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On the order and type of the entire functions associated with an indeterminate Hamburger moment problem

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0. Introduction

In the fundamental paper [8] Riesz proved that the entire functions A, B, C, D associated with an indeterminate Hamburger moment problem are of at most minimal exponential type, i.e. they all satisfy an inequality of the form

$$\forall \varepsilon > 0 \ \exists \ N_\varepsilon > 0 \ \forall z \in \mathbf{C} : |f(z)| \leq N_\varepsilon e^{\varepsilon |z|}. \tag{1}$$

In other words the order of each of the functions is at most one, and if the order is one then the type is zero.

Concerning the moment problem we shall follow the notation and terminology of Akhiezer [1]. Up to now very few examples of quadruples A, B, C, D are explicitly known. In Berg and Valent [3] appear explicit formulas for the quadruple associated with a birth and death process admitting quartic rates, and the order and type can be calculated. It turns out that all four functions have order $\frac{1}{4}$ and type $K_0/\sqrt{2}$, where $K_0 = K(1/\sqrt{2})$ is the complete elliptic integral in the lemniscatic case.

This result encouraged us to examine if the functions A, B, C, D associated with an indeterminate Hamburger moment problem always have the same order and type. We shall show that this is indeed the case, and we further show that two more functions associated with the moment problem have the same order and type, viz.

$$p(z) = \left(\sum_{k=0}^{\infty} |P_k(z)|^2 \right)^{1/2}, \tag{2}$$

$$q(z) = \left(\sum_{k=0}^{\infty} |Q_k(z)|^2 \right)^{1/2}, \tag{3}$$

where (P_k) and (Q_k) are the orthonormal polynomials and the polynomials of the second kind respectively. We finally show that Phragmén–Lindelöf's indicator functions also agree when the type is finite. To see this we prove that $\log p$ and $\log q$ are subharmonic functions.

1. Preliminaries about order and type

For a continuous function $f\colon \mathbf{C}\to\mathbf{C}$ we define the *maximum modulus*

$$M_f(r) = \sup_{|z|\le r} |f(z)|, \quad r\ge 0, \tag{4}$$

and in case $M_f(r)\to\infty$ for $r\to\infty$ we define the *order*

$$\varrho_f = \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log r} \in [0,\infty]. \tag{5}$$

If $0<\varrho_f<\infty$ we define the *type* of f as

$$\sigma_f = \limsup_{r\to\infty} \frac{\log M_f(r)}{r^{\varrho_f}} \in [0,\infty]. \tag{6}$$

If $0<\sigma_f<\infty$ we define the Phragmén–Lindelöf indicator function

$$h_f(\theta) = \limsup_{r\to\infty} \frac{\log|f(re^{i\theta})|}{r^{\varrho_f}}, \quad \theta\in\mathbf{R}. \tag{7}$$

If f is an entire function of positive and finite order and type, then it is known that h_f is a continuous periodic function. For these concepts see Markushevich [6].

We say that an entire function f is of *at most minimal exponential type*, and we write $v(f)\le(1,0)$, if either $\varrho_f<1$ or $\varrho_f=1$ and $\sigma_f=0$. Equivalently f satisfies (1).

Lemma 1.1. *Let f be an entire function such that $v(f)\le(1,0)$. Then the order of f is equal to the convergence exponent of its zeros.*

Proof. Let (z_n) denote the non-vanishing zeros of f ordered such that $|z_1|\le |z_2|\le \ldots$ and repeated according to their multiplicities.

The convergence exponent τ_f of the zeros is the infimum of the numbers $\tau>0$ for which

$$\sum \frac{1}{|z_n|^\tau} < \infty.$$

By Hadamard's first theorem $\tau_f\le\varrho_f\le 1$, and if $\varrho_f<1$ then $\tau_f=\varrho_f$ by [4, p. 24], so we can assume $\varrho_f=1$ and $\sigma_f=0$.

The rank of the zeros is the smallest integer $\varkappa \geq 0$ for which

$$\sum \frac{1}{|z_n|^{\varkappa+1}} < \infty.$$

If the rank is zero, the theorem of Lindelöf (cf. [4, p. 27]) gives that f is a canonical product. For canonical products we know by Borel's Theorem that the order is equal to the convergence exponent.

