Ark. Mat., 34 (1996), 103–117

Distribution of interpolation points

René Grothmann

Abstract. We show that interpolation to a function, analytic on a compact set E in the complex plane, can yield maximal convergence only if a subsequence of the interpolation points converges to the equilibrium distribution on E in the weak sense. Furthermore, we will derive a converse theorem for the case when the measure associated with the interpolation points converges to a measure on E, which may be different from the equilibrium measure.

1. Introduction

The problem we wish to investigate is the interpolation of analytic functions in the complex plane. Let a function

$$f: E \to \mathbf{C}$$

be given on a compact set $E \subset \mathbf{C}$ with connected complement. Assume that f is analytic on an open neighborhood U of E. Furthermore, take a sequence of sets of interpolation points

$$z_{0,n}, \dots, z_{n,n} \in E, \quad n \in \mathbf{N}. \tag{1.1}$$

Then we can interpolate f in $z_{i,n}$ with polynomials from Π_n (the set of algebraic polynomials with complex coefficients of degree not greater than n), i.e.,

$$p_n(z_{\nu,n}) = f(z_{\nu,n}), \quad \nu = 0, \dots, n.$$

In the case when the points $z_{0,n}, \dots, z_{n,n}$ are not pairwise distinct, we use Hermite interpolation. By Hermite's formula, the interpolating polynomial p_n has a representation

$$p_n(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(\zeta)(\omega_n(\zeta) - \omega_n(z))}{\omega_n(\zeta)(\zeta - z)} \, d\zeta, \quad z \in E, \tag{1.2}$$

where γ is a path in $U \setminus E$ running around every point in E exactly once (such a path always exists), and

$$\omega_n(z) := \prod_{\nu=0}^{n} (z - z_{n,\nu}). \tag{1.3}$$

By Cauchy's formula, the error can be written as

$$f(z) - p_n(z) = \frac{1}{2\pi i} \oint_\gamma \frac{\omega_n(z)}{\omega_n(\zeta)} \frac{f(\zeta)}{(\zeta - z)} \, d\zeta. \tag{1.4}$$

From these well-known facts, one can deduce that p_n converges to f maximally in a sense to be described below, whenever the sequence of points satisfies the following distribution condition.

We introduce the *point counting measures* τ_n associated with (1.1). These measures are probability measures such that

$$\tau_n(\{z_{\nu,n}\}) = \frac{c_{\nu,n}}{n+1}, \quad \nu = 0, \dots, n, \tag{1.5}$$

where $c_{\nu,n}$ is the multiplicity of $z_{\nu,n}$.

Furthermore, we denote by μ_E the equilibrium measure with respect to the logarithmic kernel on the boundary of E (cf. [9]), i.e., μ_E minimizes the logarithmic energy

$$\iint \log \frac{1}{|z - \zeta|} \, d\mu(z) \, d\mu(\zeta)$$

over all probability measures μ on the boundary of E.

To define the notion of maximally converging polynomials, we need the Green's function $G_E = G$ in

$$\Omega := \mathbf{C} \setminus E$$

with pole at infinity and set

$$E_r = \{z \in \Omega : G(z) \le \log r\} \cup E, \quad r \ge 0.$$

We will assume that the boundary ∂E of E is regular, i.e., G can be extended continuously to E such that $G = 0$ on E. Furthermore, the level lines of G are denoted Γ_r. Then Γ_r is the boundary of E_r.

Finally, if $r > 1$ and f is analytic in the interior of E_r but in no neighborhood of E_r, then we call a sequence of polynomials $p_n \in \Pi_n$ *maximally convergent* to f, if

$$\limsup_{n \to \infty} \|f - p_n\|_E^{1/n} = \frac{1}{r}. \tag{1.6}$$

Here, $\|\cdot\|_E$ denotes the supremum norm of a bounded function on E. The right hand side term in (1.6) is the minimal one for such functions f. Maximal convergence has been studied by S. N. Bernstein and by J. L. Walsh [13].

It is well known [13], that p_n will converge maximally to f, if τ_n converges to μ_E in the weak sense (i.e., one has

$$\lim_{n\to\infty}\int g\,d\tau_n=\int g\,d\mu_E$$

for any g continuous in $\mathbf{C}$ with compact support).

The purpose of this paper is to investigate the converse direction of this statement. We assume that each polynomial p_n interpolates f in $n+1$ points of E. The sequence of these points is given in (1.1). We also assume, that f is analytic in a neighborhood of E, but not entire.

