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Boundedness, compactness, and Schatten p-classes of Hankel operators between weighted Dirichlet spaces

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1. Introduction and main results

In this paper, we study the small and big Hankel operators from one weighted Dirichlet space to another. We characterize the analytic symbols for which these operators are bounded, compact or belong to the Schatten p-classes for a certain range of p. The endpoints of this range are also discussed.

Let $\mathbf{D}$ be the unit disk in the complex plane. Let $dA(z)=1/\pi\, dx\, dy$ be the normalized area measure on $\mathbf{D}$. For $\alpha<1$, set

$$dA_\alpha(z)=(2-2\alpha)(1-|z|^2)^{1-2\alpha}\, dA(z).$$

The Sobolev space $L^{2,\alpha}$ is the Hilbert space of functions $u\colon \mathbf{D}\to\mathbf{C}$, for which the norm

$$\|u\|_\alpha=\left(\left|\int_{\mathbf{D}} u(z)\, dA_\alpha(z)\right|^2+\int_{\mathbf{D}}(|\partial u/\partial z|^2+|\partial u/\partial \bar z|^2)\, dA_\alpha(z)\right)^{1/2}$$

is finite. The weighted Dirichlet space D_α is the subspace of all analytic functions in $L^{2,\alpha}$. (This scale of spaces includes the Bergman space ($\alpha=-\frac{1}{2}$), the Hardy space ($\alpha=0$) and the classical Dirichlet space ($\alpha=\frac{1}{2}$).) The orthogonal projection, P_α, from $L^{2,\alpha}$ onto D_α can be understood as the integral operator represented by

$$P_\alpha(u)(w)=\int_{\mathbf{D}} u(z)\, dA_\alpha(z)+\int_{\mathbf{D}}\frac{\partial u}{\partial z}(z)\overline{\frac{\partial}{\partial z}K_\alpha(z,w)}\, dA_\alpha(z).$$

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Here K_α, the reproducing kernel of D_α, has the expression (see [W])

$$K_\alpha(z,w)=1+\int_0^z\int_0^{\bar w}\frac{1}{(1-st)^{3-2\alpha}}\,ds\,dt.$$

Let P denote the set of all analytic polynomials on $\mathbf{D}$. For $\alpha<1$, $\gamma<\frac{1}{2}(1+\alpha)$ and $f\in L^{2,\alpha}$, the small and big Hankel operators with symbol f are defined on P, respectively, by

$$h_f^{(\gamma)}(g)=\overline{P_\gamma(f\bar g)}\quad\text{and}\quad H_f^{(\gamma)}(g)=(I-P_\gamma)(\bar f g),\quad \forall g\in P.$$

This definition is fine because of the fact that $\bar f g$ is in $L^{2,\alpha}$ if $f\in L^{2,\alpha}$ and $g\in P$, and P_γ is bounded on $L^{2,\alpha}$ if $\gamma<\frac{1}{2}(1+\alpha)$ and $\alpha<1$ (see Lemma D later).

The set P is in fact dense in D_β for any $\beta<1$. Hence we can regard the small and big Hankel operators as operators on D_β.

We will in this paper consider only analytic symbols.

Suppose H and K are Hilbert spaces. The Schatten p-class $S_p(H,K)$, $(0<p<\infty)$ is the set of all linear operators T, from H to K, for which the sequence of the singular numbers $\{\,S_k(T)=\inf\{\,\|T-R\|:\operatorname{rank}(R)\le k\,\}\,\}_{k=0}^\infty$ belongs to l^p. The $S_p(H,K)$ norm of T is defined by $\|T\|_{S_p(H,K)}=\|\{S_k(T)\}\|_{l^p}$.

Previous work has obtained necessary and sufficient conditions for the boundedness, compactness, and for the membership in the Schatten p-classes of the small and big Hankel operators acting on the Hardy and Bergman spaces. We refer the reader to [P1], [P2], [R1], [R2], [S], [A], [AFP], [J1], [Z] and their references. For the other weighted Dirichlet spaces some results for the boundedness and for the membership in the Schatten p-classes of these operators can be found in [W] and [RW1].

In this paper, we will characterize the symbols f for which $H_f^{(\gamma)}$ and $h_f^{(\gamma)}$, as operators from D_β to $L^{2,\alpha}$, are bounded, compact or belong to S_p for a certain range of p. We discuss also the endpoints of this range. (The ranges of α, β, γ and p are stated later.)

For $0<p\le\infty$ and $-\infty<s<\infty$, B_p^s, BL_p^s and b_∞^s denote the spaces of analytic functions on $\mathbf{D}$ defined as follows:

$$\begin{aligned}
B_p^s&=\{\,f:(1-|z|)^{m-s}f^{(m)}(z)\in L^p((1-|z|^2)^{-1}\,dA)\,\};\\
BL_p^s&=\left\{\,f:(1-|z|)^{m-s}f^{(m)}(z)\in L^p\Big((1-|z|^2)^{-1}\log\frac{1}{1-|z|^2}\,dA\Big)\,\right\};\\
b_\infty^s&=\{\,f:(1-|z|)^{m-s}f^{(m)}(z)\to 0,\ \text{as } |z|\to 1\,\}.
\end{aligned}$$

Here m is a nonnegative integer so that $m>s$. $B_p^{1/p}=B_p$ is the usual Besov space and $b_\infty^0=B_0$ is the little Bloch space.

A nonnegative measure μ on $\mathbf{D}$ is called an α-Carleson measure if

$$\int_{\mathbf{D}} |g(z)|^2\, d\mu(z) \le C\|g\|_\alpha^2, \quad \forall g \in D_\alpha.$$

The best constant C here, denoted by $\|\mu\|_\alpha$, is said to be the α-Carleson measure norm of μ. 0-Carleson measures are just the classical Carleson measures (see [G]). There are many equivalent characterizations on the α-Carleson measures (see for example [St] and [KS]). In this paper, however, we do not need them. The above definition seems easier to work with in our proofs.

We use $S_p^{\beta\alpha}$ to denote $S_p(D_\beta, L^{2,\alpha})$ and χ_r $(0<r<1)$ to mean the characteristic function of the set $\mathbf{D}\setminus\overline{r\mathbf{D}}=\{z: r<|z|<1\}$. Our main results are stated as follows.

Theorem 1. *Suppose $\alpha<1$, $\beta\le\frac{1}{2}$, $\alpha-\beta<1$, $\gamma<\frac{1}{2}(1+\alpha)$, $0<p<\infty$ and f is analytic on $\mathbf{D}$. Then*

(1) $H_f^{(\gamma)}$ is bounded or compact from D_β to $L^{2,\alpha}$ if and only if $\|\,|f'|^2 dA_\alpha\|_\beta<\infty$ or $\|\chi_r|f'|^2\, dA_\alpha\|_\beta\to 0$ as $r\to 1_-$, respectively;

(2) If $\beta<\frac{1}{2}$, $1/(1+\beta-\alpha)<p$ and $\beta_p<1$, then $H_f^{(\gamma)}\in S_p^{\beta\alpha}$ if and only if $f\in B_p^{1/p+\alpha-\beta}$. If $p\le 1/(1+\beta-\alpha)$, then $H_f^{(\gamma)}\in S_p^{\beta\alpha}$ if and only if f is constant.

Theorem 2. *Suppose $\alpha<1$, $\beta\le\frac{1}{2}$, $\alpha-\beta<1$, $\gamma<\frac{1}{2}(1+\alpha)$, $0<p<\infty$ and f is analytic on $\mathbf{D}$. Then*

(1) $h_f^{(\gamma)}$ is bounded or compact from D_β to $L^{2,\alpha}$ if and only if $\|\,|f'|^2\, dA_\alpha\|_\beta<\infty$ or $\|\chi_r|f'|^2\, dA_\alpha\|_\beta\to 0$ as $r\to 1_-$, respectively;

(2) If $\beta<\frac{1}{2}$ and $\beta_p<1$, then $h_f^{(\gamma)}\in S_p^{\beta\alpha}$ if and only if $f\in B_p^{1/p+\alpha-\beta}$.

Theorem 3. *If $\alpha<1$, $\frac{1}{2}<\beta<1$, $\gamma<\frac{1}{2}(1+\alpha)$ and $p>1/\beta$, then the following are equivalent:*

(1) $H_f^{(\gamma)}$ or $h_f^{(\gamma)}$ is bounded from D_β to $L^{2,\alpha}$;

(2) $H_f^{(\gamma)}$ or $h_f^{(\gamma)}$ is compact from D_β to $L^{2,\alpha}$;

(3) $H_f^{(\gamma)}$ or $h_f^{(\gamma)}$ is in $S_p^{\beta\alpha}$;

(4) f is in D_α.

Theorem 4. *Suppose $\alpha<1$ and $1<p<\infty$. Then*

(1) If $p>2$ and $f\in BL_p^\alpha$, then $H_f^{(\alpha)}$ and $h_f^{(\alpha)}\in S_p^{1/p\alpha}$;

(2) If $p<2$ and $H_f^{(\alpha)}$ or $h_f^{(\alpha)}\in S_p^{1/p\alpha}$, then $f\in BL_p^\alpha$.

