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# Criteria for validity of the maximum modulus principle for solutions of linear parabolic systems

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**Abstract.** We consider systems of partial differential equations of the first order in $t$ and of order $2s$ in the $x$ variables, which are uniformly parabolic in the sense of Petrovskii. We show that the classical maximum modulus principle is not valid in $R^n \times (0,T]$ for $s \geq 2$.

For second order systems we obtain necessary and, separately, sufficient conditions for the classical maximum modulus principle to hold in the layer $R^n \times (0,T]$ and in the cylinder $\Omega \times (0,T]$, where $\Omega$ is a bounded subdomain of $R^n$. If the coefficients of the system do not depend on $t$, these conditions coincide. The necessary and sufficient condition in this case is that the principal part of the system is scalar and that the coefficients of the system satisfy a certain algebraic inequality. We show by an example that the scalar character of the principal part of the system everywhere in the domain is not necessary for validity of the classical maximum modulus principle when the coefficients depend both on $x$ and $t$.

## Introduction

It is well-known that solutions of parabolic second order equations with real coefficients in the cylinder

$$Q_T = \{ (x,t) : x \in \Omega,\ 0 < t \leq T \}, \quad \Omega \subset R^n,$$

satisfy the maximum modulus principle. Namely, for any solution of the equation

$$\frac{\partial u}{\partial t} - \sum_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i(x,t) \frac{\partial u}{\partial x_i} + a_0(x,t) u = 0,$$

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where $((a_{ij}))$ is a positive-definite $(n \times n)$-matrix-valued function and $a_0 \geq 0$, the inequality holds

$$|u(x,t)| \leq \sup\{\, |u(y,\tau)| : (y,\tau) \in \partial Q_T,\ \tau < T \,\}.$$

This classical fact was extended to parabolic second order systems with scalar coefficients of the first and second derivatives in [10]. Maximum principles for weakly coupled parabolic systems are discussed in the books [8], [11].

Furthermore, there exists a large literature on "invariant sets" for non-linear parabolic systems (see, for example, [1], [2], [3], [4], [9], [12] and references there). However, we shall not characterize this interesting field, since here we consider only the maximum modulus principle and since the papers on invariant sets do not contain our results as special cases.

In this paper we find criteria for validity of the classical maximum modulus principle for solutions of the uniformly parabolic system in the sense of Petrovskii

$$\frac{\partial u}{\partial t} - \sum_{|\beta| \leq 2s} \mathcal{A}_\beta(x,t) \partial_x^\beta u = 0. \tag{1}$$

Here $u$ is an $m$-component vector-valued function, $\mathcal{A}_\beta$ are real or complex $(m \times m)$-matrix-valued functions, $\beta = (\beta_1, \dots, \beta_n)$ is a multiindex of order $|\beta| = \beta_1 + \dots + \beta_n$ and $\partial_x^\beta = \partial^{|\beta|}/\partial x_1^{\beta_1} \dots \partial x_n^{\beta_n}$. For $s \geq 1$ the vector-function $u$ is defined in the closure $\overline{R_T^{n+1}}$ of the layer $R_T^{n+1} = R^n \times (0,T]$. In the special case $s=1$ it will be defined also in the closure $\overline{Q}_T$ of the cylinder $Q_T = \Omega \times (0,T]$, where $\Omega$ is a bounded domain in $R^n$.

Throughout the article we make the following assumptions:

(A) The matrix-valued functions $\mathcal{A}_\beta$ are defined in $\overline{R_T^{n+1}}$ and have bounded derivatives in $x$ up to the order $|\beta|$ which satisfy the uniform Hölder condition on $\overline{R_T^{n+1}}$ with exponent $\alpha$, $0 < \alpha \leq 1$, with respect to the parabolic distance

$$d\,[(x,t),(x',t')] = (|x-x'|^2 + |t-t'|^{1/s})^{1/2}.$$

(B) For any point $(x,t) \in \overline{R_T^{n+1}}$, the real parts of the $\lambda$-roots of the equation

$$\det\left( (-1)^s \sum_{|\beta| = 2s} \mathcal{A}_\beta(x,t) \sigma^\beta - \lambda I_m \right) = 0$$

satisfy the inequality $\operatorname{Re} \lambda(x,t,\sigma) \leq -\delta |\sigma|^{2s}$, where $\delta = \text{const} > 0$ for any $\sigma \in R^n$, $I_m$ is the identity matrix of order $m$, and $|\cdot|$ is the Euclidean length of a vector.

We obtain an expression for the best constant $\mathcal{K}(R^n, T)$ in the inequality

$$|u(x,t)| \leq \mathcal{K}(R^n, T) \sup\{\, |u(y,0)| : y \in R^n \,\},$$

where $(x,t)\in R_T^{n+1}$. It is shown that $\mathcal{K}(R^n,T)>1$ for all $s\geq 2$.

For $s=1$, besides the constant $\mathcal{K}(R^n,T)$, we study the best constant $\mathcal{K}(\Omega,T)$ in the inequality

$$|u(x,t)|\leq \mathcal{K}(\Omega,T)\sup\{\,|u(y,\tau)|:(y,\tau)\in\bar{\Gamma}_T\,\},$$

where $(x,t)\in Q_T, \Gamma_T=\{\,(x,t)\in\partial Q_T : t<T\,\}$. The closure $\bar{\Gamma}_T$ of $\Gamma_T$ is called the parabolic boundary of the domain $\Omega\times(0,T)$.

Then we give separate necessary and sufficient conditions for validity of the classical maximum modulus principle (i.e. $\mathcal{K}(\Omega,T)=1$, $\mathcal{K}(R^n,T)=1$) for solutions of the parabolic second order system

$$\frac{\partial u}{\partial t}-\sum_{j,k=1}^{n}\mathcal{A}_{jk}(x,t)\frac{\partial^2 u}{\partial x_j\partial x_k}+\sum_{j=1}^{n}\mathcal{A}_j(x,t)\frac{\partial u}{\partial x_j}+\mathcal{A}_0(x,t)u=0. \tag{2}$$

If the coefficients of the system (2) do not depend on $t$, then the above mentioned necessary and sufficient conditions coincide. More precisely, the following statement concerning the system

$$\frac{\partial u}{\partial t}-\sum_{j,k=1}^{n}\mathcal{A}_{jk}(x)\frac{\partial^2 u}{\partial x_j\partial x_k}+\sum_{j=1}^{n}\mathcal{A}_j(x)\frac{\partial u}{\partial x_j}+\mathcal{A}_0(x)u=0 \tag{3}$$

holds for the case of real coefficients.

**Theorem.** *The classical maximum modulus principle is valid for solutions of the system* (3) *in* $Q_T(R_T^{n+1})$ *if and only if:*

(i) *for all* $x\in\Omega$ $(x\in R^n)$ *the equalities*

$$\mathcal{A}_{jk}(x)=a_{jk}(x)I_m,\quad 1\leq j,k\leq n,$$

*hold, where* $((a_{jk}))$ *is a positive-definite* $(n\times n)$*-matrix-valued function;*

(ii) *for all* $x\in\Omega$ $(x\in R^n)$ *and for any* $\xi_j$, $\varsigma\in R^m$, $j=1,\dots,n$, *with* $(\xi_j,\varsigma)=0$, *the inequality*

$$\sum_{j,k=1}^{n}a_{jk}(x)(\xi_j,\xi_k)+\sum_{j=1}^{n}(\mathcal{A}_j(x)\xi_j,\varsigma)+(\mathcal{A}_0(x)\varsigma,\varsigma)\geq 0$$

*is valid.*

The next assertion immediately follows from this theorem.

**Corollary.** *The classical maximum modulus principle holds for solutions of the system* (3) *in* $Q_T(R_T^{n+1})$ *if and only if condition* (i) *of the theorem is satisfied and*

(ii$'$) *for all* $x\in\Omega$ $(x\in R^n)$ *and any* $\varsigma\in R^m$, $|\varsigma|=1$ *the inequality*

$$\sum_{i,j=1}^{n} b_{ij}(x)[(\mathcal{A}_i(x)\varsigma,\varsigma)(\mathcal{A}_j(x)\varsigma,\varsigma)-(\mathcal{A}_i^*(x)\varsigma,\mathcal{A}_j^*(x)\varsigma)]+4(\mathcal{A}_0(x)\varsigma,\varsigma)\geq 0$$

*is valid. Here* $((b_{ij}))$ *is the* $(n\times n)$*-matrix-valued function inverse of* $((a_{ij}))$ *and* $^*$ *means the passage to the transposed matrix.*

In the paper we demonstrate by an example that the scalar character of the principal part of the system (2) everywhere in the domain is not necessary for validity of the maximum principle when the coefficients depend both on $x$ and $t$.

Finally, it is shown that all the facts concerning the maximum modulus principle for solutions of systems with complex coefficients are corollaries of the corresponding assertions for systems with real coefficients.

In particular, for the scalar parabolic equation with complex coefficients

$$\frac{\partial u}{\partial t}-\sum_{j,k=1}^{n} a_{jk}(x)\frac{\partial^2 u}{\partial x_j\partial x_k}+\sum_{j=1}^{n} a_j(x)\frac{\partial u}{\partial x_j}+a_0(x)u=0$$

we obtain in Subsection 2.2 that *the classical maximum modulus principle is valid in* $Q_T(R_T^{n+1})$ *if and only if:*

(i) *the* $(n\times n)$*-matrix-valued function* $((a_{jk}))$ *is real and positive-definite*;

(ii) *for all* $x\in\Omega$ $(x\in R^n)$ *the inequality*

$$4\,\mathrm{Re}\,a_0(x)\geq \sum_{j,k=1}^{n} b_{jk}(x)\,\mathrm{Im}\,a_j(x)\,\mathrm{Im}\,a_k(x)$$

*holds.*

The present paper is related to our work [6] where we considered the system (2) with constant matrix coefficients $\mathcal{A}_{jk}$ and with $\mathcal{A}_0=\mathcal{A}_1=\ldots=\mathcal{A}_n=0$. In [6] we established that the classical maximum modulus principle holds if and only if $\mathcal{A}_{jk}=a_{jk}I_m$ where $((a_{jk}))$ is a real positive-definite $(n\times n)$-matrix.

We are going to devote a special paper to extend our present results to the so called maximum norm principle, where the role of the modulus is played by the norm in a finite-dimensional normed space.

## 1. Systems of order $2s$ in $R_T^{n+1}$

### 1.1. The case of real coefficients

1.1.1. *Some notations.* We introduce the operators

$$\mathfrak{A}(x,t,\partial/\partial x)=\sum_{|\beta|\leq 2s}\mathcal{A}_\beta(x,t)\partial_x^\beta,\ \mathfrak{A}_0(x,t,\partial/\partial x)=\sum_{|\beta|=2s}\mathcal{A}_\beta(x,t)\partial_x^\beta,$$

where $\mathcal{A}_\beta$ are real $(m\times m)$-matrix-valued functions satisfying the conditions (A) and (B), formulated in the Introduction.

Below we use the following notation.

Let $S=D\times(0,\varrho]$, where $D$ is a domain in $R^n$ and $0<\varrho\leq\infty$. Let, for brevity, either $\mathcal{M}=\overline{S}$ or $\mathcal{M}$ be the parabolic boundary of the domain $D\times(0,\varrho)$. By $C(\mathcal{M})$ we denote the space of continuous and bounded $m$-component vector-valued functions on $\mathcal{M}$ with the norm

$$\|u\|=\sup\{\,|u(q)|:q\in\mathcal{M}\,\}.$$

By $C^{(k,1)}(S)$ we mean the space of $m$-component vector-valued functions $u(x,t)$ on $S$ whose derivatives with respect to $x$ up to order $k$ and first derivative with respect to $t$ are continuous. By $C^{k+\alpha}(R^n)$ we denote the space of $m$-component vector-valued functions with continuous and bounded derivatives with respect to $x$ up to order $k$ which satisfy the uniform Hölder condition with exponent $\alpha$. Finally, let $C^{k+\alpha,\alpha/2s}(\overline{R_T^{n+1}})$ denote the space of $m$-component vector-valued functions with derivatives up to order $k$ with respect to $x$ which are bounded in $\overline{R_T^{n+1}}$ and satisfy the uniform Hölder condition on $\overline{R_T^{n+1}}$ with exponent $\alpha$ with respect to the parabolic distance. For the space of $(m\times m)$-matrix-valued functions, defined on $\overline{R_T^{n+1}}$ and having similar properties, we use the notation $C_m^{k+\alpha,\alpha/2s}(\overline{R_T^{n+1}})$.

