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Geometric localization, uniformly John property and separated semihyperbolic dynamics

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0. Introduction

Different types of geometric localization are used extensively in analysis. Localization grasps fine properties of the boundary which allows one to carry out estimates of harmonic measure, Green's function, etc. In the absence of the Riemann mapping theorem, localization may serve as a weak substitute. Examples of such an approach can be found in Ancona [A1], [A2], J.-M. Wu [W] for (mainly) Lipschitz domains and in Jones [J1], Jerison and Kenig [JK] for non-tangentially accessible domains. Carleson's work [C] may be considered as a source for this approach. In this paper we are going to deal with a localization property for John domains. Our motivation is the following. Denote by $A_\infty(f)$ the domain of attraction to ∞ of a polynomial f. A recent result of L. Carleson, P. W. Jones and J. C. Yoccoz ([CJY]) shows that $A_\infty(f)$ is a John domain if and only if f is semihyperbolic. In the first section we prove localization for simply connected John domains. In Section 2 we give an example showing that localization fails for arbitrary John domains and we prove the localization at a fixed scale. The third section is devoted to some geometric properties of the Julia set of a semihyperbolic polynomial. In the fourth section we introduce separated semihyperbolic dynamics and prove that the localization of $A_\infty(f)$ is equivalent to the property of being separated semihyperbolic. For example localization works for critically finite f. In Section 5 we show that localizability is equivalent to uniformity for John domains and Section 6 provides an example of semihyperbolic but not separated semihyperbolic polynomials. In the last section, we discuss some applications of this property. Commenting on Sections 1–3 let us mention that throughout them we modify ideas virtually present in [BH], [CJY],

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[HR], [NV], [P].

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1. Localization of simply connected John domains

Let $\Omega \subseteq \mathbf{C}$ be a John domain. This means that there is a point $x_0 \in \Omega$ (called a center) such that any $x \in \Omega$ can be connected to x_0 by an arc $\gamma \subseteq \Omega$ such that:

$$\operatorname{dist}(\xi, \partial\Omega) \geq c \operatorname{dist}(\xi, x) \quad \text{for } \xi \in \gamma,$$

for some $c>0$ independent of x.

The arc γ is called a John arc and the best constant $c>0$ is the John constant of the domain Ω. A simply connected John domain is called a John disk. Let us recall that the internal metric ϱ for a domain Ω is defined by:

$$\varrho(x, y) = \inf_{\gamma} \{\operatorname{diam} \gamma : \gamma \setminus \{x, y\} \subseteq \Omega \text{ is a curve connecting } x \text{ and } y\},$$

for $x, y \in \overline{\Omega}$. If Ω is simply connected, ϱ can be extended so that (Ω^*, ϱ) becomes a complete metric space where Ω^* denotes the union of the interior points and prime ends. In what follows we denote by $B_\varrho(Q, r)$ the points in Ω which are r-close to Q in the metric ϱ.

1.1. *Definition.* Let Ω be a John domain and $Q \in \partial\Omega$, $r>0$. A finite collection of domains $\{\Omega_Q^l(r)\}_{l=1}^N$ is called a John prelocalization at the point Q on the scale r if:

(1) $\Omega_Q^l(r) \subset \Omega$ is a John domain with constant c for $l=1, \dots, N$,
(2) $\bigcup_l \Omega_Q^l(r) \supset B_\varrho(Q, r)$,
(3) $\operatorname{diam} \Omega_Q^l(r) \leq Mr$, $l=1, \dots, N$.

1.2. *Definition.* A prelocalization is called localization if in addition to (1), (2), (3) we have

(4) $\Omega_Q^i(r) \cap \Omega_Q^j(r) = \emptyset$, $i \neq j$, $Q \in \partial\Omega$, $0<r<r_0$.

1.3. *Definition.* A John disk admits a localization for all scales $r<r_0$ if there is a localization at any point $Q \in \partial\Omega$ for all scales $r<r_0$; moreover the constants N, M, c do not depend on Q and r.

The main goal of this section is to prove Theorem 1.9 but first we prove the following result.

1.4. Theorem. *A John disk admits a prelocalization.*

First we give a series of preparatory results. They are known but we include them for the convenience of the reader.

1.5. Lemma. *Let $f:\mathbf{D}\to\Omega$ be a conformal map onto a John disk Ω. If $\Omega'\subseteq\mathbf{D}$ is a John disk and* $\operatorname{dist}(f(0),\partial\Omega)\geq c\operatorname{diam}(\Omega)$, *then $f(\Omega')$ is a John disk with constant depending only on Ω, Ω' and c.*

Proof. For $z\in\mathbf{D}$, $z=re^{it}$, let us introduce

$$B(z)=B(re^{it})=\{\varrho e^{i\theta}:r\leq\varrho\leq 1,\ |\theta-t|\leq\pi(1-r)\}.$$

The proof will be based on the following characterization of John disks (see [P, p. 97]):

Theorem A. *Let C map $\mathbf{D}$ conformally onto G such that* $\operatorname{dist}(C(0),\partial G)\geq c_1\operatorname{diam}(G)$. *Then the following conditions are equivalent:*

(i) *G is a John disk with constant c,*

(ii) *there exists α, $0<\alpha\leq 1$ such that*

$$|C'(\xi)|\leq M_1|C'(z)|\left(\frac{1-|\xi|}{1-|z|}\right)^{\alpha-1}\quad \textit{for } \xi\in B(z),\ z\in\mathbf{D},$$

(iii) $\operatorname{diam} C(B(z))\leq M_2\operatorname{dist}(C(z),\partial G)$ *for $z\in G$, where the constants M_1,M_2 and α depend only on the John constant c and the constant c_1.*

To start the proof of the lemma we consider the Riemann map $g:\mathbf{D}\to\Omega'$ such that $\operatorname{dist}(g(0),\partial\Omega')\geq c_1'\operatorname{diam}(\Omega')$ where c_1' depends only on c'—the John constant of Ω'. Consider the map $h=f\circ g$, $h:\mathbf{D}\to f(\Omega')$. According to Theorem A we need to estimate

$$h'(\xi)=|f'(g(\xi))|\,|g'(\xi)|\quad\text{for } \xi\in B(z),\ z\in\mathbf{D}.$$

By Theorem A:

$$\operatorname{diam} g(B(z))\leq M_2\operatorname{dist}(g(z),\partial\mathbf{D}).\tag{1.1}$$

To estimate $|f'(g(\xi))|$ we distinguish between two cases: $\operatorname{diam} g(B(z))\geq\frac{1}{3}$ and $\operatorname{diam} g(B(z))<\frac{1}{3}$. In the first case define $\tilde z=0$ and in the second case define $\tilde z=g(z)(1-2\operatorname{diam} g(B(z)))$. In the first case by (1.1) we have that $\operatorname{dist}(\tilde z,g(z))=\operatorname{dist}(0,g(z))\leq 1-1/M_2$. Then by the distortion theorem we have

$$|f'(\tilde z)|=|f'(0)|\leq M_3|f'(g(z))|.\tag{1.2}$$

In the second case $\operatorname{dist}(\tilde{z}, g(z)) = 2 \operatorname{diam} g(B(z)) \le M_2 \operatorname{dist}(g(z), \partial\mathbf{D})$ and again by the distortion theorem:

$$|f'(\tilde{z})| \le M_3 |f'(g(z))|. \tag{1.3}$$

Without loss of generality assume that $\Omega' \subseteq \mathbf{D}$ is included in a half disk. Then if we consider $B(\tilde{z}) \subseteq \mathbf{D}$, we observe that in both cases $g(B(z)) \subseteq B(\tilde{z})$. Now we apply Theorem A for the function f, and for $g(\xi) \in B(\tilde{z})$:

$$|f'(g(\xi))| \le M_4 |f'(\tilde{z})| \left(\frac{1-|g(\xi)|}{1-|\tilde{z}|} \right)^{\alpha'-1}.$$

Because $1-|\tilde{z}| \le M_5(1-|g(z)|)$ by the way $\tilde{z}$ was defined we get:

$$|f'(g(\xi))| \le M_6 |f'(\tilde{z})| \left(\frac{1-|g(\xi)|}{1-|g(z)|} \right)^{\alpha'-1}.$$

By the Koebe distortion theorem (considering two cases: either $\operatorname{dist}(g(z), \partial\Omega) \le$ or $\ge \varepsilon(1-|g(z)|)$)

$$|f'(g(\xi))| \le M_7 |f'(\tilde{z})| \left| \frac{g'(\xi)}{g'(z)} \right|^{\alpha'-1} \left(\frac{1-|\xi|}{1-|z|} \right)^{\alpha'-1}.$$

If we apply Theorem A for g we obtain:

$$|f'(g(\xi))| \, |g'(\xi)| \le M_8 |f'(\tilde{z})| \left(\frac{1-|\xi|}{1-|z|} \right)^{\alpha\alpha'-1} |g'(z)|,$$

and by (1.2) or (1.3) we get:

$$|h'(\xi)| \le M_9 |h'(z)| \left(\frac{1-|\xi|}{1-|z|} \right)^{\alpha\alpha'-1},$$

which is the estimate we need to conclude that

$$h(\mathbf{D}) = f(\Omega') \text{ is a John domain.}$$

Notice that the constant M_9 in the final estimate depends only on the constants of f and g and hence the John constant of $f(\Omega')$ depends only on c and the constants of Ω and Ω' respectively. $\square$

Following [NV] a John disk Ω has locally connected boundary. If $f: \mathbf{D} \to \Omega$ is the Riemann map, we can consider the continuous extension $f: \overline{\mathbf{D}} \to \overline{\Omega}$. The set of prime ends $\partial^*\Omega$ is extremely simple. The impression of each prime end contains one point and moreover the prime ends are just the accessible points.

The following lemma says more about the boundary of Ω.

1.6. Lemma. *Let Ω be a John disk with constant $c>0$, and $f:\mathbf{D}\to\Omega$ a Riemann map extended continuously to $f:\overline{\mathbf{D}}\to\overline{\Omega}$. Then for $Q\in\partial\Omega$,*

$$\#(f^{-1}(Q))\leq K \quad \textit{where } K=K(c).$$

Proof. Let $Q\in\partial\Omega$, $z_i\in f^{-1}(Q)$ for $i=1,\dots,K$. Consider the geodesics $\gamma_i=f([0,z_i])$. Let us use the convention $\gamma_{K+1}=\gamma_1$. Then $\gamma_i\cup\gamma_{i+1}$ are closed Jordan curves for $i=1,\dots,K$. Because γ_i is not homotopic to γ_{i+1}, there exists $y_i\in\operatorname{int}(\gamma_i\cup\gamma_{i+1})\cap\Omega^c$. Since Ω^c is connected, there is a continuum $E_i\subset\Omega^c\cap\operatorname{int}(\gamma_i\cup\gamma_{i+1})$ connecting y_i and Q.

Let $r=\min_i|y_i-Q|$. Clearly E_i intersects $\partial B(Q,r)$ in certain points which we denote again by y_i. Similarly let us denote by $\bar{y}_i$ the intersection $\gamma_i\cap\partial B(Q,r)$.

Consequently we have obtained $2K$ points $\bar{y}_1$, y_1, $\bar{y}_2$, $y_2,\dots,\bar{y}_K$, y_K situated in this order on $\partial B(Q,r)$.

