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Respectful quasiconformal extension from dimension $n-1$ to n

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1. Introduction

Tukia and Väisälä proved in [20] that every quasiconformal self-homeomorphism f of $\mathbf{R}^{n-1}$ with $n \geq 2$ can be extended to a quasiconformal self-homeomorphism F of $\mathbf{R}^n_+ = \mathbf{R}^{n-1} \times [0, \infty)$ which, in addition, in the hyperbolic metric of $H^n = \mathbf{R}^{n-1} \times (0, \infty)$ is bi-Lipschitz and uniformly approximates arbitrarily closely a natural homeomorphic extension F_f of f. The main result of this paper, Theorem 3.1, is that if X_0 is the subset $\mathbf{R}^p$ $(0 \leq p < n)$ or $\mathbf{R}^p_+$ $(1 \leq p < n)$ of $\mathbf{R}^{n-1}$ and if f respects X_0, i.e., maps it onto itself, then F can be chosen to respect $X = X_0 \times [0, \infty)$. Following Siebenmann, we call this extension theorem respectful (to X). An easy consequence, Theorem 4.1, is that if we forgo the properties of F involving the hyperbolic metric, F can be prescribed on X. The respectful quasiconformal Schoenflies extension theorem allows us to use Theorem 3.1 in Section 5 to show that every locally quasiconformal (LQC) self-homeomorphism of $\mathbf{R}^{n-1}$ respecting X_0 can be extended to an LQC self-homeomorphism of $\mathbf{R}^n_+$ respecting X. Moreover, the extension can be prescribed on X. This result, which generalizes the non-respectful version of it proved by the author in [10], is needed in [13] when proving that a self-homeomorphism of an LQC manifold M which respects a closed locally LQC flat LQC submanifold Q of M can be respectfully approximated by LQC homeomorphisms, i.e., by ones respecting Q, also in the case where Q meets the boundary of M (in this approximation theorem dimension four must necessarily be excluded).

In the proof of our main result we follow the simplified version of the proof of [20] as indicated in [19; 7.1]. For their part, Tukia and Väisälä were inspired by Carleson's [4] quasiconformal extension method in the case $n \leq 4$. We first decompose H^n into similar pieces (parallelotopes) and give each piece an index in $\{1, \ldots, 2^n\}$ such that pieces of the same index are disjoint. From the quasiconformality of f it follows that the restrictions of the homeomorphism F_f to slightly larger parallelotopes, when suitably normalized, belong to a compact family of embeddings. This makes it possible to construct the approximation F of F_f on a neighbourhood of the union of the pieces of index $\leq i$ inductively with respect to i. In the ith stage local bi-Lipschitz approximations of F_f are provided by a result on respectful Lipschitz approximation of homeomorphisms which follows from known results in piecewise-linear topology and differential topology. This result is known to be false if dimension four is present in it, but fortunately the formula of F_f, recalled in 2.1, allows us to apply the result in such a way that no dimensional restrictions follow. We glue the local approximations to the approximation produced by the $(i-1)$th stage by using a respectful deformation theorem for Lipschitz embeddings due to Siebenmann and Sullivan or rather the version of it, proved by the author in [11], where bi-Lipschitz constants are under control. This deformation theorem is the substitute for Sullivan's Lipschitz deformation theorem used in [20].

2. Preliminaries

2.1. Notation and terminology

For integers $0\leq p\leq n$, we identify $\mathbf{R}^p$ with $\mathbf{R}^p\times 0\subset\mathbf{R}^n$, and writing $x=(x_1,\ldots,x_n)$ for a point $x\in\mathbf{R}^n$, we let $\mathbf{R}^n_+=\{x\in\mathbf{R}^n\,|\,x_n\geq 0\}$ and $H^n=\{x\in\mathbf{R}^n\,|\,x_n>0\}$ if $n\geq 1$, $\mathbf{R}^{n,p}=\{x\in\mathbf{R}^n\,|\,x_i=0 \text{ if } i\leq n-p\}$, $\mathbf{R}^{n,p}_+=\{x\in\mathbf{R}^{n,p}\,|\,x_n\geq 0\}$ if $p\geq 1$, and $\mathbf{R}^{n,p}_{++}=\{x\in\mathbf{R}^{n,p}_+\,|\,x_{n-1}\geq 0\}$ if $p\geq 2$. Then $\mathbf{R}^n_+=\mathbf{R}^{n-1}\cup H^n$.