It is known that a necessary condition for maximal convergence in the class of *all* functions, which are analytic on E_r $(r>1)$, is the weak convergence of the distribution of the interpolation points to the equilibrium distribution. In fact, it suffices to take the class of functions of the form $f_a(z)=1/(z-a)$ with $a\notin E_r$. Then the interpolating polynomial for f_a is

$$p_n(z)=\frac{1}{z-a}-\frac{\omega_n(z)}{\omega_n(a)(z-a)}.$$

We will prove a converse of the above mentioned result for a single function f (rather than a class of functions).

A special consequence of our result is the fact that maximal convergence cannot be achieved even for a single function unless at least a subsequence of the interpolation points converges weakly to the equilibrium measure.

The second aim of this paper is to study the relation between the speed of convergence and analyticity, even if the sequence of interpolation points is not distributed according to the equilibrium measure. To make this more precise, we assume that for the measures introduced in (1.5), we have

$$\tau_n\to\sigma,$$

where σ is some unit measure on E. Since U^σ is lower semi-continuous, it will attain its minimum on E. Let

$$-\log\lambda_0=\min_{z\in E}U^\sigma(z). \tag{1.7}$$

Then

$$E\subseteq\{z\in\mathbf{C}\setminus E:U^\sigma(z)\geq-\log\lambda_0\}.$$

We define for all $\lambda \in \mathbf{R}$

$$\mathcal{E}_\lambda = \mathcal{E}_\lambda^\sigma := \{z \in \mathbf{C} \setminus E : U^\sigma(z) \geq -\log \lambda\} \cup E. \tag{1.8}$$

Then E is contained in the open interior of $\mathcal{E}_\lambda$ for $\lambda > \lambda_0$.

It is well known (see [7], [10]–[12]) that, if f is analytic inside $\mathcal{E}_\lambda$ for some $\lambda > \lambda_0$ and $L_n(f)$ denotes the interpolating polynomial to f in the points $\{z_{k,n}\}$, then

$$\limsup_{n \to \infty} \|f - L_n(f)\|_E^{1/n} \leq \frac{\lambda_0}{\lambda}. \tag{1.9}$$

The proof of this statement uses the Hermite formula (1.2). The case $\sigma = \mu_E$ is due to Bernstein and Walsh.

Our aim is to prove the converse of this result, i.e., we prove that (1.9) implies the analyticity of f in the interior of $\mathcal{E}_\lambda$. There seems to be no straightforward proof of this result. In the case $\sigma = \mu_E$ the standard proof uses the Bernstein–Walsh estimate, which states that

$$\|p_n\|_{E_r} \leq r^n \|p_n\|_E \tag{1.10}$$

for all polynomials of degree at most n and $r > 1$. Such a result is no longer available in our case. So we have to find another way of proving that f is analytic in a neighborhood of $\mathcal{E}_{\lambda_0}$. This will suffice to show that f is analytic in E_λ. Note, that the analyticity in a neighborhood of E follows from the classical result. But $\mathcal{E}_{\lambda_0}$ is larger than E, unless σ is μ_E.

A well-known special example is the case when all points $z_{k,n} = z_0$. In this case

$$\lambda_0 = \sup_{z \in E} |z - z_0|,$$

and it is clear from the theory of the Taylor expansion that p_n converges inside the disk of radius λ_0 around z_0. In this case, the rest of the theorem follows from the Bernstein–Walsh estimate by classical arguments.

2. Statement of results

Since we introduced the necessary terms in the last section, we can immediately formulate our first theorem.

2.1. Theorem. *Let $f\colon E\to\mathbf{C}$ be a continuous function such that there exists a maximally convergent sequence of polynomials $p_n\in\Pi_n$ with*

$$\limsup_{n\to\infty}\|f-p_n\|_E^{1/n}=\frac{1}{r}$$

for $r>1$ $(r<\infty)$ and such that $f-p_n$ has $n+1$ zeros

$$z_{0,n}\,,\dots\,,z_{n,n}\in\partial E,$$

where ∂E denotes the boundary of E. Then the sequence of measures τ_n associated to these points (according to (1.5)) has μ_E as a weak limit point.

Consequently, if f is analytic in the interior of E_r $(r>1)$ but not in any open neighborhood of E_r, then maximal convergence can only occur if τ_n has μ_E as weak limit point.