By the fact that the geodesics γ_i are John arcs (see [GHM], [NV]) we obtain that $B_i=B(\bar{y}_i,cr)\subset\Omega$.

Finally if $K>[2\pi/c]+1$, then we necessarily have that $y_i\in B_i$ for some i, contrary to the fact that $B_i\subset\Omega$. $\square$

Before stating the next lemma, we introduce the notation $\Omega_{r,Q}$ for the union of the components of $\Omega\cap B(Q,r)$ which have Q in their boundary. Furthermore notice the inclusions

$$\Omega_{r,Q}\subset B_\varrho(Q,2r)\subset\Omega_{2r,Q}.$$

Before our next result let us introduce some notation.

Given $d>0$ let $\{z_i\}_{i=1}^K=f^{-1}(Q)$ for $Q\in\partial\Omega$, $\tilde{z}_i=r_iz_i$, $0<r_i<1$ where r_i is chosen such that $\operatorname{dist}(f(\tilde{z}_i),\partial\Omega)=d$, $i=1,\dots,K$. Let $B(\tilde{z}_i)$ be as in Lemma 1.5 and $T_i=\partial f(B(\tilde{z}_i))\setminus\partial\Omega$. With this notation we have:

1.7. Lemma. *There exists $c_1>0$, $c_1=c_1(\Omega)$ such that:*

$$\Omega_{c_1d,Q}\subset\bigcup_i f(B(\tilde{z}_i)).$$

Before the proof let us state the following

1.8. Corollary. *There exists $c_1>0$, $c_1=c_1(\Omega)$ such that:*

$$B_\varrho(Q,c_1d)\subset\bigcup_i f(B(\tilde{z}_i)).$$

Proof of Lemma 1.7. It would suffice to show that $T_i=\partial f(B(\tilde{z}_i))\setminus\partial\Omega$ is far from Q. Unfortunately this is not the case but we will construct a subdomain

$D_i \subseteq B(\tilde{z}_i)$ such that $\widetilde{T}_i = \partial f(D_i) \setminus \partial\Omega$ is far from Q. To do this denote by I_1, I_2 the two halves of $\partial B(\tilde{z}_i) \cap \partial\mathbf{D}$. Because $\operatorname{diam} f(I_j) \geq c_2 d$ (see [P, Prop. 4.19]) for $j=1,2$; there exist $Q_1 \in f(I_1)$, $Q_2 \in f(I_2)$ such that:

$$|Q_j - Q| \geq \tfrac{1}{4} c_2 d, \quad j = 1, 2. \tag{1.4}$$

Denote by $P_j \in f^{-1}(Q_j)$ the closest preimage to $\tilde{z}_i$, $j=1,2$, and consider the ray from P_j to the boundary of $B(\tilde{z}_i)$. In this way we construct a subdomain $D_i \subseteq B(\tilde{z}_i)$ bounded by the two rays R_1, R_2 and arcs C_1, C_2, where $C_1 \subset \partial B(\tilde{z}_i) \setminus \partial\mathbf{D}$ and $C_2 \subseteq \partial\mathbf{D}$. Take a point $x \in C_1$ then:

$$\operatorname{dist}(f(x), Q) \geq \operatorname{dist}(f(x), \partial\Omega) \geq c_1 d.$$

Suppose $x \in R_1$. Then:

$$\operatorname{dist}(f(x), Q) \geq \operatorname{dist}(Q_1, Q) - \operatorname{dist}(f(x), Q_1). \tag{1.5}$$

On the other hand $f(R_1)$ is a John arc ([GHM], [NV]) and therefore:

$$\operatorname{dist}(f(x), Q_1) \leq \frac{1}{c} \operatorname{dist}(f(x), Q).$$

This estimate and (1.5) imply:

$$\operatorname{dist}(f(x), Q) \geq c_3 \operatorname{dist}(Q_1, Q)$$

and by (1.4) we get:

$$\operatorname{dist}(f(x), Q) \geq c_1 d.$$

The estimate is the same for $x \in R_2$ and we are done. □

We are now in position to give the:

Proof of Theorem 1.3. The proof is a combination of the previous results. Let $f: \mathbf{D} \to \Omega$ be a Riemann mapping. For $Q \in \partial\Omega$ consider $\{z_i\}_{i=1}^K = f^{-1}(Q)$ as in Lemma 1.5. If c_1 is the constant from Lemma 1.7, put $d = r/c_1$ and define: $\Omega_Q^i(r) = f(B(\tilde{z}_i))$, $i = 1, \dots, K$. By Lemma 1.5 each is an $\Omega_Q^i(r)$ John domain with constant c depending only on Ω.

It is clear that $\{\Omega_Q^l(r)\}_{l=1}^K$ satisfies (1) and (3) with $N=K$, $M=1/c_1$. By Corollary 1.8 property (2) holds also. □

The following is an easy consequence of the previous results:

1.9. Theorem. *A John disk admits a localization.*

Proof. The idea is to enclose two subdomains which intersect each other by a slightly bigger domain. So let us start with $\Omega^i_Q(r)$, $\Omega^j_Q(r)$ such that $\Omega^i_Q(r)\cap\Omega^j_Q(r)\neq\emptyset$. Then with the notation of Lemma 1.7 we have $B(\tilde z_i)\cap B(\tilde z_j)\neq\emptyset$. Suppose $\operatorname{diam} B(\tilde z_i)\geq\operatorname{diam} B(\tilde z_j)$ and enlarge $B(\tilde z_i)$ to obtain a new domain $B(z)$ with $B(z)\supseteq B(\tilde z_i)\cup B(\tilde z_j)$ and $\operatorname{diam} B(z)\leq 4\operatorname{diam} B(\tilde z_i)$. It is clear that $\operatorname{dist}(f(z),Q)\leq M_1 r$.

Define a new collection $\{\Omega^l_Q(r)\}_{l=1}^{N-1}$ where we replace $\Omega^i_Q(r)$ and $\Omega^j_Q(r)$ by $f(B(z))$. The new collection has all the properties (1), (2), (3) but now the constant M needs to be replaced by M_1M. If the domains are disjoint in this new collection, we stop; if not, we replace two by one as above. After at most N steps we either stop when we get disjoint domains or we obtain just one domain.

The consequence of this process is a localization since the domains will be disjoint and we changed the constant in (3) to at most MM_1^N. $\square$

In the next section we deal with arbitrary John domains, not just John disks.

2. Localization for arbitrary John domains at a fixed scale

At the end of this section we comment on the comparison of our localization and the one in [CJY]. First we present an example which shows that arbitrary John domains do not admit localization.

2.1. *Example.* Let $\Omega_0=\{x+iy:x\in[-1,1],\ y\in[0,2]\}$ and consider $\Omega=\Omega_0\setminus\bigcup C_n$ where $C_n=\{iy:y^1_n\leq y\leq y^2_n\}$. We choose y^1_n, y^2_n such that $y^2_n>y^1_n>y^2_{n+1}>y^1_{n+1}$ and $y^2_n-y^1_n=2^{-n}$, $y^1_n-y^2_{n+1}=2^{-n^2}$.

It is easy to see that Ω does not admit a good localization at the point $0\in\partial\Omega$ because the John constant will tend to zero as the scale tends to zero.

However, an arbitrary John domain admits a localization if the scale $\varepsilon>0$ is fixed.

Namely we have the following:

2.1. Theorem. *Let $\varepsilon>0$ be fixed and Ω be an arbitrary John domain. Then Ω admits localization at the scale $\varepsilon>0$.*

Proof. Since we have a localization in the simply connected case, it is natural to add boundary pieces to $\partial\Omega$ to obtain a simply connected John domain $\widetilde\Omega$. Using a result due to Jones (see [J2, Theorem 2]) we can do this without changing the John constant of Ω too much.

Applying Theorem 1.9 to $\widetilde{\Omega}$ we obtain a collection $\{\Omega^l_Q(\varepsilon)\}_{l=1}^N$ of simply connected John domains with these properties:

(1) $\operatorname{diam}\Omega^l_Q(\varepsilon)\le M_0\varepsilon$, $l=1,\dots,N$,

(2) $\bigcup\overline{\Omega^l_Q(\varepsilon)}\supseteq B_\varrho(Q,\varepsilon)$,

(3) $\Omega^i_Q(\varepsilon)\cap\Omega^j_Q(\varepsilon)=\emptyset$ for $i\ne j$.

Next we are going to eliminate the boundary pieces we have added for the construction of $\widetilde{\Omega}$. Let us introduce some notation to indicate the presence of these boundary pieces. Let c^i be a center of $\Omega^i_Q(\varepsilon)$. For a pair (i,j) of indices introduce:

$$\Gamma_{(i,j),Q,M}=\{\gamma\subseteq\Omega:\gamma \text{ connects } c^i,\ c^j,\ \text{centers of } \Omega^i_Q(\varepsilon),\ \Omega^j_Q(\varepsilon),\ \operatorname{diam}\gamma\le 2M\varepsilon\}.$$

Furthermore we define:

$$c_{(i,j)}(Q,M)=\begin{cases}0 & \text{if } \Gamma_{(i,j),Q,M}=\emptyset,\\ \sup\limits_{\gamma\in\Gamma_{(i,j),Q,M}}\ \inf\limits_{\xi\in\gamma}\operatorname{dist}(\xi,\partial\Omega), & \text{otherwise}\end{cases}$$

and let $c(Q,M)=\max_{(i,j)} c_{(i,j)}(Q,M)$.

Let us put $M=M_0$ with M_0 from (1). The indicator $c(Q,M_0)$ shows whether there might be boundary pieces we need to eliminate. In particular if $c(Q,M_0)=0$, we have:

$$(\partial\Omega^i_Q(\varepsilon)\cap\partial\Omega^j_Q(\varepsilon))\setminus\partial\Omega=\emptyset$$

for any pair (i,j).

Let $F=\{Q\in\partial\Omega:c(Q,M_0)>0\}$. The proof of the theorem is based on the following simple fact:

2.2. Lemma. *There is $\alpha=\alpha(\Omega)>0$ such that $c(Q,2M_0)>\alpha$ for any $Q\in F$.*

Proof. To see this let $Q_n\in F$ be such that $c(Q_n,2M_0)\to 0$. Let $\Omega^i_{Q_n}(\varepsilon)$, $\Omega^j_{Q_n}(\varepsilon)$ be such that $c_{ij}(Q_n,M_0)>0$ and c^i_n, c^j_n be the corresponding centers. We can assume that $c^i_n\to c^i$, $c^j_n\to c^j$. Then there is n_0 such that $\operatorname{dist}(c^i_{n_0},c^i_n)<\frac12 c_1\varepsilon$, $\operatorname{dist}(c^j_{n_0},c^j_n)<\frac12 c_1\varepsilon$ for $n\ge n_0$. Here c^i, c^j may not be the centers.

Let $\beta=c(Q_{n_0},M_0)$ and consider the curve $\gamma_n=[c^i_{n_0},c^i_n]\cup\gamma_{n_0}\cup[c^j_{n_0},c^j_n]$ where γ_{n_0} connects $c^i_{n_0}$ to $c^j_{n_0}$ with $\operatorname{diam}\gamma_{n_0}\subseteq 2M_0\varepsilon$, and $\operatorname{dist}(\xi,\partial\Omega)\ge\frac12\beta$ for $\xi\in\gamma_{n_0}$. Then γ_n connects c^i_n to c^j_n, $\operatorname{diam}\gamma_n\le 4M_0\varepsilon$ and $\operatorname{dist}(\xi,\partial\Omega)\ge\tilde\beta$ where $\tilde\beta=\min\left(\frac12\beta,\frac12 c_1\varepsilon\right)$. Therefore $\gamma_n\in\Gamma_{(i,j),Q_n,2M_0}$ and $c(Q_n,2M_0)\ge\tilde\beta$ contrary to $c(Q_n,2M_0)\to 0$. □

Proof of Theorem 2.1. We can now proceed with the elimination of the extra boundary pieces. This algorithm will be used in later sections where we do localization at any scale.