For $s\geq 1$ we put

$$\mathcal{K}(R^n,T)=\sup\frac{\|u\|_{C(\overline{R_T^{n+1}})}}{\|u|_{t=0}\|_{C(R^n)}},\tag{1.1}$$

where the supremum is taken over all functions in the class

$$C^{(2s,1)}(R_T^{n+1})\cap C(\overline{R_T^{n+1}})$$

satisfying the system

$$\frac{\partial u}{\partial t}-\mathfrak{A}(x,t,\partial/\partial x)u=0.$$

Let $R_+^{n+1}=\{(x,t)\in R^{n+1}:t>0\}$. We introduce one more constant

$$\mathcal{K}_0(y)=\sup\frac{\|u\|_{C(\overline{R_+^{n+1}})}}{\|u|_{t=0}\|_{C(R^n)}},\tag{1.2}$$

where the supremum is taken over all functions in the class

$$C^{(2s,1)}(R_+^{n+1})\cap C(\overline{R_+^{n+1}})$$

satisfying the system

$$\frac{\partial u}{\partial t}-\mathfrak{A}_0(y,0,\partial/\partial x)u=0$$

and $y\in R^n$ plays the role of a parameter.

1.1.2. *Representations for the constants $\mathcal{K}(R^n,T)$ and $\mathcal{K}_0(y)$.* According to [5] there exists one and only one function in the class

$$C^{(2s,1)}(R_T^{n+1})\cap C(\overline{R_T^{n+1}})$$

which is the solution of the Cauchy problem

$$\frac{\partial u}{\partial t}-\mathfrak{A}(x,t,\partial/\partial x)u=0 \text{ in } R_T^{n+1},\quad u\,|_{t=0}=\psi,\tag{1.3}$$

with $\psi\in C(R^n)$. This solution can be represented in the form

$$u(x,t)=\int_{R^n}G(t,0,x,\eta)\psi(\eta)\,d\eta.\tag{1.4}$$

Here $G(t,\tau,x,\eta)$ is the Green matrix (or the fundamental matrix of solutions of the Cauchy problem (1.3)). The Green matrix for the system

$$\frac{\partial u}{\partial t}-\mathfrak{A}(y,t,\partial/\partial x)u=0$$

will be denoted by $\mathcal{G}(t,\tau,x-\eta;y)$.

The Green matrix $\mathcal{G}_0(t-\tau,x-\eta;y)$ for the system

$$\frac{\partial u}{\partial t}-\mathfrak{A}_0(y,0,\partial/\partial x)u=0$$

has the following representation

$$\mathcal{G}_0(t-\tau,x-\eta;y)=(2\pi)^{-n}\int_{R^n}\exp\Big[(-1)^s\sum_{|\beta|=2s}\mathcal{A}_\beta(y,0)\sigma^\beta(t-\tau)\Big]e^{i(x-\eta,\sigma)}\,d\sigma,$$

where $\sigma=(\sigma_1,\dots,\sigma_n)\in R^n$. This implies

$$\mathcal{G}_0(t-\tau,x-\eta;y)=(t-\tau)^{-n/2s}P\left(\frac{x-\eta}{(t-\tau)^{1/2s}};y\right), \tag{1.5}$$

with

$$P(x;y)=(2\pi)^{-n}\int_{R^n}e^{i(x,\sigma)}\exp\left[(-1)^s\sum_{|\beta|=2s}\mathcal{A}_\beta(y,0)\sigma^\beta\right]d\sigma.$$

When discussing the system (1) with coefficients depending only on $t$ we use the notation

$$\mathcal{A}_\beta(t),\ \mathfrak{A}(t,\partial/\partial x),\ \mathfrak{A}_0(t,\partial/\partial x),\ \mathcal{G}(t,\tau,x-\eta),\ \mathcal{G}_0(t-\tau,x-\eta),\ P(x).$$

**Theorem 1.1.** *The following formula is valid*

$$\mathcal{K}(R^n,T)=\sup_{x\in R^n}\sup_{0<t\le T}\sup_{|z|=1}\int_{R^n}|G^*(t,0,x,\eta)z|\,d\eta, \tag{1.6}$$

*where the* $^*$ *denotes passage to the transposed matrix.*

*In particular,*

$$\mathcal{K}_0(y)=\sup_{|z|=1}\int_{R^n}|P^*(\eta;y)z|\,d\eta. \tag{1.7}$$

*Proof.* Let $(x,t)$ be a fixed point in $R^{n+1}_T$. We find the norm $|u(x,t)|$ of the mapping $C(R^n)\ni\psi\to u(x,t)\in R^m$, where $u$ is defined by (1.4). Using the properties of the inner product in $R^m$ and the fact that the supremum operations commute, we obtain

$$\begin{aligned}|u(x,t)|&=\sup_{|\psi|\le1}\left|\int_{R^n}G(t,0,x,\eta)\psi(\eta)\,d\eta\right|\\&=\sup_{|\psi|\le1}\sup_{|z|=1}\left(z,\int_{R^n}G(t,0,x,\eta)\psi(\eta)\,d\eta\right)\\&=\sup_{|z|=1}\sup_{|\psi|\le1}\int_{R^n}(z,G(t,0,x,\eta)\psi(\eta))\,d\eta\\&=\sup_{|z|=1}\sup_{|\psi|\le1}\int_{R^n}(G^*(t,0,x,\eta)z,\psi(\eta))\,d\eta.\end{aligned} \tag{1.8}$$

Let $N_{(x,t)}(z)$ denote the set of points $\eta\in R^n$ on which

$$G^*(t,0,x,\eta)z=0.$$

By (1.8) we have

$$|u(x,t)| = \sup_{|z|=1} \sup_{|\psi|\leq 1} \int_{R^n\setminus N_{(x,t)}(z)} (G^*(t,0,x,\eta)z, \psi(\eta))\, d\eta.$$

Clearly, the inside supremum of the last integral is attained at the vector-valued function

$$\frac{G^*(t,0,x,\eta)z}{|G^*(t,0,x,\eta)z|}.$$

Consequently,

$$|u(x,t)| = \sup_{|z|=1} \int_{R^n\setminus N_{(x,t)}(z)} |G^*(t,0,x,\eta)z|\, d\eta = \sup_{|z|=1} \int_{R^n} |G^*(t,0,x,\eta)z|\, d\eta.$$

Using the definition (1.1) of the constant $\mathcal{K}(R^n,T)$, we obtain

$$\begin{aligned}
\mathcal{K}(R^n,T) &= \sup\{ \|u\|_{C(\overline{R_T^{n+1}})} : \|u|_{t=0}\|_{C(R^n)} \leq 1,\\
&\qquad \partial u/\partial t - \mathfrak{A}(x,t,\partial/\partial x)u = 0 \text{ in } R_T^{n+1}\}\\
&= \sup_{x\in R^n} \sup_{0<t\leq T} \sup\{ |u(x,t)| : \|u|_{t=0}\|_{C(R^n)} \leq 1,\\
&\qquad \partial u/\partial t - \mathfrak{A}(x,t,\partial/\partial x)u = 0 \text{ in } R_T^{n+1} \}\\
&= \sup_{x\in R^n} \sup_{0<t\leq T} |u(x,t)|\\
&= \sup_{x\in R^n} \sup_{0<t\leq T} \sup_{|z|=1} \int_{R^n} |G^*(t,0,x,\eta)z|\, d\eta
\end{aligned} \tag{1.9}$$

which gives the representation (1.6).

Substituting $\mathcal{G}_0(t-\tau, x-\eta; y)$ from (1.5) into (1.6) in place of $G(t,0,x,\eta)$, we arrive at the representation of $\mathcal{K}_0(y)$ in the form (1.7). □

1.1.3. *Necessity of the condition $s=1$ for the maximum modulus principle in $R_T^{n+1}$.* Henceforth all positive constants with non-significant value will be denoted by $c$ with various indices.

**Lemma 1.1.** *The inequality*

$$\mathcal{K}(R^n,T) \geq \sup\{ \mathcal{K}_0(y) : y \in R^n \} \tag{1.10}$$

*is valid.*

*Proof.* From (1.6) it follows

$$\mathcal{K}(R^n,T) \geq \lim_{r\to 0} \sup_{y\in R^n} \sup_{0<t\leq T} \sup_{|z|=1} \int_{B_r(y)} |G^*(t,0,y,\eta)z|\, d\eta, \tag{1.11}$$

where $B_r(y)=\{x\in R^n : |x-y|<r\}$.

We denote by $\|L\|$ the norm

$$\sup\{\,|Lz| : z\in R^m,\ |z|=1\,\}$$

of the $(m\times m)$-matrix $L$. Since $\mathcal{A}_\beta\in C_m^{\alpha,\alpha/2s}(\overline{R_T^{n+1}})$, then, according to estimates given in [5], the Green matrix $G(t,\tau,y,\eta)$ for $\tau=0$ admits the representation

$$G(t,0,y,\eta)=\mathcal{G}_0(t,y-\eta;\,y)+[\mathcal{G}_0(t,y-\eta;\eta)-\mathcal{G}_0(t,y-\eta;y)]+W(t,y,\eta), \tag{1.12}$$

where

$$\begin{aligned}\|\mathcal{G}_0(t,y-\eta;\eta)-\mathcal{G}_0(t,y-\eta;y)\|&\\ \le c_1|y-\eta|^\alpha t^{-n/2s}\exp&\left[-c_2\left(\frac{|y-\eta|}{t^{1/2s}}\right)^{2s/(2s-1)}\right],\end{aligned} \tag{1.13}$$

$$\|W(t,y,\eta)\|\le c_3t^{-(n-\alpha)/2s}\exp\left[-c_4\left(\frac{|y-\eta|}{t^{1/2s}}\right)^{2s/(2s-1)}\right]. \tag{1.14}$$

Using (1.12)–(1.14) and the representation (1.5), from (1.11) we obtain

$$\begin{aligned}\mathcal{K}(R^n,T)&\ge \sup_{y\in R^n}\sup_{|z|=1}\lim_{r\to 0}\overline{\lim_{t\to+0}}\left\{\int_{B_r(y)}|\mathcal{G}_0^*(t,y-\eta;y)|z\,d\eta-c_5r^\alpha-c_6t^{\alpha/2s}\right\}\\ &=\sup_{y\in R^n}\sup_{|z|=1}\lim_{r\to 0}\overline{\lim_{t\to+0}}\int_{B_r(y)}t^{-n/2s}\left|P^*\left(\frac{y-\eta}{t^{1/2s}};y\right)z\right|d\eta\\ &=\sup_{y\in R^n}\sup_{|z|=1}\lim_{r\to 0}\lim_{t\to+0}\int_{B_{rt^{-1/2s}}}|P^*(x;y)z|\,dx\\ &=\sup_{y\in R^n}\sup_{|z|=1}\int_{R^n}|P^*(x;y)z|\,dx.\end{aligned}$$

This and (1.7) yield the inequality for $\mathcal{K}(R^n,T)$ given in the statement of the lemma. □

**Lemma 1.2.** *If the classical maximum modulus principle is valid for solutions of the system*

$$\frac{\partial u}{\partial t}-\sum_{1\le|\beta|\le 2s}\mathcal{A}_\beta(t)\partial_x^\beta u=0 \tag{1.15}$$

*in* $R_T^{n+1}$, *then* $\mathcal{A}_\beta(t)=a_\beta(t)I_m$, *where* $a_\beta$ *are scalar functions.*

*Proof.* 1. *The structure of the matrix* $\mathcal{G}$. From (1.8) and (1.9) it follows that

$$\mathcal{K}(R^n,T) \geq \sup_{|z|=1} \sup_{|\psi|\leq 1} \int_{R^n} (\mathcal{G}^*(t,0,x-\eta)z, \psi(\eta))\, d\eta, \tag{1.16}$$

where $(x,t)$ is an arbitrary point of the layer $R_T^{n+1}$.

Let $z$ be a fixed unit vector in $R^m$. Since

$$\int_{R^n} \mathcal{G}(t,0,x-\eta)\, d\eta = I_m, \tag{1.17}$$

(see [5]), then the set $R^n \setminus N_{(x,t)}(z)$ has non-zero measure for any fixed $z\in R^m$, $|z|=1$, and $(x,t)\in R_T^{n+1}$.

