-hand side of (6), we make the change of variable $t=\phi(\theta)$ in the integral. In view of (4), this gives

$$\frac{\alpha}{2\pi}\int_0^{2\pi}\log\frac{1}{1-|z(e^{i\theta})|}\,d\theta\le\frac{1}{1-\eta}\,\frac{\alpha}{2\pi}\int_0^{2\pi}\log\frac{1}{1-\varrho(t)}\,dt.$$

An explicit computation, using the definition of ϱ, then gives

$$\frac{\alpha}{2\pi}\int_0^{2\pi}\log\frac{1}{1-|z(e^{i\theta})|}\,d\theta\le\frac{q\alpha}{1-\eta}\widehat{\varkappa}(F)+C(q,\varepsilon). \tag{7}$$

To estimate the left-hand side of (6), let $z\neq 0$ be some point in G_F, and let ζ be the point on $\Gamma_{F;q,\varepsilon}$ for which $\arg(\zeta)=\arg(z)$. From (3) it follows that

$$w(z)=w(\zeta)-(\zeta-z)+\xi(\zeta-z)=\omega+\xi(\zeta-z), \tag{8}$$

[1] It is well-known that $z'(w)$ extends to a continuous function on $\bar{U}$ so that the expressions make sense; see, e.g., [4, Theorem 3.5, p. 48].

where

$$|\xi| \le \frac{\eta}{1-\eta} = \gamma.$$

Using (3), we find that

$$\cos\left(\arg \frac{\zeta}{w(\zeta)}\right) \ge 1-\eta^2,$$

so that

$$|\omega|^2 \le 1+|z-\zeta|^2 - 2|z-\zeta|(1-\eta^2) \le (1-(1-\sqrt{2}\eta)|z-\zeta|)^2.$$

Because of this and since $1-|\zeta| \le \eta(1-|z|)$, we obtain from (8) that

$$|w(z)| \le 1-(1-3\gamma)(|\zeta|-|z|) \le |z|+4\gamma(1-|z|) \le |z|^{1-8\gamma},$$

assuming that $|z| \ge \frac{1}{2}$ and $1-8\gamma>0$. Combining (7) with this estimate, we get from (6)

$$\sigma_F(Z) - \frac{q\alpha}{(1-8\gamma)(1-\eta)} \widehat{\varkappa}(F) \le C'(q,\varepsilon).$$

This proves (2) since the number $q/((1-8\gamma)(1-\eta))$ can be made arbitrarily close to 1 by appropriate choices of q and ε.

It seems natural now to go one step further and ask what connection there may be between the condition

$$\sup_F (\sigma_F(Z) - \alpha \widehat{\varkappa}(F)) < \infty$$

and the property that Z be the zero sequence of some $f \in A^{-\alpha}$. We do not know if there is an implication in any direction.

References

1. Beurling, A. and Malliavin, P., On the closure of characters and the zeros of entire functions, *Acta Math.* **118** (1967), 79–93.
2. Horowitz, C., Zeros of functions in the Bergman spaces, *Duke Math. J.* **41** (1974), 693–710.
3. Korenblum, B., An extension of the Nevanlinna theory, *Acta Math.* **135** (1975), 187–219.
4. Pommerenke, C., *Boundary Behaviour of Conformal Maps, Grundlehren der mathematischen Wissenschaften* **299**, Springer-Verlag, Berlin–Heidelberg, 1992.
5. Seip, K., Beurling type density theorems in the unit disk, *Invent. Math.* **113** (1993), 21–39.
6. Seip, K., Cartwright's theorem in the unit disk, *Manuscript* (1993).

7. SPECHT, E. J., Estimates on the mapping function and its derivatives in conformal mapping of nearly circular regions, *Trans. Amer. Math. Soc.* **71** (1951), 183–196.

Received January 22, 1993

Kristian Seip
Department of Mathematical Sciences
Norwegian Institute of Technology
N-7034 Trondheim
Norway