If $c(Q,M_0)=0$, there is nothing to do for this particular $Q\in\partial\Omega$. If $c(Q,M_0)>0$, take a pair (i,j) such that $c_{(i,j)}(Q,M_0)>0$ and hence $c_{(i,j)}(Q,2M_0)\geq\alpha$. Let γ_{ij} be a curve connecting c^i and c^j in Ω such that $\operatorname{diam}\gamma_{ij}\leq 4M_0\varepsilon$ and $\operatorname{dist}(\xi,\partial\Omega)\geq\frac{1}{2}\alpha$ for $\xi\in\gamma_{ij}$. Let H_{ij} be a neighborhood of γ_{ij} of thickness $\frac{1}{4}\alpha$. Then $H_{ij}\subseteq\Omega$ and we can define:

$$\Omega_Q^{i,j}(\varepsilon)=\operatorname{int}\left((\bar\Omega_Q^i(\varepsilon)\cup H_{ij}\cup\bar\Omega_Q^j(\varepsilon))\setminus\partial\Omega\right).$$

It is clear that $\Omega_Q^{i,j}(\varepsilon)$ is a John domain with the correct constant and by this step we have eliminated the common boundary $(\partial\Omega_Q^i(\varepsilon)\cap\partial\Omega_Q^j(\varepsilon))\setminus\partial\Omega$. Consider the new system of local domains where $\Omega_Q^i(\varepsilon)$ and $\Omega_Q^j(\varepsilon)$ are replaced by $\Omega_Q^{i,j}(\varepsilon)$ and denote this again by $\{\Omega_Q^l(\varepsilon)\}_l$. The new system has all the properties of the old one; the only difference is that M_0 in (1) is replaced by $5M_0$ and we reduced the number of domains by one.

If $c(Q,5M_0)=0$ for the new system, then we stop. If $c(Q,5M_0)>0$, we perform the above construction again reducing the number of domains. We repeat everything at most N times and as a result either we will stop when $c(Q,5^kM_0)=0$ for some k, $0\leq k\leq N$ or we will have just one domain left. In either case we obtain a collection of local domains $\{\Omega_Q^l(\varepsilon)\}_l$ which is a localization at scale ε. $\square$

2.3. *Remark.* In the proof of Lemma 2.2 we had implicitly that $\alpha=\alpha(\Omega,\varepsilon)$ and therefore the John constant also depends on ε. This in fact happens for the domain in Example 2.1.

2.4. *Remark.* It is interesting to compare the localizations in Theorem 1.9, Theorem 2.1 and the one in (4.1)–(4.3) of [CJY]. Basically (4.1), (4.2), are obtained in Theorem 1.9. But not (4.3). The localization of Theorem 4.9 below is much finer than (4.1)–(4.2), but again does not touch (4.3). However, one can prove (and we use it in subsequent works [BV1], [BV2]) that the localizations of Theorems 1.9, 2.1, 4.9 satisfy (4.3) of [CJY]. But here we do not use harmonic measure or Green's function, concentrating only on geometry.

In the next section we present some geometric properties of the Julia set J of a semihyperbolic polynomial. A polynomial is called *semihyperbolic* if it has no parabolic periodic points and for any critical point $\omega\in J$ we have $\operatorname{dist}(\omega,\sigma(\omega))>0$ where $\sigma(\omega)$ denotes the forward orbit of ω.

Let $A_\infty(f)$ be the domain of attraction to ∞ for f. A recent result of Carleson, Jones and Yoccoz states that $A_\infty(f)$ is a John domain if and only if f is semihyperbolic.

Our next purpose is to show that in some cases $A_\infty(f)$ admits localization and to characterize all such cases.

3. Geometric properties of the Julia set

Let $f: U \to V$ be a polynomial where $\overline{U} \subset V$ are topological disks. Then $J \subseteq U$ and we always assume

$$(*) \qquad J = \bigcap_{n=0}^{\infty} f^{-n}(U).$$

Condition $(*)$ means that the Julia set of the polynomial does not split the plane.

In what follows we are going to describe briefly a method used by Branner and Hubbard in studying the structure of cubic polynomials (see [BH]).

Let γ_1, γ_2 be analytic Jordan curves such that $U \subseteq U_1 \subseteq U_2 \subseteq V$ where $U_1 = \operatorname{int}(\gamma_1)$, $U_2 = \operatorname{int}(\gamma_2)$. Without loss of generality we assume that if ω is a critical point for f, then either $\omega \in J$ or $\omega \in V \setminus U_2$.

For $x \in J$ denote by C_x the connected component of J containing x. Let $P_n(x)$ be the component of $f^{-n}(U_2)$ such that $C_x \subseteq P_n(x)$. It is clear that $f: P_n(x) \to P_{n-1}(f(x))$ is a branched or regular covering and we have the chain:

$$C_x \subseteq P_n(x) \subseteq \ldots \subseteq P_1(x) \subseteq U_2.$$

By $(*)$ it is clear that $C_x = \bigcap_{n=0}^{\infty} P_n(x)$.

By the way, this proves that for $\varepsilon > 0$ there is an analytic curve $\gamma = \gamma_\varepsilon$ disjoint from J and surrounding C_x such that $\operatorname{diam} \gamma \leq \operatorname{diam} C_x + \varepsilon$. In fact, for given $\varepsilon > 0$ we put $\widetilde{\gamma} = \partial P_N(x)$, and take N to be sufficiently large.

Let $A_0 = U_2 \setminus U_1$ and $A_n(x)$ be the annulus surrounding x, $A_n(x) = P_n(x) \cap f^{-n}(A_0)$. We are going to call $A_n(x)$ critical if a critical point ω is surrounded by $A_n(x)$. In this situation we have $A_n(x) = A_n(\omega)$. Furthermore $f: A_n(x) \to A_{n-1}(f(x))$ and we have the chain

$$A_n(x) \to A_{n-1}(f(x)) \to \ldots \to A_1(f^{n-1}(x)) \to A_0.$$

The map $f: A_{n-i}(f^i(x)) \to A_{n-i-1}(f^{i+1}(x))$ is univalent if $A_{n-i}(f^i(x))$ is not critical. If this happens, we have that $\operatorname{mod} A_{n-i}(f^i(x)) = \operatorname{mod} A_{n-i-1}(f^{i+1}(x))$. If this is the case for any i, $1 \leq i \leq n$, we get $\operatorname{mod} A_n(x) = \operatorname{mod} A_0$.

In case $A_{n-i}(f^i(x))$ is critical, the map

$$f: A_{n-i}(f^i(x)) \to A_{n-i-1}(f^{i+1}(x))$$

is a regular covering of a certain degree d_i and we have

$$\operatorname{mod} A_{n-i}(f^i(x)) = (1/d_i) \operatorname{mod} A_{n-i-1}(f^{i+1}(x)).$$

For example if there is just one critical point w_0 of multiplicity 1, we have that

$$\operatorname{mod} A_{n-i}(f^i(x)) = \tfrac{1}{2}\operatorname{mod} A_{n-i-1}(f^{i+1}(x))$$

in case $A_{n-i}(f^i(x))=A_{n-i}(\omega_0)$. If k denotes the number of critical annuli among $A_{n-i}(f^i(x))$, then $\operatorname{mod} A_n(x)=(1/2^k)\operatorname{mod} A_0$.

The following result is very useful in deciding whether $C_x=\{x\}$ for a certain x (see [BH]).

3.1. Lemma. *Suppose A is a bounded open annulus and A_n is an infinite set of disjoint open annuli $A_n \subseteq A$ each one winding around the bounded component of $\mathbf{C}\setminus A$. If $\sum_n \operatorname{mod} A_n=\infty$, then the bounded component of $\mathbf{C}\setminus A$ equals a single point.*

To apply this lemma notice that since $f^{-1}(U_2)\subset U_1$, we have that $A_n(x)\subseteq \operatorname{int} A_{n-1}(x)$ where $\operatorname{int} A_{n-1}(x)$ denotes the bounded component of $\mathbf{C}\setminus A_{n-1}(x)$. Therefore $\{A_n(x)\}_n$ is a sequence of disjoint, nested annuli surrounding C_x.

If there is no critical point in J, then $\operatorname{mod} A_n(x)=\operatorname{mod} A_0$ and $\sum_n \operatorname{mod} A_n(x)=\infty$. By Lemma 3.1 we obtain $C_x=\{x\}$. This shows that the hyperbolic Julia set which is disconnected is always a Cantor set. Another application of Lemma 3.1 is the following result.

3.2. Lemma. *Let f be semihyperbolic and $C_\omega=\{\omega\}$ for any critical point ω. Then $C_x=\{x\}$ for $x\in J$.*

Proof. The idea is to show that $\operatorname{mod}(A_n(x))\geq c$ and then apply Lemma 3.1.

Let $n\in\mathbf{N}$ and $i(n)$ be the number of indices i such that $A_{n-i}(f^i(x))=A_{n-i}(\omega)$ for some critical point ω. We claim that there exists $D\in\mathbf{N}$ independent of n such that $i(n)\leq D$ for any $n\in\mathbf{N}$. If this is true, then we can decrease the modulus at most D times and hence $\operatorname{mod}(A_n(x))\geq c$.

Suppose our claim is not true. Then there is a critical point ω_0 such that if $i(n)$ denotes the number of indices i for which $A_{n-i}(f^i(x))=A_{n-i}(\omega_0)$, we have that $i\to\infty$ as $n\to\infty$. Let $i_1<i_2<...<i_{i(n)}$ be the indices such that

$$A_{n-i_j}(f^{i_j}(x)) = A_{n-i_j}(\omega_0), \quad i_j \in \{i_1, ..., i_{i(n)}\}.$$

Because $A_{n-i_1}(f^{i_1}(x))=A_{n-i_1}(\omega_0)$ we get that $A_{n-i_2}(f^{i_2}(x))=A_{n-i_2}(f^{i_2-i_1}(\omega_0))$. On the other hand $A_{n-i_2}(f^{i_2}(x))=A_{n-i_2}(w_0)$. Hence $f^{i_2-i_1}(\omega_0)\in P_{n-i_2}(\omega_0)$. Let $j_n=i_2-i_1$. Then $f^{j_n}(\omega_0)\in P_{n-i_2}(\omega_0)$. It is clear that $n-i_2\to\infty$ as $n\to\infty$ and $P_{n-i_2}(\omega_0)\to C_{\omega_0}$. But $C_{\omega_0}=\{\omega_0\}$, and consequently there is a sequence of iterates $\{f^{j_n}(\omega_0)\}_n$ such that $f^{j_n}(\omega_0)\to\omega_0$ contrary to semihyperbolicity. □

To deal with the critical points ω for which $C_\omega \neq \{\omega\}$ we need the following characterization of semihyperbolicity due to Carleson, Jones and Yoccoz (see [CJY]).