Suppose $\mathcal{K}(R^n,T)=1$ and let there exist a unit $m$-dimensional vector $z_0$ and the set $M\subset R^n\setminus N_{(x,t)}(z_0)$, $\mathrm{mes}_n\, M>0$, such that for all $\eta\in M$ the inequality

$$z_0 \neq \mathcal{G}^*(t,0,x-\eta)z_0/|\mathcal{G}^*(t,0,x-\eta)z_0|$$

holds. Then, using (1.16) and (1.17), we obtain

$$\begin{aligned}
1=\mathcal{K}(R^n,T) &\geq \sup_{|\psi|\leq 1} \int_{R^n} (\mathcal{G}^*(t,0,x-\eta)z_0, \psi(\eta))\, d\eta \\
&= \sup_{|\psi|\leq 1} \int_{R^n\setminus N_{(x,t)}(z_0)} (\mathcal{G}^*(t,0,x-\eta)z_0, \psi(\eta))\, d\eta \\
&= \int_{R^n\setminus N_{(x,t)}(z_0)} \frac{(\mathcal{G}^*(t,0,x-\eta)z_0,\ \mathcal{G}^*(t,0,x-\eta)z_0)}{|\mathcal{G}^*(t,0,x-\eta)z_0|}\, d\eta \\
&> \int_{R^n\setminus N_{(x,t)}(z_0)} (\mathcal{G}^*(t,0,x-\eta)z_0, z_0)\, d\eta \\
&= \int_{R^n} (\mathcal{G}^*(t,0,x-\eta)z_0, z_0)\, d\eta = 1.
\end{aligned}$$

Consequently, if $\mathcal{K}(R^n,T)=1$, then for all $z\in R^m$ with $|z|=1$ and for almost all $\eta\in R^n\setminus N_{(x,t)}(z)$ one has

$$z=\mathcal{G}^*(t,0,x-\eta)z/|\mathcal{G}^*(t,0,x-\eta)z|. \tag{1.18}$$

Let $g_{jk}$, $j,k=1,\dots,m$, denote the elements of the matrix $\mathcal{G}$. Setting $z_1=(1,0,\dots,0)^*,\dots,z_m=(0,0,\dots,1)^*$ successively instead of $z$ in (1.18) and taking into account the continuity of the Green matrix $\mathcal{G}(t,\tau,x-\eta)$ for $t>\tau$, we find that $g_{jk}(t,0,x-\eta)=0$ for $j\neq k$ for all $\eta\in R^n\setminus N_{(x,t)}(z_k)$.

Since $g_{jk}(t,0,x-\eta)=0$ for $j=1,2,...,m$ and for $\eta\in N_{(x,t)}(z_k)$, $k=1,2,...,m$, we conclude that $g_{jk}(t,0,x-\eta)=0$ for $j\neq k$ and for all $\eta\in R^n$. Now we put $z'=m^{-1/2}(1,1,...,1)^*$ instead of $z$ in (1.18). Then

$$[g_{11}^2(t,0,x-\eta)+...+g_{mm}^2(t,0,x-\eta)]^{1/2}=m^{1/2}g_{jj}(t,0,x-\eta)$$

for all $j=1,2,...,m$ and for $\eta\in R^n\setminus N_{(x,t)}(z')$. Hence making use of the equalities $g_{jj}(t,0,x-\eta)=0$ for $j=1,2,...,m$ and for $\eta\in N_{(x,t)}(z')$, we get

$$g_{11}(t,0,x-\eta)=g_{22}(t,0,x-\eta)=...=g_{mm}(t,0,x-\eta)$$

for all $\eta\in R^n$.

Let $g(t,0,x-\eta)=g_{jj}(t,0,x-\eta)$, $1\leq j\leq m$, and assume that $\mathcal{K}(R^n,T)=1$. Then the solution of the Cauchy problem for the system (1.15) has the form

$$u(x,t)=\int_{R^n} g(t,0,x-\eta)\psi(\eta)\,d\eta, \tag{1.19}$$

where $\psi(\eta)=u(\eta,0)$.

2. *The structure of the operator* $\mathfrak{A}$. By $\psi_0$ we denote a scalar function that is continuous and bounded on $R^n$. Let

$$u_0(x,t)=\int_{R^n} g(t,0,x-\eta)\psi_0(\eta)\,d\eta. \tag{1.20}$$

According to (1.19) the vector-valued function $h_z(x,t)=u_0(x,t)z$, with $z\in R^m$, is a solution of the Cauchy problem

(1.21)
$$\frac{\partial h_z}{\partial t}-\mathfrak{A}(t,\partial/\partial x)h_z=\frac{\partial h_z}{\partial t}-\sum_{1\leq|\beta|\leq 2s}\mathcal{A}_\beta(t)\partial_x^\beta h_z=0 \text{ in } R_T^{n+1},\quad h_z\,|_{t=0}=\psi_0 z.$$

Setting $z_1=(1,0,...,0)^*,...,z_m=(0,0,...,1)^*$ successively instead of $z$ in (1.21), we obtain $m^2$ boundary value problems

$$\delta_{p,q}\frac{\partial u_0}{\partial t}-\sum_{1\leq|\beta|\leq 2s}\mathcal{A}_\beta^{(p,q)}(t)\partial_x^\beta u_0=0 \text{ in } R_T^{n+1},\quad u_0\,|_{t=0}=\psi_0. \tag{1.22}$$

Here $\delta_{p,q}$ is the Kronecker symbol, $\mathcal{A}_\beta^{(p,q)}$ is the element of the matrix $\mathcal{A}_\beta$ placed at the intersection of the $p$th row and the $q$th column, $p,q=1,2,...,m$.

Suppose the initial function $\psi_0(\eta)$ in (1.20) has a compact support. Since the Green matrix $g(t,\tau,x-\eta)I_m$ satisfies the inequality

$$\|g(t,\tau,x-\eta)I_m\| = |g(t,\tau,x-\eta)| \leq c_1(t-\tau)^{-n/2s}\exp\left[-c_2\left(\frac{|x-\eta|^{2s}}{t-\tau}\right)^{1/(2s-1)}\right]$$

(see [5]), then (1.20) implies the estimate

$$|u_0(x,t)| \leq c_3(t)\exp(-c_4(t)|x|^{2s/(2s-1)})$$

for any fixed $t>0$.

Applying the Fourier transform with respect to the variables $x_1,...,x_n$ to the equation (1.22), we get

$$\delta_{p,q}\frac{d(Fu_0)}{dt}-(Fu_0)\sum_{1\leq|\beta|\leq 2s} i^{|\beta|}\mathcal{A}_\beta^{(p,q)}(t)\sigma^\beta = 0. \tag{1.23}$$

Let $p\neq q$. Since the function $\psi_0$ determining $u_0$ by (1.20) is arbitrary, the last equality yields $\mathcal{A}_\beta^{(p,q)}(t)=0$ for all $p\neq q$ and all multiindices $\beta$, $1\leq|\beta|\leq 2s$.

Suppose now that $p=q$. After integrating the equation (1.23) with account taken of the initial condition $Fu_0=F\psi_0$ for $t=0$, we find

$$Fu_0 = F\psi_0\exp\left[\sum_{1\leq|\beta|\leq 2s} i^{|\beta|}\sigma^\beta\int_0^t \mathcal{A}_\beta^{(p,q)}(\tau)\,d\tau\right].$$

Therefore,

$$F\psi_0\exp\left[\sum_{1\leq|\beta|\leq 2s} i^{|\beta|}\sigma^\beta\int_0^t \mathcal{A}_\beta^{(p,p)}(\tau)\,d\tau\right]$$
$$= F\psi_0\exp\left[\sum_{1\leq|\beta|\leq 2s} i^{|\beta|}\sigma^\beta\int_0^t \mathcal{A}_\beta^{(q,q)}(\tau)\,d\tau\right],$$

where $p,q=1,2,...,m$. Hence $\mathcal{A}_\beta^{(p,p)}(t)=\mathcal{A}_\beta^{(q,q)}(t)$ for all $p,q=1,2,...,m$ and for all multiindices $\beta$, $1\leq|\beta|\leq 2s$. Thus, if $\mathcal{K}(R^n,T)=1$, then the operator $\partial/\partial t-\mathfrak{A}(t,\partial/\partial x)$ in the left-hand side of (1.15) satisfies the equality

$$\frac{\partial u}{\partial t}-\mathfrak{A}(t,\partial/\partial x)u = \frac{\partial u}{\partial t}-\sum_{1\leq|\beta|\leq 2s} a_\beta(t)\partial_x^\beta u,$$

where $a_\beta$ are scalar functions. $\square$

**Lemma 1.3.** *Let the parabolic system have the form*

$$\frac{\partial u}{\partial t} - \sum_{|\beta|=2s} a_\beta \partial_x^\beta u = 0, \tag{1.24}$$

*where $a_\beta$ are constant scalar coefficients. Then the equality $\mathcal{K}(R^n, T)=1$ is valid if and only if $s=1$.*

*Proof. The sufficiency* of the condition $s=1$ for the equality

$$\mathcal{K}(R^n, T) = 1$$

follows from the positivity of the fundamental solution $g_0(t-\tau, x-\eta)$ of the Cauchy problem for the parabolic second order equation with constant real coefficients

$$\frac{\partial u}{\partial t} - \sum_{j,k=1}^{n} a_{jk} \frac{\partial^2 u}{\partial x_j \partial x_k} = 0$$

and from the formula

$$u(x,t) = \int_{R^n} g_0(t, x-\eta)\psi(\eta)\, d\eta \tag{1.25}$$

for the solution of the Cauchy problem

$$\frac{\partial u}{\partial t} - \sum_{j,k=1}^{n} a_{jk} \frac{\partial^2 u}{\partial x_j \partial x_k} = 0 \text{ in } R_T^{n+1}, \quad u(x,0) = \psi(x),$$

where $u$ and $\psi$ are $m$-component vector-valued functions, $\psi \in C(R^n)$.

*Necessity.* The solution of the Cauchy problem for the system (1.24) can be expressed by (1.25) for any $s$, where

$$g_0(t-\tau, x-\eta) = (t-\tau)^{-n/2s} P\left(\frac{x-\eta}{(t-\tau)^{1/2s}}\right)$$

is the fundamental solution of the Cauchy problem for the scalar equation (1.24), i.e. for $m=1$. Consequently, the constant

$$\mathcal{K}(R^n, T) = \int_{R^n} |g_0(t, x-\eta)|\, d\eta = \int_{R^n} |P(\eta)|\, d\eta$$

does not depend on $m$ and one can put $m=1$.

We note that the solution of the Cauchy problem

$$\frac{\partial v}{\partial t} - a_{1\dots 1}\frac{\partial^{2s} v}{\partial x_1^{2s}} = 0 \text{ in } R_T^2,\ v(x_1,0)=\psi(x_1), \tag{1.26}$$

where sign $a_{1\dots 1}=(-1)^{s-1}$ and $v=v(x_1,t)$ is a scalar function, also satisfies the Cauchy problem for the scalar equation (1.24) with the initial condition $u(x,0)=\psi(x_1)$. Therefore,

$$\mathcal{K}(R^n,T) \geq k = \sup(\|v\|_{C(\overline{R_T^2})}/\|v\,|_{t=0}\,\|_{C(R^1)}),$$

where the supremum is taken over all solutions of (1.26) in the class

$$C^{(2s,1)}(R_T^2)\cap C(\overline{R_T^2}).$$

Henceforth in this lemma $x_1$ is denoted by $x$. We set

$$a = |a_{1\dots 1}|^{1/2s}.$$

The solution of the Cauchy problem

$$\frac{\partial v}{\partial t} + (-1)^s a^{2s}\frac{\partial^{2s} v}{\partial x^{2s}} = 0 \text{ in } R_T^2,\ v(x,0)=\psi(x),$$

$\psi\in C(R^1)$, has the form

$$v(x,t) = t^{-1/2s}\int_{-\infty}^{\infty} P\left(\frac{x-\eta}{t^{1/2s}}\right)\psi(\eta)\,d\eta, \tag{1.27}$$

where

$$\begin{aligned} P(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix\sigma}\exp(-a^{2s}|\sigma|^{2s})\,d\sigma \\ &= \frac{1}{\pi}\int_0^{\infty}\cos(x\sigma)\exp(-a^{2s}\sigma^{2s})\,d\sigma \\ &= \frac{1}{\pi a}\int_0^{\infty}\cos\left(\frac{x\theta}{a}\right)\exp(-\theta^{2s})\,d\theta. \end{aligned}$$

From (1.27) it follows that

$$k = \int_{-\infty}^{+\infty} |P(\eta)|\,d\eta.$$

Since

$$t^{-1/2s}\int_{-\infty}^{\infty} P\left(\frac{x-\eta}{t^{1/2s}}\right) d\eta = \int_{-\infty}^{\infty} P(\eta)\, d\eta = 1,$$

then $k>1$ in case $P(\eta)$ changes sign.

We show that $P(\eta)$ is a function with alternating signs. Put

$$\mathcal{F}_\lambda(r) = \int_0^\infty \cos(r\theta)\exp(-\theta^\lambda)\, d\theta, \tag{1.28}$$

where $r\geq 0$, $\lambda\geq 1$. Then $P(x)=(\pi a)^{-1}\mathcal{F}_{2s}(x/a)$. Since $\mathcal{F}_\lambda(0)>0$, then $\mathcal{F}_\lambda(r)>0$ with $r\in(0,\varepsilon]$ for some $\varepsilon>0$. We verify that $\mathcal{F}_\lambda\left(\frac{3}{2}\pi\right)<0$ for $\lambda\geq 4$.