If the rank is one we have $\Sigma |z_n|^{-1} = \infty$ and hence $\tau_f = 1$, so in this case $\tau_f = \varrho_f = 1$. □

2. The indeterminate moment problem

In the following μ denotes a probability measure on $\mathbf{R}$ with moments of any order, and we assume that μ is *indeterminate*. Let $(P_n)_{n \geq 0}$ denote the corresponding orthonormal polynomials chosen such that P_n has a positive leading coefficient for each n. The polynomials $(Q_n)_{n \geq 0}$ of the second kind are given by

$$Q_n(x) = \int \frac{P_n(x) - P_n(y)}{x - y} \, d\mu(y). \tag{8}$$

Since μ is indeterminate, the series in (2) and (3) converge uniformly on compact subsets of $\mathbf{C}$, and we define the entire functions

$$\begin{cases} A(z) = z \sum_{k=0}^{\infty} Q_k(0) Q_k(z) \\ B(z) = -1 + z \sum_{k=0}^{\infty} Q_k(0) P_k(z) \\ C(z) = 1 + z \sum_{k=0}^{\infty} P_k(0) Q_k(z) \\ D(z) = z \sum_{k=0}^{\infty} P_k(0) P_k(z) \end{cases} \tag{9}$$

cf. [1, p. 54].

We recall that the Nevanlinna extremal solutions $(\mu_t)_{t \in \mathbf{R} \cup \{\infty\}}$ are given by the following formula

$$\int \frac{d\mu_t(x)}{z - x} = \frac{A(z)t - C(z)}{B(z)t - D(z)}, \quad z \in \mathbf{C} \setminus \mathbf{R}, \tag{10}$$

where we use the convention that $At-C$ (resp. $Bt-D$) shall be interpreted as A (resp. B) for $t=\infty$.

The polynomial sequences $y_n=P_n(z)$ and $y_n=Q_n(z)$, $n\geq 0$ satisfy the second order difference equation

$$zy_n=b_ny_{n+1}+a_ny_n+b_{n-1}y_{n-1},\quad n\geq 1, \tag{11}$$

where

$$a_n=\int xP_n^2(x)\,d\mu(x),\quad b_n=\int xP_n(x)P_{n+1}(x)\,d\mu(x),\quad n\geq 0.$$

The sequence $(P_n(z))$ (resp. $Q_n(z)$) is uniquely determined by (11) and the initial conditions

$$y_0=1,\quad y_1=\frac{1}{b_0}(z-a_0),\quad (\text{resp. } y_0=0,\quad y_1=\frac{1}{b_0}). \tag{12}$$

Replacing (a_n) and (b_n) in (11) and (12) by the shifted sequences

$$\tilde{a}_n=a_{n+1},\quad \tilde{b}_n=b_{n+1}$$

we get solutions $(\widetilde{P}_n(z))$ and $(\widetilde{Q}_n(z))$ satisfying

$$\widetilde{P}_n(z)=b_0Q_{n+1}(z), \tag{13}$$

$$\widetilde{Q}_n(z)=P_1(z)Q_{n+1}(z)-\frac{1}{b_0}P_{n+1}(z), \tag{14}$$

cf. Pedersen [7]. By Favard's theorem $(\widetilde{P}_n)$ are the orthonormal polynomials associated with some probability $\tilde{\mu}$, and $(\widetilde{Q}_n)$ are the corresponding polynomials of the second kind. The measure $\tilde{\mu}$ is indeterminate like μ because of (13). The corresponding functions $\widetilde{A},\widetilde{B},\widetilde{C},\widetilde{D},\tilde{p},\tilde{q}$ are derived in [7], and we shall use the following:

Proposition 2.1 (Pedersen [7]).