First we introduce some notation. For $z \in \mathbf{C}$ let $B_n(x,\varepsilon)$ denote the connected component of $f^{-n}(B(z,\varepsilon))$ containing x. By the maximum principle $B_n(x,\varepsilon)$ is simply connected and $f^n: B_n(x,\varepsilon) \to B(z,\varepsilon)$ defines a branched or regular covering and we denote its degree by $\mathrm{dg}(B_n(x,\varepsilon))$. With this notation we have:

Theorem B. *The following are equivalent:*
(A) *f is semihyperbolic,*
(B) *there exists $\varepsilon>0$, $c>0$, $0<\theta<1$ and $D<\infty$ such that for all $x \in J$ and $n \in \mathbf{N}$*

$$\mathrm{dg}(B_n(x,\varepsilon)) \leq D \quad \textit{and} \quad \mathrm{diam}\, B_n(x,\varepsilon) \leq c\theta^n.$$

Using this result we can prove:

3.3. Lemma. *Let f be semihyperbolic and $x \in J$ such that $C_x \neq \{x\}$. Then $\{C_{f^k(x)}\}_k$ is either periodic or preperiodic.*

Proof. Let $x \in J$ such that $C_x \neq \{x\}$. First we show that there exists $\delta>0$ such that $\mathrm{diam}\, C_{f^k(x)} \geq \delta$ for $k \in \mathbf{N}$. For if not, then there exists a subsequence k_i such that $\lim_{i\to\infty} \mathrm{diam}\, C_{f^{k_i}(x)} = 0$. Then there is an index i_0 such that $\mathrm{diam}\, C_{f^{k_i}(x)} \leq \frac{1}{3}\varepsilon$ for $i \geq i_0$. This implies that $C_{f^{k_i}(x)} \subseteq B(f^{k_i}(x),\varepsilon)$ for $i \geq i_0$. By Theorem B $\mathrm{diam}\, B_{k_i}(x,\varepsilon) \leq c\theta^{k_i}$ and hence $\lim_{i\to\infty} \mathrm{diam}\, B_{k_i}(x,\varepsilon) = 0$. On the other hand $B_{k_i}(x,\varepsilon) \supseteq C_x$ which is a contradiction.

Suppose next that $\{C_{f^k(x)}\}_k$ is an infinite sequence of distinct components. Take a subsequence k_i such that $C_{f^{k_i}(x)}$ are all different and choose a further subsequence k_{i_j} such that $f^{k_{i_j}}(x) \to x_0$. In this way we obtain a point x_0 such that there is a sequence of distinct components of J of fixed size δ accumulating at x_0. On the other hand $U_2 \setminus J$ is a John domain (see [CJY]) and this leads to a contradiction. □

A consequence of this lemma is:

3.4. Corollary. *Let $C_x \neq \{x\}$ be a component of a semihyperbolic Julia set J. Then C_x is equal to or is a preimage of a periodic critical component C_ω.*

Proof. Let us consider the annuli $A_n(x)$ as in Lemma 3.2. From the proof of Lemma 3.2 we see that the number of critical annuli $i(n)$ tends to infinity as $n \to \infty$, otherwise we would have $C_x = \{x\}$. Then there exists a critical point ω_0 such that $A_{n-i_k}(f^{i_k}(x)) = A_{n-i_k}(\omega_0)$ for $k \in \{1, \dots, i(n)\}$. Among $\{C_{f^n(x)}\}_{n\geq 0}$ there are p distinct components and let $X_1, \dots, X_p$ be them. Among $\{C_{f^n\omega_0}\}_{n\geq 0}$ there are q distinct components and let $C_1 = C_{\omega_0}, \dots, C_q = C_{f^{q-1}\omega_0}$ be them. Among

$f^{i_1}(x), \dots, f^{i_{p+1}}(x)$ there are two points, say $f^{i_{k_1}}(x)$, $f^{i_{k_2}}(x)$, $i_{k_1}<i_{k_2}$, belonging to the same $X_{p(n)}$, $1\le p(n)\le p$. Notice that i_{k_1} and i_{k_2} depend on n. Then $A_{n-i_{k_1}}(f^{i_{k_1}}x)$ and $A_{n-i_{k_2}}(f^{i_{k_2}}x)$ encircle $X_{p(n)}$. Let us denote the first of these annuli by A_1^n and the second by A_2^n; the corresponding pieces are called P_1^n and P_2^n. Then $X_{p(n)}\subseteq P_1^n\subseteq P_2^n$. Now let $\mathcal{N}$ be an infinite set of indices n for which $p(n)=p_0$, for a certain p_0, $1\le p_0\le p$, which clearly exists.

We can write the chain of equalities

$$\begin{aligned} A_2^n &= A_{n-i_{k_2}}(f^{i_{k_2}}x) = f^{i_{k_2}-i_{k_1}}(A_{n-i_{k_1}}(f^{i_{k_1}}x)) \\ &= f^{i_{k_2}-i_{k_1}}(A_{n-i_{k_1}}(\omega_0)) = A_{n-i_{k_2}}(f^{i_{k_2}-i_{k_1}}\omega_0). \end{aligned}$$

This means that $f^{i_{k_2}-i_{k_1}}\omega_0\in P_2^n$. But $A_2^n=A_{n-i_{k_2}}(\omega_0)$ and so also $\omega_0\in P_2^n$. Thus both C_{ω_0} and $C_{f^{i_{k_2}-i_{k_1}}\omega_0}$ lie in P_2^n. Because $n-i_{k_2}\ge n-i_{p+1}\to\infty$ we can write

$$X_{p_0} = \bigcap_{n\in\mathcal{N}} P_2^n = C_{\omega_0}.$$

It remains to prove that C_{ω_0} is periodic. If for a certain n we have $f^{i_{k_2}-i_{k_1}}\omega_0\in C_{\omega_0}$, we are done. Otherwise for infinitely many $n\in\mathcal{N}$, $f^{i_{k_2}-i_{k_1}}\omega_0\in C_{q_0}$ for a certain q_0, $2\le q_0\le q$. Thus C_{q_0} lies in P_2^n for those n and so $C_{f^{q_0-1}\omega_0}=C_{q_0}=\bigcap_{n\in\mathcal{N}} P_2^n=C_{\omega_0}$ and we are done. □

3.5. *Remark.* At this point we can see that a conjecture of Branner and Hubbard is true for semihyperbolic polynomials. Namely we can conclude that J is a Cantor set if and only if there are no periodic critical components.

4. Localization of $A_\infty(f)$

Theorem 4.9 (obtained by the first author) is the main result of this section. But before proving it we are going to give independent proofs of its particular cases. This seems to us to be illustrative. In what follows we are going to use the distortion properties of d-valent functions. First let us introduce some notation. Given a topological disk W and a closed set $F\subseteq W$ we say that F is α-admissible for $\alpha>0$ if

$$\operatorname{diam} F \ge \alpha \operatorname{diam} W.$$

For a given $\alpha>0$ we say that a domain W is α-thick at a point $z\in W$ if

$$\operatorname{dist}(z, \partial W) \ge \alpha \operatorname{diam} W.$$

We will use the following lemma which is in the spirit of Proposition 2.1 of [HR].

Lemma C. *Let* (W_1, V_1), (W_0, V_0) *be two pairs of topological disks* $V_i \subset W_i$, $i=0,1$ *and* $0<\beta<\operatorname{mod}(W_0 \setminus V_0)$. *Fix* $\alpha>0$, $d\in\mathbf{N}$ *and let* $f: W_1 \to W_0$ *be a branched covering of degree* d *such that* V_1 *is a component of* $f^{-1}(V_0)$. *Then the following holds*:

(a) *for any two connected* α*-admissible sets* F', F'' *in* V_1:

$$\frac{\operatorname{diam}(F')}{\operatorname{diam}(F'')} \sim \frac{\operatorname{diam} f(F')}{\operatorname{diam} f(F'')},$$

(b) *the same holds for* F', F'' *such that* $f(F')$, $f(F'')$ *are* α*-admissible and connected,*

(c) *for a point* $x_0 \in V_1$ *and a closed set* $F \subseteq V_1$ *we have* $\operatorname{dist}(x_0, F) \sim \operatorname{diam}(V_1)$ *if* $\operatorname{dist}(f(x_0), f(F)) \sim \operatorname{diam}(V_0)$,

(d) *for a point* $x_0 \in V_1$ *and a closed set* $F_0 \subset V_0$ *we have* $\operatorname{dist}(f(x_0), F_0) \sim \operatorname{diam} V_0$ *if* $\operatorname{dist}(x_0, f^{-1}(F_0) \cap V_1) \sim \operatorname{diam} V_1$.

4.1. *Remark.* A consequence is that V_1 is α-thick at a point $x_0 \in V_1$ if and only if V_0 is α'-thick at $f(x_0)$.

Proof of Lemma C. Clearly $\operatorname{mod}(W_1 \setminus V_1) \geq \beta' = \beta'(\beta, d) > 0$. This means that we can always assume that W_0, W_1 are unit disks by conformal changes in the image and in the preimage. These changes do not affect the conclusion of the lemma because collars $W_0 \setminus V_0$, $W_1 \setminus V_1$ have moduli uniformly bounded away from zero. In particular f is just a Blaschke product with d zeros (counting multiplicity). A sequence of such functions can converge (uniformly on compact sets) only to a Blaschke product with at most d zeros. If we let B_0^d denote the family of Blaschke products f with at most d zeros such that $f(0)=0$, then we notice that B_0^d is closed in the topology of uniform convergence on compact subsets of $\mathbf{D}$.

Notice also that we can assume that the modulus of the collar $W_0 \setminus V_0$ is bounded from above by an absolute constant. If $W_0 = \mathbf{D}$ (as we assumed) and $0 \in V_0$, we can see that $\operatorname{diam} V_1 \leq 1 - C(\beta') < 1$.

If $0 \in V_1$, we use the fact that $\sup\{|f'(z)| : |z| \leq 1 - c(\beta'),\ f \in B_0^d\} \leq K(\beta) < \infty$ to conclude that

$$\operatorname{diam} V_1 \geq k(\beta') \operatorname{diam} V_0 \geq k\tau_0.$$

Now let us prove (c). We can assume that $x_0 = 0 = f(x_0)$. Let $x_n \in V_1^n$, $f_n \in B_0^d$ be sequences such that $|x_n| / \operatorname{diam} V_1^n \to 0$ but $|f(x_n)| / \operatorname{diam} V_0^n \geq \gamma > 0$. Our remarks show $|f(x_n)| \geq \gamma\tau_0$. But the derivatives of f_n are uniformly bounded on $B(0, \operatorname{diam} V_1) \subset B(0, 1 - c(\beta'))$ and we come to a contradiction.

To prove (d) we connect $0 = f(x_0)$ with the closest point y_0 of $F_0 \subset V_0$ by a segment γ_0. Then there is a lifting γ_1 of γ_0 to V_1 that connects $x_0 = 0$ with a $y_1 \in f^{-1}(y_0) \in f^{-1}(F_0) \cap V_1$.

Suppose that we have sequences $f_n \in B_0^d$ and y_0, y_1 as above and such that $|y_0^n|/\operatorname{diam} V_0^n \to 0$, $|y_1^n| \geq k\tau_0$.