Integrating by parts in (1.28), we find

$$\begin{aligned}
\mathcal{F}_\lambda(r) &= \frac{1}{r}\int_0^\infty \exp(-\theta^\lambda)\, d(\sin(r\theta)) \\
&= \frac{1}{r}\exp(-\theta^\lambda)\sin(r\theta)\Big|_0^\infty + \frac{\lambda}{r}\int_0^\infty \theta^{\lambda-1}\exp(-\theta^\lambda)\sin(r\theta)\, d\theta \\
&= \frac{\lambda}{r}\int_0^\infty \theta^{\lambda-1}\exp(-\theta^\lambda)\sin(r\theta)\, d\theta = \frac{\lambda}{r}(J_\lambda(r)+L_\lambda(r)),
\end{aligned}$$

where

$$\begin{aligned}
J_\lambda(r) &= \int_0^2 \theta^{\lambda-1}\exp(-\theta^\lambda)\sin(r\theta)\, d\theta, \\
L_\lambda(r) &= \int_2^\infty \theta^{\lambda-1}\exp(-\theta^\lambda)\sin(r\theta)\, d\theta.
\end{aligned}$$

The maximum value of the function $f(\theta)=\theta^{\lambda-1}\exp(-\theta^\lambda)$ is attained at $\theta_0=((\lambda-1)/\lambda)^{1/\lambda}<1$. Hence $L_\lambda\left(\frac{3}{2}\pi\right)<0$ for $\lambda\geq 1$, so it suffices to show that $J_\lambda\left(\frac{3}{2}\pi\right)<0$ when $\lambda\geq 4$.

Estimating each of four integrals in the equality

$$\begin{aligned}
J_\lambda\left(\tfrac{3}{2}\pi\right) &= \int_0^{2/3} \theta^{\lambda-1}\exp(-\theta^\lambda)\sin\left(\tfrac{3}{2}\pi\theta\right) d\theta \\
&\quad + \int_{2/3}^1 \theta^{\lambda-1}\exp(-\theta^\lambda)\sin\left(\tfrac{3}{2}\pi\theta\right) d\theta + \int_1^{4/3} \theta^{\lambda-1}\exp(-\theta^\lambda)\sin\left(\tfrac{3}{2}\pi\theta\right) d\theta \\
&\quad + \int_{4/3}^2 \theta^{\lambda-1}\exp(-\theta^\lambda)\sin\left(\tfrac{3}{2}\pi\theta\right) d\theta,
\end{aligned}$$

we get

$$\begin{aligned}
J_\lambda\left(\tfrac{3}{2}\pi\right) &\le \int_0^{2/3} \theta^{\lambda-1}\exp(-\theta^\lambda)\,d\theta + \left(\tfrac{2}{3}\right)^{\lambda-1}\exp\left(-\left(\tfrac{2}{3}\right)^\lambda\right)\int_{2/3}^1 \sin\left(\tfrac{3}{2}\pi\theta\right)d\theta \\
&\quad + \left(\tfrac{4}{3}\right)^{\lambda-1}\exp\left(-\left(\tfrac{4}{3}\right)^\lambda\right)\int_1^{4/3}\sin\left(\tfrac{3}{2}\pi\theta\right)d\theta + \int_{4/3}^2 \theta^{\lambda-1}\exp(-\theta^\lambda)\,d\theta \\
&= \lambda^{-1}\left(1-\exp\left(-\left(\tfrac{2}{3}\right)^\lambda\right)\right) - \tfrac{2}{3\pi}\left(\tfrac{2}{3}\right)^{\lambda-1}\exp\left(-\left(\tfrac{2}{3}\right)^\lambda\right) \\
&\quad - \tfrac{2}{3\pi}\left(\tfrac{4}{3}\right)^{\lambda-1}\exp\left(-\left(\tfrac{4}{3}\right)^\lambda\right) + \lambda^{-1}\left(\exp\left(-\left(\tfrac{4}{3}\right)^\lambda\right) - \exp(-2^\lambda)\right) \\
&\le \left(\tfrac{2}{3}\right)^\lambda\left(\lambda^{-1} - \pi^{-1}\exp\left(-\left(\tfrac{2}{3}\right)^\lambda\right)\right) \\
&\quad + \left(\lambda^{-1} - \tfrac{2}{3\pi}\left(\tfrac{4}{3}\right)^{\lambda-1}\right)\exp\left(-\left(\tfrac{4}{3}\right)^\lambda\right) - \lambda^{-1}\exp(-2^\lambda).
\end{aligned}$$

The functions

$$f_1(\lambda) = \lambda^{-1} - \pi^{-1}\exp\left(-\left(\tfrac{2}{3}\right)^\lambda\right), \quad f_2(\lambda) = \lambda^{-1} - \tfrac{2}{3\pi}\left(\tfrac{4}{3}\right)^{\lambda-1}$$

are monotone decreasing as $\lambda$ increases. Since $f_1(4)<0$, $f_2(4)<0$, then $J_\lambda\left(\frac{3}{2}\pi\right)<0$ for $\lambda \ge 4$. Taking into account that

$$\mathcal{F}_\lambda(r) = (\lambda/r)(J_\lambda(r)+L_\lambda(r)), \quad P(x) = (\pi a)^{-1}\mathcal{F}_{2s}(x/a),$$
$$L_\lambda\left(\tfrac{3}{2}\pi\right) < 0 \quad \text{for } \lambda \ge 1, \text{ and } \mathcal{F}_\lambda(0) > 0,$$

we arrive at the conclusion that $P$ changes sign for $s \ge 2$. Thus, $\mathcal{K}(R^n,T) \ge k > 1$ for $s \ge 2$. □

**Theorem 1.2.** *The classical maximum modulus principle is not valid for solutions of the system*

$$\frac{\partial u}{\partial t} - \mathfrak{A}(x,t,\partial/\partial x)u = 0 \quad \textit{in } R_T^{n+1}$$

*if* $s>1$.

*Proof.* Lemma 1.2 implies that the equality $\mathcal{K}_0(y)=1$ is valid for the system

$$\frac{\partial u}{\partial t} - \sum_{|\beta|=2s} \mathcal{A}_\beta(y,0)\partial_x^\beta u = 0$$

only if $\mathcal{A}_\beta(y,0) = a_\beta(y,0)I_m$.

By Lemma 1.3 we have $\mathcal{K}_0(y)>1$ for the system

$$\frac{\partial u}{\partial t} - \sum_{|\beta|=2s} a_\beta(y,0)\partial_x^\beta u = 0,$$

with $s>1$ which together with (1.10) completes the proof of the theorem. □

### 1.2. The case of complex coefficients

In this subsection we extend basic results of Subsection 1.1 to the systems (1) with complex coefficients and with solutions $u=v+iw$, where $v$ and $w$ are $m$-component vector-valued functions with real-valued components.

For the spaces of vector-valued functions with complex components we retain the same notation as in the case of real components but use bold. The same relates notation for the spaces of matrix-valued functions.

We introduce the operators

$$\mathfrak{L}(x,t,\partial/\partial x)=\sum_{|\beta|\le 2s}\mathcal{A}_\beta(x,t)\partial_x^\beta,\quad \mathfrak{L}_0(x,t,\partial/\partial x)=\sum_{|\beta|=2s}\mathcal{A}_\beta(x,t)\partial_x^\beta,$$

where $\mathcal{A}_\beta$ are $(m\times m)$-matrix-valued functions with complex elements satisfying the conditions (A) and (B), formulated in Introduction.

Let $\mathcal{R}_\beta$ and $\mathcal{H}_\beta$ be real $(m\times m)$-matrix-valued functions defined on $\overline{R_T^{n+1}}$ such that $\mathcal{A}_\beta(x,t)=\mathcal{R}_\beta(x,t)+i\mathcal{H}_\beta(x,t)$. We use the following notation

$$\mathfrak{R}(x,t,\partial/\partial x)=\sum_{|\beta|\le 2s}\mathcal{R}_\beta(x,t)\partial_x^\beta,\quad \mathfrak{H}(x,t,\partial/\partial x)=\sum_{|\beta|\le 2s}\mathcal{H}_\beta(x,t)\partial_x^\beta,$$

$$\mathfrak{R}_0(x,t,\partial/\partial x)=\sum_{|\beta|=2s}\mathcal{R}_\beta(x,t)\partial_x^\beta,\quad \mathfrak{H}_0(x,t,\partial/\partial x)=\sum_{|\beta|=2s}\mathcal{H}_\beta(x,t)\partial_x^\beta.$$

Separating the real and imaginary parts of the system

$$\frac{\partial u}{\partial t}-\mathfrak{L}(x,t,\partial/\partial x)u=0,$$

we get the system with real coefficients, which like the original system is uniformly parabolic in $\overline{R_T^{n+1}}$

$$\frac{\partial v}{\partial t}-\mathfrak{R}(x,t,\partial/\partial x)v+\mathfrak{H}(x,t,\partial/\partial x)w=0,$$
$$\frac{\partial w}{\partial t}-\mathfrak{H}(x,t,\partial/\partial x)v-\mathfrak{R}(x,t,\partial/\partial x)w=0.$$

*Remark* 1. The preservation of the uniform parabolicity in the sense of Petrovskii under the passage from a system with complex coefficients to the system with real coefficients is a corollary of the following simple algebraic property.

Let $Q$ be an $(m\times m)$-matrix with $R$ and $H$ being its real and imaginary parts, respectively. Let the eigenvalues $\lambda$ of $Q$ satisfy $\operatorname{Re}\lambda<-\delta$, $\delta>0$. Then for the eigenvalues $\mu$ of the matrix

$$\Lambda=\begin{pmatrix} R & -H\\ H & R\end{pmatrix}$$

the inequality $\operatorname{Re}\mu < -\delta$ holds.

In fact,

$$\det(\Lambda-\mu I_{2m}) = \det\begin{pmatrix} R-\mu I_m & -H \\ H & R-\mu I_m \end{pmatrix} = \det\begin{pmatrix} Q-\mu I_m & i(Q-\mu I_m) \\ H & R-\mu I_m \end{pmatrix}$$
$$= \det\begin{pmatrix} Q-\mu I_m & 0 \\ H & \overline{Q}-\mu I_m \end{pmatrix} = \det(Q-\mu I_m)\det(\overline{Q}-\mu I_m),$$

where $\overline{Q}$ is a matrix whose eleme ts are complex conjugate of corresponding elements of $Q$. Since

$$\det(\overline{Q}-\mu I_m) = \det(\overline{Q-\bar{\mu} I_m}) = \overline{\det(Q-\bar{\mu} I_m)},$$

then $\operatorname{Re}\mu < -\delta$.

We introduce the matrix differential operators

$$\mathfrak{K}(x,t,\partial/\partial x) = \begin{pmatrix} \mathfrak{R}(x,t,\partial/\partial x) & -\mathfrak{H}(x,t,\partial/\partial x) \\ \mathfrak{H}(x,t,\partial/\partial x) & \mathfrak{R}(x,t,\partial/\partial x) \end{pmatrix},$$
$$\mathfrak{K}_0(x,t,\partial/\partial x) = \begin{pmatrix} \mathfrak{R}_0(x,t,\partial/\partial x) & -\mathfrak{H}_0(x,t,\partial/\partial x) \\ \mathfrak{H}_0(x,t,\partial/\partial x) & \mathfrak{R}_0(x,t,\partial/\partial x) \end{pmatrix}.$$

Let $G'(t,\tau,x,\eta)$ and $\mathcal{G}'_0(t-\tau,x-\eta;y)$ denote fundamental matrices of solutions of the Cauchy problem for the systems

$$\left(\frac{\partial}{\partial t} - \mathfrak{K}(x,t,\partial/\partial x)\right)\{v,w\} = 0 \quad \text{and} \quad \left(\frac{\partial}{\partial t} - \mathfrak{K}_0(y,0,\partial/\partial x)\right)\{v,w\} = 0,$$

respectively. Further, let $P'(x,y)$ be the $(2m\times 2m)$-matrix-valued function in the representation

$$\mathcal{G}'_0(t-\tau,x-\eta;y) = (t-\tau)^{-n/2s} P'\left(\frac{x-\eta}{(t-\tau)^{1/2s}}; y\right).$$

Let $\mathcal{M}$ be the set in the Euclidean space introduced in Subsection 1.1. The norm in the space $\mathbf{C}(\mathcal{M})$ of continuous and bounded on $\mathcal{M}$ vector-valued functions $u=v+iw$ with $m$ complex components is defined by the equality

$$\|u\| = \sup\{\, (|v(q)|^2 + |w(q)|^2)^{1/2} : q\in\mathcal{M}\,\}.$$

As in the definition of $\mathcal{K}(R^n,T)$ we put

$$\mathcal{K}'(R^n,T) = \sup \frac{\|u\|_{\mathbf{C}(\overline{R^{n+1}_T})}}{\|u\,|_{t=0}\,\|_{\mathbf{C}(R^n)}},$$

where the supremum is taken over all functions $u=v+iw$ in the class $\mathbf{C}^{(2s,1)}(R_T^{n+1})\cap \mathbf{C}(\overline{R_T^{n+1}})$ satisfying the system

$$\frac{\partial u}{\partial t}-\mathfrak{L}(x,t,\partial/\partial x)u=0.$$

We define one more constant

$$\mathcal{K}_0'(y)=\sup\frac{\|u\|_{\mathbf{C}(\overline{R_+^{n+1}})}}{\|u\,|_{t=0}\,\|_{\mathbf{C}(R^n)}},$$

where the supremum is taken over all solutions $u=v+iw$ of the system

$$\frac{\partial u}{\partial t}-\mathfrak{L}_0(y,0,\partial/\partial x)u=0$$

in the class $\mathbf{C}^{(2s,1)}(R_+^{n+1})\cap\mathbf{C}(\overline{R_+^{n+1}})$ and $y\in R^n$ plays the role of a parameter.