Some of our results above can be reduced to the results in [P1], [S], [AFP], [J1], [W] and [RW2]. The main work in this paper is to characterize the boundedness and compactness of the small Hankel operators and the Schatten p-classes of the big Hankel operators and to provide a proof for Theorem 4.

We take up some basic results in Section 2 and prove Theorems 1, 2, 3 and 4 in Sections 3, 4, 5 and 6 respectively. In Section 6, we discuss some examples which show that the conditions in Theorem 4 are "sharp" but not sufficient.

Throughout this paper, the notation "$\asymp$" means comparable and C means a positive constant which may vary at each occurrence. We will use $\langle\cdot,\cdot\rangle_\alpha$ to denote the inner product in $L^{2,\alpha}$. More precisely, for $u, v\in L^{2,\alpha}$

$$\langle u,v\rangle_\alpha=\int_{\mathbf{D}} u(z)\,dA_\alpha(z)\int_{\mathbf{D}} v(z)\,dA_\alpha(z)+\int_{\mathbf{D}}\frac{\partial u}{\partial z}(z)\overline{\frac{\partial v}{\partial z}(z)}\,dA_\alpha(z)$$
$$+\int_{\mathbf{D}}\frac{\partial u}{\partial \bar z}(z)\overline{\frac{\partial v}{\partial \bar z}(z)}\,dA_\alpha(z).$$

2. Preliminaries

For $\alpha<1$, the space $A^{2,1-2\alpha}$ is the subspace of all analytic functions in $L^2(dA_\alpha)$. It is easy to check that $A^{2,1-2\alpha}=D_{\alpha-1}$ (their norms are different but equivalent). The orthogonal projection from $L^2(dA_\alpha)$ onto $A^{2,1-2\alpha}$ is the integral operator defined by (see [Z] for example)

$$\widetilde{P}_\alpha(u)(w)=\int_{\mathbf{D}}\frac{u(z)}{(1-\bar z w)^{3-2\alpha}}\,dA_\alpha(z).$$

The Hankel operator with symbol f from $A^{2,1-2\beta}$ into $L^2(dA_\alpha)$ is densely defined by

$$\widetilde{H}_f(g)=(I-\widetilde{P}_\alpha)(\bar f g),\quad \forall g\in P.$$

The following theorem, which can be found in [J1], is needed in Section 3.

Theorem A. *Suppose $\alpha,\beta,\alpha-\beta<1$ and f is analytic on $\mathbf{D}$. Regard $\widetilde{H}_f$ as an operator from $A^{2,1-2\beta}$ to $L^2(dA_\alpha)$. Then*

(1) *$\widetilde{H}_f$ is bounded or compact if and only if f is in $B_\infty^{\alpha-\beta}$ or in $b_\infty^{\alpha-\beta}$ respectively;*

(2) *If $1/(1+\beta-\alpha)<p$, then $\widetilde{H}_f\in S_p$ if and only if $f\in B_p^{1/p+\alpha-\beta}$;*

(3) *If $p\le 1/(1+\beta-\alpha)$, then $\widetilde{H}_f\in S_p$ if and only if f is constant.*

The following result, which is needed in Section 4, can be found in [RW1].

Theorem B. *Suppose g is analytic on $\mathbf{D}$, $\beta\le\frac12$, $\sigma,\tau>-1$ and $\min(\sigma,\tau)+2\beta>-1$. Then*

$$\int_{\mathbf{D}}\int_{\mathbf{D}}\frac{|g(z)-g(w)|^2}{|1-\bar z w|^{3+\sigma+\tau+2\beta}}(1-|z|^2)^\sigma(1-|w|^2)^\tau\,dA(z)\,dA(w)\asymp\int_{\mathbf{D}}|g'(z)|^2\,dA_\beta(z).$$

For a fixed $w\in\mathbf{D}$, let φ_w be the function defined by $\varphi_w(z)=(w-z)/(1-\bar{w}z)$. We know $\varphi_w:\mathbf{D}\to\mathbf{D}$ is an analytic, one to one, and onto map. The hyperbolic distance on $\mathbf{D}$, which is Möbius invariant, is defined by

$$d(z,w)=\log\frac{1+|\varphi_w(z)|}{1-|\varphi_w(z)|}.$$

A sequence $\{z_j\}_0^\infty$ in $\mathbf{D}$ is called a d-lattice (see [R2]), if every point of $\mathbf{D}$ is within hyperbolic distance $5d$ of some z_j and no two points of this sequence are within hyperbolic distance $d/5$ of each other. Associated to each d-lattice $\{z_j\}_0^\infty$, there exists a disjoint decomposition $\{D_j\}_0^\infty$ of $\mathbf{D}$ such that

$$\{\, z\in\mathbf{D}: d(z,z_j)<d/10\,\}\subseteq D_j\subseteq\{\, z\in\mathbf{D}: d(z,z_j)<10d\,\}$$

and

$$\int_{D_j} dA(z)\asymp(1-|z_j|^2)^2$$

(see [CR] for details).

For a measure μ on $\mathbf{D}$, the Toeplitz operator, $T_\mu^{(\beta)}$, with symbol μ is defined on D_β by

$$\langle T_\mu^{(\beta)}(g),h\rangle_\beta=\int_{\mathbf{D}} g(z)\overline{h(z)}\,d\mu(z).$$

For the following theorem, proofs of the parts (a), (b), (c) and (d) can be found in [St], [RW2], [L], and [W], respectively.

Theorem C. *Suppose $\beta<1$, μ is a nonnegative measure on $\mathbf{D}$, $\{z_j\}_0^\infty$ is a d-lattice in $\mathbf{D}$ and $\{D_j\}_0^\infty$ is the corresponding disjoint decomposition of $\mathbf{D}$. Then*

(a) *$T_\mu^{(\beta)}$ is bounded if and only if μ is a β-Carleson measure.*

(b) *$T_\mu^{(\beta)}$ is compact if and only if μ is a β-Carleson measure and*

$$\|\chi_r\mu\|_\beta\to 0 \qquad \text{as } r\to 1_-.$$

(c) *If $\beta<\frac{1}{2}$, $p>0$ and $\beta_p<\frac{1}{2}$, then $T_\mu^{(\beta)}$ belongs to S_p if and only if*

$$\sum_0^\infty\bigl(\mu(D_j)(1-|z_j|^2)^{2\beta-1}\bigr)^p<\infty.$$

(d) *If $\beta>\frac{1}{2}$, $p>0$ and $\beta p>\frac{1}{2}$, then $T_\mu^{(\beta)}$ is bounded, compact or belongs to S_p if and only if μ is a finite measure on $\mathbf{D}$, i.e. $\mu(\mathbf{D})<\infty$.*

Lemma D. *Suppose $\beta>-1$ and $b>1+\frac{1}{2}(1+\beta)$. Then the operator defined by*

$$F\mapsto\int_{\mathbf{D}}\frac{F(z)}{|1-\bar{z}w|^{b}}(1-|z|^{2})^{b-2}\,dA(z)$$

is bounded on $L^{2}((1-|z|^{2})^{\beta}\,dA)$.

This result can be found in, for example, [R2] for $b>2$. For the full range, it can be proved in a similar way. See also [Z].

Remark. A consequence of Lemma D is that P_γ is bounded on $L^{2,\alpha}$ if $\gamma<\frac{1}{2}(1+\alpha)<1$. In fact $\partial u/\partial z$ is in $L^{2}(dA_\alpha)$ if $u\in L^{2,\alpha}$ and

$$\frac{\partial}{\partial w}P_\gamma(u)(w)=\int_{\mathbf{D}}\frac{\frac{\partial u}{\partial z}(z)}{(1-\bar{z}w)^{3-2\gamma}}\,dA_\gamma(z),\quad \forall u\in L^{2,\alpha}.$$

Applying Lemma D with $b=3-2\gamma$, we get the desired result.

3. Proof of Theorem 1

Because of the independent interest, we break the proof of Theorem 1 into several lemmas.

Lemma 3.1. *Suppose $\alpha,\beta<1$, $\gamma<\frac{1}{2}(1+\alpha)$ and f is analytic on $\mathbf{D}$. Then*

$$H_f^{(\gamma)}\colon D_\beta\to L^{2,\alpha}$$

is bounded, compact or belongs to S_p $(0<p<\infty)$ if and only if

$$H_f^{(\alpha)}\colon D_\beta\to L^{2,\alpha}$$

is bounded, compact or belongs to S_p $(0<p<\infty)$ respectively (compare the result in [J2]*).*

Proof. By the assumption on α and γ and the remark following Lemma D, we have that P_α and P_γ are both bounded on $L^{2,\alpha}$. Clearly

$$P_\alpha P_\gamma=P_\gamma\quad\text{and}\quad P_\gamma P_\alpha=P_\alpha.$$

Hence for $g\in P$, we have the following identity:

$$\begin{aligned}H_f^{(\alpha)}(g)-H_f^{(\gamma)}(g)&=(\bar{f}g-P_\alpha(\bar{f}g))-(\bar{f}g-P_\gamma(\bar{f}g))\\&=P_\gamma(\bar{f}g)-P_\alpha(\bar{f}g)=P_\gamma(\bar{f}g-P_\alpha(\bar{f}g))=-P_\alpha(\bar{f}g-P_\gamma(\bar{f}g)).\end{aligned}$$

This yields that

$$H_f^{(\alpha)}=H_f^{(\gamma)}-P_\alpha H_f^{(\gamma)}\quad\text{and}\quad H_f^{(\gamma)}=H_f^{(\alpha)}-P_\gamma H_f^{(\alpha)}.$$

The desired result follows. □

For a function f, define M_f to be the multiplication by f.