Note, that a subsequence may well converge faster. As an example, let

$$f(z)=\sum_{\nu=0}^{\infty}z^{n!},$$

set $E=\left[-\frac{1}{2},\frac{1}{2}\right]$ and denote by s_n the nth partial sum of the power series of f around 0. Then

$$\lim_{n\to\infty}\|f-s_{n!}\|_E^{1/n!}=0.$$

Theorem 2.1 requires that there exist interpolation points. If $E=[a,b]$ this may be guaranteed for certain sequences of best approximating polynomials. For instance, if p_n^* denotes the polynomial of best uniform approximation to a real-valued function f on E,

$$\|f-p_n^*\|_E=e_n(f)=\inf\{\|f-q_n\|_E : q_n\in\Pi_n\},$$

then it is well known that $r_n^*=f-p_n^*$ has an alternation set of $n+2$ points. These points are interlaced by interpolation points of p_n^* to f. Since polynomials of best approximation are of course maximally convergent, the theorem can be applied. A result of Kadec [6] gives more precise information on the location of the alternation points, which has been sharpened by Blatt [1]. If r is big enough even more precise results can be achieved [5].

However, we may apply Theorem 2.1 in a more general situation.

In [9] Saff and Shekhtman investigated interpolation properties of best $L_p(\sigma)$-approximants. In their setup, a measure σ is given on $[-1,1]$, the support of which contains infinitely many points, and $s_{n,p}$ is the best $L_p(\sigma)$-approximant to a given function $f\in C[-1,1]$ with respect to Π_n $(1\leq p<\infty)$.

Since then

$$\int |f(x)-s_{n,p}(x)|^{p-1}\operatorname{sign}(f(x)-s_{n,p}(x))q_n(x)\,d\sigma(x)=0 \tag{2.1}$$

for all $q_n\in\Pi_n$, one can see that $s_{n,p}$ necessarily interpolates f in at least $n+1$ points, where the error $f-s_{n,p}$ changes its sign. Saff and Shekhtman investigated the denseness of these points. Their main result says, that, if f is not σ-a.e. equal to a polynomial, then in any interval $[a,b]\subseteq[-1,1]$ there are infinitely many such interpolation points. They then boldly conjecture that a subsequence of the sequence of nth Fekete subsets of the set of interpolation points distributes like the equilibrium measure.

Using Theorem 2.1, we can prove their conjecture for analytic functions. However, we pose a restriction on σ. We will assume that the function

$$\omega(\sigma,\delta):=\sup\{\sigma([x-\delta,x+\delta]):x\in\mathbf{R}\}$$

does not rapidly tend to 0 when $\delta\to 0$. Furthermore, we require that the support E of σ is an interval.

2.2. Corollary. *Let σ have compact support $E=[a,b]$. Let $1\leq p<\infty$ and suppose that*

$$\liminf_{n\to\infty}\omega(\sigma,1/n^2)^{1/n}>0.$$

If f is analytic on E, but not entire, and $x_{0,n}<\ldots<x_{n,n}$ are zeros of $f-s_{n,p}$, then the corresponding sequence of counting measures τ_n has μ_E as a weak limit point.

If f is not assumed to be analytic on E, then the question remains open. However, in the above mentioned paper, Saff and Shekhtman have shown that in this case the interpolation points are dense in E. Furthermore, Blatt has proven the result in the case of L_2 approximation [2].

We wish to note, that we are not restricted to interpolation on the boundary of E. If the interpolation points are situated in all of E, we need to introduce the *balayage* of τ_n to the boundary of E. It is defined as a measure $\tilde{\tau}_n$ on the boundary of E such that

$$U^{\tilde{\tau}_n}(z)=U^{\tau_n}(z),\quad \text{for all } z\in\mathbf{C}\setminus E.$$

By the following Lemma 2.3, which will be useful for us for other reasons, this measure is unique. For the existence of the balayage of a measure see [9].

The following result is classical. A simple proof is implicitly contained in a paper of Carleson [3] (see the proof of Lemma 3). For an older reference see the paper of Deny [4].

2.3. Lemma. *Let μ_1 and μ_2 be two measures supported on the boundary of E, such that $U^{\mu_1}(z)=U^{\mu_2}(z)$ for all $z\in\Omega$. Then $\mu_1=\mu_2$.*

Now, we generalize Theorem 2.1 to the balayage of the distribution of interpolation points.

2.4. Theorem. *Let the sequence $(p_n)_{n\in\mathbf{N}}$ $(p_n\in\Pi_n)$ of interpolating polynomials be maximally convergent to f. Assume that the interpolation points are in E. Then the sequence $(\tilde{\tau}_n)$ has μ_E as a weak limit point.*

An example of this is the simple fact that the Taylor series of a function, which is analytic in a circle of radius $r>1$ around 0, taken in a point z_0 in the unit circle D_1 can only converge with rate $1/r$ in D_1, if $z_0=0$.