Then for at least half of $r \in [0, k\tau_0]$ the circle $\mathbf{T}_r = \{z : |z| = r\}$ intersects γ_1^n at a point where $|f_n'| \leq 10\varepsilon_n$. But $\{f_n'\}$ are holomorphic functions, uniformly bounded on compact sets. As f_n converge to a certain function from B_0^d which cannot be a constant, we come to a contradiction.

It is now easy to see that (b) and (a) follow from (c) and (d) respectively. □

We are grateful to the referee for pointing out a mistake in the original statement of this lemma.

There are a number of particular cases when we can achieve a localization. The first case is:

4.2. Theorem. *Let f be semihyperbolic and assume that $C_\omega = \{\omega\}$ for any critical point $\omega \in J$. Then $A_\infty(f)$ admits a localization.*

Proof. By Lemma 3.2 J is a Cantor set in this case. Moreover the annuli $A_{n-k}(f^k(x))$ can be critical only for a finite number of indices k. Let $\widetilde{P}_n(x)$ be the component of $f^{-n}(U_1)$ containing x. The degree of the map $f^n : P_n(x) \to U_2$ is bounded by D, where $D \in \mathbf{N}$ is independent of n and x. It is easy to see that $P_n(x)$ is α-thick at any point $z \in \partial \widetilde{P}_n(x)$. Therefore $\operatorname{dist}(\partial \widetilde{P}_n(x), \partial P_n(x)) \sim \operatorname{diam} P_n(x)$. Consider an analytic Jordan curve γ_n in $A_n(x)$ such that $\operatorname{dist}(\gamma_n, \partial A_n(x)) \sim \operatorname{diam} P_n(x)$ and let $\Omega_n(x)$ be the domain containing x with $\partial \Omega_n(x) = \gamma_n$. It is clear that $\Omega_n(x) \setminus J$ is a John domain with the right constant. From Lemma C it follows also that $\operatorname{diam} P_n(x) \sim \operatorname{diam} P_{n+1}(x)$ and hence $\operatorname{diam} \Omega_n(x) \sim \operatorname{diam} \Omega_{n+1}(x)$. Now for given $x \in J$, $r > 0$ put $\Omega_x(r) = \Omega_n(x) \setminus J$ where n is chosen to be the largest integer for which $\operatorname{diam} \Omega_n(x) \geq r$. □

The next case is $C_\omega \neq \{\omega\}$ for all the critical points $\omega \in J$. In this situation we add extra boundary pieces to obtain a simply connected domain $\widetilde{\Omega}$ as in Section 2. We consider the collection $\{\Omega_Q^l(r)\}_{l=1}^N$ of simply connected John domains and introduce:

$$\Gamma_{(i,j),Q,M,r} = \{\gamma : \gamma \text{ connects } c^i,\ c^j, \text{ centers of } \Omega_Q^i(r),\ \Omega_Q^j(r),\ \operatorname{diam} \gamma \leq 2Mr\}$$

As in Section 2 we introduce the control:

$$c_{(i,j)}(Q, r, M) = \begin{cases} 0 & \text{if } \Gamma_{(i,j),Q,M,r} = \emptyset, \\ \sup\limits_{\gamma \in \Gamma_{(i,j)Q,M,r}} \inf\limits_{\xi \in \gamma} \operatorname{dist}(\xi, J)/r, & \text{otherwise} \end{cases}$$

and put $c(Q, r, M) = \max_{(i,j)} c_{(i,j)}(Q, r, M)$.

Let $F(r, M) = \{Q \in J : c(Q, r, M) > 0\}$ and we have the following:

4.3. Lemma. *Let f be semihyperbolic and assume that $C_\omega \neq \{\omega\}$ for each critical point ω. Then there exist $M'>0$, $\alpha>0$ independent of Q and r such that*

$$c(Q,r,M')>\alpha \quad \textit{for any } Q\in F(r,M).$$

4.4. Theorem. *Let f be semihyperbolic and assume that $C_\omega\neq\{\omega\}$ for any critical point $\omega\in J$. Then $A_\infty(f)$ admits localization.*

Proof. Lemma 4.3 is a key step and the localization now follows by the algorithm described in Theorem 2.1. $\square$

Proof of Lemma 4.3. Suppose the statement is false. Then there is a sequence $\{Q_n\}_n, \{r_n\}_n$ of points and scales such that $Q_n\in F(r_n, M)$ and $c(Q_n,r_n,n)\to 0$ as $n\to\infty$. Let c_n^1, c_n^2 be centers of $\Omega^1_{Q_n}(r_n)$, $\Omega^2_{Q_n}(r_n)$ and γ_n be a curve connecting c_n^1, c_n^2 with $\operatorname{diam}\gamma_n\leq 2Mr_n$. Let $\delta>0$ be small and $M_1>M_2>2M_3$ large constants. Denote by W_n^i the component of $f^{-k_n}(B(f^{k_n}(Q_n),M_i\delta))$ which contains Q_n and k_n is the largest integer such that $\operatorname{diam}(W_n^3)\geq KMr_n$. By the semihyperbolicity of f we can choose δ so small that the map $f^{k_n}\colon W_n^1\to B(f^{k_n}(Q_n),M_1\delta)$ has degree $\leq D$ where D is independent of Q_n. We can apply Lemma C and conclude that W_n^3 is α'-thick at Q_n. Now choosing K big we conclude that $\gamma_n\subseteq W_n^3$.

Without loss of generality we can assume that $f^{k_n}(Q_n)\to P$ and let $W^i_{n,P}$, $i=1,2,3$, be the component of $f^{-k_n}(B(P,2M_i\delta))$ containing Q_n. Let $b_n^1=f^{k_n}(c_n^1)$, $b_n^2=f^{k_n}(c_n^2)$ and without loss of generality again we say that $b^1=\lim b_n^1$, $b^2=\lim b_n^2$. If n is large enough, say $n\geq n_0$, $W_n^i\subseteq W^i_{n,P}$, $i=1,2,3$, and by Lemma C $\operatorname{dist}(P,b_n^1)\sim \operatorname{dist}(P,b_n^2)\sim\delta$, $\operatorname{dist}(b_n^1,J)\sim\delta\sim\operatorname{dist}(b_n^2,J)$. Let $\Gamma_n=f^{k_n}(\gamma_n)$ connect b_n^1 to b_n^2. As $\gamma_n\subseteq W_n^3\subseteq W^3_{n,P}$, we have $\Gamma_n\subseteq B(P,2M_3\delta)$ and hence $\operatorname{diam}\Gamma_n\leq 4M_3\delta$. By the algorithm of Lemma 2.2 there exists $\widetilde{\Gamma}_n$ connecting b_n^1 and b_n^2 $(n\geq n_0)$ such that $\operatorname{diam}\widetilde{\Gamma}_n\leq 8M_3\delta$ and $\operatorname{dist}(\xi,J)\geq\alpha''$ for $\xi\in\widetilde{\Gamma}_n$.

Let $L_n=\Gamma_n\cup\widetilde{\Gamma}_n$. Let us recall that by Lemma 3.3 there exists $\delta'>0$ such that $\operatorname{diam}(C_{f^k(\omega)})>\delta'$ for any $k\in\mathbf{N}$ and any critical point ω. Choosing δ small with respect to δ' we obtain that the index of L_n with respect to any critical value is 0. This implies that there is an f^{k_n}-lifting $\widetilde{\gamma}_n$ of $\widetilde{\Gamma}_n$ which has endpoints c_n^1 and c_n^2. On the other hand by Lemma C $\operatorname{diam}\widetilde{\gamma}_n\leq 2M'r_n$ and $\operatorname{dist}(\xi,J)\geq\alpha r_n$, $\xi\in\widetilde{\gamma}_n$, which contradicts the fact that $c(Q_n,r_n,n)\to 0$. $\square$

4.5. Corollary. *Let f be semihyperbolic with only one critical point on J. Then $A_\infty(f)$ admits a localization.*

4.6. Corollary. *If f is a cubic semihyperbolic polynomial, then $A_\infty(f)$ admits a localization.*

In fact, there are three possibilities: (1) both critical points escape to infinity and then hyperbolicity implies localization; (2) both critical points are on J, so J

is connected and localization follows from [CJY] and Theorem 1.9; (3) one critical point is on J, another escapes, then Corollary 4.5 implies localization.

4.7. *Remark.* In Section 5 we introduce the notion of a uniformly John domain and prove that uniformly John domains are exactly localizable John domains. After that we can reformulate the results above as follows: under certain condition (e.g. only one critical point on J) $A_\infty(f)$ is a John domain if and only if it is a uniformly John domain.

4.8. Theorem. *Let f be critically finite. Then $A_\infty(f)$ admits a localization.*

Proof. We use the notation $c(Q,r,M)$, $\mathcal{F}(r,M)$ of Lemma 4.3. And we are going to prove as before that

$$\exists M',\ \alpha > 0 : c(Q,r,M') > \alpha \quad \text{for any } Q \in \mathcal{F}(r,M).$$

Recall that this statement implies Theorem 4.8 by the algorithm used in Theorem 2.1.

First we repeat the proof of Lemma 4.3 word by word. If the statement is false, then there is a sequence $\{Q_n\}$, $\{r_n\}$, $Q_n \in \mathcal{F}(r_n,M)$ and $c(Q_n,r_n,n)\to 0$ as $n\to\infty$. We consider c_n^1, c_n^2, $\Omega^1_{Q_n}(r_n)$, $\Omega^2_{Q_n}(r_n)$ and γ_n connecting c_n^1, c_n^2 with $\operatorname{diam}\gamma_n \le 2M_n r_n$. We construct k_n and W_n^i, $i=1,2,3$, with the help of $M_1 > M_2 > 2M_3$ and a small δ which we are going to choose from the condition that

in any disk of radius $2M_1\delta$ centered at J there is at most one critical value of f.

We introduce $W^i_{n,P}$, $i=1,2,3$, exactly as before and $b^1 = \lim b_n^1$, $b^2 = \lim b_n^2$ where $b_n^1 = f^{k_n}(c_n^1)$, $b_n^2 = f^{k_n}(c_n^2)$. By Lemma C $\operatorname{dist}(b_n^1, J) \sim \delta \sim \operatorname{dist}(b_n^2, J)$. Let $\Gamma_n = f^{k_n}(\gamma_n)$. It connects b_n^1 to b_n^2. As $\gamma_n \subseteq W_n^3 \subseteq W^3_{n,P}$, we have $\Gamma_n \subseteq B(P, 2M_3\delta)$ and hence $\operatorname{diam}\Gamma_n \le 4M_3\delta$. Modify Γ_n to $\Gamma_n' = [b^1, b_n^1] \cup \Gamma_n \cup [b_n^2, b^2]$, curves connecting b^1 and b^2. Then $\operatorname{diam}\Gamma_n' \le 5M_3\delta$, $n \ge n_0$.

Let y be the (possible) critical value of f in $B(P, 2M_1\delta)$. For the sake of simplicity we assume in what follows that all critical points of f are simple and their orbits do not intersect. However the reader can easily see how to modify the considerations to obtain the general case.