Lemma 3.2. *Suppose $\alpha,\beta<1$ and f is analytic on* $\mathbf{D}$. *Then $H_f^{(\alpha)}\colon D_\beta\to L^{2,\alpha}$ is bounded, compact or belongs to S_p if and only if both of the operators*

$$M_{f'}\colon D_\beta\to A^{2,1-2\alpha} \quad \text{and} \quad \widetilde{H}_f\colon A^{2,1-2\beta}\to L^2(dA_\alpha)$$

are bounded, compact or belong to S_p respectively.

Proof. Let $g\in P$. By the definition of the big Hankel operator, we have

$$\begin{aligned} H_f^{(\alpha)}(g)(w) &= (I-P_\alpha)(\bar{f}g)(w)\\ &= \overline{f(w)}g(w)-\int_{\mathbf{D}}\overline{f(z)}g(z)\,dA_\alpha(z)-\int_{\mathbf{D}}\overline{f(z)}g'(z)\overline{\frac{\partial}{\partial z}K_\alpha(z,w)}\,dA_\alpha(z).\end{aligned}$$

With this formula, it is easy to check

$$\int_{\mathbf{D}} H_f^{(\alpha)}(g)\,dA_\alpha(z)=0, \qquad \frac{\partial H_f^{(\alpha)}(g)}{\partial\overline{w}}(w)=\overline{f'(w)}g(w)$$

and

$$\frac{\partial H_f^{(\alpha)}(g)}{\partial w}(w)=\overline{f(w)}g'(w)-\int_{\mathbf{D}}\overline{f(z)}g'(z)\overline{\frac{\partial^2}{\partial z\partial\overline{w}}K_\alpha(z,w)}\,dA_\alpha(z)=\widetilde{H}_f(g')(w).$$

Notice that $g,h\in D_\beta$ if and only if $g',h'\in A^{2,1-2\beta}$. The desired result follows from the following computation:

$$\begin{aligned} \langle (H_f^{(\alpha)})^*H_f^{(\alpha)}(g),h\rangle_\alpha &= \langle H_f^{(\alpha)}(g),H_f^{(\alpha)}(h)\rangle_\alpha\\ &= \langle \widetilde{H}_f(g'),\widetilde{H}_f(h')\rangle_{L^2(dA_\alpha)}+\langle \bar{f}'g,\bar{f}'h\rangle_{L^2(dA_\alpha)}\\ &= \langle (\widetilde{H}_f)^*\widetilde{H}_f(g'),(h')\rangle_{A^{2,1-2\beta}}+\langle (M_{f'})^*M_{f'}(g),h\rangle_\beta. \ \square\end{aligned}$$

Lemma 3.3. *Let $\alpha,\beta<1$ and $0<p<\infty$. Regard $M_{f'}$ as an operator from D_β to $A^{2,1-2\alpha}$. Then*

(a) *$M_{f'}$ is bounded or compact if and only if $\|\,|f'|^2dA_\alpha\|_\beta<\infty$ or*

$$\|\chi_r|f'|^2dA_\alpha\|_\beta\to 0 \quad \text{as } r\to 1_-,$$

respectively;

(b) *If $\beta<\frac{1}{2}$, $\alpha-\beta<1$, $1/(1+\beta-\alpha)<p$ and $\beta_p<1$, then $M_{f'}\in S_p$ if and only if $f\in B_p^{1/(p+\alpha-\beta)}$.*

(c) *If $\beta<\frac{1}{2}$, $\alpha-\beta<1$ and $p\le 1/(1+\beta-\alpha)$, then $M_{f'}\in S_p$ if and only if f is constant.*

Proof. For $g,h\in P$, we have clearly

$$\langle (M_{f'})^* M_{f'}(g),h\rangle_\beta=\langle f'g,f'h\rangle_{L^2(dA_\alpha)}=\int_{\mathbf{D}} g(z)\overline{h(z)}|f'(z)|^2\,dA_\alpha(z).$$

This is equivalent to

$$(M_{f'})^* M_{f'}=T^{(\beta)}_{|f'|^2 dA_\alpha}. \tag{3.1}$$

A direct consequence of (3.1) is that $M_{f'}$, as an operator from D_β to $A^{2,1-2\alpha}$, is bounded, compact or belongs to S_p if and only if $T^{(\beta)}_{|f'|^2 dA_\alpha}$ is bounded, compact or belongs to $S_{p/2}$ respectively. Part (a) is then a consequence of Theorem C, (a) and (b).

To prove (b) and (c), let $\{z_j\}_0^\infty$ be a d-lattice in $\mathbf{D}$. Theorem C, (c) says that, for $\beta p<1$, the Toeplitz operator $T^{(\beta)}_{|f'|^2 dA_\alpha}$ is in $S_{p/2}$ if and only if

$$\sum_0^\infty\left(\int_{D_j}|f'(z)|^2\,dA_\alpha(z)(1-|z_j|^2)^{2\beta-1}\right)^{p/2}<\infty.$$

For d small enough the above inequality is equivalent to

$$\sum_0^\infty \sup_{D_j}\{|f'(z)|^p\}(1-|z_j|^2)^{p\beta-p\alpha+p}<\infty.$$

And this is a discrete version of

$$\int_{\mathbf{D}}|f'(z)|^p(1-|z|^2)^{p\beta-p\alpha+p-2}\,dA(z)<\infty. \tag{3.2}$$

If $p>1/(1+\beta-\alpha)$, then (3.2) is just $\|f\|^p_{B_p^{1/p+\alpha-\beta}}<\infty$. If $p\le 1/(1+\beta-\alpha)$, then (3.2) is finite for the analytic function f' if and only if $f'=0$, that is $f=$ constant. □

By Theorem A, Lemmas 3.1, 3.2 and 3.3, we see that to complete the proof of Theorem 1 it is enough to show the following (compare a similar result in [RW1]).

Lemma 3.4. *Suppose $\alpha,\beta,\alpha-\beta<1$ and f is analytic on $\mathbf{D}$. Regard $M_{f'}$ as an operator from D_β to $A^{2,1-2\alpha}$. Then*

(a) *$M_{f'}$ is bounded or compact implies f is in $D_\alpha\cap B_\infty^{\alpha-\beta}$ or in $D_\alpha\cap b_\infty^{\alpha-\beta}$, respectively.*

(b) *If $\beta<\frac{1}{2}$ and $p>1/(1+\beta-\alpha)$, then $M_{f'}\in S_p$ implies $f\in B_p^{1/p+\alpha-\beta}$.*

Proof. The constant 1 is in D_β, hence the boundedness of $M_{f'}$ implies $f' \in A^{2,1-2\alpha}$, that is $f \in D_\alpha$.

For a fixed $w \in \mathbf{D}$, let

$$e_w(z) = \frac{(1-|w|^2)^{3/2-\alpha}}{(1-\bar{w}z)^{3-2\alpha}} \quad \text{and} \quad f_w(z) = (1-|w|^2)^{1/2} \frac{z}{(1-\bar{w}z)^{1-\beta}}.$$

Straightforward computations show that the norm estimates $\|e_w\|_{A^{2,1-2\alpha}} \asymp 1$ and $\|f_w\|_\beta \asymp 1$ are independent of w.

Since $(1-\bar{w}z)^{2\alpha-3}$ is the reproducing kernel of $A^{2,1-2\alpha}$, we have

$$\begin{aligned}\langle M_{f'}(f_w), e_w \rangle_{A^{2,1-2\alpha}} &= (1-|w|^2)^{2-\alpha} \int_{\mathbf{D}} \frac{z f'(z)\, dA_\alpha(z)}{(1-\bar{w}z)^{1-\beta}(1-w\bar{z})^{3-2\alpha}} \\ &= w f'(w)(1-|w|^2)^{1+\beta-\alpha}.\end{aligned}$$

This yields that

$$|f'(w)|(1-|w|^2)^{1+\beta-\alpha} \le C \|M_{f'}(f_w)\|_{A^{2,1-2\alpha}}.$$

Hence, if $M_{f'}$ is bounded, we get

$$|f'(w)|(1-|w|^2)^{1+\beta-\alpha} \le C\|M_{f'}\|.$$

This is $f \in B_\infty^{\alpha-\beta}$.