Next we ask if a subsequence of $(\tilde{\tau}_n)$, which does not converge to μ_E, is exceptional in some sense. To make this more precise, we need to introduce a distance between measures on E. We still assume that ∂E is regular with respect to Ω (see above). In this case, we define

$$\mathcal{M}_{\partial E}:=\{\mu:\mu \text{ is a unit measure on } \partial E\},$$

and, for $r>1$,

$$d_r(\mu_1,\mu_2)=\|U^{\mu_1}-U^{\mu_2}\|_{\Gamma_r},\quad \mu_1,\mu_2\in\mathcal{M}_{\partial E},$$

where $\|\cdot\|_K$ denotes the supremum norm on a compact set K. If $d_r(\mu_1,\mu_2)=0$, then by the maximum principle $U^{\mu_1}=U^{\mu_2}$ on $\mathbf{C}\setminus E_r$ and thus on Ω. By Lemma 2.3, $\mu_1=\mu_2$. Clearly, d_r satisfies the other prerequisites for a metric. Moreover, convergence with respect to this metric is equivalent to weak convergence. For if (μ_n) converges weakly to $\mu\in\mathcal{M}_{\partial E}$, then the potentials (U^{μ_n}) converge uniformly on Γ_r to U^μ, since this set has a positive distance to E; i.e.,

$$\lim_{n\to\infty}\int\log\frac{1}{|z-\zeta|}\,d\mu_n(\zeta)=\int\log\frac{1}{|z-\zeta|}\,d\mu(\zeta),$$

uniformly for $z\in\Gamma_r$. Thus $d_r(\mu_n,\mu)\to 0$. On the other hand, if $d_r(\mu_n,\mu)$ converges to 0, then any weak limit point σ of (μ_n) is equal to μ by the same argument. Since any subsequence of (μ_n) has a weak limit point, (μ_n) converges to μ weakly.

Though d_r can be defined for all measures on E, it is not a distance on this enlarged set of measures. Clearly

$$d_r(\tau,\tilde{\tau})=0$$

if τ is a unit measure on E and $\tilde{\tau}$ is the balayage of τ to the boundary of E. However, we will define d_r for all unit measures on E.

2.5. Theorem. *If under the conditions of Theorem 2.4, for* $\varepsilon > 0$

$$M_\varepsilon = \{n_1 < n_2 < \dots\} \subset \mathbf{N}$$

is the set of indices, such that

$$d_r(\mu_E, \tilde{\tau}_{n_k}) > \varepsilon \quad \textit{for all } k \in \mathbf{N}$$

with any fixed $r > 1$, *then*

$$\limsup_{k\to\infty} \frac{n_{k+1}}{n_k} > 1. \tag{2.2}$$

Relation (2.2) implies that the density of M_ε in $\mathbf{N}$ defined as

$$\liminf_{k\to\infty} \frac{\text{cardinality } \{\nu \le k : \nu \in M_\varepsilon\}}{k}$$

is less than 1. However, there are sets with density 0 such that (2.2) is not satisfied.

We now proceed to state the second main result of this paper. Again, we have already introduced the necessary definitions.

2.6. Theorem. *Let the counting measures* τ_n *(see (1.5)), associated with the interpolation points, weakly converge to a measure* σ *supported on* E *and assume that we have a geometric rate of convergence, i.e., for the sequence* p_n *of interpolating polynomials to* f, *we have with some* $\lambda_1 > \lambda_0$

$$\limsup_{k\to\infty} \|f - p_{n_k}\|_E^{1/n_k} = \frac{\lambda_1}{\lambda_0},$$

where λ_0 *is defined as in (1.7). Additionally, assume that*

$$\lim_{k\to\infty} \frac{n_{k+1}}{n_k} = 1.$$

Then f *is analytic in the open interior of* $\mathcal{E}_{\lambda_1}^\sigma$ *(see (1.8)).*

We remark once more that the converse is a well-known result, which follows directly from the Hermite formula and the weak convergence. We already mentioned the case when σ is a point measure and all interpolation takes place in that point. In this case, we get a well-known result about the Taylor series. If $\sigma = \mu_E$, we obtain another well-known result due to Bernstein and Walsh. There is also a relatively simple proof in the case when each set of interpolation points differs from the previous one by just a single point.

3. Proofs

It is clear that Theorems 2.1 and 2.4 are corollaries to Theorem 2.5. So we will restrict our attention to the proof of this theorem.