$$A(z)=\frac{1}{b_0^2}\widetilde{D}(z)$$

$$C(z)=-\frac{a_0}{b_0^2}\widetilde{D}(z)-\widetilde{B}(z)$$

$$q(z)=\frac{1}{b_0}\tilde{p}(z)$$

$$\widetilde{C}(z)=-B(z)+(z-a_0)A(z)$$

$$b_0^2\widetilde{A}(z)+a_0\widetilde{C}(z)=D(z)-(z-a_0)C(z).$$

Proposition 2.2. *For $z=x+iy$, $y\neq 0$ we have*

$$|A(z)|\leq \frac{1}{|y|}|B(z)|, \tag{15}$$

$$|C(z)|\leq \frac{1}{|y|}|D(z)|. \tag{16}$$

Proof. From (10) we get for z as above and $t\in\mathbf{R}\cup\{\infty\}$

$$\left|\frac{A(z)t-C(z)}{B(z)t-D(z)}\right|\leq\int\frac{d\mu_t(u)}{|z-u|}\leq\frac{1}{|y|},$$

and (15) and (16) follow for $t=\infty$ and $t=0$. □

Proposition 2.3. *For $z=x+iy$, $y\neq 0$ we have*

$$|B(z)|\leq\left(\frac{b_0^2}{|y|}+|z-a_0|\right)|A(z)|, \tag{17}$$

$$|D(z)|\leq\left(\frac{b_0^2}{|y|}+|z-a_0|\right)|C(z)|. \tag{18}$$

Proof. By (16) we have

$$|\widetilde{C}(z)|\leq\frac{1}{|y|}|\widetilde{D}(z)|,$$

and inserting the expressions for $\widetilde{C}$ and $\widetilde{D}$ gives (17).

As in the proof of Proposition 2.2 we similarly have

$$\left|\frac{\widetilde{A}(z)t-\widetilde{C}(z)}{\widetilde{B}(z)t-\widetilde{D}(z)}\right|\leq\frac{1}{|y|},$$

and choosing $t=-b_0^2/a_0$ ($=\infty$ if $a_0=0$) we get from Proposition 2.1

$$\left|\frac{-D(z)+(z-a_0)C(z)}{b_0^2C(z)}\right|\leq\frac{1}{|y|},$$

and (18) follows. □

3. The order of A, B, C, D

Proposition 3.1. *The functions $tB-D, t\in\mathbf{R}\cup\{\infty\}$ all have the same order. In particular $\varrho_B=\varrho_D$.*

Proof. It is known that the functions in question have countably many zeros which are all real and simple, cf. [1, p. 101]. For $t,x\in\mathbf{R}$ we have

$$(tB-D)'(x)B(x)-(tB-D)(x)B'(x)=B'(x)D(x)-B(x)D'(x)=p^2(x),$$

cf. [1, p. 114], showing that there is exactly one zero of $tB-D$ between two consecutive zeros of B and vice-versa. Therefore, if B has infinitely many positive zeros, say $\lambda_1<\lambda_2<\dots$, then so has $tB-D$, and if they are denoted $\mu_1<\mu_2<\dots$, we have either $0<\mu_1<\lambda_1<\mu_2<\lambda_2<\dots$ or $0<\lambda_1<\mu_1<\lambda_2<\mu_2<\dots$. In both cases the series

$$\sum_{n=1}^{\infty}\frac{1}{\lambda_n^\tau},\quad \sum_{n=1}^{\infty}\frac{1}{\mu_n^\tau}$$

converge for the same numbers $\tau>0$. This argument together with a similar one, if B has infinitely many negative zeros, shows that the zeros of B and $tB-D$ have the same convergence exponent. By Lemma 1.1 the orders of B and $tB-D$ are equal. □

Proposition 3.2. *The functions $tA-C$, $t\in\mathbf{R}\cup\{\infty\}$ all have the same order as B. In particular $\varrho_A=\varrho_C=\varrho_B$.*

Proof. The zeros of $At-C$ and $Bt-D$ are all real and simple and they are interlacing because of (10). Therefore the convergence exponents and hence the orders are equal by Lemma 1.1. □

Theorem 3.3. *The functions A,B,C,D,p,q all have the same order.*

Proof. From the formula (9) for D we get by the Cauchy–Schwarz inequality

$$|D(z)|\le|z|p(0)p(z), \tag{19}$$

hence

$$M_D(r)\le p(0)rM_p(r), \tag{20}$$

and $\varrho_D\le\varrho_p$ follows.