Notice that $f_w \to 0$ in D_β as $|w| \to 1_-$. Hence the compactness of $M_{f'}$ yields that

$$|f'(w)|(1-|w|^2)^{1+\beta-\alpha} \le C\|M_{f'}(f_w)\|_{A^{2,1-2\alpha}} \to 0 \quad \text{as } |w| \to 1_-.$$

Thus $f \in b_\infty^{\alpha-\beta}$.

Part (b) is true for $1/(1+\beta-\alpha) < p \le 2$ $(<1/\beta)$ by Lemma 3.3. For $p>2$, we need to estimate the $B_p^{1/p+\alpha-\beta}$norm of f. The discrete version of this norm is easy to work with in our case. For d small enough, we choose a d-lattice $\{z_j\}_0^\infty$ in $\mathbf{D}$ such that

$$\begin{aligned}\|f\|^p_{B_p^{1/p+\alpha-\beta}} &= \int_{\mathbf{D}} |f'(z)|^p (1-|z|^2)^{p+p\beta-p\alpha-2}\, dA(z) \\ &\asymp \sum_0^\infty |f'(z_j)|^p (1-|z_j|^2)^{p+p\beta-p\alpha}.\end{aligned}$$

On the other hand, a result in [R2] says that if $\{z_j\}_0^\infty$ is a d-lattice in $\mathbf{D}$, then $\{f_{z_j}\}_0^\infty$ and $\{e_{z_j}\}_0^\infty$ are, respectively, the images of some orthonormal sequences in D_β and $A^{2,1-2\alpha}$ under bounded maps. Hence if $p \ge 1$, then (see [RS2])

$$\sum_0^\infty |\langle M_{f'}(f_{z_j}), e_{z_j} \rangle_{A^{2,1-2\alpha}}|^p \le C\|M_{f'}\|_p^p.$$

This, together with the previous computation, yields

$$\sum_{0}^{\infty} |f'(z_j)|^p (1-|z_j|^2)^{p+p\beta-p\alpha} \leq C\|M_{f'}\|_p^p.$$

And thus $f\in B_p^{1/p+\alpha-\beta}$ for $p>2$. $\square$

4. Proof of Theorem 2

In this section, we assume, for convenience, that f has the expansion $f(z)=\sum_0^\infty f_k z^k$.

Lemma 4.1. *Suppose $\alpha,\beta,\alpha-\beta<1$, $\gamma<\frac{1}{2}(1+\alpha)$ and f is analytic on $\mathbf{D}$. Then $h_f^{(\gamma)}: D_\beta \to \overline{D_\alpha}$ is bounded or compact implies f is in $D_\alpha \cap B_\infty^{\alpha-\beta}$ or in $D_\alpha \cap b_\infty^{\alpha-\beta}$, respectively.*

Proof. The following proof is similar to the proof of Lemma 3.4. The constant 1 is in D_β and it is clear that $h_f^{(\gamma)}=f$, hence the boundedness of $h_f^{(\gamma)}$ implies $f\in D_\alpha$.

Let $[\beta]$ be the greatest integer in β and set $n=1-[\beta]$. For a fixed $w\in\mathbf{D}$, let

$$f_w(z)=\frac{(1-|w|^2)^{-1/2+\beta+n} z^n}{(1-\bar{w}z)^n} \quad \text{and} \quad g_w(z)=\frac{(1-|w|^2)^{3/2-\alpha}}{\bar{w}(2-2\alpha)}\left(\frac{1}{(1-\bar{w}z)^{2-2\alpha}}-1\right).$$

Straightforward computation yields that the estimates $\|f_w\|_\beta \asymp 1$ and $\|g_w\|_\alpha \asymp 1$ are independent of w.

It is easy to check

$$\begin{aligned}
\langle h_f^{(\gamma)}(f_w), \overline{g_w}\rangle_\alpha &= \overline{\langle P_\gamma(f\overline{f_w}), g_w\rangle_\alpha} \\
&= \overline{\langle [P_\gamma(f\overline{f_w})]', g_w'\rangle_{L^2(dA_\alpha)}} = (1-|w|^2)^{3/2-\alpha}\overline{P_\gamma(f\overline{f_w})'(w)} \\
&= \frac{(1-|w|^2)^{n+1+\beta-\alpha}}{(3-2\gamma)(4-2\gamma)\dots(n+2-2\gamma)}\overline{f^{(n+1)}(w)}.
\end{aligned}$$

This yields the estimate

$$|f^{(n+1)}(w)|(1-|w|^2)^{n+1+\beta-\alpha} \leq C\|h_f^{(\gamma)}(f_w)\|_\alpha.$$

Hence, if $h_f^{(\gamma)}$ is bounded, then

$$|f^{(n+1)}(w)|(1-|w|^2)^{n+1+\beta-\alpha} \leq C\|h_f^{(\gamma)}\|.$$

That is $f\in B_\infty^{\alpha-\beta}$.

Notice that $f_w\to 0$ weakly, as $|w|\to 1_-$. Hence the compactness of $h_f^{(\gamma)}$ yields

$$\|h_f^{(\gamma)}(f_w)\|_\alpha\to 0 \quad \text{as } |w|\to 1_-.$$

This implies

$$|f^{(n+1)}(w)|(1-|w|^2)^{n+1+\beta-\alpha}\to 0 \quad \text{as } |w|\to 1_-.$$

Thus $f\in b_\infty^{\alpha-\beta}$. $\square$

We will see in the following discussion that the second part of Theorem 2 can be reduced to the results by Peller, [P1] ($p\geq 1$) and Semmes, [S] ($p<1$) (see also [R2]).

For $\sigma<1$ and $k=0,1,2,\ldots$, set $\gamma_{k,\sigma}=\|z^k\|_\sigma$, and find explicitly: $\gamma_{0,\sigma}=1$ and for $k\geq 1$

$$\gamma_{k,\sigma}^2=k^2\int_{\mathbf{D}}|z^{k-1}|^2\,dA_\sigma(z)=\frac{k^2\Gamma(k)\Gamma(3-2\alpha)}{\Gamma(k+2-2\alpha)}.$$

Thus

$$\gamma_{k,\sigma}\asymp(k+1)^\sigma,\quad k=0,1,2,\ldots\,. \tag{4.1}$$

The sequences $\{z^m/\gamma_{m,\beta}\}_0^\infty$ and $\{\bar z^n/\gamma_{n,\alpha}\}_0^\infty$ are clearly orthonormal bases of D_β and $\overline{D_\alpha}$ respectively. The matrix elements of $h_f^{(\gamma)}\colon D_\beta\to\overline{D_\alpha}$ related to these bases can be computed as follows.

$$\begin{aligned}
\langle h_f^{(\gamma)}(z^m/\gamma_{m,\beta})(w),\overline{w}^n/\gamma_{n,\alpha}\rangle_\alpha &= \gamma_{m,\beta}^{-1}\gamma_{n,\alpha}^{-1}\overline{\langle P_\gamma(f(z)\bar z^m)(w),w^n\rangle_\alpha}\\
&=\gamma_{m,\beta}^{-1}\gamma_{n,\alpha}^{-1}\biggl(\int_{\mathbf{D}}\overline{f(z)}z^m\,dA_\gamma(z)\int_{\mathbf{D}}\overline{w}^n\,dA_\alpha(w)\\
&\quad+\int_{\mathbf{D}}\int_{\mathbf{D}}\frac{n\overline{f'(z)}z^m w^{n-1}}{(1-z\overline{w})^{3-2\gamma}}\,dA_\gamma(z)\,dA_\alpha(w)\biggr)\\
&=\gamma_{m,\beta}^{-1}\gamma_{n,\alpha}^{-1}\biggl(\int_{\mathbf{D}}\overline{f(z)}z^m\,dA_\gamma(z)\int_{\mathbf{D}}\overline{w}^n\,dA_\alpha(w)\\
&\quad+\frac{n\gamma_{n,\alpha}^2}{\gamma_{n,\gamma}^2}\int_{\mathbf{D}}\overline{f'(z)}z^{m+n-1}\,dA_\gamma(z)\biggr)\\
&=\frac{\gamma_{m+1,\gamma}^2}{(m+1)^2\gamma_{m,\beta}\gamma_{n,\alpha}}\delta_{n,0}\overline{f_m}\\
&\quad+\frac{n\gamma_{m+n,\gamma}^2\gamma_{n,\alpha}}{(m+n)\gamma_{m,\beta}\gamma_{n,\gamma}^2}\overline{f_{m+n}}.
\end{aligned}$$