Proof of Theorem 2.5

For abbreviation, set $\mu := \mu_E$. Furthermore, we denote by $\mathbf{C}_\infty$ the compactified complex plane.

We assume now that (2.2) does not hold; i.e.,

$$\lim_{k\to\infty} \frac{n_{k+1}}{n_k} = 1.$$

By the maximum principle applied in $\mathbf{C}\setminus E_R$,

$$d_R(\mu, \tilde{\tau}_{n_k}) > d_r(\mu, \tilde{\tau}_{n_k}) > \varepsilon$$

for $1 < R < r$.

We fix the maximal $\varrho > 0$ such that

$$\limsup_{n\to\infty} \|f - p_n\|_E^{1/n} \le \frac{1}{\varrho}.$$

We then choose $1 < R < \varrho_0 < \varrho_1 < \varrho$, where $R < r$. Our aim is to show

$$\limsup_{n\to\infty} \|f - p_n\|_{E_{\varrho_0}}^{1/n} < \frac{\varrho_0}{\varrho}. \tag{3.1}$$

Next, we claim that there is a $\delta > 0$ (depending on ε) with the following property: If for $\sigma \in \mathcal{M}_{\partial E}$ with $d_R(\mu, \sigma) > \varepsilon$, we construct a harmonic function ϕ_σ on

$$U := \operatorname{int}(E_{\varrho_1} \setminus E_R)$$

such that ϕ_σ extends continuously to Γ_R and to Γ_{ϱ_1}, $\phi_\sigma(z) = 0$ for $z \in \Gamma_{\varrho_1}$, and

$$\phi_\sigma(z) = \min\{U^\mu(z) - U^\sigma(z) - c_\sigma, 0\}, \quad z \in \Gamma_R,$$

where

$$c_\sigma := \min_{z\in\Gamma_{\varrho_1}} (U^\mu(z) - U^\sigma(z)) = -\log \varrho_1 - \log \operatorname{cap} E + \min_{z\in\Gamma_{\varrho_1}} (-U^\sigma(z)),$$

then

$$\phi_\sigma(z) < -\delta, \quad z \in \Gamma_{\varrho_0}. \tag{3.2}$$

Assume there is no such δ. Then there exists a weakly converging sequence (σ_n) in $\mathcal{M}_{\partial E}$ such that

$$\max_{z \in \Gamma_{\varrho_0}} \phi_{\sigma_n}(z) > -\frac{1}{n}.$$

By the uniform convergence of the sequence (U^{σ_n}) on Γ_R, we get for the weak limit point σ of (σ_n)

$$\max_{z \in \Gamma_{\varrho_0}} \phi_\sigma(z) \geq 0.$$

Together with the boundary condition, this implies that $\phi_\sigma = 0$. It follows that $U^\mu(z) - U^\sigma(z) \geq c_\sigma$ on Γ_R. But, since $U^\mu(z) - U^\sigma(z)$ is a harmonic function in $\mathbf{C}_\infty \setminus E$, we get $U^\mu = U^\sigma$ by the minimum principle. Thus by Lemma 2.3 $\mu = \sigma$. This is a contradiction to $d_R(\mu, \sigma) > \varepsilon$, which proves the existence of our δ.

Now it is well known, that p_n converges maximally to f on all E_s with $1 \leq s < \varrho$, i.e.

$$\limsup_{n \to \infty} \|p_n - f\|_{E_s} = \frac{s}{\varrho}.$$

This implies the following two facts. Fix $\eta > 0$. Then we can choose $N_\eta \in \mathbf{N}$ such that, for all $z \in E_R$,

$$\frac{1}{n} \log |f(z) - p_n(z)| \leq \log R - \log \varrho + \eta, \quad n \geq N_\eta, \tag{3.3}$$

and, for all $z \in E_{\varrho_1}$,

$$\frac{1}{n} \log |f(z) - p_n(z)| \leq \log \varrho_1 - \log \varrho + \eta, \quad n \geq N_\eta. \tag{3.4}$$

Additionally, we can use formula (1.4) for the function $f(z) - p_n(z)$ (interpolating in the interpolation points) and choose N_η so large that, for all $z \in E_R$, $n > N_\eta$

$$\begin{aligned} \frac{1}{n} \log |f(z) - p_n(z)| &\leq -U^\sigma(z) - \inf_{w \in \Gamma_{\varrho_1}} (-U^\sigma(w)) + \frac{1}{n} \log \|f - p_n\|_{E_{\varrho_1}} + \eta \\ &\leq -U^\sigma(z) - \inf_{w \in \Gamma_{\varrho_1}} (-U^\sigma(w)) + \log \varrho_1 - \log \varrho + 2\eta \\ &= (U^\mu(z) - U^\sigma(z)) - c_\sigma + \log R - \log \varrho + 2\eta, \end{aligned} \tag{3.5}$$

where we used (3.4).