For $1\leq p\leq n\geq 2$ we let $\mathcal{X}(n,p)=\{\mathbf{R}^{n,p}_+,\mathbf{R}^{n,p}_{++}\}$ if $p\geq 2$ and $\mathcal{X}(n,1)=\{\mathbf{R}^{n,1}_+\}$ if $p=1$. For $X\in\mathcal{X}(n,p)$ we set $X_0=X\cap\mathbf{R}^{n-1}$. Then $X_0=\mathbf{R}^{n-1,p-1}$ or $X_0=\mathbf{R}^{n-1,p-1}_+$, and $(\mathbf{R}^n_+,X)=(\mathbf{R}^{n-1},X_0)\times\mathbf{R}^1_+$.

For a homeomorphism $f\colon\mathbf{R}^{n-1}\to\mathbf{R}^{n-1}$ with $n\geq 2$ we define, as in [20], a homeomorphism $F_f\colon\mathbf{R}^n_+\to\mathbf{R}^n_+$ extending f by $F_f(x,t)=\bigl(f(x),\tau_f(x,t)\bigr)$ where $x\in\mathbf{R}^{n-1}$, $t\geq 0$, and $\tau_f(x,t)=\max\bigl\{|f(x)-f(y)|\,\big|\,y\in\mathbf{R}^{n-1},\ |x-y|=t\bigr\}$. If X is as above, then $fX_0=X_0$ implies $F_fX=X$.

We say that a function $f\colon S\to T$ respects a set Y if $f^{-1}[Y\cap T]=Y\cap S$. If f is bijective, this is equivalent to $f[Y\cap S]=Y\cap T$; then we also write $f\colon(S,Y\cap S)\to(T,Y\cap T)$. By id we denote various inclusion maps.

The boundary of a manifold Y is denoted by ∂Y. We set $I^n(r)=[-r,r]^n$, $J^n(r)=(-r,r)^n$, $B^n(r)=\bigl\{x\in\mathbf{R}^n\,\big|\,|x|<r\bigr\}$, $\bar{B}^n(r)=\bigl\{x\in\mathbf{R}^n\,\big|\,|x|\leq r\bigr\}$, $B^n_+(r)=B^n(r)\cap\mathbf{R}^n_+$, and $C^n(r)=\bigl\{(x,t)\in\mathbf{R}^n_+\,\big|\,|x|+t<r\bigr\}$ for $r>0$, and $B^n=B^n(1)$, $\bar{B}^n=\bar{B}^n(1)$, and $S^{n-1}=\partial\bar{B}^n$. The standard basis of $\mathbf{R}^n$ is written as $(e_1,\ldots,e_n)$. We let $\dot{\mathbf{R}}^n=\mathbf{R}^n\cup\{\infty\}$ be the one-point compactification of $\mathbf{R}^n$, and for $A\subset\mathbf{R}^n$ we set $\dot{A}=A\cup\{\infty\}\subset\dot{\mathbf{R}}^n$.

We let d denote the Euclidean metric. On the domains H^n, $\mathbf{R}^{n+k}\setminus\mathbf{R}^{n-1}$ with $k\geq 1$, and B^n we also use the hyperbolic metric σ, defined by the element of length $|dx|/d(x,\mathbf{R}^{n-1})$ in the first two cases and $2|dx|/(1-|x|^2)$ in the third case. Then every Möbius homeomorphism $(H^n,\sigma)\to(B^n,\sigma)$ is isometric. If $f\colon S\to T$ and $f'\colon S'\to T$ are maps to a metric space (T,ϱ) and $A\subset S\cap S'$, we write $\varrho(f,f';A)=\sup\{\varrho(f(x),f'(x))\,|\,x\in A\}$, with $\varrho(f,f')=\varrho(f,f';A)$ whenever $A=S=S'$.