Clearly, the constant $\mathcal{K}'(R^n,T)$ for the system

$$\frac{\partial u}{\partial t}-\mathfrak{L}(x,t,\partial/\partial x)u=0$$

coincides with the constant $\mathcal{K}(R^n,T)$ for the system

$$\left(\frac{\partial}{\partial t}-\mathfrak{K}(x,t,\partial/\partial x)\right)\{v,w\}=0.$$

Therefore, all the assertions concerning $\mathcal{K}'(R^n,T)$ are immediate corollaries of analogous assertions about $\mathcal{K}(R^n,T)$. Taking this into account, we obtain the assertions marked below by primes from Theorems 1.1, 1.2 and Lemma 1.1.

**Theorem 1.1′.** *The following formula is valid*

$$\mathcal{K}'(R^n,T)=\sup_{x\in R^n}\ \sup_{0<t\le T}\ \sup_{\{z\in R^{2m}:|z|=1\}}\int_{R^n}|(G')^*(t,0,x,\eta)z|\,d\eta,$$

*where the* $^*$ *means passage to the transposed matrix.*

*In particular,*

$$\mathcal{K}_0'(y)=\sup_{\{z\in R^{2m}:|z|=1\}}\int_{R^n}|(P')^*(\eta;y)z|\,d\eta.$$

**Lemma 1.1′.** *The inequality*

$$\mathcal{K}'(R^n,T)\ge\sup\{\,\mathcal{K}_0'(y):y\in R^n\,\}$$

*holds.*

**Theorem 1.2′.** *The classical maximum modulus principle is not valid for solutions of the system*

$$\frac{\partial u}{\partial t}-\mathfrak{L}(x,t,\partial/\partial x)u=0\quad \textit{in } R_T^{n+1}$$

*if* $s>1$.

## 2. Second order systems

### 2.1. The case of real coefficients

2.1.1. *Necessary conditions.* In this subsection we study the validity of the classical maximum modulus principle for the system (2) with coefficients $\mathcal{A}_{jk}$, $\mathcal{A}_j$, $\mathcal{A}_0$ $(1\leq j, k\leq n)$ that are real $(m\times m)$-matrix-valued functions in the layer $\overline{R_T^{n+1}}$ and in the cylinder $\overline{Q}_T=\overline{\Omega}\times[0,T]$, where $\Omega$ is a bounded domain in $R^n$.

We retain all notations introduced in Section 1 for arbitrary $s\geq 1$ and introduce one more constant

$$\mathcal{K}(\Omega,T)=\sup\frac{\|u\|_{C(\overline{Q}_T)}}{\|u\,|_{\bar{\Gamma}_T}\|_{C(\bar{\Gamma}_T)}}, \tag{2.1}$$

where the supremum is taken over all solutions of the system (2) in the class $C^{(2,1)}(Q_T)\cap C(\overline{Q}_T)$.

**Lemma 2.1.** *The inequality*

$$\mathcal{K}(\Omega,T)\geq\sup\{\,\mathcal{K}_0(y):y\in\Omega\,\} \tag{2.2}$$

*holds.*

*Proof.* Let $y$ be an arbitrary point of $\Omega$ and let the radius of the ball $B_r(y)$ be so small that $\overline{B_r(y)}\subset\Omega$. Further, let

$$\psi_\varepsilon\in C(R^n),\quad |\psi_\varepsilon(x)|\leq 1,\quad \operatorname{supp}\psi_\varepsilon\subset B_\varepsilon(y),\quad 0<\varepsilon\leq r/2.$$

The vector-valued function

$$u_\varepsilon(x,t)=\int_{R^n}G(t,0,x,\eta)\psi_\varepsilon(\eta)\,d\eta=\int_{B_\varepsilon(y)}G(t,0,x,\eta)\psi_\varepsilon(\eta)\,d\eta \tag{2.3}$$

is the solution of the Cauchy problem

$$\frac{\partial u_\varepsilon}{\partial t}-\sum_{j,k=1}^{n}\mathcal{A}_{jk}(x,t)\frac{\partial^2 u_\varepsilon}{\partial x_j\partial x_k}+\sum_{j=1}^{n}\mathcal{A}_j(x,t)\frac{\partial u_\varepsilon}{\partial x_j}+\mathcal{A}_0(x,t)u_\varepsilon=0\quad\text{in } R_T^{n+1},$$
$$u_\varepsilon(x,0)=\psi_\varepsilon(x).$$

Since $\mathcal{A}_{jk},\mathcal{A}_j,\mathcal{A}_0\in C_m^{\alpha,\alpha/2}(\overline{R_T^{n+1}})$, then, according to [5], one has

$$\|G(t,0,x,\eta)\|\leq c_1t^{-n/2}\exp\left(-c_2\frac{|x-\eta|^2}{t}\right). \tag{2.4}$$

The last estimate used for $(x,t)\in(\overline{\Omega}\backslash B_r(y))\times(0,T]$, yields

$$\begin{aligned}|u_\varepsilon(x,t)| &\le \int_{B_\varepsilon(y)} \|G(t,0,x,\eta)\|\,d\eta \le c_1 t^{-n/2}\int_{B_\varepsilon(y)} \exp\left(-c_2\frac{|x-\eta|^2}{t}\right)d\eta\\ &\le c_3\varepsilon^n t^{-n/2}\exp\left(-c_4\frac{r^2}{t}\right)\end{aligned}$$

which implies

$$|u_\varepsilon(x,t)| \le c_5\varepsilon^n,$$

where $|u_\varepsilon(x,t)|$ is the norm of the mapping $\psi_\varepsilon \to u_\varepsilon(x,t)$. Hence, the following estimate is valid for sufficiently small $\varepsilon$

$$\sup\{\,|u_\varepsilon(x,t)| : (x,t)\in\partial\Omega\times[0,T]\,\}\le 1. \tag{2.5}$$

From this and the definition (2.1) of the constant $\mathcal{K}(\Omega,T)$ we get

$$\mathcal{K}(\Omega,T)\ge \sup_{y\in\Omega}\overline{\lim_{\varepsilon\to 0}}\,\overline{\lim_{t\to+0}}\,|u_\varepsilon(y,t)|. \tag{2.6}$$

Using (2.3) and (2.4), in the same way as in the proof of Theorem 1.1, we find

$$|u_\varepsilon(y,t)| = \sup_{|z|=1}\int_{B_\varepsilon(y)}|G^*(t,0,y,\eta)z|\,d\eta.$$

The last equality and (2.6) yield

$$\begin{aligned}\mathcal{K}(\Omega,T)&\ge \sup_{y\in\Omega}\lim_{\varepsilon\to 0}\,\overline{\lim_{t\to+0}}\,\sup_{|z|=1}\int_{B_\varepsilon(y)}|G^*(t,0,y,\eta)z|\,d\eta\\ &\ge \sup_{y\in\Omega}\sup_{|z|=1}\lim_{\varepsilon\to 0}\,\overline{\lim_{t\to+0}}\int_{B_\varepsilon(y)}|G^*(t,0,y,\eta)z|\,d\eta.\end{aligned} \tag{2.7}$$

It was shown in Lemma 1.1 that

$$\begin{aligned}&\sup_{|z|=1}\lim_{\varepsilon\to 0}\,\overline{\lim_{t\to+0}}\int_{B_\varepsilon(y)}|G^*(t,0,y,\eta)z|\,d\eta\\ &\qquad=\sup_{|z|=1}\lim_{\varepsilon\to 0}\,\overline{\lim_{t\to+0}}\int_{B_\varepsilon(y)}|\mathcal{G}_0^*(t,y-\eta;y)z|\,d\eta=\mathcal{K}_0(y)\end{aligned}$$

which together with (2.7) gives the lower estimate for $\mathcal{K}(\Omega,T)$ in the statement of the lemma. □

**Theorem 2.1.** *The classical maximum modulus principle is valid for solutions of the system*

$$\frac{\partial u}{\partial t}-\sum_{j,k=1}^{n}\mathcal{A}_{jk}(t)\frac{\partial^2 u}{\partial x_j\partial x_k}+\sum_{j=1}^{n}\mathcal{A}_j(t)\frac{\partial u}{\partial x_j}=0 \tag{2.8}$$

*in* $R_T^{n+1}$ *if and only if the equalities*

$$\mathcal{A}_{jk}(t)=a_{jk}(t)I_m,\quad \mathcal{A}_j(t)=a_j(t)I_m,\quad 1\le j,k\le n,$$

*hold, where* $((a_{jk}))$ *is a positive-definite* $(n\times n)$*-matrix-valued function and* $a_j$ *are scalar functions.*

*Proof.* The necessity of the above equalities follows from Lemma 1.2. We show that $\mathcal{K}(R^n,T)=1$ under the conditions of the theorem.

Consider the Cauchy problem

$$\frac{\partial u}{\partial t}-\sum_{j,k=1}^{n}a_{jk}(t)\frac{\partial^2 u}{\partial x_j\partial x_k}+\sum_{j=1}^{n}a_j(t)\frac{\partial u}{\partial x_j}=0\quad\text{in } R_T^{n+1}, \tag{2.9}$$

$u|_{t=0}=\psi$, where $\psi\in C(R^n)$. The solution has the form

$$u(x,t)=\int_{R^n}g(t,0,x-\eta)\psi(\eta)\,d\eta,$$

where $g(t,\tau,x-\eta)$ is the fundamental solution of the Cauchy problem for the equation (2.9) in which $u$ is a scalar function.

Substituting $g(t,0,x-\eta)I_m$ in place of $G(t,0,x-\eta)$ in (1.6), we obtain

$$\mathcal{K}(R^n,T)=\sup_{x\in R^n}\sup_{0<t\le T}\int_{R^n}|g(t,0,x-\eta)|\,d\eta=\sup_{0<t\le T}\int_{R^n}|g(t,0,\xi)|\,d\xi$$

which means that $\mathcal{K}(R^n,T)$ does not depend on $m$. Since $\mathcal{K}(R^n,T)=1$ for $m=1$ we arrive at the sufficiency of conditions of the theorem. □

**Theorem 2.2.** *Let the classical maximum modulus principle be valid for solutions of the system* (2) *in* $Q_T(R_T^{n+1})$. *Then:*

(i) *for all* $x\in\Omega$ $(x\in R^n)$ *the equalities*

$$A_{jk}(x,0)=a_{jk}(x)I_m,\quad 1\le j,k\le n,$$

*hold, where* $((a_{jk}))$ *is a positive-definite* $(n\times n)$*-matrix-valued function;*

(ii) *for all* $x\in\Omega$ $(x\in R^n)$ *and for all* $\xi_j$, $\varsigma\in R^m$, $j=1,\dots,n$, *with* $(\xi_j,\varsigma)=0$ *the inequality*

$$\sum_{j,k=1}^{n}a_{jk}(x)(\xi_j,\xi_k)+\sum_{j=1}^{n}(\mathcal{A}_j(x,0)\xi_j,\varsigma)+(\mathcal{A}_0(x,0)\varsigma,\varsigma)\ge 0$$

*is valid.*

*Proof.* We assume for brevity that $\Omega$ is a bounded subdomain of $R^n$ or $\Omega = R^n$. From Lemmas 1.1, 2.1 it follows that the equality $\mathcal{K}(\Omega, T)=1$ is valid only if $\mathcal{K}_0(y)=1$ for all $y\in\Omega$, where $\mathcal{K}_0(y)$ is the constant (1.2) defined for the system

$$\frac{\partial u}{\partial t}-\sum_{j,k=1}^{n}\mathcal{A}_{jk}(y,0)\frac{\partial^2 u}{\partial x_j \partial x_k}=0.$$

By Theorem 2.1 the equality $\mathcal{K}_0(y)=1$ takes place if and only if (i) is satisfied.