Next, we define the function

$$h_n(z):=\frac{1}{n}\log|f(z)-p_n(z)|-(G(z)-\log\varrho).$$

This function is subharmonic in Ω and thus obeys the maximum principle in this region.

Take z with $G(z)=\log R$, $n\geq N_\eta$. Then, if $\phi_\sigma(z)<0$, we have by (3.5)

$$h_n(z)\leq(U^\mu(z)-U^\sigma(z))-c_\sigma+\log R-\log\varrho-(G(z)-\log\varrho)+2\eta=\phi_\sigma(z)+2\eta.$$

If $\phi_\sigma(z)=0$, we get by (3.3)

$$h_n(z)\leq\eta\leq\phi_\sigma(z)+2\eta.$$

For z with $G(z)=\log\varrho_1$, $n\geq N_\eta$, we have $\phi_\sigma(z)=0$ and thus, by (3.4),

$$h_n(z)\leq\eta\leq\phi_\sigma(z)+2\eta.$$

Using the maximum principle, we see that, for $\log R\leq G(z)\leq\log\varrho_1$,

$$h_n(z)\leq\phi_\sigma(z)+2\eta,\quad n\geq N_\eta.$$

With (3.2), for $z\in\Gamma_{\varrho_0}$,

$$h_n(z)<\phi_\sigma(z)+2\eta<-\delta+2\eta$$

and thus

$$\frac{1}{n}\log|f(z)-p_n(z)|\leq\log\varrho_0-\log\varrho+2\eta-\delta<\log\varrho_0-\log\varrho-\delta/2,\quad n\geq N_\eta,\ n\in M_\varepsilon$$

if we choose $\eta>0$ small enough. By the maximum principle, the last estimate holds for $z\in E_{\varrho_0}$.

Using (2.2), we see that for any $\alpha>0$ there is a constant K_α such that

$$\|p_{n_{k+1}}-p_{n_k}\|_{E_{\varrho_1}}\leq\left(\frac{(1+\alpha)\varrho_0}{e^{\delta/2}\varrho}\right)^{n_k}$$

for $k>K_\alpha$, which is equivalent to (3.1).

By the Bernstein–Walsh inequality and (2.2), there is an $s>\varrho$ and $\gamma<1$ such that

$$\|p_{n_{k+1}}-p_{n_k}\|_{E_s}\leq\gamma^{n_k}.$$

Thus f is analytic in E_s, which is a contradiction. $\square$

Proof of Corollary 2.2

We may assume $[a,b]=[-1,1]$. From (2.1) we know that there are at least $n+1$ zeros of $f-s_{n,p}$ in $E=[-1,1]$. Since these zeros serve as interpolation points, we only have to show that $s_{n,p}$ converges maximally to f in the supremum norm. Let the maximal geometric rate of uniform convergence be $1/\varrho<1$.

If we denote by p_n^* the best uniform approximation to f on E, then for each $\eta>0$ there exists an $N_\eta\in\mathbf{N}$ such that

$$\begin{aligned}(3.6)\qquad \left(\int_E |f-s_{n,p}|^p\,d\sigma\right)^{1/p} &\le \left(\int_E |f-p_n^*|^p\,d\sigma\right)^{1/p}\\ &\le \sigma(E)^{1/p}\|f-p_n^*\|_E \le \left(\frac{1+\eta}{\varrho}\right)^n,\end{aligned}$$

for all $n\ge N_\eta$. Thus $s_{n,p}$ converges maximally in the $L_p(\sigma)$-sense.

Next we claim that for $q\in\Pi_n$

$$(3.7)\qquad \left(\int_E |q|^p\,d\sigma\right)^{1/p} \ge \frac{\|q\|_E\,\omega(\sigma,1/n^2)^{1/p}}{2}.$$

Let $x\in E$ be the point, where $|q|$ attains its maximum. We may assume $q(x)=1$. Then by the Markov inequality, $|q'(x)|<n^2$. Thus

$$\begin{aligned}\left(\int_E |q|^p\,d\sigma\right)^{1/p} &\ge \left(\int_E \max\{(1-n^2|x-t|)^p,0\}\,d\sigma\right)^{1/p}\\ &\ge \frac{\sigma([x-n^{-2}/2,x+n^{-2}/2])^{1/p}}{2} \ge \frac{\omega(\sigma,1/n^2)^{1/p}}{2}.\end{aligned}$$

This proves (3.7).