On the other hand by formula 2.33 in [1] we have

$$p^2(z)\le\frac{|B(z)|\,|D(z)|}{|y|}\quad\text{for }\ z=x+iy,\ y\neq 0, \tag{21}$$

from which we conclude

$$M_p^2(r) \le M_B(1+r)M_D(1+r). \tag{22}$$

In fact for $|z| \le r$ with $y \ge 1$ we have

$$p^2(z) \le |B(z)|\,|D(z)| \le M_B(r)M_D(r).$$

We next use that $t \mapsto p^2(x+it)$ is increasing for $t \ge 0$. (The function $t \mapsto |P(x+it)|^2$ is increasing for any polynomial P having only real roots.) Therefore we get for $z=x+iy$, $0 \le y \le 1$, $|z| \le r$

$$p^2(x+iy) \le p^2(x+i) \le M_B(1+r)M_D(1+r).$$

Finally using $p(\bar{z})=p(z)$ we get (22).

From (22) we get $\varrho_p \le \varrho_D$ since $\varrho_B=\varrho_D$, hence $\varrho_p=\varrho_D$. Similarly $\widetilde{D}$ and $\tilde{p}$ have the same order, so by Proposition 2.1 we finally get $\varrho_q=\varrho_A$. □

4. The type of A, B, C, D

In the following we assume that the common order ϱ of the functions A, B, C, D, p, q is positive, hence $0<\varrho\le 1$, so the types of these functions are well-defined. We shall show that they have the same type σ, and this shall be the conclusion of Theorem 4.2. If $\varrho=1$ it is known by the theorem of Riesz that all six functions have type $\sigma=0$. We shall therefore suppose $0<\varrho<1$.

Proposition 4.1. *We have* $\sigma_p=\sigma_B=\sigma_D$.

Proof. From (20) we get $\sigma_D \le \sigma_p$.

By (9) we get from the Cauchy–Schwarz inequality

$$|B(z)| \le 1+|z|q(0)p(z) \tag{23}$$

and hence

$$M_B(r) \le 1+rq(0)M_p(r), \tag{24}$$

from which we similarly get $\sigma_B \le \sigma_p$. On the other hand (22) implies

$$\sigma_p \le \tfrac{1}{2}(\sigma_B+\sigma_D).$$

If $\sigma_p<\infty$ the above inequalities imply $\sigma_p=\sigma_B=\sigma_D$, and if $\sigma_p=\infty$ at least one of the numbers σ_B, σ_D must be infinite. That they are both infinite can be seen by Theorem 2.9.5 in [4] because $0<\varrho<1$ so

$$\sigma_B<\infty \quad \Leftrightarrow \quad n_B(r) \in O(r^{\varrho})$$

and similarly for D. Here $n_B(r)$ denotes the number of zeros of B in the disc $|z| \le r$. Since the zeros of B and D are interlacing, cf. the proof of Proposition 3.1, we have $n_B(r) \sim n_D(r)$. □

Theorem 4.2. *The functions A, B, C, D, p, q have the same type.*

Proof. Applying Proposition 4.1 to the shifted problem and using Proposition 2.1 we obtain

$$\sigma_{\tilde{B}} = \sigma_{\tilde{D}} = \sigma_A = \sigma_{\tilde{p}} = \sigma_q. \tag{25}$$

From (15) we deduce

$$M_A(r) \leq M_B(1+r). \tag{26}$$

In fact, for $z=x+iy$, $y\geq 1$, $|z|\leq r$ we have

$$|A(z)| \leq |B(z)| \leq M_B(1+r),$$

and as in the proof of Theorem 3.3 we get for $|z|\leq r$, $0\leq y\leq 1$

$$|A(z)| \leq |A(x+i)| \leq |B(x+i)| \leq M_B(1+r).$$

To conclude we use that $A(\bar{z})=\overline{A(z)}$.

From (17) we similarly deduce

$$M_B(r) \leq (b_0^2+|a_0|+r) M_A(1+r). \tag{27}$$

Using (26) and (27) we find $\sigma_A=\sigma_B$.