(Here $\delta_{n,0}=1$ if $n=0$; 0 otherwise.) Hence $h_f^{(\gamma)}\in S_p^{\beta\alpha}$ if and only if the matrix

$$(4.2)\qquad \left(\frac{\gamma_{m+1,\gamma}^2}{(m+1)^2\gamma_{m,\beta}\gamma_{n,\alpha}}\delta_{n,0}\overline{f_m}\right)_{m,n=0}^{\infty}+\left(\frac{n\gamma_{m+n,\gamma}^2\gamma_{n,\alpha}}{(m+n)\gamma_{m,\beta}\gamma_{n,\gamma}^2}\overline{f_{m+n}}\right)_{m,n=0}^{\infty}$$

defines an S_p operator on l^2. The first matrix of (4.2) is a rank one operator (because all columns are zero except the first one) and its S_p norm estimate is

$$\begin{aligned}\left\|\left(\frac{\gamma_{m+1,\gamma}^2}{(m+1)^2\gamma_{m,\beta}\gamma_{n,\alpha}}\delta_{n,0}\overline{f_m}\right)_{m,n=0}^{\infty}\right\|_{S_p}^2 &= \left\|\left(\frac{\gamma_{m+1,\gamma}^2}{(m+1)^2\gamma_{m,\beta}\gamma_{n,\alpha}}\delta_{n,0}\overline{f_m}\right)_{m,n=0}^{\infty}\right\|^2\\
&\le C\sum_0^\infty \frac{|f_m|^2}{(m+1)^{4+2\beta-4\gamma}}\\
&\le C\int_{\mathbf{D}}|f'(z)|^2(1-|z|^2)^{5+2\beta-4\gamma}\,dA(z)\\
\left(\text{since } \gamma<\frac{1+\alpha}{2}\right)\quad &\le C\int_{\mathbf{D}}|f'(z)|^2(1-|z|^2)^{2+2\beta-2\alpha}\,dA(z)\\
&\le C\|f\|_{B_\infty^{\alpha-\beta}}^2.\end{aligned}$$

Using (4.1), we know that the second matrix of (4.2) defines an S_p operator on l^2 if and only if the matrix

$$\left((m+1)^{-\beta}(n+1)^{1+\alpha-2\gamma}\frac{\Gamma(m+n)\Gamma(3-2\gamma)}{\Gamma(m+n+2-2\gamma)}(m+n)f_{m+n}\right)_{m,n=0}^{\infty}$$

does. It is then a consequence of the results by Peller [P1], and Semmes [S], that for $\beta<\frac12$, $\gamma<\frac12(1+\alpha)$, $p>0$ and $\beta p<1$, the above matrix defines an S_p operator on l^2 if and only if $\sum_0^\infty kf_kz^k\in B_p^{1/p+\alpha-\beta-1}$, i.e., $f\in B_p^{1/p+\alpha-\beta}$. Theorem 2, (2) is hence obtained by these facts and Lemma 4.1.

To prove the first part of Theorem 2, we notice that $h_f^{(\gamma)}\colon D_\beta\to L^{2,\alpha}$ is bounded or compact if and only if the operator defined on l^2 by the second matrix of (4.2) is bounded or compact respectively. This matrix operator is clearly corresponding to the operator $\frac{\partial}{\partial\bar z}h_f^{(\gamma)}$ which maps D_β to $\overline{A^{2,1-2\alpha}}$. It is easy to verify that $\overline{\frac{\partial}{\partial\bar z}h_f^{(\gamma)}}$ has the integral expression

$$\overline{\frac{\partial}{\partial\bar z}h_f^{(\gamma)}(g)(w)}=\int_{\mathbf{D}}\frac{f'(z)\overline{g(z)}}{(1-\bar zw)^{3-2\gamma}}\,dA_\gamma(z)=\widetilde{P}_\gamma(f'\bar g)=\widetilde{P}_\gamma M_{f'}(\bar g),\quad \forall g\in P.$$

Clearly $M_{f'}$ is bounded or compact from $\overline{D_\beta}$ to $L^2(dA_\alpha)$ if and only if $M_{f'}$ is bounded or compact from D_β to $A^{2,1-2\alpha}$ respectively. Since $\widetilde{P}_\gamma$ is a bounded operator on $L^2(dA_\alpha)$ if $\gamma<\frac12(1+\alpha)<1$ (see the remark following Lemma D), we get the "if" part immediately by Theorem C, (a) and (b).

We need the following lemma to continue our discussion.

Lemma 4.2. *Suppose $\alpha, \beta<1$ and $\gamma<\frac{1}{2}(1+\alpha)$. If the operator $\widetilde{P}_\gamma M_{f'}\colon \overline{D_\beta}\to A^{2,1-2\alpha}$ is bounded or compact, then the operator $\widetilde{P}_{\gamma-1/2}M_{f'}\colon \overline{D_\beta}\to A^{2,1-2\alpha}$ is bounded or compact, respectively.*

Proof. We will prove the lemma by showing that the boundedness or compactness of $\widetilde{P}_\gamma M_{f'}$ implies the boundedness or compactness of $\widetilde{P}_{\gamma-1/2}M_{f'}-\widetilde{P}_\gamma M_{f'}$, respectively.

By previous computation for obtaining matrix (4.2), we see that the matrix of the operator $\widetilde{P}_\sigma M_{f'}\colon \overline{D_\beta}\to A^{2,1-2\alpha}$ related to the bases $\{\bar{z}^m/\gamma_{m,\beta}\}_0^\infty$ in $\overline{D_\beta}$ and $\{(n+1)z^n/\gamma_{n+1,\alpha}\}_0^\infty$ in $A^{2,1-2\alpha}$ is

$$\mathbf{M}_\sigma\left(\frac{(n+1)\gamma^2_{m+n+1,\sigma}\gamma_{n+1,\alpha}}{(m+n+1)\gamma_{m,\beta}\gamma^2_{n+1,\sigma}}f_{m+n+1}\right)_{m,n=0}^{\infty}.$$

Transferring the difference of the operators $\widetilde{P}_{\gamma-1/2}M_{f'}$ and $\widetilde{P}_\gamma M_{f'}$ into the difference of their corresponding matrices, we have

$$\mathbf{M}_{\gamma-1/2}-\mathbf{M}_\gamma=\left(\left(\frac{\gamma^2_{m+n+1,\gamma-1/2}\gamma^2_{n+1,\gamma}}{\gamma^2_{n+1,\gamma-1/2}\gamma^2_{m+n+1,\gamma}}-1\right)\frac{(n+1)\gamma^2_{m+n+1,\gamma}\gamma_{n+1,\alpha}}{(m+n+1)\gamma_{m,\beta}\gamma^2_{n+1,\gamma}}f_{m+n+1}\right)_{m,n=0}^{\infty}.$$

Using the identity

$$\begin{aligned}\frac{\gamma^2_{m+n+1,\gamma-1/2}\gamma^2_{n+1,\gamma}}{\gamma^2_{n+1,\gamma-1/2}\gamma^2_{m+n+1,\gamma}}-1&=\frac{n+4-2\gamma}{m+n+4-2\gamma}-1\\&=\frac{m}{m+n+1}\left(-1+\frac{3-2\gamma}{m+n+4-2\gamma}\right)\\&=\frac{m}{m+n+1}\left(-1+\frac{\gamma^2_{m+n+1,\gamma-1/2}}{\gamma^2_{m+n+1,\gamma}}\right),\end{aligned}$$

we get

$$\begin{aligned}\mathbf{M}_{\gamma-1/2}-\mathbf{M}_\gamma&=\left(\frac{-m(n+1)\gamma_{n+1,\alpha}}{\gamma_{m,\beta}\gamma^2_{n+1,\gamma}}\frac{\gamma^2_{m+n+1,\gamma}}{(m+n+1)^2}f_{m+n+1}\right)_{m,n=0}^{\infty}\\&\quad+\left(\frac{m(n+1)\gamma_{n+1,\alpha}}{\gamma_{m,\beta}\gamma^2_{n+1,\gamma}}\frac{\gamma^2_{m+n+1,\gamma-1/2}}{(m+n+1)^2}f_{m+n+1}\right)_{m,n=0}^{\infty}.\end{aligned}$$

By Peller's result, [P1], the first matrix is bounded or compact on l^2 if and only if f is in $B_\infty^{\alpha-\beta}$ or in $b_\infty^{\alpha-\beta}$, respectively; and the second matrix is bounded or compact on l^2 if and only if f is in $B_\infty^{\alpha-\beta-1}$ or in $b_\infty^{\alpha-\beta-1}$, respectively. Notice that $B_\infty^{\alpha-\beta}\subset B_\infty^{\alpha-\beta-1}$

and $b_\infty^{\alpha-\beta}\subset b_\infty^{\alpha-\beta-1}$, we have hence $\widetilde{P}_{\gamma-1/2}M_{f'}-\widetilde{P}_\gamma M_{f'}\colon \overline{D_\beta}\to A^{2,1-2\alpha}$ is bounded or compact if and only if f belongs to $B_\infty^{\alpha-\beta}$ or $b_\infty^{\alpha-\beta}$, respectively.