Let (S,ϱ) and (T,ϱ') be metric spaces and $f\colon S\to T$ an embedding. If there is $L\geq 1$ such that $\varrho(x,y)/L\leq\varrho'(f(x),f(y))\leq L\varrho(x,y)$ for all $x,y\in S$, we say that f is bi-Lipschitz (abbreviated BL) or also L-BL. If f is only a map satisfying the right-hand inequality for some $L\geq 0$, we say that f is (L-)Lipschitz. If there is a homeomorphism $\eta\colon\mathbf{R}^1_+\to\mathbf{R}^1_+$ such that $t'\leq\eta(t)$ whenever $a,b,x\in S$, $b\neq x$, $t=\varrho(a,x)/\varrho(b,x)$, and $t'=\varrho'(f(a),f(x))/\varrho'(f(b),f(x))$, we say that f is quasisymmetric (abbreviated QS) or also η-QS. The basic theory of quasisymmetric embeddings is given in [18] and [24]. We say that f is LIP, locally L-BL, or LQS if each point of S has a neighbourhood on which f is, respectively, BL, L-BL, or QS. Locally Lipschitz maps are defined similarly. If f is QS, if $s>0$, and if $t\leq 1/s$ implies $t'\leq t+s$ whenever t and t' are as above, we say as in [26] that f is s-QS. We let 0-QS mean id-QS.

Let $n\geq 1$, let $A\subset\mathbf{R}^n$ be a set with $A\subset\operatorname{cl}\operatorname{int}A$, and let $f\colon A\to\mathbf{R}^n$ be an embedding. If there is $K\geq 1$ such that for each component D of $\operatorname{int}A$ the homeomorphism $D\to fD$ defined by f is K-quasiconformal in the sense of [23] whenever $n\geq 2$ or K-quasisymmetric in the sense of [9] (though possibly decreasing) whenever $n=1$, we say that f is quasiconformal (abbreviated QC) or also K-QC. We say that f is LQC if each point of A has an open neighbourhood in A on which f is QC. For the proofs of the following two facts, see [24; Section 2] and [23; 35.2] if $n\geq 2$ and [18; 2.16] if $n=1$. A self-homeomorphism f of $\mathbf{R}^n$ or of $\mathbf{R}^n_+$ is K-QC if and only if f is η-QS, with K and η depending only on each other and n. If A is open in $\mathbf{R}^n$ or in $\mathbf{R}^n_+$ and if in the latter case f respects $\mathbf{R}^n_+$ and $\mathbf{R}^{n-1}$, then f is LQC if and only if f is LQS. By [21; 2.6] (which is valid also if $n=1$), a homeomorphism $f\colon\mathbf{R}^n\to\mathbf{R}^n$ is K-QC if and only if f is s-QS, with K and s depending only on each other and n and such that $K\to 1$ if and only if $s\to 0$.

Suppose that $D\subset\mathbf{R}^m$ is a domain for which we have defined the hyperbolic metric σ, that $A\subset D$ is open, and that $f\colon A\to D$ is an embedding. If f is BL, L-BL, or locally L-BL with respect to σ, we say that f is BLH, L-BLH, or locally L-BLH, respectively. From now on assume $D\neq B^m$. We let $L_\sigma(x,f)$ and $l_\sigma(x,f)$ denote the upper and lower limit, respectively, of the quotient $\sigma(f(x),f(y))/\sigma(x,y)$ as $y\to x$ in A. Note that $L_\sigma(f(x),f^{-1})=l_\sigma(x,f)^{-1}$. For the expression of these quantities in terms of the corresponding quantities $L_d(x,f)$ and $l_d(x,f)$ in the Euclidean metric, see [28; 4.5]. If $f\colon D\to D$ is a homeomorphism and if each point $x\in\overline{D}$ has an open neighbourhood U such that $f|U\cap D$ is BLH, we say that f is LIPH. If $g\colon \bar{D}\to\bar{D}$ is a homeomorphism which defines a LIPH homeomorphism $D\to D$ and if $m\geq 2$, then g is LQS by [23; 34.2 and 35.1]. The following fact is needed in 5.5. Suppose that $A=B^m(r)\cap D$. Then, since every two points $x,y\in A$ can be joined by an arc in A of hyperbolic length $\sigma(x,y)$, we have by [28; 4.4] that f is L-Lipschitz with respect to σ if and only if $L_\sigma(x,f)\leq L$ for each $x\in A$. From this it also easily follows that if a homeomorphism $f\colon D\to D$ extends to a LIP homeomorphism $\bar{D}\to\bar{D}$, then f is LIPH.

2.2. Solidity

The rest of Section 2 is needed only for the proof of Theorem 3.1. First we recall two terms introduced and observed to be related in [19].