Now, we combine (3.6) and (3.7) and get

$$\|s_{n,p}-s_{n-1,p}\|_E \le \left(\frac{1+\eta}{\varrho}\right)^n \frac{4}{\omega(\sigma,1/n^2)^{1/p}}.$$

By the assumption of the corollary, we have that $s_{n,p}$ converges to some function g uniformly with geometric rate $1/\varrho$. Clearly, this function is identical to f on E, since $s_{n,p}$ converges to f in the $L_p(\sigma)$-sense. This completes the proof of the corollary. □

Proof of Theorem 2.6

It is well known, that f is analytic in $U:=\operatorname{int} E_{\lambda_1/\lambda_0}$, and $E\subset U$. Denote by

$$\lambda_2 := \sup\{\lambda : f \text{ is analytic in a neighborhood of } \mathcal{E}_\lambda^\sigma\}.$$

Clearly, $\lambda_2>0$, since by definition U^σ is bounded on $\Omega\backslash U$, and thus $\mathcal{E}_\lambda^\sigma\subseteq\operatorname{int} E_{\lambda_1/\lambda_0}$ for $\lambda>0$ small enough.

Assume $\lambda_2<\lambda_1$. Reasoning as in the proof of the converse result to Theorem 2.6, we construct for some fixed small $\eta>0$ a path in

$$(\mathcal{E}_{\lambda_2}^\sigma\cup E_{\lambda_1/\lambda_0})\backslash(\mathcal{E}_{\lambda_2-\eta}^\sigma\cup E_{\lambda_1/\lambda_0-\eta}),$$

such that the path winds around every point in E exactly once. This is possible, because of the following inclusions

$$E\subset\mathcal{E}_{\lambda_2}^\sigma\cup E_{\lambda_1/\lambda_0}\subset\operatorname{int}(\mathcal{E}_{\lambda_2-\eta}^\sigma\cup E_{\lambda_1/\lambda_0-\eta}).$$

Using the Hermite formula, we get

$$\limsup_{n\in M}|f(z)-p_n(z)|^{1/n}\le\frac{\lambda}{\lambda_2},\quad z\in\mathcal{E}_\lambda^\sigma\backslash E,\ \lambda<\lambda_2,$$

and this limit is uniform in $\mathcal{E}_\lambda^\sigma\backslash\operatorname{int} E_\tau$, $\lambda<\lambda_2$, $\tau>1$. Furthermore, for any $\lambda_1/\lambda_0>\tau>1$, we have uniformly for $z\in\Gamma_\tau$,

$$\limsup_{n\in M}|f(z)-p_n(z)|^{1/n}\le\frac{\lambda_0\tau}{\lambda_1}.$$

This follows from the Bernstein–Walsh inequality (1.10) in the classical way.

Thus, uniformly on Γ_τ, $1<\tau<\lambda_1/\lambda_0$,

$$\limsup_{n\in M}\frac{1}{n}\log|f(z)-p_n(z)|\le\min\{-U^\sigma(z)-\log\lambda_2,\ \log\lambda_0-\log\lambda_1+\log\tau\}.$$

Since $\lim n_{k+1}/n_k=1$ by assumption, we get

(3.8)
$$\limsup_{k\to\infty}\frac{1}{n_{k+1}}\log|p_{n_{k+1}}(z)-p_{n_k}(z)|\le\min\{-U^\sigma(z)-\log\lambda_2,\ \log\lambda_0-\log\lambda_1+\log\tau\}.$$

Now

$$h_k(z):=\frac{1}{n_{k+1}}\log|p_{n_{k+1}}(z)-p_{n_k}(z)|+U^\sigma(z)+\log\lambda_2$$

is subharmonic in $\mathbf{C}_\infty\backslash E$. We choose $\tau>1$ such that

$$4\log\tau<\log\lambda_1-\log\lambda_2.$$

Let z_0 be the point in Γ_τ, where U^σ is minimal. Then $U^\sigma(z_0) < -\log\lambda_0$ and, by (3.8), there is a constant $K_\tau \in \mathbf{N}$, such that

$$\begin{aligned} h_k(z_0) &< U^\sigma(z_0) + \log\lambda_2 + \log\lambda_0 - \log\lambda_1 + 2\log\tau \\ &< \log\lambda_2 - \log\lambda_1 + 2\log\tau < \tfrac{1}{2}(\log\lambda_2 - \log\lambda_1) =: c \end{aligned}$$

for $k \geq K_\tau$.