An implicit consequence of the proof of Lemma 4.1 is that if $\widetilde{P}_\gamma M_{f'}\colon \overline{D_\beta}\to A^{2,1-2\alpha}$ or $\widetilde{P}_{\gamma-1/2}M_{f'}\colon \overline{D_\beta}\to A^{2,1-2\alpha}$ is bounded or compact, then f is in $B_\infty^{\alpha-\beta}$ or in $b_\infty^{\alpha-\beta}$, respectively. Lemma 4.2 hence follows. $\square$

To prove the "only if" part of Theorem 2, (2), we compare $\frac{\partial}{\partial\bar z}h_f^{(\gamma-1/2)}$ and $M_{\overline{f'}}$ by considering their difference

$$M_{\overline{f'}}-\frac{\partial}{\partial\bar z}h_f^{(\gamma-1/2)}\colon D_\beta\to L^2(dA_\alpha).$$

We need to estimate this difference.

Let $g\in D_\beta$, $0\le r,s<1$, we have

$$M_{\overline{f'}}(g)(z)-\frac{\partial}{\partial\bar z}h_f^{(\gamma-1/2)}(g)(z)=\overline{\int_{\mathbf D}\frac{f'(z)(\overline{g(w)}-\overline{g(z)})}{(1-\bar zw)^{4-2\gamma}}\,dA_{\gamma-1/2}(z)}.$$

Then

$$\begin{aligned}\left|M_{\overline{f'}}(g)(z)-\frac{\partial}{\partial\bar z}h_f^{(\gamma-1/2)}(g)(z)\right|^2&=\left|\int_{\mathbf D}\frac{f'(z)(\overline{g(w)}-\overline{g(z)})}{(1-\bar zw)^{4-2\gamma}}\,dA_{\gamma-1/2}(z)\right|^2\\&=\left|\int_{\mathbf D\setminus r\mathbf D}+\int_{r\mathbf D}\right|^2\le 2\left(\left|\int_{\mathbf D\setminus r\mathbf D}\right|^2+\left|\int_{r\mathbf D}\right|^2\right),\end{aligned}$$

hence

$$\begin{aligned}\left\|M_{\overline{f'}}(g)-\frac{\partial}{\partial\bar z}h_f^{(\gamma-1/2)}(g)\right\|^2_{L^2(dA_\alpha)}&\le 2\left(\int_{\mathbf D}\left|\int_{\mathbf D\setminus r\mathbf D}\right|^2+\int_{\mathbf D}\left|\int_{r\mathbf D}\right|^2\right)\\&=2\left(\int_{\mathbf D}\left|\int_{\mathbf D\setminus r\mathbf D}\right|^2+\int_{\mathbf D\setminus s\mathbf D}\left|\int_{r\mathbf D}\right|^2+\int_{s\mathbf D}\left|\int_{r\mathbf D}\right|^2\right)\\&=2(\mathrm{I}(r,g)+\mathrm{II}(r,s,g)+\mathrm{III}(r,s,g)).\end{aligned}$$

We will gather the estimates of the $L^2(dA_\alpha)$ norm of $M_{\overline{f'}}(g)-\frac{\partial}{\partial\bar z}h_f^{(\gamma-1/2)}(g)$ in the following lemma (compare the result in [RW1]).

Lemma 4.3. *Suppose $\alpha<1$, $\beta\le\frac12$, $\alpha-\beta<1$, $\gamma<\frac12(1+\alpha)$, $0\le r,s<1$, $f\in B_\infty^{\alpha-\beta}$ and $g\in D_\beta$. Then*

$$\left\|M_{\overline{f'}}(g)-\frac{\partial}{\partial\bar z}h_f^{(\gamma-1/2)}(g)\right\|^2_{L^2(dA_\alpha)}\le 2(\mathrm{I}(r,g)+\mathrm{II}(r,s,g)+\mathrm{III}(r,s,g)),$$

where

$$\begin{aligned}\mathrm{I}(r,g)&=\sup_{|z|\geq r}\{|f'(z)|^2(1-|z|^2)^{2+2\beta-2\alpha}\}\|g\|_\beta^2,\\ \mathrm{II}(r,s,g)&=C(1-r)^{-8+2\gamma-2\beta-2\alpha}(1-s)^{2-2\alpha}\|f\|^2_{B_\infty^{\alpha-\beta}}\|g\|_\beta^2,\\ \mathrm{III}(r,s,g)&\leq C(1-r)^{-6}(1-s)^{-6}\|f\|^2_{B_\infty^{\alpha-\beta}}(\sup_{|w|\leq s}\{|g(w)|^2\}+\sup_{|z|\leq r}\{|g(z)|^2\}).\end{aligned}$$

Remark. If $r=0$, then II=III=0.

The following lemma will be employed to estimate II.

Lemma 4.4. *Suppose $\alpha-\beta<1$, $g\in D_\beta$ and $0<s<1$. Then*

$$\int_{\mathbf{D}\setminus s\mathbf{D}}|g(w)|^2\,dA_\alpha(w)\leq C(1-s)^{2-2\alpha+2\beta}\|g\|_\beta^2.$$

Proof. Let $g\in D_\beta$ and $g(z)=\sum_0^\infty g_nz^n$. It is easy to check

$$\|g\|_\beta^2\asymp\sum_0^\infty(1+n)^{2\beta}|g_n|^2.$$

Hence we get the pointwise estimate

$$|g(w)|^2\leq\left(\sum_0^\infty(1+n)^{2\beta}|g_n|^2\right)\left(\sum_0^\infty(1+n)^{-2\beta}|w_n|^2\right)\leq C(1-|w|^2)^{-1+2\beta}\|g\|_\beta^2.$$

This is enough to obtain the desired result. □

Proof of Lemma 4.3. By the Schwarz inequality, we have

$$\begin{aligned}\mathrm{I}(r,g)&=\int_{\mathbf{D}}\left|\int_{\mathbf{D}\setminus r\mathbf{D}}\frac{f'(z)(\overline{g(w)}-\overline{g(z)})}{(1-\bar zw)^{4-2\gamma}}\,dA_{\gamma-1/2}(z)\right|^2dA_\alpha(w)\\ &\leq\int_{\mathbf{D}}\left\{\int_{\mathbf{D}\setminus r\mathbf{D}}\frac{|f'(z)|^2}{|1-\bar zw|^{4-2\gamma}}(1-|z|^2)^{3-2\gamma-\alpha+2\beta}\,dA(z)\right.\\ &\quad\left.\times\int_{\mathbf{D}\setminus r\mathbf{D}}\frac{|g(w)-g(z)|^2}{|1-\bar zw|^{4-2\gamma}}(1-|z|^2)^{1+\alpha-2\beta-2\gamma}\,dA(z)\right\}dA_\alpha(w)\\ &\leq\sup_{|z|\geq r}\{|f'(z)|^2(1-|z|^2)^{2+2\beta-2\alpha}\}\int_{\mathbf{D}}\left\{\int_{\mathbf{D}}\frac{(1-|z|^2)^{1+\alpha-2\gamma}}{|1-\bar zw|^{4-2\gamma}}\,dA(z)\right.\\ &\quad\left.\times\int_{\mathbf{D}}\frac{|g(w)-g(z)|^2}{|1-\bar zw|^{4-2\gamma}}(1-|z|^2)^{1+\alpha-2\beta-2\gamma}\,dA(z)\right\}dA_\alpha(w).\end{aligned}$$

Since

$$\int_{\mathbf{D}} \frac{(1-|z|^2)^{1+\alpha-2\gamma}}{|1-\bar{z}w|^{4-2\gamma}}\, dA(z) \asymp (1-|w|^2)^{-1+\alpha},$$

we can continue the estimate by

$$\begin{aligned}
&\le \sup_{|z|\ge r} \{|f'(z)|^2(1-|z|^2)^{2+2\beta-2\alpha}\} \\
&\quad\times \int_{\mathbf{D}}\int_{\mathbf{D}} \frac{|g(w)-g(z)|^2}{|1-\bar{z}w|^{4-2\gamma}}(1-|z|^2)^{1+\alpha-2\beta-2\gamma}\, dA(z)(1-|w|^2)^{-\alpha}\, dA(w).
\end{aligned}$$

The double integral above is comparable to $\|g\|_\beta^2$ by Theorem B. Hence we get the desired estimate for $I(r,s,g)$.