For an open set $U\subset\mathbf{R}^n$ we let $E(U,\mathbf{R}^n)$ denote the set of all embeddings of U into $\mathbf{R}^n$, equipped with the compact-open topology, and $H(U)$ denote the group of self-homeomorphisms of U. If $Y\subset\mathbf{R}^n$ is closed, $E_Y(U,\mathbf{R}^n)$ denotes the closed subset of $E(U,\mathbf{R}^n)$ consisting of the embeddings respecting Y. As in [19; 3.8], a set $\mathcal{F}\subset E(U,\mathbf{R}^n)$ is said to be solid if its closure in $E(U,\mathbf{R}^n)$, denoted by $\mathrm{cl}_E\,\mathcal{F}$, is compact.

Let $f\colon H^n\to H^n$ be a homeomorphism. If there is a homeomorphism $\varphi\colon\mathbf{R}^1_+\to\mathbf{R}^1_+$ such that $\varphi^{-1}\bigl(\sigma(x,y)\bigr)\leq\sigma\bigl(f(x),f(y)\bigr)\leq\varphi\bigl(\sigma(x,y)\bigr)$ for all $x,y\in H^n$, we say as in [19; 6.10] that f is φ-solid. By [8; Theorem 3], this is the case if f is K-QC and $n\geq 2$, with φ depending only on n and K. The following lemma gives the important fact that although $F_f|H^n$ with f QC is possibly not QC itself, it is solid, however. The converse is also known to hold; see [29; 7.1 and 7.9].

2.3. Lemma. *Let $n\geq 2$ and $K\geq 1$. Then there is a homeomorphism $\varphi\colon\mathbf{R}^1_+\to\mathbf{R}^1_+$ such that $F_f|H^n$ is φ-solid for every K-QC homeomorphism $f\colon\mathbf{R}^{n-1}\to\mathbf{R}^{n-1}$.*

This fact was claimed in [19; 7.1] without proof. We give in 2.4–2.6 a proof, based on solid sets of homeomorphisms, whose idea is mentioned in [27; p. 162]. Especially, Lemma 2.5 is an analogue of [20; 2.13]. In [29; 7.26] Lemma 2.3 is given an elementary but lengthy direct proof.

2.4. *Notation.* Fix $n\geq 2$ and $K\geq 1$. If $z=(\bar{z},z_n)\in H^n$, let $\alpha_z\colon\mathbf{R}^n_+\to\mathbf{R}^n_+$ be the similarity homeomorphism $x\mapsto\bar{z}+z_nx$; then $\alpha_z(e_n)=z$. If $f\colon\mathbf{R}^{n-1}\to\mathbf{R}^{n-1}$ is a homeomorphism, $z\in H^n$, and $z'=F_f(z)$, let $\beta^f_z=\alpha^{-1}_{z'}$. Define homeomorphisms $f_z=\beta^f_z f\alpha_z\colon\mathbf{R}^{n-1}\to\mathbf{R}^{n-1}$ and $F^f_z=\beta^f_z F_f\alpha_z\colon\mathbf{R}^n_+\to\mathbf{R}^n_+$. Then $F^f_z=F_{f_z}$ [20; (2.5)]. Let $\mathcal{G}=\{f\in H(\mathbf{R}^{n-1})\,|\,f\text{ is }K\text{-QC}\}$, $\mathcal{F}=\bigl\{F_f|H^n\,\big|\,f\in\mathcal{G}\bigr\}$, $\mathcal{G}_0=\{f_z\,|\,f\in\mathcal{G},\ z\in H^n\}\subset\mathcal{G}$, and $\mathcal{F}_0=\bigl\{F^f_z|H^n\,\big|\,f\in\mathcal{G},\ z\in H^n\bigr\}\subset\mathcal{F}$.

2.5. Lemma. *The set $\mathcal{F}_0$ is solid.*

Proof. Let $f\in\mathcal{G}$ and $z\in H^n$. Then $f_z\in\mathcal{G}$ and $f_z(0)=0$. Since f is η-QS with $\eta=\eta_{\mathcal{G}}$, we have

$$\frac{1}{\eta(1)}\leq|f_z(e_1)|=\frac{|f(\bar{z}+z_ne_1)-f(\bar{z})|}{\tau_f(\bar{z},z_n)}\leq 1.$$