Consider the loops $L_n = \Gamma'_{n_0} \cup \Gamma'_n$. If there is a subsequence $\{L_{n_j}\}$ of loops with even index at y, then we come to a contradiction because this would mean there exists an $f^{k_{n_j}}$-lifting $\tilde{\gamma}_{n_j}$ of $\tilde{\Gamma}_{n_j} \overset{\text{def}}{=} [b^1_{n_j}, b^1] \cup \Gamma'_{n_0} \cup [b^2, b^2_{n_j}]$ which has endpoints $c^1_{n_j}$ and $c^2_{n_j}$. On the other hand by Lemma C, $\operatorname{diam}\tilde{\gamma}_{n_j} \le M' r_{n_j}$ and $\operatorname{dist}(\xi, J) \ge \alpha r_{n_j}$ which contradicts the assumption that $c(Q_n, r_n, n) \to 0$. Otherwise all indices of L_n at y are odd, $n \ge m$. Consider $L_n^m = L_n - L_m$ and observe that L_n^m has even index at y for $n\geq m$. Put $\widetilde{\Gamma}_n=[b_n^1,b^1]\cup\Gamma'_m\cup[b^2,b_n^2]$, $n\geq m$. The fact that the index of L_n^m at y is even shows that there exists an f^{k_n}-lifting $\widetilde{\gamma}_n$ of $\widetilde{\Gamma}_n$, $n\geq m$, which has endpoints c_n^1 and c_n^2. But by Lemma C $\operatorname{diam}\widetilde{\gamma}_n\leq M'r_n$ and $\operatorname{dist}(\xi,J)\geq\alpha r_n$, $\xi\in\widetilde{\gamma}_n$, $n\geq m$, since $\operatorname{dist}(\xi,J)\geq\alpha'$, $\xi\in\widetilde{\Gamma}_n=[b_n^1,b^1]\cup\Gamma'_m\cup[b^2,b_n^2]$, $n\geq m$.

Thus the assumption $c(Q_n,r_n,n)\to 0$ leads to a contradiction in any case and this proves Theorem 4.8. □

All these results above served as illustrations. Now Theorem 4.9 gives a necessary and sufficient condition for the localization property of $A_\infty(f)$. This criterion was obtained by the first author. First we introduce some notation:

$$\Omega_1=\{\omega\in J:\omega \text{ is a critical point and } \mathcal{C}_\omega=\{\omega\}\},$$
$$\Omega_2=\{\omega\in J:\omega \text{ is a critical point and } \mathcal{C}_\omega\neq\{\omega\}\}.$$

With this notation we have the following definition.

Definition. Let $f:\widehat{\mathbf{C}}\to\widehat{\mathbf{C}}$ be a polynomial. We say that f is separated if there exists $\beta>0$ such that

$$\operatorname{dist}(f^k(\omega_1),\mathcal{C}_{\omega_2})>\beta \quad \text{for } k\in\mathbf{N},\ \omega_1\in\Omega_1 \text{ and } \omega_2\in\Omega_2.$$

Here is the characterization of the uniformly John property for $A_\infty(f)$:

4.9. Theorem. *Let $f:\widehat{\mathbf{C}}\to\widehat{\mathbf{C}}$ be semihyperbolic. Then $A_\infty(f)$ admits a localization if and only if f is separated semihyperbolic.*

Proof. Suppose f is not separated semihyperbolic. Then there exists $\omega_1\in\Omega_1$, $\omega_2\in\Omega_2$ and a sequence $\{n_k\}_k$ such that

$$\operatorname{dist}(f^{n_k}(\omega_1),\mathcal{C}_{\omega_2})\to 0 \quad \text{as } k\to\infty.$$

Without loss of generality we can assume that $f^{n_k}(\omega_1)\to y$ where $y\in\mathcal{C}_{\omega_2}$. Let $B=B\bigl(y,\frac{1}{2}\varepsilon\bigr)$ and $\mathcal{C}=B\cap\mathcal{C}_{\omega_2}$. Denote by W_k the component of $f^{-n_k}(B)$ which contains ω_1. Because ω_1 is a critical point and $f^{n_k}(\omega_1)\notin\mathcal{C}_{\omega_2}$, there are at least two disjoint components $\{\mathcal{C}_k^i\}_i$ of $f^{-n_k}(\mathcal{C})$ contained in W_k. The degree of the map:

$$f^{n_k}:W_k\to B$$

is D by Theorem B. Consequently $\operatorname{diam}\mathcal{C}_k^i\sim\operatorname{diam}W_k$ for $i=1,2$ by Lemma C. Also by Lemma C we have that for certain two disjoint components $\mathcal{C}_k^1$, $\mathcal{C}_k^2$:

$$\operatorname{dist}(\mathcal{C}_k^1,\mathcal{C}_k^2)\leq C(k)\operatorname{diam}W_k$$

where $C(k)\to 0$ as $k\to\infty$.

Now it is easy to see that in this situation the uniformly John property is violated (see Section 5). So localizability fails.

Conversely, suppose that f is separated semihyperbolic. We show that $A_\infty(f)$ is localizable. The idea is to consider two cases which are similar to the situation in Theorem 4.2 respectively Theorem 4.4 above. First we introduce some notation and prove a somewhat stronger version of the separation property. For $\omega\in\Omega_1$, $t>0$ and $n\in\mathbf{N}$ we introduce:

$$T_n^\omega(t)=\{\mathcal{C}\subseteq J:\mathcal{C} \text{ is a connected component of } J,\ \operatorname{dist}(f^n(\omega),\mathcal{C})\le t\}$$

and set

$$d_n^\omega(t)=\sup\{\operatorname{diam}\mathcal{C}:\mathcal{C}\in T_n^\omega(t)\}.$$

With the above notation we have the following claim:

Claim. If f is separated semihyperbolic, then there exists $K>0$ such that for $t<\varepsilon$, $\omega\in\Omega_1$, and $k\in\mathbf{N}$,

$$d_\omega^k(t)<Kt. \tag{4.1}$$

Proof of the claim. Assume the statement is false. Then there exists $\omega_1\in\Omega_1$ and sequences $\{n_k\}_k$, $\{t_k\}_k$ and $\{\mathcal{C}_k\}_k$, $n_k\in\mathbf{N}$, $t_k>0$ and $\mathcal{C}_k\subseteq J$ such that

$$\frac{\operatorname{dist}(f^{n_k}(\omega_1),\mathcal{C}_k)}{\operatorname{diam}\mathcal{C}_k}<\frac{1}{k}\quad\text{for } k\in\mathbf{N}.$$

Let l_k denote the smallest integer (which exists by Lemma 3.3) such that $f^{l_k}(\mathcal{C}_k)=\mathcal{C}_{\omega_2}$ where $\omega_2\in\Omega_2$. Without loss of generality we can assume that ω_2 is the same for each k. Since f is separated semihyperbolic, there exists $N_0\in\mathbf{N}$ such that $P_{N_0}(\omega_2)$ does not contain any critical values except for the ones situated on $\mathcal{C}_{\omega_2}$. Then there is an f^{l_k}-lifting denoted by $P_{l_k+N_0}$ of $P_{N_0}(\omega_2)$ such that $\mathcal{C}_k\subset P_{l_k+N_0}$ and the map:

$$f^{l_k}:P_{l_k+N_0}\to P_{N_0}(\omega_2) \text{ is univalent.}$$

By the distortion theorem for univalent maps we have

$$\frac{\operatorname{dist}(\partial P_{l_k+N_0},\mathcal{C}_k)}{\operatorname{diam}\mathcal{C}_k}\sim\frac{\operatorname{dist}(\partial P_{N_0},\mathcal{C}_{\omega_2})}{\operatorname{diam}\mathcal{C}_{\omega_2}}=\alpha>0,$$

where α does not depend on k. Thus $f^{n_k}(\omega_1)\in P_{l_k+N_0}$ if k is large enough. Again by the distortion theorem:

$$\frac{\operatorname{dist}(f^{n_k}(\omega_1),\mathcal{C}_k)}{\operatorname{diam}\mathcal{C}_k}\sim\frac{\operatorname{dist}(f^{n_k+l_k}(\omega_1),\mathcal{C}_{\omega_2})}{\operatorname{diam}\mathcal{C}_{\omega_2}}.$$

This implies that $\operatorname{dist}(f^{n_k+l_k}(\omega_1),\mathcal{C}_2)\to 0$ which is contrary to the assumption that the dynamics is separated semihyperbolic. The claim is proved. $\square$

To continue the proof we introduce some notation:

$$\eta=\inf\{\operatorname{diam}\ \mathcal{C}_{f^n(\omega)}:\omega\in\Omega_2,\ n\in\mathbf{N}\}.$$

By Lemma 3.3 we have that $\eta>0$. For a reason which will become clear we choose

$$\delta=\min\Big(\frac{\varepsilon}{4K},\frac{\eta}{2}\Big). \tag{4.2}$$

We also fix a large constant $M_1>0$ which will be determined later.

For $x\in J$ and $r>0$ introduce:

$$n(r,x)=\max\{n:\operatorname{diam} W_n\geq M_1 r\}, \tag{4.3}$$

where W_n is the component of $f^{-k}(B(f^n(x),\delta))$ which contains x. Denote by $W_{n,k}$ the component of $f^{-k}(B(f^n(x),\delta))$ which contains $f^{n-k}(x)$ and let $y=f^{n(r,x)}(x)$. To separate into cases similar to the ones in Theorem 4.2 and respectively Theorem 4.4, for $r>0$ we introduce:

$$J_2(r)=\{x\in J:W_{n(r,x),k}\cap\Omega_1=\emptyset \text{ for any } k=1,2,\dots,n(r,x)\},$$

and put $J_1(r)=J\setminus J_2(r)$.

Take first $x\in J_1(r)$. This is the case similar to the one in Theorem 4.2.

Let us recall the topological disks $U\subset U_1\subset U_2\subset f(U)$ from Section 3. Denote by $\widetilde{P}_n(z)$, $P_n(z)$ the components of $f^{-n}(U_1)$ and $f^{-n}(U_2)$ respectively which contain the point $z\in J$. For $z\in J$ let $N=N(z)$ denote the first integer n with the property:

$$\operatorname{diam} P_n(z)\leq\operatorname{diam}\mathcal{C}_z+\tfrac{1}{2}\delta. \tag{4.4}$$

Let us consider the open covering of J by the sets $\{\widetilde{P}_N(z)\}_{z\in J}$. Since J is compact, we can choose a finite subcovering $\{\widetilde{P}_{N_i}(z_i)\}_{i=1}^{L}$. There is an index $i_0\in\{1,\dots,L\}$ such that $f^{n(r,x)}(x)=y\in\widetilde{P}_{N_{i_0}}(z_{i_0})$. We claim that in this first case (i.e., $x\in J_1(r)$) we have:

$$\operatorname{diam} P_{N_{i_0}}(z_{i_0})<\tfrac{1}{2}\varepsilon.$$

To see this, recall that there exist $\omega\in\Omega_1$ and $k\leq n(r,x)$ such that $\operatorname{dist}(y,f^k(\omega))<\delta$, $\operatorname{diam} P_{N_{i_0}}(z_{i_0})\leq\operatorname{diam}\mathcal{C}_{z_{i_0}}+\frac{1}{2}\delta$, and therefore

$$\operatorname{dist}(f^k(\omega),\mathcal{C}_{z_{i_0}})<\tfrac{3}{2}\delta.$$

By condition (4.1) we obtain: $\operatorname{diam}\mathcal{C}_{z_{i_0}} \leq K\frac{3}{2}\delta$. Finally, we use (4.4) and (4.2) to get:

$$\operatorname{diam} P_{N_{i_0}}(z_{i_0}) < (3K+1)\tfrac{1}{2}\delta < \tfrac{1}{2}\varepsilon,$$

which proves the claim.