Again by the Schwarz inequality, we have

$$\begin{aligned}
\mathrm{II}(r,s,g) &= \int_{\mathbf{D}\setminus s\mathbf{D}} \left| \int_{r\mathbf{D}} \frac{f'(z)(\overline{g(w)}-\overline{g(z)}\,)}{(1-\bar{z}w)^{4-2\gamma}}\, dA_{\gamma-1/2}(z) \right|^2 dA_\alpha(w) \\
&\le \sup_{|z|\le r} \{|f'(z)|^2(1-|z|^2)^{2+2\beta-2\alpha}\} \\
&\quad\times \int_{\mathbf{D}\setminus s\mathbf{D}}\int_{r\mathbf{D}} \frac{|g(w)-g(z)|^2}{(1-|z|)^{8-4\gamma}}(1-|z|^2)^{-2\gamma-2\beta-2\alpha}\, dA(z)\, dA_\alpha(w) \\
&\le \sup_{|z|\le r} \{|f'(z)|^2(1-|z|^2)^{2+2\beta-2\alpha}\}(1-r)^{-8+2\gamma-2\beta-2\alpha} \\
&\quad\times \int_{\mathbf{D}\setminus s\mathbf{D}}\int_{r\mathbf{D}} (|g(w)|^2+|g(z)|^2)\, dA(z)\, dA_\alpha(w).
\end{aligned}$$

Using Lemma 4.4, we get

$$\begin{aligned}
&\int_{\mathbf{D}\setminus s\mathbf{D}}\int_{r\mathbf{D}} (|g(w)|^2+|g(z)|^2)\, dA(z)\, dA_\alpha(w) \\
&\qquad\le \int_{\mathbf{D}\setminus s\mathbf{D}} \left(|g(w)|^2 + \int_{r\mathbf{D}} |g(z)|^2 \right) dA(z)\, dA_\alpha(w) \\
&\qquad\le C((1-s)^{2-2\alpha+2\beta}\|g\|_\beta^2 + (1-s)^{2-2\alpha}\|g\|_\beta^2) \\
&\qquad\le C(1-s)^{2-2\alpha}\|g\|_\beta^2,
\end{aligned}$$

hence

$$\mathrm{II}(r,s,g) \le C(1-r)^{-8+2\gamma-2\beta-2\alpha}(1-s)^{2-2\alpha}\|f\|^2_{B_\infty^{\alpha-\beta}}\|g\|_\beta^2.$$

Similarly

$$\begin{aligned}\mathrm{III}(r,s,g)&=\int_{s\mathbf{D}}\left|\int_{r\mathbf{D}}\frac{f'(z)(\overline{g(w)}-\overline{g(z)})}{(1-\bar{z}w)^{4-2\gamma}}\,dA_{\gamma-1/2}(z)\right|^2 dA_\alpha(w)\\&\leq\sup_{|z|\leq r}\{|f'(z)|^2(1-|z|^2)^{2+2\beta-2\alpha}\}\\&\quad\times\int_{s\mathbf{D}}\int_{r\mathbf{D}}\frac{|g(w)-g(z)|^2}{(1-rs)^{8-4\gamma}}(1-|z|^2)^{-2\gamma-2\beta-2\alpha}\,dA(z)\,dA_\alpha(w)\\&\leq C\|f\|^2_{B_\infty^{\alpha-\beta}}(1-r)^{-6}(1-s)^{-6}\big(\sup_{|w|\leq s}\{|g(w)|^2\}+\sup_{|z|\leq r}\{|g(z)|^2\}\big).\end{aligned}$$

The proof is complete. □

The "only if" part of Theorem 2, (2) can now be proved as follows. Suppose first $h_f^{(\gamma)}$ is bounded from D_β to $L^{2,\alpha}$. Then by Lemma 4.2, $\frac{\partial}{\partial\bar{z}}h_f^{(\gamma-1/2)}$ is bounded from D_β to $L^2(dA_\alpha)$. By Lemma 4.3, ($r{=}0$), we have for any $g\in D_\beta$

$$\begin{aligned}\|M_{\overline{f'}}(g)\|_{L^2(dA_\alpha)}&\leq\left\|M_{\overline{f'}}(g)-\frac{\partial}{\partial\bar{z}}h_f^{(\gamma-1/2)}(g)\right\|_{L^2(dA_\alpha)}+\left\|\frac{\partial}{\partial\bar{z}}h_f^{(\gamma-1/2)}(g)\right\|_{L^2(dA_\alpha)}\\&\leq[I(0,g)]^{1/2}+\|h_f^{(\gamma-1/2)}\|\,\|g\|_\beta\leq C(\|f\|_{B_\infty^{\alpha-\beta}}+\|h_f^{(\gamma)}\|)\|g\|_\beta.\end{aligned}$$

Hence by Lemma 4.1, we get

$$\int_{\mathbf{D}}|g(z)|^2|f'(z)|^2\,dA_\alpha(z)=\|M_{\overline{f'}}(g)\|^2_{L^2(dA_\alpha)}\leq C\|h_f^{(\gamma)}\|^2\|g\|^2_\beta.$$

Now suppose $h_f^{(\gamma)}$ is compact from D_β to $L^{2,\alpha}$ and $\{g_j\}_{j\geq0}$ is a sequence in D_β with $\|g_j\|_\beta\leq1$ and $g_j\to0$ weakly in D_β as $j\to\infty$. We want to show that $M_{\overline{f'}}(g_j)\to0$ in $L^2(dA_\alpha)$ as $j\to\infty$. Notice that

$$\|M_{\overline{f'}}(g_j)\|_{L^2(dA_\alpha)}\leq\left\|M_{\overline{f'}}(g_j)\frac{\partial}{\partial\bar{z}}h_f^{(\gamma-1/2)}(g_j)\right\|_{L^2(dA_\alpha)}+\left\|\frac{\partial}{\partial\bar{z}}h_f^{(\gamma-1/2)}(g_j)\right\|_{L^2(dA_\alpha)}.$$

By Lemma 4.3, we get

$$\begin{aligned}\|M_{\overline{f'}}(g_j)\|^2_{L^2(dA_\alpha)}&\leq4(I(r,g_j)+II(r,s,g_j)+III(r,s,g_j))\\&\quad+2\left\|\frac{\partial}{\partial\bar{z}}h_f^{(\gamma-1/2)}(g_j)\right\|^2_{L^2(dA_\alpha)}.\end{aligned}$$

By Lemma 4.2, the compactness of $h_f^{(\gamma)}$ from D_β to $L^{2,\alpha}$ implies the compactness of $h_f^{(\gamma-1/2)}$ from D_β to $L^{2,\alpha}$. Hence the last term of the above inequality converges to zero as $j\to\infty$. By Lemmas 4.1 and 4.2, we have $I(r,g_j)\to0$ uniformly for j as $r\to1_-$. For fixed $r\in(0,1)$, we have clearly $II(r,s,g_j)\to0$ uniformly for j as $s\to1_-$. Since $g_j\to0$ weakly in D_β as $j\to\infty$, we have $III(r,s,g_j)\to0$, for fixed r and s in $(0,1)$, as $j\to\infty$. These facts show that

$$\|M_{\overline{f'}}(g_j)\|^2_{L^2(dA_\alpha)}\to0\quad\text{as }j\to\infty.$$

5. Proof of Theorem 3

We know from Lemmas 3.1, 3.2 and 3.4 that $H_f^{(\gamma)}: D_\beta \to L^{2,\alpha}$ is bounded, compact or belongs to S_p if and only if the Toeplitz operator $T^{(\beta)}_{|f'|^2 dA_\alpha}$ is bounded, compact or belongs to $S_{p/2}$, respectively. We know also by Theorem C (d), for $\beta, p\beta > \frac{1}{2}$ that these properties are equivalent to

$$\int_{\mathbf{D}} |f'(z)|^2 \, dA_\alpha(z) < \infty,$$

which means that $f \in D_\alpha$.

For the small Hankel operator, from Section 4, we know that $h_f^{(\alpha)}: D_\beta \to L^{2,\alpha}$ is bounded, compact or belongs to S_p if and only if the operator defined on l^2 by the matrix (4.2) is bounded, compact or belongs to S_p. As we saw in Section 4, the S_p norm of the operator corresponding to the first matrix of (4.2) is dominated by

$$C \int_{\mathbf{D}} |f'(z)|^2 (1-|z|^2)^{2+2\beta-2\alpha} \, dA(z).$$

This can be further estimated by (since we assume $\frac{1}{2} < \beta < 1$ in Theorem 3)

$$\leq C \int_{\mathbf{D}} |f'(z)|^2 (1-|z|^2)^{3-2\alpha} \, dA(z) \leq C \int_{\mathbf{D}} |f'(z)|^2 (1-|z|^2)^2 \, dA(z).$$

Thus if f is in D_α, then the operator corresponding to the first matrix of (4.2) is in S_p for any $0 < p < \infty$. If $p\beta > 1$, a result in [W] yields that the operator defined on l^2 by the second matrix of (4.2) is bounded, compact or belongs to S_p if and only if $f \in D_\alpha$.