Hence, $\mathrm{cl}_E\,\mathcal{G}_0$ is a compact set in $\mathcal{G}$ by [23; 19.4(1), 20.5, 21.7, and 37.3] if $n\geq 3$ and by, e.g., [18; 3.4, 3.6, and 3.7] if $n=2$. Since the map $H(\mathbf{R}^{n-1})\to H(H^n)$, $f\mapsto F_f|H^n$, is continuous [20; (2.6)] and since it maps $\mathcal{G}_0$ onto $\mathcal{F}_0$ and $\mathcal{G}$ onto $\mathcal{F}$, it follows that $\mathrm{cl}_E\,\mathcal{F}_0$ is a compact set in $\mathcal{F}$. □

2.6. *Completion of the proof of* 2.3. Let $\varepsilon>0$ be given. Then find by 2.5 a number $\delta=\delta_\varepsilon=\delta_\varepsilon(n,K)>0$ such that if $F_0\in\mathcal{F}_0$ and $z\in H^n$, then $\sigma(e_n,z)\leq\delta$ implies $\sigma(F_0(e_n),F_0(z))\leq\varepsilon$, and $\sigma(F_0(e_n),F_0(z))\leq\delta$ implies $\sigma(e_n,z)\leq\varepsilon$. Given $F\in\mathcal{F}$ and $x,y\in H^n$, choose $f\in\mathcal{G}$ with $F=F_f|H^n$ and let $z=\alpha_x^{-1}(y)$; then $\sigma(x,y)=\sigma(e_n,z)$ and $\sigma(F(x),F(y))=\sigma(F_x^f(e_n),F_x^f(z))$. Thus, $\sigma(x,y)\leq\delta$ implies $\sigma(F(x),F(y))\leq\varepsilon$, and $\sigma(F(x),F(y))\leq\delta$ implies $\sigma(x,y)\leq\varepsilon$.

Let $\omega(t)=\sup\{\sigma(F(x),F(y))\,|\,F\in\mathcal{F},\,x,y\in H^n,\,\sigma(x,y)\leq t\}$ for $t\geq 0$. Since $\omega(\delta_\varepsilon)\leq\varepsilon$ for each $\varepsilon>0$, there is a homeomorphism $\psi\colon\mathbf{R}^1_+\to\mathbf{R}^1_+$ such that $\psi(t)\geq\omega(t)$ if $t\leq\delta_1$ and $\psi(t)=2t/\delta_1$ if $t\geq\delta_1$. Consider $F\in\mathcal{F}$, $x,y\in H^n$, $t=\sigma(x,y)$, and $t'=\sigma(F(x),F(y))$. We show that $t'\leq\psi(t)$. This is obvious if $t\leq\delta_1$. Suppose $t>\delta_1$. Choose successive points $x=z_0$, z_1, ..., $z_k=y$ in the hyperbolic geodesic joining x and y such that $\sigma(z_{j-1},z_j)=\delta_1$ if $j<k$ and $\sigma(z_{k-1},z_k)\leq\delta_1$. Then $t'\leq k=(\sigma(z_0,z_{k-1})+\delta_1)/\delta_1\leq\psi(t)$. Find in a similar way a homeomorphism $\psi'\colon\mathbf{R}^1_+\to\mathbf{R}^1_+$ depending only on (n,K) such that $t\leq\psi'(t')$. Then $\varphi=\max(\psi,\psi')$ satisfies 2.3. □

In the next two lemmas the cases $Y=\emptyset$ and $Y=\mathbf{R}^n$ are in fact the same. The first of these results deals with extension of locally L-BL approximations. It is a respectful version of a part of [19; 3.9].

2.7. Lemma. *Let $n\geq 1$, let U, U', V, W be open sets in $\mathbf{R}^n$ such that $W\subset V\subset U$, $\overline{W}\cap U\subset V$, $\overline{U}'\subset U$, and $\overline{U}'$ is compact, let either $Y=\emptyset$ or $Y=\mathbf{R}^p$ with $0\leq p\leq n$ or $Y=\mathbf{R}^p_+$ with $1\leq p\leq n$, let $\mathcal{F}$ be a solid subset of $E_Y(U,\mathbf{R}^n)$ whose members are approximable by* LIP *embeddings in $E_Y(U,\mathbf{R}^n)$, and let $\varepsilon>0$. Then there is $\delta>0$ such that for every $L\geq 1$ there is $L'\geq 1$ with the following property: If $g\in\mathcal{F}$ and if $h\in E_Y(V,\mathbf{R}^n)$ is locally L-*BL *such that $d(h,g;V)\leq\delta$, then there is an L'-*BL *embedding $h'\in E_Y(U',\mathbf{R}^n)$ such that $d(h',g;U')\leq\varepsilon$ and $h'=h$ on $W\cap U'$.*

Proof. The proof is otherwise the same as that of [19; 3.9] but in place of [19; 3.6], the quantitative version of a result due to Sullivan, we refer to [11; 5.7]. □

To be able to apply 2.7 we need the following respectful LIP approximation result, Lemma 2.9 on extension of homeomorphic approximations, and the elementary LIP approximation result 2.10.