Let us consider now the annulus $A_{N_i} = P_{N_i}(z_i) \setminus \widetilde{P}_{N_i}(z_i)$ and an analytic arc $\gamma_i \subset A_{N_i}$ with the property that:

$$\operatorname{dist}(\gamma_i, \partial A_{N_i}) \geq \delta' > 0 \quad \text{for } i = 1, \dots, L,$$

where $\delta' > 0$ is a fixed constant. Denote by Ω_i the topological disk with $\partial\Omega_i = \gamma_i$, $i = 1, \dots, L$.

Let us summarize the chain of constructions:

$$(x, r) \mapsto n(r, x) \mapsto y = f^{n(r,x)}(x) \mapsto i_0 = i_0(r, x) \mapsto \Omega_{i_0(r,x)}.$$

Let $\widetilde{\Omega}_x(r)$ be the component of $f^{-n(r,x)}(\Omega_{i_0(r,x)})$ which contains x. Furthermore we denote by $U_x(r)$ the component of $f^{-n(r,x)}(P_{N_{i_0}(r,x)}(z_{i_0}(r,x)))$ which contains x. We have that $\widetilde{\Omega}_x(r) \subseteq U_x(r)$. Because $\operatorname{diam} P_{N_{i_0(r,x)}}(z_{i_0}(r,x)) < \frac{1}{2}\varepsilon$, the covering:

$$f^{n(r,x)} \colon U_x(r) \to P_{N_{i_0(r,x)}}(z_{i_0}(r,x)) \text{ has degree } \leq D,$$

by Theorem B.

We can now apply Lemma C to conclude that:

(a_1) $\operatorname{diam}\widetilde{\Omega}_x(r) \sim r$,

(a_2) $\widetilde{\Omega}_x(r)$ is α'-thick at x for $\alpha' = \alpha'(f)$,

(a_3) $\operatorname{dist}(J \cap \widetilde{\Omega}_x(r), \partial\widetilde{\Omega}_x(r)) \sim r$.

As in the proof of Theorem 4.2 we define $\Omega_x(r) = \widetilde{\Omega}_x(r) \setminus J$. By [CJY] and by the properties (a_1), (a_2), and (a_3) we see that $\Omega_x(r)$ is a local John domain at $x \in J_1(r)$ with uniform John constant.

The second case is $x \in J_2(r)$. The proof is similar to that of Theorem 4.4. We take $x \in \mathcal{F}(r, M) \cap J_2$ and we are going to prove the statement of Lemma 4.3. Namely we will show that for any M there are M_1 (defining $J_2(r)$) and M', $\alpha > 0$ independent of x and r such that

$$c(Q, r, M') > \alpha r \quad \text{for } x \in \mathcal{F}(r, M) \cap J_2(r). \tag{4.5}$$

If (4.5) is proved for all $M \leq M(N, M_0, D)$, where N, M_0 are from (1) of Theorem 2.1, then localization follows by the algorithm described in Theorem 2.1.

Suppose (4.5) is false. Then there exist sequences $\{x_k\}_k$, $\{r_k\}_k$ of points and scales such that $x_k \in F(r_k, M) \cap J_2(r_k)$ and $c(x_k, r_k, k) \to 0$ as $k \to \infty$. Let c_k^1, c_k^2 be centers of $\Omega^1_{x_k}(r_k)$, $\Omega^2_{x_k}(r_k)$ which are domains of prelocalization described in (1), (2), (3) of Theorem 2.1. Let γ_k be a curve with $\operatorname{diam}\gamma_k<2Mr_k$. By our assumption given any positive α for all sufficiently large k, there is no curve $\widetilde{\gamma}_k$ connecting c^1_k and c^2_k with the properties:

$$\operatorname{diam}\widetilde{\gamma}_k \le 2kr_k, \tag{4.6}$$

$$\operatorname{dist}(\widetilde{\gamma}_k, J) \ge \alpha r_k. \tag{4.7}$$

We obtain a contradiction and we will be done if we construct $\widetilde{\gamma}_k$ with properties (4.6) and (4.7).

Let $n_k=n(r_k,x_k)$ and $y_k=f^{n_k}(x_k)$. Consider fixed small constants δ_1, δ_2 with $\delta_2<\delta_1<\delta$. We denote by W^2_k, W^1_k and W_k the components of $f^{-n_k}(B(y_k,\delta_2))$, $f^{-n_k}(B(y_k,\delta_1))$ and $f^{-n_k}(B(y_k,\delta))$ which contain x_k. It is clear that the map

$$f^{n_k}\colon W_k \to B(y_k,\delta) \text{ has degree } \le D.$$

For any $k\in\mathbf{N}$ the disks W^2_k, W^1_k and W_k are α'-thick at x_k for some $\alpha'>0$. We choose M_1 large enough (depending on M) so that we have

$$B(x_k, 4Mr_k) \subseteq W^2_k.$$

Then $\gamma_k\subseteq W^2_k$, and we define $\Gamma_k=f^{n_k}(\gamma_k)$, $b^i_k=f^{n_k}(c^i_k)$, $i=1,2$. By Lemma C we obtain that for any k

$$c_1\delta_2 \le \operatorname{dist}(b^i_k, J) \le 2\delta_2,$$

for some $c_1=c_1(f)>0$.

Then the curves Γ_k connect b^1_k and b^2_k and $\Gamma_k\subseteq B(y_k,\delta_2)$. For the same reason as in the proof of Lemma 2.2 we can find a curve $\widetilde{\Gamma}_k\subseteq B(y_k,\delta_1)$ connecting b^1_k and b^2_k with the property:

$$\operatorname{dist}(\widetilde{\Gamma}_k, J) > \alpha \quad \text{where } \alpha=\alpha(f)>0.$$

Because $x_k\in J_2(r_k)$, the ball $B(y_k,\delta)$ is free from critical values coming from Ω_1.

On the other hand:

$$\operatorname{diam}(\Gamma_k\cup\widetilde{\Gamma}_k) < 2\delta_1 < \eta$$

by (4.2), and therefore the index of $\Gamma_k\cup\widetilde{\Gamma}_k$ with respect to critical values coming from Ω_2 is zero. Consequently there is an f^{n_k}-lifting $\widetilde{\gamma}_k$ of Γ_k connecting c^1_k and c^2_k. Another application of Lemma C shows that $\widetilde{\gamma}_k$ satisfies (4.6) and (4.7). This contradiction completes the proof. □

Remark. Obviously critically finite polynomials are separated semihyperbolic and thus $A_\infty(f)$ is uniformly John if f is critically finite. We preferred to provide an independent proof of the last assertion for illustrative purposes.

5. Uniformly John domains

Recall that ϱ means the internal metric in a domain.

5.1. *Definition.* A domain Ω is called a *uniformly John* domain if there are constants c_1 and c_2 such that for any two points $x_1, x_2 \in \Omega$ there exists a curve $\gamma = \gamma_{x_1,x_2}$ connecting x_1 to x_2 and lying in Ω which has two properties:

(i) $\forall \xi \in \gamma : \operatorname{dist}(\xi, \partial\Omega) \geq c_1 \operatorname{dist}(\xi, \{x_1, x_2\})$;

(ii) $\operatorname{diam} \gamma \leq c_2 \varrho(x_1, x_2)$.

5.2. *Remark.* Condition (i) is just the John type condition; that is, the existence of a "cigar" connecting x_1 and x_2 inside Ω. Condition (ii) means that we can connect x_1 to x_2 by a path which is the best up to a constant.

5.3. Theorem. *A John domain is localizable if and only if it is uniformly John.*

Proof. We have seen that in the simply connected case localization holds. It turns out that in this case the uniformly John property holds as well. This is based on the fact that we can choose γ with property (i) to be the hyperbolic geodesics (see [NV] or [GHM]). On the other hand geodesics satisfy (ii) by a theorem of Gehring and Hayman. So we are going to deal with the non-simply connected case.

(a) Let Ω be a localizable John domain and $x_1, x_2 \in \Omega$. Let Q_1 be a point of $\partial\Omega$ which is a closest point to x_1 and let $r_1 = |Q_1 - x_1|$. If $\varrho(x_1, x_2) \leq \frac{1}{2} r_1$, one can choose γ to be $[x_1, x_2]$. So assume that $r = \varrho(x_1, x_2) \geq \frac{1}{2} r_1$. Let $\Omega_{Q_1}(4r)$ be a local John domain which contains the curves γ_{x_1,x_2}, $\operatorname{diam} \gamma_{x_1,x_2} \leq 1.01 r$, and $[x_1, Q_1]$. Let c be a center of $\Omega_{Q_1}(4r)$ and γ^i, $i=1,2$, be two John arcs in $\Omega_{Q_1}(4r)$ connecting c with x_i, $i=1,2$, respectively. Put $\gamma = \gamma^1 \cup \gamma^2$. Property (i) follows immediately because the John constants of local domains are uniformly bounded. As for (ii) we have a simple estimate

$$\operatorname{diam} \gamma \leq \operatorname{diam} \gamma^1 + \operatorname{diam} \gamma^2 \leq 2 \operatorname{diam} \Omega_{Q_1}(4r) \leq 8Mr = 8M\varrho(x_1, x_2).$$

(b) Let Ω be a uniformly John domain. First we use only the John property. Applying Theorem 1.9 to a simply connected John domain $\widetilde{\Omega} \subset \Omega$ constructed in [J2] we obtain a collection $\{\Omega_Q^l(r)\}$ of simply connected John domains with John constants uniformly bounded by C_0 which satisfy the properties:

(1) $\operatorname{diam} \Omega_Q^l(r) \leq M_0 r$, $l = 1, \dots, N$;

(2) $\bigcup \overline{\Omega_Q^l(r)} \supseteq B_\varrho(Q, r)$;

(3) $\Omega_Q^i(r) \cap \Omega_Q^j(r) = \emptyset$, for $i \neq j$.

As was done several times before, we introduce the control $c(Q, r, M)$, $M \geq M_0$, and the sets $\mathcal{F}(r, M) = \{Q : c(Q, r, M) > 0\}$.

Now we are going to prove the statement:

$\exists M',\ \alpha>0$ independent of Q and $r: c(Q,r,M')>\alpha$ for any $Q\in\mathcal{F}(r,M)$.

Recall that this statement implies that Ω is localizable by the algorithm of Theorem 2.1.

So let $Q\in\mathcal{F}(r,M)$, which gives a curve $\gamma\subseteq\Omega$ connecting centers c^i, c^j of $\Omega^i_Q(r)$, $\Omega^j_Q(r)$ respectively and such that $\operatorname{diam}\gamma\le 2Mr$. Now use the fact that Ω is a uniformly John domain. We know that $\varrho(c^i,c^j)\le\operatorname{diam}\gamma\le 2Mr$. Using (i) and (ii) we obtain $\tilde{\gamma}, \operatorname{diam}\tilde{\gamma}\le 2C_2Mr$ such that

$$\operatorname{dist}(\xi,\partial\Omega)\ge c_1 r.$$

The statement above is proved with $M'=C_2M$ and $\alpha=c_1$. □

5.4. Corollary. *Let f be a polynomial. Suppose J does not split the plane. Assume that either $C_\omega=\{\omega\}$ or $C_\omega\neq\{\omega\}$ for all critical points $\omega\in J$ simultaneously. Then the following assertions are equivalent*:

(a) *$A_\infty(f)$ is a John domain*;

(b) *$A_\infty(f)$ is a uniformly John domain.*

Proof. One needs to check only (a) $\Rightarrow$ (b). By [CJY] (a) implies that f is semihyperbolic. An application of Theorem 4.3 or 4.5 combined with Theorem 5.3 finishes the proof of (b). □

5.5. Corollary. *Let f be a separated polynomial. Suppose J does not split the plane. Then the following assertions are equivalent*:

(a) *$A_\infty(f)$ is a John domain*;

(b) *$A_\infty(f)$ is a uniformly John domain.*

5.6. *Remark.* It seems rather probable that uniformly John domains are Gromov hyperbolic. At least from discussions with Juha Heinonen and Mario Bonk we strongly believe this assertion for $A_\infty(f)$ of separated semihyperbolic f.