Now we prove the necessity of (ii). Let $y$ be an arbitrary fixed point of $\Omega$, and let radius of the ball $B_r(y)$ be so small that $\overline{B_r(y)}\subset\Omega$. We introduce the vector-valued function

$$v(x)=\left(\sum_{j=1}^{n}\xi_j x_j+\varsigma\right)\left(\left|\sum_{j=1}^{n}\xi_j x_j\right|^2+|\varsigma|^2\right)^{-1/2},$$

where $\xi_j\in R^m$, $\varsigma\in R^m\backslash\{0\}$, $(\xi_j,\varsigma)=0$, $j=1,\dots,n$. Further, let

$$\chi_\varepsilon\in C^\infty(R^n),\quad \chi_\varepsilon(x)=1 \text{ for } |x|\le\varepsilon/2,\quad \chi_\varepsilon(x)=0 \text{ for } |x|\ge\varepsilon$$

and

$$0\le\chi_\varepsilon(x)\le 1\quad\text{for all } x\in R^n,$$

where $\varepsilon\le r$.

The vector-valued function

$$u_\varepsilon(x,t)=\int_{R^n}G(t,0,x,\eta)\chi_\varepsilon(\eta-y)v(\eta-y)\,d\eta, \tag{2.10}$$

where $G(t,\tau,x,\eta)$ is the Green matrix of the system (2), is the solution of the Cauchy problem for the system (2) with the initial data $u_\varepsilon(x,0)=v_\varepsilon(x)$, where $v_\varepsilon(x)=\chi_\varepsilon(x-y)v(x-y)$. If $\Omega$ is a bounded subdomain of $R^n$, then by (2.5) the value of $\varepsilon$ can be chosen so small that

$$\sup\{\,|u_\varepsilon(x,t)|:(x,t)\in\partial\Omega\times[0,T]\,\}\le 1.$$

Then $\|u_\varepsilon|_{\bar\Gamma_T}\|_{C(\bar\Gamma_T)}=1$, and, consequently

$$\mathcal{K}(\Omega,T)\ge\sup_{0<t\le T}|u_\varepsilon(y,t)|. \tag{2.11}$$

In the case $\Omega=R^n$ the last inequality is obvious.

Suppose the condition (i) is satisfied. We take the scalar product of the system (2) with $u$ and transform the equality

$$\left(\frac{\partial u}{\partial t}, u\right) - \sum_{j,k=1}^{n} a_{jk}(x)\left(\frac{\partial^2 u}{\partial x_j \partial x_k}, u\right) + \sum_{j=1}^{n}\left(\mathcal{A}_j(x,t)\frac{\partial u}{\partial x_j}, u\right)$$
$$+(\mathcal{A}_0(x,t)u,u) - \sum_{j,k=1}^{n}\left([\mathcal{A}_{jk}(x,t)-a_{jk}(x)I_m]\frac{\partial^2 u}{\partial x_j \partial x_k}, u\right) = 0,$$

where $a_{jk}(x)I_m = \mathcal{A}_{jk}(x,0)$. As a result we find

$$\begin{aligned}\frac{1}{2}\frac{\partial |u|^2}{\partial t} &= \frac{1}{2}\sum_{j,k=1}^{n} a_{jk}(x)\frac{\partial^2 |u|^2}{\partial x_j \partial x_k} - \sum_{j,k=1}^{n} a_{jk}(x)\left(\frac{\partial u}{\partial x_j}, \frac{\partial u}{\partial x_k}\right)\\ &-\sum_{j=1}^{n}\left(\mathcal{A}_j(x,t)\frac{\partial u}{\partial x_j}, u\right) - (\mathcal{A}_0(x,t)u,u)\\ &+\sum_{j,k=1}^{n}\left([\mathcal{A}_{jk}(x,t)-a_{jk}(x)I_m]\frac{\partial^2 u}{\partial x_j \partial x_k}, u\right).\end{aligned} \tag{2.12}$$

Since the coefficients of the system (2) belong to the class $C_m^{\alpha,\alpha/2}(\overline{R_T^{n+1}})$ and since $v_\varepsilon \in C^{2+\alpha}(R^n)$ then, according to [5], $u_\varepsilon \in C^{2+\alpha,\alpha/2}(\overline{R_T^{n+1}})$. So after substitution of $u_\varepsilon$ defined by (2.10) into (2.12) in place of $u$, we obtain

$$\lim_{(x,t)\to(y,+0)} \frac{\partial |u_\varepsilon|^2}{\partial t} = 2\left\{\frac{1}{2}\sum_{j,k=1}^{n} a_{jk}(y)\frac{\partial^2 |v_\varepsilon|^2}{\partial x_j \partial x_k} - \sum_{j,k=1}^{n} a_{jk}(y)\left(\frac{\partial v_\varepsilon}{\partial x_j}, \frac{\partial v_\varepsilon}{\partial x_k}\right)\right.$$
$$\left.-\sum_{j=1}^{n}\left(\mathcal{A}_j(y,0)\frac{\partial v_\varepsilon}{\partial x_j}, v_\varepsilon\right) - (\mathcal{A}_0(y,0)v_\varepsilon, v_\varepsilon)\right\}\Bigg|_{x=y}.$$

Hence, using the equalities

$$\frac{\partial^2 |v_\varepsilon|^2}{\partial x_j \partial x_k}\Bigg|_{x=y} = 0, \quad \frac{\partial v_\varepsilon}{\partial x_j}\Bigg|_{x=y} = \frac{\xi_j}{|\varsigma|}, \quad v_\varepsilon(y) = \frac{\varsigma}{|\varsigma|},$$

where $j,k = 1,\dots,n$, we find

$$\begin{aligned}\lim_{t\to+0}\frac{\partial |u_\varepsilon(y,t)|^2}{\partial t} = -\frac{2}{|\varsigma|^2}\Bigg[\sum_{j,k=1}^{n} a_{jk}(y)(\xi_j,\xi_k) + \sum_{j=1}^{n}(\mathcal{A}_j(y,0)\xi_j,\varsigma)\\ +(\mathcal{A}_0(y,0)\varsigma,\varsigma)\Bigg].\end{aligned} \tag{2.13}$$

The function $|u_\varepsilon(y,t)|^2$ is continuous on $[0,T]$ and differentiable at all points of the interval $(0,T)$, its derivative $\partial|u_\varepsilon(y,t)|^2/\partial t$ tends to a finite limit as $t\to+0$. Therefore, the function $|u_\varepsilon(y,t)|^2$ has the right-hand derivative $\partial_+|u_\varepsilon(y,t)|^2/\partial t$ at $t=0$, and by virtue of (2.13) we have

$$\begin{aligned}\frac{\partial_+|u_\varepsilon(y,t)|^2}{\partial t}\bigg|_{t=0}=-\frac{2}{|\varsigma|^2}\bigg[\sum_{j,k=1}^{n} a_{jk}(y)(\xi_j,\xi_k)+\sum_{j=1}^{n}(\mathcal{A}_j(y,0)\xi_j,\varsigma)\\ +(\mathcal{A}_0(y,0)\varsigma,\varsigma)\bigg].\end{aligned}\tag{2.14}$$

Suppose $\mathcal{K}(\Omega,T)=1$ and let there exist a point $y\in\Omega$ and vectors $\xi_j\in R^m$, $\varsigma\in R^m\backslash\{0\}$ with $(\xi_j,\varsigma)=0$, $j=1,...,n$, for which

$$\sum_{j,k=1}^{n} a_{jk}(y)(\xi_j,\xi_k)+\sum_{j=1}^{n}(\mathcal{A}_j(y,0)\xi_j,\varsigma)+(\mathcal{A}_0(y,0)\varsigma,\varsigma)<0.\tag{2.15}$$

Since $|u_\varepsilon(y,0)|=|v(0)|=1$, then (2.14) and (2.15) imply the existence of a constant $\delta>0$ such that $|u_\varepsilon(y,t)|>1$ for $0<t\le\delta$. From this and (2.11) it follows that $\mathcal{K}(\Omega,T)>1$ which contradicts our assumption on the validity of the classical maximum modulus principle.

Thus, if $\mathcal{K}(\Omega,T)=1$, then for all $x\in\Omega$ and for all vectors $\xi_j\in R^m$, $\varsigma\in R^m\backslash\{0\}$ with $(\xi_j,\varsigma)=0$, $j=1,...,n$, the inequality

$$\sum_{j,k=1}^{n} a_{jk}(x)(\xi_j,\xi_k)+\sum_{j=1}^{n}(\mathcal{A}_j(x,0)\xi_j,\varsigma)+(\mathcal{A}_0(x,0)\varsigma,\varsigma)\ge 0$$

holds. The condition $\varsigma\in R^m\backslash\{0\}$ can be omitted since the inequality

$$\sum_{j,k=1}^{n} a_{jk}(x)(\xi_j,\xi_k)\ge 0$$

is true due to the necessity of the condition (i) for validity of the classical maximum modulus principle. □

*Remark* 2. In what follows we show that conditions (i), (ii) of Theorem 2.2, are necessary and sufficient for validity of the classical maximum modulus principle for second order systems with coefficients depending only on $x$. But in general, when the coefficients depend on $x$ and $t$ these conditions are not sufficient.

Consider, for example, the parabolic system

$$\frac{\partial u}{\partial t} - \sum_{j,k=1}^{n} \mathcal{A}_{jk}(x,t)\frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{j=1}^{n} \mathcal{A}_j(x,t)\frac{\partial u}{\partial x_j} = 0 \tag{2.16}$$

in $R_T^{n+1}$, where $\mathcal{A}_{jk} \in C_m^{2+\alpha,\alpha/2}(\overline{R_T^{n+1}})$, $\mathcal{A}_j \in C_m^{1+\alpha,\alpha/2}(\overline{R_T^{n+1}})$. Suppose the coefficients of the system (2.16) do not depend on $x$ in the layer $R_\delta^{n+1}$, $0<\delta<T$, and let

$$\mathcal{A}_{jk}(x,0) = a_{jk} I_m, \quad \mathcal{A}_j(x,0) = a_j I_m,$$

where $((a_{jk}))$ is a positive-definite $(n \times n)$-matrix and $a_j$ are scalars. Let the matrix $A_1(x,t)$ be non-diagonal for all $(x,t) \in R_\delta^{n+1}$. Then, according to Theorem 2.1 the classical maximum modulus principle is not valid for the system (2.16) in $R_\delta^{n+1}$ (the more so in $R_T^{n+1}$) whereas the conditions (i), (ii) of Theorem 2.2 are satisfied. □

2.1.2. *Sufficient conditions for systems with scalar principal part.* Next we present a theorem on a sufficient condition for validity of the classical maximum modulus principle for second order systems with scalar principal part.

**Theorem 2.3.** *Let the coefficients of the parabolic system*

$$\frac{\partial u}{\partial t} - \sum_{j,k=1}^{n} a_{jk}(x,t)\frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{j=1}^{n} \mathcal{A}_j(x,t)\frac{\partial u}{\partial x_j} + \mathcal{A}_0(x,t)u = 0 \tag{2.17}$$

*satisfy the condition:*

(j) *for all $(x,t) \in Q_T$ $((x,t) \in R_T^{n+1})$ and for all vectors $\xi_j, \varsigma \in R^m$ with $(\xi_j, \varsigma) = 0$, $j = 1, \ldots, n$, the inequality*

$$\sum_{j,k=1}^{n} a_{jk}(x,t)(\xi_j, \xi_k) + \sum_{j=1}^{n} (\mathcal{A}_j(x,t)\xi_j, \varsigma) + (\mathcal{A}_0(x,t)\varsigma, \varsigma) \geq 0 \tag{2.18}$$

*holds. Then $\mathcal{K}(\Omega, T) = 1$ $(\mathcal{K}(R^n, T) = 1)$.*

*Proof.* Suppose first that for all $(x,t) \in Q_T$ and for all $\xi_j \in R^m$, $\varsigma \in R^m \setminus \{0\}$, $j = 1, \ldots, n$, with $(\xi_j, \varsigma) = 0$ we have

$$\sum_{j,k=1}^{n} a_{jk}(x,t)(\xi_j, \xi_k) + \sum_{j=1}^{n} (\mathcal{A}_j(x,t)\xi_j, \varsigma) + (\mathcal{A}_0(x,t)\varsigma, \varsigma) > 0. \tag{2.19}$$

We show that for any non-trivial solution $u \in C^{(2,1)}(Q_T) \cap C(\overline{Q}_T)$ of the system (2.17) the function $|u(x,t)|$ can not attain its global maximum at a point

$(x,t)\in Q_T$. This will imply that if the system (2.17) has a regular solution in $Q_T$ and if (2.19) holds, then the function $|u(x,t)|$ takes its maximum value on $\bar{\Gamma}_T$.