In a ball B around z_0, we get

$$h_k(z) < c < 0, \quad k \geq M_\tau.$$

Furthermore, by (3.8), we have

$$h_k \leq \varepsilon, \quad z \in \Gamma_\tau, \ k \geq N_\varepsilon,$$

for arbitrary $\varepsilon > 0$. Since h_k is subharmonic in $\mathbf{C}_\infty \setminus E_\tau$, we can construct a harmonic majorant ϕ_ε on $\mathbf{C}_\infty \setminus E_\tau$ with

$$\phi_\varepsilon(z) = \begin{cases} \varepsilon, & \text{for } z \in \Gamma_\tau \setminus B, \\ c, & \text{for } z \in \Gamma_\tau \cap B. \end{cases}$$

In fact, if we set $\phi = \phi_0$, we have

$$\phi_\varepsilon(z) = \frac{\phi(z)(|c| + \varepsilon)}{|c|} + \varepsilon.$$

Furthermore, $\phi(z) < C_0 < 0$ in any compact subset of $\mathbf{C} \setminus E$. With $\varepsilon > 0$ small enough, it follows that $h_k(z) < C_1 < 0$ in a neighborhood V of the set

$$\{z \in \Omega \setminus E_{\lambda_1/\lambda_0} : U^\sigma(z) = -\log\lambda_2\}.$$

Thus $(p_n)_{n \in M}$ converges in V to an analytic continuation of f. This contradiction completes the proof. $\square$

References

1. Blatt, H.-P., On the distribution of simple zeros of polynomials, *J. Approx. Theory* **69** (1992), 250–268.
2. Blatt, H.-P., The distribution of sign changes of the error function in best L^p-approximation, in *Computational Methods and Function Theory* (Ali, R. M., Ruscheweyh, St. and Saff, E. B., eds.), pp. 75–87, World Scientific Publishing, London, 1995.

3. CARLESON, L., Mergelyan's theorem on uniform polynomial approximation, *Math. Scand.* **15** (1964), 167–175.
4. DENY, J., Systèmes totaux de fonctions harmoniques, *Ann. Inst. Fourier* (*Grenoble*) **1** (1950), 103–113.
5. GROTHMANN, R., On the real CF-approximation for polynomial approximation and strong uniqueness constants, *J. Approx. Theory* **55** (1988), 86–103.
6. KADEC, M. I., On the distribution of points of maximal deviation in the approximation of continuous functions by polynomials, *Uspekhi Mat. Nauk* **15** (1960), 199–202. English transl.: *Trans. Amer. Math. Soc.* **26** (1960), 231–234.
7. KRYLOW, V. J., *Approximate Calculation of Integrals*, Macmillan Company, New York, 1962.
8. SAFF, E. B. and SHEKHTMAN, B., Interpolatory properties of best L_2-approximants, *Indag. Math.* (*N. S.*) **1** (1990), 489–498.
9. TSUJI, M., *Potential Theory in Modern Function Theory*, Maruzen Co. Ltd., Tokyo, 1959. Also: Chelsea Publ. Co., New York, 1975.
10. WALLIN, H., Rational interpolation to meromorphic functions, in *Padé Approximation and its Applications, Proc. Conf. Amsterdam 1980,* (Bruin, M. G. and van Rossum, H., eds.), Lecture Notes in Math. **888**, pp. 371–383, Springer-Verlag, Berlin–Heidelberg, 1981.
11. WALLIN, H., Divergence of multipoint Padé approximation, in *Complex Analysis, Proc. 5th Rom.-Finn. Seminar Bukarest 1981, Part 2,* (Cazacu, A., Boboc, N. and Jurchescu, M., eds.), Lecture Notes in Math. **1014**, pp. 246–255, Springer-Verlag, Berlin–Heidelberg, 1983.
12. WALLIN, H., Convergence and divergence of multipoint Padé approximants of meromorphic functions, in *Rational Approximation and Interpolation, Proc. Conf. Tampa* (Graves-Morris, P. R., Saff, E. B. and Varga, R. S., eds.), Lecture Notes in Math. **1105**, pp. 272–284, Springer-Verlag, Berlin–Heidelberg, 1984.
13. WALSH, J. L., *Interpolation and Approximation by Rational Functions in the Complex Domain,* Amer. Math. Soc. Colloq. **20**, Amer. Math. Soc., Providence, R. I., 1935.

Received August 7, 1995

René Grothmann
Katholische Universität Eichstätt
D-85071 Eichstätt
Germany