6. Proof of Theorem 4 and some further discussion

For $\alpha < 1$, $\beta = \frac{1}{2}$ and $\gamma < \frac{1}{2}(1+\alpha)$, it is easy to compute and get that

$$\sum_0^\infty \left\| H_f^{(\gamma)}\left(\frac{z^k}{\gamma_{k,\beta}}\right)\right\|_\alpha^2, \sum_0^\infty \left\| h_f^{(\gamma)}\left(\frac{z^k}{\gamma_{k,\beta}}\right)\right\|_\alpha^2 \asymp \int_{\mathbf{D}} |f'(z)|^2 (1-|z|^2)^{\beta-\alpha} \log\frac{1}{1-|z|^2} \, dA(z).$$

Hence we have (see also [W]):

Lemma 6.1. *Suppose $\alpha<1$, $\beta=\frac{1}{2}$ and $\gamma<\frac{1}{2}(1+\alpha)$. Then $H_f^{(\gamma)}$ or $h_f^{(\gamma)}$ is in $S_2^{\beta\alpha}$ if and only if $f\in BL_2^\alpha$.*

To prove Theorem 4, (1), we apply the interpolation theory to the map

$$f\mapsto H_f^{(\gamma)} \text{ (or } h_f^{(\gamma)}\text{)}: D_{1/p}\to L^{2,\alpha}, \quad p=2 \text{ or } \infty. \tag{6.1}$$

We know, by Lemma 6.1, that this map takes functions in BL_2^α to operators (from $D_{1/2}$ to $L^{2,\alpha}$ in S_2 and, by Theorems 1 and 2, this map takes functions in

$$\mathrm{BMO}_\alpha=\{\, g \text{ is analytic in } \mathbf{D}: |g'(z)|^2\, dA_\alpha(z) \text{ is a 0-Carleson measure}\,\}$$

to the bounded operators (bounded from D_0 to $L^{2,\alpha}$). Interpolation theory then insures that the map (6.1) takes functions in the spaces between BL_2^α and BMO_α to the operators (which map the corresponding spaces between $D_{1/2}$ and D_0 to $L^{2,\alpha}$) in the corresponding spaces between S_2 and the space of the bounded operators.

In [RW1], one can find the atomic decomposition theorem for BMO_α (for $\alpha=0$, see also [RS1]). The result of the atomic decomposition for BL_2^α is similar to the same type of result for B_p (see [R2]). Using these results and the methods in [RS1] and [R2], one can prove easily that the spaces intermediate between BL_2^α and BMO_α are the spaces BL_p^α, $2<p<\infty$. The spaces intermediate between $D_{1/2}$ and D_0 are clearly the spaces $D_{1/p}$, $2<p<\infty$. And it is a result from the theory of Schatten p-classes that the spaces between S_2 and the space of bounded operators are the spaces S_p, $2<p<\infty$. Hence

$$H_f^{(\gamma)} \text{ or } h_f^{(\gamma)}: D_{1/p}\to L^{2,\alpha}$$

is in S_p, $2<p<\infty$, if f is in BL_p^α.

Let $1<p<\infty$ and q be the conjugate of p: $1/q+1/p=1$. It is easy to see, by a standard dual argument, that the dual of BL_p^α is BL_q^α under the pairing

$$\langle f,g\rangle=\int_{\mathbf{D}} f'(z)\overline{g'(z)}(1-|z|^2)^{1-2\alpha}\log\frac{1}{1-|z|^2}\,dA(z), \quad f\in BL_q^\alpha \text{ and } g\in BL_p^\alpha.$$

The second part of Theorem 4 is in fact a "dual" result of the first one. To see this we suppose, for example, that $H_f^{(\gamma)}$ is in $S_p^{1/p\alpha}$, $1<p<2$. We know from Section 3 that $H_f^{(\gamma)}$ is in $S_p^{1/p\alpha}$, $1<p<2$, if and only if $M_{f'}: D_{1/p}\to A^{2,1-2\alpha}$ is in S_p. Let q be the conjugate of p and g be in BL_q^α. Then, by the first part of Theorem 4, we have $M_{g'}: D_{1/q}\to A^{2,1-2\alpha}$ is in S_q. Notice that

$$\log\frac{1}{1-|z|^2}=\sum_1^\infty\frac{|z|^{2k}}{k}=\sum_1^\infty\frac{z^k}{k^{1/p}}\frac{\bar{z}^k}{k^{1/q}}.$$

By using (4.1), we have

$$\begin{aligned}
|\langle f,g\rangle| &= \left|\int_{\mathbf{D}} f'(z)\overline{g'(z)}(1-|z|^2)^{1-2\alpha}\log\frac{1}{1-|z|^2}\,dA(z)\right| \\
&= \left|\sum_1^\infty \left\langle M_{f'}\left(\frac{z^k}{k^{1/p}}\right), M_{g'}\left(\frac{z^k}{k^{1/p}}\right)\right\rangle_{A^{2,1-2\alpha}}\right| \\
&\le C\|(M_{g'})^* M_{f'}\|_{S_1} \\
&\le C\|M_{f'}\|_{S_p}\|M_{g'}\|_{S_q} \\
&\le C\|M_{f'}\|_{S_p}\|g\|_{BL_q^\alpha}.
\end{aligned}$$

This implies that f is in BL_p^α. A similar argument will prove that the same result is true if $h_f^{(\gamma)}$ is in $S_p^{1/p\alpha}$, $1<p<2$.

We now discuss the conditions for the symbols in Theorem 4.

Let $f_1(z)=z^k$ and $f_2(z)=(1-\bar{a}z)^{2\alpha-2}$. We first estimate their BL_p^α norms.

$$\begin{aligned}
\|f_1\|_{BL_p^\alpha}^p &= k^p\int_0^1 r^{(k-1)p}(1-r)^{(1-\alpha)p-1}\log\frac{1}{1-r}\,dr \\
&= k^p\sum_1^\infty \frac{1}{j}\int_0^1 r^{(k-1)p}(1-r)^{(1-\alpha)p-1+j}\,dr \\
&\asymp (k+1)^{\alpha p}\log(k+1);
\end{aligned}$$

$$\begin{aligned}
\|f_2\|_{BL_p^\alpha}^p &= C\int_{\mathbf{D}}(1-|z|^2)^{(1-\alpha)p-1}|1-\bar{a}z|^{(2\alpha-3)p}\log\frac{1}{1-|z|^2}\,dA(z) \\
&\asymp \sum_0^\infty (j+1)^{(3-2\alpha)p-2}|a|^{2j}\int_0^1 r^j(1-r)^{(1-\alpha)p-1}\log\frac{1}{1-r}\,dr \\
&\asymp \sum_0^\infty (j+1)^{(3-2\alpha)p-2}|a|^{2j}\sum_1^\infty n^{-1}(j+n)^{(\alpha-1)p} \\
&\asymp \sum_0^\infty (j+1)^{(2-\alpha)p-2}\log(j+1)|a|^{2j} \\
&\asymp (1-|a|^2)^{(\alpha-2)p+1}\log\frac{1}{1-|a|^2}.
\end{aligned}$$

We then estimate the S_p norms of $h_{f_1}^{(\alpha)}$ and $H_{f_1}^{(\alpha)}$. We know that $h_{f_1}^{(\alpha)}$ is a rank $k+1$

operator and hence

$$\begin{aligned}\|h_{f_1}^{(\alpha)}\|_p^p &= \left|\frac{\gamma_{k+1,\alpha}^2}{(k+1)^2\gamma_{k,1/p}\gamma_{0,\alpha}}\right|^p + \sum_{m+n=k}\left|\frac{n\gamma_{k,\alpha}^2}{(k+1)\gamma_{m,1/p}\gamma_{n,\alpha}}\right|^p \\ \text{(by (4.1))} &\asymp \left|\frac{(k+2)^{2\alpha}}{(k+1)^2(k+1)^{1/p}}\right|^p + \sum_{m+n=k}\left|\frac{n(k+1)^{2\alpha-1}}{(m+1)^{1/p}(n+1)^{\alpha}}\right|^p \\ &\asymp (k+1)^{\alpha p}\log(k+1).\end{aligned}$$

And by Lemmas 3.2 and 3.3, we have

$$\|H_{f_1}^{(\alpha)}\|_p^p \asymp \|M_{f_1'}\|_p^p + \|\widetilde{H}_{f_1}\|_p^p \asymp \|M_{f_1'}\|_p^p.$$

Here we regard $M_{f_1'}$ as an operator from $D_{1/p}$ to $A^{2,1-2\alpha}$. Clearly $M_{f_1'}$ has the singular values

$$\left\{\frac{k^2\gamma_{k+m,\alpha}}{(k+m)^2\gamma_{m,1/p}}\right\}_{m=0}^{\infty}.$$

Therefore

$$\|M_{f_1'}\|_p^p = \sum_0^{\infty}\left|\frac{k^2\gamma_{k+m,\alpha}}{(k+m)^2\gamma_{m,1/p}}\right|^p \asymp \sum_0^{\infty}\frac{k^{2p}(k+m+1)^{\alpha p}}{(k+m)^{2p}(m+1)} \asymp (k+1)^{\alpha p}\log(k+1).$$

Hence

$$\|h_{f_1}^{(\alpha)}\|_p \asymp \|H_{f_1}^{(\alpha)}\|_p \asymp \|f_1\|_{BL_p^{\alpha}}.$$

These estimates show that the conditions of the symbols in Theorem 4 are sharp.

However, the following estimate on the S_p norm of $h_{f_2}^{(\alpha)}$ shows that the conditions of the symbols in Theorem 4 are not sufficient at least for $h_f^{(\alpha)}$ in S_p.

It is easy to check that $h_{f_2}^{(\alpha)}$ is, at most, a rank two operator and hence

$$\|h_{f_2}^{(\alpha)}\|_p \asymp \|h_{f_2}^{(\alpha)}\|_2 \asymp \|f_2\|_{B_2^{1/2+\alpha-1/p}} \asymp (1-|a|^2)^{\alpha+1/p-2}.$$

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