In the same way we see that $\sigma_C=\sigma_D$, and combined with (25) and Proposition 4.1 the proof is completed. □

5. The indicators of A, B, C, D

In this section we assume that the common order ϱ and common type σ of the functions A, B, C, D, p, q satisfy $0<\varrho<1$, $0<\sigma<\infty$. In this case we can define the Phragmén–Lindelöf indicator functions $h_A, ..., h_q$ of the above functions, cf. (7).

We shall need that h_p and h_q are continuous periodic functions like the indicators for entire functions. In fact, the crucial property for deriving this, cf. [6], is the trigonometric convexity which depends on the Phragmén–Lindelöf theorem. It is known that the ordinary Phragmén–Lindelöf theorem holds for *logarithmically subharmonic functions*, i.e. functions f for which $\log f$ is subharmonic. Such functions are necessarily subharmonic, and they appear as a generalization of the important class of log-convex functions. Concerning this extension of Phragmén–Lindelöf's theorem see [5, p. 26].

Proposition 5.1. *The functions* $\log p$ *and* $\log q$ *are subharmonic.*

This follows immediately from the following result:

Proposition 5.2. *Let* G *be a domain in* $\mathbf{C}$ *and let* $f_1, \dots, f_n$ *be holomorphic functions, not all identically zero. Then*

$$h(z) = \log\left(\sum_{i=1}^{n} |f_i(z)|^2\right), \quad z \in G$$

is subharmonic.

Proof. The statement is classical for $n=1$, and we proceed by induction. Assume that the statement holds for $n-1$ functions. If one of the functions, say f_n, is non-zero in a subdomain G_1 we write in G_1

$$\log\left(\sum_{i=1}^{n} |f_i|^2\right) = 2\log|f_n| + \log\left(1 + \sum_{i=1}^{n-1} \left|\frac{f_i}{f_n}\right|^2\right), \tag{28}$$

but

$$\varphi = \log\left(\sum_{i=1}^{n-1} \left|\frac{f_i}{f_n}\right|^2\right)$$

is subharmonic in G_1 by the induction hypothesis, and so is $\log(1+\exp(\varphi))$, since $\log(1+\exp x)$ is increasing and convex. This shows that (28) is subharmonic in G_1.

If all the functions f_i have a common zero $z_0 \in G$, and if k is the smallest order of z_0 as a zero of $f_i, i=1, \dots, n$, then we can write

$$f_i(z) = (z - z_0)^k g_i(z), \quad i = 1, \dots, n,$$

where g_i is holomorphic in G and $g_i(z_0) \neq 0$ for at least one i. Then we have

$$h(z) = 2k \log|z - z_0| + \log\left(\sum_{i=1}^{n} |g_i(z)|^2\right),$$

which is subharmonic in a neighbourhood of z_0. □

Remark 5.3. Using Proposition 5.2 it is easy to establish the following fundamental result:

If f *is holomorphic in a domain* G *with values in a Hilbert space* H*, then* $\log \|f\|$ *is subharmonic.* (This result also holds for Banach space valued holomorphic functions, see e.g. Aupetit [2, p. 52].)

Of course Proposition 5.1 is a special case of this result since $P\colon z \mapsto (P_k(z))_{k\geq 0}$ is an entire function with values in the Hilbert space l_2.

Theorem 5.4. *The functions A, B, C, D, p, q have the same Phragmén–Lindelöf indicator which is non-negative.*

Proof. From (19) and (23) we clearly get $h_D, h_B \le h_p$. Using (21) we obtain

$$h_p(\theta) \le \tfrac{1}{2}(h_B(\theta) + h_D(\theta)) \qquad \text{for } \theta \in]0, 2\pi[\setminus \{\pi\},$$

so for these θ we get $h_p(\theta) = h_B(\theta) = h_D(\theta)$. This holds finally for all $\theta \in [0, 2\pi]$ by continuity of the indicator functions.

We next apply this equality to the shifted problem and use Proposition 2.1 to obtain

$$h_q = h_{\widetilde{B}} = h_{\widetilde{D}} = h_A.$$

From (15) and (17) we get $h_A(\theta) = h_B(\theta)$ for $\theta \in]0, 2\pi[\setminus \{\pi\}$ so by continuity $h_A = h_B$. Similarly we get $h_C = h_D$ from (16) and (18).

The indicator h_p is ≥ 0 because $p \ge 1$. □

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