5.7. *Remark.* The condition $(*)$ (that is "J does not split the plane") is not essential. One can replace components of J by components of the filled-in Julia set K_f everywhere and get the same results for $A_\infty(f)=\overline{\mathbf{C}}\setminus K_f$.

6. Example of a semihyperbolic polynomial which is not separated semihyperbolic

Let U_0, U_1, U_2 be topological disks such that $U_i\subset U_0$, $i=1,2$ and $U_1\cap U_2=\emptyset$. Let $f_i\colon U_i\to U_0$, $i=1,2$ be branched coverings of degree 2. We can take f_i to be second degree polynomials with critical points ω_i and Julia sets $J_i \subseteq U_i$ for $i=1,2$. Choose f_2 in such a way that $\omega_2 \in J_2$ and $\operatorname{orb}(\omega_2)$ is finite. Hence J_2 is a dendrite. Choose f_1 such that $f_1^n(\omega_1) \to \infty$ and therefore J_1 is a Cantor set.

Using quasiconformal surgery we find a fourth degree polynomial f_3 which is conjugated by a quasiconformal mapping φ to our dynamics. Two of the critical points of f_3 are escaping and the third one is in J_3 with finite forward orbit. Let us denote this critical point by ω_3. It is clear that $\omega_3 = \varphi(\omega_2)$.

Before we continue the construction let us state the following:

Claim. There exist $y_0 \in J_3$ and $\{n_k\}_k \subseteq \mathbf{N}$ such that $\mathcal{C}_{y_0} = \{y_0\}$ and

$$\operatorname{dist}(f_3^{n_k}(y_0), \mathcal{C}_{\omega_3}) \to 0 \quad \text{as } k \to \infty.$$

Assume for the moment that the claim is true. Choose topological disks $U_3 \subseteq U_4$ such that $J_3 \subseteq U_3$ and $f_3: U_3 \to U_4$ is a branched covering of degree 4. Let us take another topological disk $U_5 \subseteq U_4$ with $U_3 \cap U_5 = \emptyset$ and a quadratic polynomial f_4 which is a branched covering $f_4: U_5 \to U_4$ with $J_4 \subset U_5$ and the critical ω_4 escaping. More exactly we are going to choose f_4 in a way that $f_4(\omega_4) = y_0$. Making another surgery we obtain a polynomial f of degree 6 with three escaping critical points. Let us denote by $\bar{\omega}_1$ and $\bar{\omega}_2$ the other two critical points which are in $J = J(f)$.

If ψ denotes the new quasiconformal conjugacy, we have $\bar{\omega}_1 = \psi(\omega_4)$ and $\bar{\omega}_2 = \psi(\omega_3)$. It is clear that $\operatorname{orb}(\bar{\omega}_1) \subseteq \psi(U_3)$ and we have that $\operatorname{dist}(\bar{\omega}_1, \operatorname{orb}(\bar{\omega}_1)) > 0$ since $\bar{\omega}_1 \in \psi(U_5)$. Consequently f is semihyperbolic.

By our claim we obtain that

$$\operatorname{dist}(f^{n_k+1}(\bar{\omega}_1), \mathcal{C}_{\bar{\omega}_2}) \to 0 \quad \text{as } k \to \infty,$$

which shows that f is not separated semihyperbolic.

This finishes the construction, we only need to prove the claim.

Proof of the Claim. Instead of f_3, ω_3, J_3 we are going to use the notation f, ω, J and let V, W be two topological disks (corresponding to $\varphi(U_2)$, $\varphi(U_0)$ in previous notations) such that $\omega \in V \cap J$ and

$$f: V \to W \text{ is a branched covering of degree two.}$$

Also, without loss of generality we can assume that the escaping critical points of f are in $W \setminus V$. Fix $x_0 \in V$ and consider the sequence $\{\mu_k\}_k$ of probability measures:

$$\mu_k = \frac{1}{4^k} \sum_{y \in f^{-k}(x_0)} \delta_y. \tag{6.1}$$

A well-known result of Lyubich [L1], [L2] and Freire, Lopes and Mañé [FLM] states that $\{\mu_k\}_k$ converges weakly to the measure of maximal entropy m of the polynomial f.

As before let us denote by $P_N(\omega)$ the component of $f^{-N}(V)$ which contains ω. Then the map

$$f^N: P_N(\omega) \to V \text{ is a branched covering of degree } 2^N.$$

Using (6.1) we have

$$\mu_k(P_N(\omega)) = \frac{\#\{y \in P_N(\omega) : f^k(y) = x_0\}}{4^k}.$$

Put $k=lN$ and observe:

$$\#\{y \in P_N(\omega) : f^{lN}(y) = x_0\} = 2^N \#\{y : f^{(l-1)N}(y) = x_0\}.$$

Consequently we have:

$$\mu_{lN}(P_N(\omega)) = \frac{2^N \#\{y : f^{(l-1)N}(y) = x_0\}}{4^{lN}} = \frac{1}{2^N}.$$

Because $\mu_{lN} \to m$ weakly, we obtain that

$$m(P_N(\omega)) = \frac{1}{2^N} \quad \text{for } N \in \mathbf{N}. \tag{6.2}$$

If we change ω to any $x \in J$, we obtain similarly:

$$m(P_N(x)) \le \frac{1}{2^N} \quad \text{for } N \in \mathbf{N} \text{ and } x \in J. \tag{6.3}$$

Let $J_1 = \{x \in J : \mathcal{C}_x = \{x\}\}$ and $J_2 = J \setminus J_1$. A consequence of (6.3) and Corollary 3.4 is that $m(J_2)=0$. By Birkhoff's ergodic theorem, there exists a set $X_N \subseteq J$ such that $m(X_N)=1$ and for any $y \in X_N$ we have:

$$\lim_{n\to\infty} \frac{\#\{k : k \le n,\ f^k(y) \in P_N(\omega)\}}{n} = \frac{1}{2^N}. \tag{6.4}$$

Since $P_{N+1}(\omega) \subseteq P_N(\omega)$, we obtain that $X_{N+1} \subseteq X_N$. Let $X^* = \bigcap_{N\in\mathbf{N}} X_N$. Then $m(X^*)=1$ is not empty. Let $y_0 \in X^* \cap J_1$. Then $\mathcal{C}_{y_0} = \{y_0\}$. On the other hand $\mathcal{C}_\omega = \bigcap_{N\in\mathbf{N}} P_N(\omega)$ and for $y=y_0$ in (6.4) we obtain a subsequence $\{n_k\}_k$ such that

$$\operatorname{dist}(f^{n_k}(y_0), \mathcal{C}_\omega) \to 0 \quad \text{as } k \to \infty.$$

We are done. $\square$

7. More general holomorphic repellers

Let V be an open finitely connected set and U be a finite union of open finitely connected sets, $\bar{U}\subseteq V$. Let $f\colon U\to V$ be a branched or regular covering and we always assume

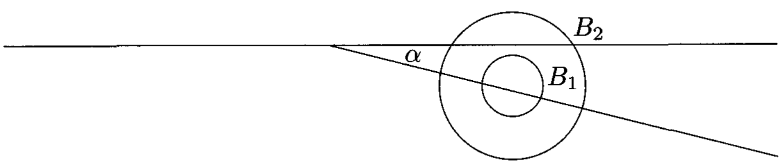
$$(*)\qquad J\overset{\text{def}}{=}\bigcap_{n\geq 0} f^{-n}(U) \text{ does not split the plane.}$$

We call (f,U,V) a semihyperbolic holomorphic repeller (HR) if $\operatorname{dist}(\omega,\sigma(\omega))>0$ for each critical point $\omega\in J$ of f. We call (HR) critically finite if the orbit of each critical point $\omega\in J$ is finite.

With obvious changes of statements Theorems 4.2, 4.4, 4.8, 4.9 and Corollaries 5.4, 5.5 hold for (HR).

These results have many applications to the study of harmonic measure on J. First of all let w_Ω denote harmonic measure of Ω evaluated at ∞ (we always consider Ω such that $\partial\Omega$ does not split the plane). One of the basic good properties of such a measure is the doubling property. In [JK] it was proved that harmonic measure of NTA domains in $\mathbf{R}^n$ (simply quasidisks on the plane) has the doubling property.

This statement can be very easily proved on the plane but there is no hope to have it generalized for John disks. Here is a simple picture.



In this picture Ω is the complement of three segments meeting at the origin. Then $w_\Omega(B_1)\sim r^{\pi/(\pi-\alpha)}$ and $w_\Omega(B_2)\sim r\gg r^{\pi/(\pi-\alpha)}$.

However harmonic measure of John disks satisfies a certain doubling condition. One needs to replace the balls in Euclidean metric by the balls in internal metric:

$$Q\in\partial\Omega,\ B_\varrho(Q,r)=\{x\in\bar{\Omega}:\exists\gamma_{Q,x} \text{ such that } \gamma_{Q,x}\setminus\{Q,x\}\subset\Omega \text{ and } \operatorname{diam}\gamma_{Q,x}\leq r\}.$$

7.1. Theorem. *Let Ω be a John disk. Then $\exists C_\Omega<\infty$ such that*

$$\forall Q\in\partial\Omega\ \forall r>0: w_\Omega(B_\varrho(Q,2r))\leq C_\Omega w_\Omega(B_\varrho(Q,r)).$$

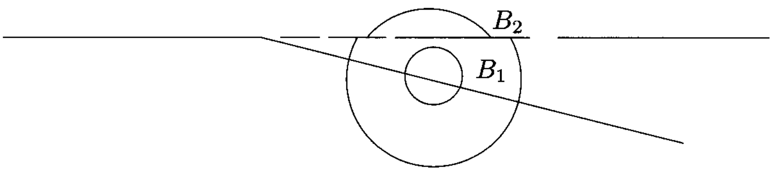
This theorem is a particular case of

7.2. Theorem. *Let Ω be a localizable John domain. Then the conclusion of Theorem* 7.1 *holds* (*that is harmonic measure has the doubling property with respect to internal metric*).

The proof is quite technical (because of possible infinite connectivity of Ω and because of the use of ϱ instead of the usual metric) and it is carried out in [BV2].

Again there is no hope that this holds for arbitrary John domains. A change of metric helps only if we can localize Ω.

The next example is an easy modification of the example above. We are grateful to Juha Heinonen for pointing out this example to us. Our initial one was more complicated.



The domain Ω now is the complement of three segments united with $\bigcup(x_n^1, x_n^2)$, where $(x_n^2-x_n^1)/x_n^1\to 0$. It is a John domain and it is not localizable. The picture shows why its harmonic measure does not have the doubling property with respect to the internal metric.

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