From (2.17) we have

$$\begin{aligned}\frac{1}{2}\frac{\partial|u|^2}{\partial t}=\frac{1}{2}\sum_{j,k=1}^{n}a_{jk}(x,t)\frac{\partial^2|u|^2}{\partial x_j\partial x_k}-\sum_{j=1}^{n}a_{jk}(x,t)\left(\frac{\partial u}{\partial x_j},\frac{\partial u}{\partial x_k}\right)\\ -\sum_{j=1}^{n}\left(\mathcal{A}_j(x,t)\frac{\partial u}{\partial x_j},u\right)-(\mathcal{A}_0(x,t)u,u).\end{aligned}\tag{2.20}$$

Suppose the function $|u(x,t)|$ takes its global maximum at a point $(x_0,t_0)\in Q_T$. Then

$$\left.\frac{\partial|u|^2}{\partial x_j}\right|_{(x_0,t_0)}=2\left.\left(\frac{\partial u}{\partial x_j},u\right)\right|_{(x_0,t_0)}=0,\tag{2.21}$$

$$\left.\frac{\partial|u|^2}{\partial t}\right|_{(x_0,t_0)}=2\left.\left(\frac{\partial u}{\partial t},u\right)\right|_{(x_0,t_0)}\geq 0,\tag{2.22}$$

$$\sum_{j,k=1}^{n}a_{jk}(x_0,t_0)\left.\frac{\partial^2|u|^2}{\partial x_j\partial x_k}\right|_{(x_0,t_0)}\leq 0.\tag{2.23}$$

By (2.20)–(2.23) we have

$$\left\{\sum_{j,k=1}^{n}a_{jk}(x_0,t_0)\left(\frac{\partial u}{\partial x_j},\frac{\partial u}{\partial x_k}\right)\right.\\ \left.\left.+\sum_{j=1}^{n}\left(\mathcal{A}_j(x_0,t_0)\frac{\partial u}{\partial x_j},u\right)+(\mathcal{A}_0(x_0,t_0)u,u)\right\}\right|_{(x_0,t_0)}\leq 0,$$

which contradicts (2.19) and hence $|u(x,t)|$ can not attain its global maximum at $(x_0,t_0)\in Q_T$. This implies

$$\|u\|_{C(\bar{Q}_T)}=\|u\,|_{\bar{\Gamma}_T}\|_{C(\bar{\Gamma}_T)}.$$

Suppose now that (2.18) holds under the conditions in the statement of the theorem. Let $\varepsilon=\text{const}>0$ and let $u$ be the solution of the system (2.17) in the class $C^{(2,1)}(Q_T)\cap C(\bar{Q}_T)$. The vector-valued function $v(x,t)=u(x,t)\exp(-\varepsilon t)$ is the solution of the system

$$\frac{\partial v}{\partial t}-\sum_{j,k=1}^{n}a_{jk}(x,t)\frac{\partial^2 v}{\partial x_j\partial x_k}+\sum_{j=1}^{n}\mathcal{A}_j(x,t)\frac{\partial v}{\partial x_j}+\mathcal{A}_0(x,t)v+\varepsilon v=0.$$

According to our assumption, for all $(x,t)\in Q_T$ and all $\xi_j\in R^m$, $\varsigma\in R^m\backslash\{0\}$ with $(\xi_j,\varsigma)=0$, $1\le j\le n$, we have

$$\sum_{j,k=1}^{n} a_{jk}(x,t)(\xi_j,\xi_k)+\sum_{j=1}^{n}(\mathcal{A}_j(x,t)\xi_j,\varsigma)+(\mathcal{A}_0(x,t)\varsigma,\varsigma)+\varepsilon|\varsigma|^2>0.$$

By what we proved above

$$\|v\|_{C(\overline{Q}_T)}=\|v\,|_{\bar{\Gamma}_T}\,\|_{C(\bar{\Gamma}_T)},$$

and hence

$$\max_{\overline{Q}_T}|e^{-\varepsilon t}u(x,t)|=\max_{\bar{\Gamma}_T}|e^{-\varepsilon t}u(x,t)|.$$

Consequently,

$$\|u\|_{C(\overline{Q}_T)}\le e^{\varepsilon T}\|u\,|_{\bar{\Gamma}_T}\,\|_{C(\bar{\Gamma}_T)}.$$

Since $\varepsilon$ is arbitrary, the best constant in the last inequality is equal to one.

Now we turn to the constant $\mathcal{K}(R^n,T)$. Contrary to the case of the bounded domain, one can not immediately conclude that $\mathcal{K}(R^n,T)=1$ because of the absence of the global maximum of the function $|u(x,t)|$ in $R_T^{n+1}$.

Suppose the inequality (2.18) holds for all $(x,t)\in R_T^{n+1}$ and all $\xi_j,\varsigma\in R^m$ with $(\xi_j,\varsigma)=0$, $j=1,...,n$, and that the classical maximum modulus principle is not valid for solutions of the system (2.17) in $R_T^{n+1}$. Then there exists a point $(x_0,t_0)\in R_T^{n+1}$ and a vector-valued function $\phi\in C(R^n)$, $|\phi(x)|\le 1$ such that

$$|u(x_0,t_0)|=\left|\int_{R^n}G(t_0,0,x_0,\eta)\phi(\eta)\,d\eta\right|>1. \tag{2.24}$$

By (2.4) one can assume that $\phi$ has a compact support, $\operatorname{supp}\phi\subset B_\varrho(x_0)$. If $|x-x_0|\ge R>\varrho$ then (2.4) implies

$$\left|\int_{R^n}G(t,0,x,\eta)\phi(\eta)\,d\eta\right|\le c_1\exp\left(-c_2\frac{(R-\varrho)^2}{T}\right),$$

where $0<t\le T$, $x\in R^n\backslash B_R(x_0)$.

Applying the assertion of the present theorem for a cylinder with a bounded base, we get

$$|u(x,t)|=\left|\int_{R^n}G(t,0,x,\eta)\phi(\eta)\,d\eta\right|\le 1,$$

where $(x,t)\in\overline{B_R(x_0)}\times[0,T]$ and $R$ is sufficiently large. The last inequality contradicts (2.24) which proves the validity of the classical maximum modulus principle in $R_T^{n+1}$. $\square$

2.1.3. *Necessary and sufficient conditions.* Theorems 2.2 and 2.3 immediately imply the following assertion.

**Theorem 2.4.** *The classical maximum modulus principle is valid for solutions of the system* (3) *in* $Q_T(R_T^{n+1})$ *if and only if:*

(i) *for all* $x\in\Omega$ $(x\in R^n)$ *the equalities*

$$\mathcal{A}_{jk}(x)=a_{jk}(x)I_m, \quad 1\le j,k\le m,$$

*hold, where* $((a_{jk}))$ *is a positive-definite* $(n\times n)$*-matrix-valued function;*

(ii) *for all* $x\in\Omega$ $(x\in R^n)$ *and all* $\xi_j$, $\varsigma\in R^m$, $j=1,\dots,n$, *with* $(\xi_j,\varsigma)=0$, *the inequality*

$$\sum_{j,k=1}^{n} a_{jk}(x)(\xi_j,\xi_k)+\sum_{j=1}^{n}(\mathcal{A}_j(x)\xi_j,\varsigma)+(\mathcal{A}_0(x)\varsigma,\varsigma)\ge 0 \tag{2.25}$$

*is valid.*

Theorem 2.4 implies

**Corollary 2.1.** *The classical maximum modulus principle is valid for solutions of the system*

$$\frac{\partial u}{\partial t}-\sum_{j,k=1}^{n}\mathcal{A}_{jk}(x)\frac{\partial^2 u}{\partial x_j\partial x_k}+\sum_{j=1}^{n}\mathcal{A}_j(x)\frac{\partial u}{\partial x_j}=0$$

*in* $Q_T(R_T^{n+1})$ *if and only if for all* $x\in\Omega$ $(x\in R^n)$ *the equalities*

$$\mathcal{A}_{jk}(x)=a_{jk}(x)I_m, \quad \mathcal{A}_j(x)=a_j(x)I_m$$

*hold, where* $((a_{jk}))$ *is a positive-definite* $(n\times n)$*-matrix-valued function and* $a_j$ *are scalar functions.*

*Proof.* Putting $\mathcal{A}_0=0$ in (2.25) we get

$$\sum_{j,k=1}^{n} a_{jk}(x)(\xi_j,\xi_k)+\sum_{j=1}^{n}(\mathcal{A}_j(x)\xi_j,\varsigma)\ge 0,$$

which can be valid for all $x\in\Omega$ $(x\in R^n)$ and for all $\xi_j,\varsigma\in R^m$ with $(\xi_j,\varsigma)=0$, $j=1,\dots,n$, only provided $(\mathcal{A}_j(x)\xi_j,\varsigma)=0$. Consequently, $\mathcal{A}_j(x)=a_j(x)I_m$, where

$$a_j(x)=\mathcal{A}_j^{(1,1)}(x)=\mathcal{A}_j^{(2,2)}(x)=\dots=\mathcal{A}_j^{(m,m)}(x), \quad j=1,\dots,n. \quad \square$$

*Remark* 3. Minimizing the left-hand side of (2.25) over

$$\xi=(\xi_1,\dots,\xi_n), \quad \xi_j\in R^m,$$

for a fixed $\varsigma \in R^m$ with $(\xi_j, \varsigma)=0$, $j=1,\dots,n$, one can write the condition (ii) of Theorem 2.4 in another form. This was used in [7], where the maximum modulus principle was studied for elliptic systems with scalar principal part.

One may assume that $\varsigma \in R^m \setminus \{0\}$, since the inequality

$$\sum_{j,k=1}^{n} a_{jk}(x)(\xi_j, \xi_k) \geq 0$$

is provided by the condition (i) of Theorem 2.4 for all $\xi_j \in R^m$.

Let

$$\mathcal{F}_\varsigma(\xi_1, \dots, \xi_n) = \sum_{j,k=1}^{n} a_{jk}(x)(\xi_j, \xi_k) + \sum_{j=1}^{n} (\mathcal{A}_j(x)\xi_j, \varsigma) + (\mathcal{A}_0(x)\varsigma, \varsigma)$$

or, which is the same

$$\mathcal{F}_\varsigma(\xi_1, \dots, \xi_n) = \sum_{j,k=1}^{n} \sum_{i=1}^{m} a_{jk}(x)\xi_j^{(i)}\xi_k^{(i)} + \sum_{j=1}^{n} \sum_{i,k=1}^{m} \mathcal{A}_j^{(i,k)}(x)\xi_j^{(k)}\varsigma^{(i)}$$
$$+ \sum_{i,k=1}^{m} \mathcal{A}_0^{(i,k)}(x)\varsigma^{(k)}\varsigma^{(i)},$$

where $\xi_j^{(k)}$ and $\varsigma^{(k)}$ are components of the vectors $\xi_j$ and $\varsigma$, respectively. At a point of the constraint extremum of the function $\mathcal{F}_\varsigma(\xi_1, \dots, \xi_n)$ one has

$$\begin{aligned}\frac{\partial}{\partial \xi_k^{(i)}} \left( \mathcal{F}_\varsigma(\xi_1, \dots, \xi_n) - \sum_{j=1}^{n} \sum_{i=1}^{m} \lambda_j \xi_j^{(i)} \varsigma^{(i)} \right) \\ = 2\sum_{j=1}^{n} a_{jk}(x)\xi_j^{(i)} + \sum_{j=1}^{m} \mathcal{A}_k^{(j,i)}(x)\varsigma^{(j)} - \lambda_k \varsigma^{(i)} = 0,\end{aligned} \tag{2.26}$$

where $k=1,2,\dots,n$, $i=1,\dots,m$ and the following constraint relations are valid

$$\sum_{i=1}^{m} \xi_j^{(i)} \varsigma^{(i)} = 0, \quad j=1,2,\dots,n.$$

Multiplying (2.26) by $\varsigma^{(i)}$ and summing up over $i$ from 1 to $m$ we obtain

$$\lambda_k = |\varsigma|^{-2}(\mathcal{A}_k^*(x)\varsigma, \varsigma),$$

where * means passage to the transposed matrix. Consequently, the conditions (2.26) defining $\xi_j$ can be written in the form

$$2\sum_{j=1}^{n} a_{jk}(x)\xi_j^{(i)}+\sum_{j=1}^{m} \mathcal{A}_k^{(j,i)}(x)\varsigma^{(j)}-|\varsigma|^{-2}(\mathcal{A}_k^*(x)\varsigma,\varsigma)\varsigma^{(i)}=0.$$

Taking into account the symmetry of the matrix $((a_{jk}(x)))$, we find

$$\xi_j=\frac{1}{2}\sum_{k=1}^{n} b_{jk}(x)[|\varsigma|^{-2}(\mathcal{A}_k^*(x)\varsigma,\varsigma)\varsigma-\mathcal{A}_k^*(x)\varsigma], \tag{2.27}$$

where $j=1,2,\dots n$ and $((b_{jk}(x)))$ is the inverse matrix of $((a_{jk}(x)))$.

The function $\mathcal{F}_\varsigma(\xi_1,\dots,\xi_n)$ attains its constraint minimum at the vectors (2.27) because of the positive-definiteness of the matrix $((a_{jk}(x)))$. Calculating the value of $\mathcal{F}_\varsigma(\xi_1,\dots,\xi_n)$ at vectors (2.27), we obtain

$$\begin{aligned}\min\{\, &\mathcal{F}_\varsigma(\xi_1,\dots,\xi_n):\xi_1,\dots,\xi_n\in R^m,(\xi_1,\varsigma)=0,\dots,(\xi_n,\varsigma)=0\,\}\\ &=\frac{1}{4}|\varsigma|^{-2}\sum_{i,j=1}^{n} b_{ij}(x)(\mathcal{A}_i(x)\varsigma,\varsigma)(\mathcal{A}_j(x)\varsigma,\varsigma)\\ &-\frac{1}{4}\sum_{i,j=1}^{n} b_{ij}(x)(\mathcal{A}_i^*(x)\varsigma,\mathcal{A}_j^*(x)\varsigma)+(\mathcal{A}_0(x)\varsigma,\varsigma).\end{aligned}$$

Thus Theorem 2.4 implies the following assertion.

**Corollary 2.2.** *The classical maximum modulus principle is valid for solutions of the system* (3) *in* $Q_T(R_T^{n+1})$ *if and only if the condition* (i) *of Theorem* 2.4 *is satisfied and*

(ii′) *for all* $x\in\Omega$ $(x\in R^n)$ *and for any* $\varsigma\in R^m$, $|\varsigma|=1$ *the inequality*

$$\sum_{i,j=1}^{n} b_{ij}(x)[(\mathcal{A}_i(x)\varsigma,\varsigma)(\mathcal{A}_j(x)\varsigma,\varsigma)-(\mathcal{A}_i^*(x)\varsigma,\mathcal{A}_j^*(x)\varsigma)]+4(\mathcal{A}_0(x)\varsigma,\varsigma)\geq 0$$

*holds, where* $((b_{ij}))$ *is the* $(n\times n)$*-matrix-valued function inverse of* $((a_{ij}))$ *and* $\mathcal{A}_j^*(x)$ *is the matrix transposed of* $\mathcal{A}_j(x)$.

*Remark* 4. We give an example of a system whose principal part is not a scalar differential operator in the whole domain and for which the classical maximum modulus principle is valid in $R_T^{n+1}$.

Consider the parabolic system

$$(2.28)\qquad \frac{\partial v}{\partial t}-\sum_{j,k=1}^{n}\mathcal{A}_{jk}(x,t)\frac{\partial^2 v}{\partial x_j\partial x_k}+\sum_{j=1}^{n}\mathcal{A}_j(x,t)\frac{\partial v}{\partial x_j}+\mathcal{A}_0(x,t)v+\lambda^2 v=0$$

in $R_T^{n+1}$, where $\mathcal{A}_{ij}\in C_m^{2+\alpha,\alpha/2}(\overline{R_T^{n+1}})$, $\mathcal{A}_j\in C_m^{1+\alpha,\alpha/2}(\overline{R_T^{n+1}})$, $\mathcal{A}_0\in C_m^{\alpha,\alpha/2}(\overline{R_T^{n+1}})$ and $\lambda=$const. Suppose the coefficients of the system (2.28) do not depend on $t$ in the layer $R_\delta^{n+1}$, $0<\delta<T$. Let the matrix-valued functions $\mathcal{A}_{jk}(x,0)$, $\mathcal{A}_j(x,0)$, $\mathcal{A}_0(x,0)$, denoted by $\mathcal{A}_{jk}(x)$, $\mathcal{A}_j(x)$, $\mathcal{A}_0(x)$, respectively, satisty the conditions (i), (ii) of Theorem 2.4. Suppose further that $|v(x,0)|\leq 1$. Then $|v(x,t)|\leq 1$ in $R_\delta^{n+1}$. By $M$ we denote the value $\mathcal{K}(R^n,T)$ for the system (2). Since $u(x,t)=v(x,t)\exp(\lambda^2 t)$ is the solution of (2), then

$$\sup\{\,|v(x,t)|:(x,t)\in\overline{R_T^{n+1}}\backslash R_\delta^{n+1}\,\}\leq M\exp(-\lambda^2\delta).$$

Thus, the solution of the Cauchy problem for the system (2.28) satisfies the classical maximum modulus principle for sufficiently large values of $\lambda$. $\square$

## 2.2. The case of complex coefficients

We can extend the results of the first subsection to the systems (2), (3) with complex coefficients and solutions $u=v+iw$, where $v$ and $w$ are $m$-component vector-valued functions with real components. The results are obtained by application of the corresponding assertions on the maximum modulus for the real case to systems obtained by the separation of real and imaginary parts (see Subsection 1.2). Thus we can formulate analogous theorems and corollaries as in Subsection 2.1. By $C^m$ we denote the complex linear $m$-dimensional space with the inner product $\langle\cdot,\cdot\rangle$.

We retain the notations of Subsection 1.2 and use them putting $s=1$. By analogy with the definition (2.1) of the constant $\mathcal{K}(\Omega,T)$ let

$$\mathcal{K}'(\Omega,T)=\sup\frac{\|u\|_{\mathbf{C}(\overline{Q}_T)}}{\|u\,|_{\bar\Gamma_T}\|_{\mathbf{C}(\bar\Gamma_T)}},$$

where the supremum is taken over all vector-valued functions $u=v+iw$ in the class $\mathbf{C}^{(2,1)}(Q_T)\cap\mathbf{C}(\overline{Q}_T)$ that satisfy the system (2) with complex coefficients. Here $v$ and $w$ are $m$-component vector-valued functions with real components.

**Theorem 2.1′.** *The classical maximum modulus principle holds for solutions of the system with complex coefficients*

$$\frac{\partial u}{\partial t}-\sum_{j,k=1}^{n}\mathcal{A}_{jk}(t)\frac{\partial^2 u}{\partial x_j\partial x_k}+\sum_{j=1}^{n}\mathcal{A}_j(t)\frac{\partial u}{\partial x_j}=0$$

*in* $R_T^{n+1}$ *if and only if the equalities*

$$\mathcal{A}_{jk}(t)=a_{jk}(t)I_m,\ \ \mathcal{A}_j(t)=a_j(t)I_m,\quad 1\le j,k\le n,$$

*are valid, where* $((a_{jk}))$ *is a real positive-definite* $(n\times n)$*-matrix-function and* $a_j$ *are real scalar functions.*

**Theorem 2.2′.** *Let the classical maximum modulus principle be valid for the system* (2) *with complex coefficients in* $Q_T(R_T^{n+1})$*. Then:*

(i) *for all* $x\in\Omega$ $(x\in R^n)$ *the equalities*

$$\mathcal{A}_{jk}(x,0)=a_{jk}(x)I_m,\quad 1\le j,k\le n,$$

*hold, where* $((a_{jk}))$ *is a real positive-definite* $(n\times n)$*-matrix-valued function;*

(ii) *for all* $x\in\Omega$ $(x\in R^n)$ *and all* $\xi_j,\varsigma\in C^m$, $j=1,\dots,n$, *with* $\mathrm{Re}\langle\xi_j,\varsigma\rangle=0$, *the inequality*

$$\mathrm{Re}\left\{\sum_{j,k=1}^n a_{jk}(x)\langle\xi_j,\xi_k\rangle+\sum_{j=1}^n\langle\mathcal{A}_j(x,0)\xi_j,\varsigma\rangle+\langle\mathcal{A}_0(x,0)\varsigma,\varsigma\rangle\right\}\ge 0$$

*is valid.*

**Theorem 2.4′.** *The classical maximum modulus principle is valid for solutions of the system* (3) *with complex coefficients in* $Q_T(R_T^{n+1})$ *if and only if*

(i) *for all* $x\in\Omega$ $(x\in R^n)$ *the equalities*

$$\mathcal{A}_{jk}(x)=a_{jk}(x)I_m,\quad 1\le j,k\le n,$$

*hold, where* $((a_{jk}))$ *is a real positive-definite* $(n\times n)$*-matrix-valued function;*

(ii) *for all* $x\in\Omega$ $(x\in R^n)$ *and for all* $\xi_j$, $\varsigma\in C^m$, $j=1,\dots,n$, *with* $\mathrm{Re}\langle\xi_j,\varsigma\rangle=0$, *the inequality*

$$\mathrm{Re}\left\{\sum_{j,k=1}^n a_{jk}(x)\langle\xi_j,\xi_k\rangle+\sum_{j=1}^n\langle\mathcal{A}_j(x)\xi_j,\varsigma\rangle+\langle\mathcal{A}_0(x)\varsigma,\varsigma\rangle\right\}\ge 0$$

*is valid.*

**Corollary 2.1′.** *The classical maximum modulus principle is valid for the system*

$$\frac{\partial u}{\partial t}-\sum_{j,k=1}^n\mathcal{A}_{jk}(x)\frac{\partial^2 u}{\partial x_j\partial x_k}+\sum_{j=1}^n\mathcal{A}_j(x)\frac{\partial u}{\partial x_j}=0$$

*with complex coefficients in* $Q_T(R_T^{n+1})$ *if and only if for all* $x\in\Omega$ $(x\in R^n)$ *the equalities*

$$\mathcal{A}_{jk}(x)=a_{jk}(x)I_m,\quad \mathcal{A}_j(x)=a_j(x)I_m,\quad 1\le j,k\le n,$$

*hold, where* $((a_{jk}))$ *is a real positive-definite* $(n\times n)$*-matrix-valued function and* $a_j$ *are real scalar functions.*

**Corollary 2.2′.** *The classical maximum modulus principle is valid for solutions of the system* (3) *with complex coefficients in* $Q_T(R_T^{n+1})$ *if and only if the condition* (i) *of Theorem* 2.4′ *is satisfied and*

(ii′) *for all* $x\in\Omega$ $(x\in R^n)$ *and for any* $\varsigma\in C^m$, $|\varsigma|=1$ *the inequality*

$$\sum_{i,j=1}^{n} b_{ij}(x)\operatorname{Re}\langle\mathcal{A}_i(x)\varsigma,\varsigma\rangle\operatorname{Re}\langle\mathcal{A}_j(x)\varsigma,\varsigma\rangle-\sum_{i,j=1}^{n} b_{ij}(x)\langle\mathcal{A}_i^*(x)\varsigma,\mathcal{A}_j^*(x)\varsigma\rangle +4\operatorname{Re}\langle\mathcal{A}_0(x)\varsigma,\varsigma\rangle\geq 0$$

*holds. Here* $((b_{jk}(x)))$ *is the* $(n\times n)$*-matrix inverse of* $((a_{jk}(x)))$ *and* $\mathcal{A}_j^*(x)$ *is the adjoint matrix of* $\mathcal{A}_j(x)$.

We remark that the second sum is real by the symmetry of the matrix $((b_{ij}(x))$.

In particular, the next assertion follows from Corollary 2.2′ for the scalar parabolic equation with complex coefficients

$$\frac{\partial u}{\partial t}-\sum_{j,k=1}^{n} a_{jk}(x)\frac{\partial^2 u}{\partial x_j\partial x_k}+\sum_{j=1}^{n} a_j(x)\frac{\partial u}{\partial x_j}+a_0(x)u=0. \tag{2.29}$$

**Corollary 2.3′.** *The classical maximum modulus principle is valid for* (2.29) *in* $Q_T(R_T^{n+1})$ *if and only if:*

(i) *the* $(n\times n)$*-matrix-valued function* $((a_{jk}(x)))$ *is real and positive-definite*

(ii) *for all* $x\in\Omega$ $(x\in R^n)$ *the inequality*

$$4\operatorname{Re}a_0(x)\geq\sum_{j,k=1}^{n} b_{jk}(x)\operatorname{Im}a_j(x)\operatorname{Im}a_k(x)$$

*holds.*

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