2.8. Lemma. *Let $n\geq 1$, let either $Y=\emptyset$ or $Y=\mathbf{R}^p$ with $0\leq p\leq n$ or $Y=\mathbf{R}^p_+$ with $1\leq p\leq n$, let $f\colon \mathbf{R}^n\to\mathbf{R}^n$ be a homeomorphism respecting Y, and let $\varepsilon\colon\mathbf{R}^n\to(0,\infty)$ be continuous. Suppose that $n\neq 4$ and that at least one of the following conditions holds*: (a) *$f|Y$ is* LIP, (b) *$p\neq 4$ and $f|\partial Y$ is* LIP, (c) *$p\neq 4,5$. Then there is a* LIP *homeomorphism $g\colon\mathbf{R}^n\to\mathbf{R}^n$ respecting Y such that $|g(x)-f(x)|<\varepsilon(x)$ for each $x\in\mathbf{R}^n$ and such that $g|Z=f|Z$ whenever $Z\in\{Y,\partial Y\}$ and $f|Z$ is* LIP.

Lemma 2.8 reduces to the special case where we have in (a) that $f|Y=\mathrm{id}$ and in (b) that $p\neq 4$ and $f|\partial Y=\mathrm{id}$. In fact, if $Z\in\{Y,\partial Y\}$ and $f|Z$ is LIP, extending $f|Z$ to a LIP homeomorphism $f_1\colon(\mathbf{R}^n,Y)\to(\mathbf{R}^n,Y)$ and replacing f by $f_1^{-1}f$ we may assume that $f|Z=\mathrm{id}$. This special case of the lemma then follows from various known piecewise-linear and smooth approximation results. For details we refer to [13].

By [5; Corollary on p. 183], the dimensional restrictions in 2.8 cannot be omitted. The proof of 2.8 makes use of the deep stable homeomorphism theorem due to Kirby unless we know that the homeomorphisms f and either $f|Y$ if $\partial Y=\emptyset$ or $f|\partial Y$ if $\partial Y\neq\emptyset$, whenever arranged to be sense-preserving, are stable.

2.9. Lemma. *Let $1\leq p\leq n$, let $Y=\mathbf{R}^p$ or $Y=\mathbf{R}^p_+$, let $f\colon\mathbf{R}^n\to\mathbf{R}^n$ be a homeomorphism respecting Y, and let $\varepsilon\colon\mathbf{R}^n\to(0,\infty)$ be continuous. Then there is a continuous $\delta\colon Y\to(0,\infty)$ with the following property*: *If $g\colon Y\to Y$ is a homeomorphism with $|g(x)-f(x)|<\delta(x)$ for each $x\in Y$, then there is a homeomorphism $g^*\colon\mathbf{R}^n\to\mathbf{R}^n$ extending g such that $|g^*(x)-f(x)|<\varepsilon(x)$ for each $x\in\mathbf{R}^n$ and such that, if, moreover, $Y=\mathbf{R}^p$, $p<n$, and f respects $\mathbf{R}^{p+1}_+$, then g^* respects $\mathbf{R}^{p+1}_+$.*

Proof. Since f is continuous, by a well-known fact (cf. [17]) there is a continuous $\eta\colon\mathbf{R}^n\to(0,\infty)$ such that $|f(x)-f(y)|<\varepsilon(x)$ if $x,y\in\mathbf{R}^n$ and $|x-y|<\eta(x)$. Writing $\mathbf{R}^n=\mathbf{R}^p\times\mathbf{R}^{n-p}$ define a continuous $\omega\colon\mathbf{R}^p\to(0,\infty)$ by $\omega(x)=\min\{\eta(x,y)\,\big|\,|y|\leq 1\}$. By [15; Theorem 5.6.4] there is a continuous $\varrho\colon\mathbf{R}^p\to(0,\infty)$ such that if $h\colon\mathbf{R}^p\to\mathbf{R}^p$ is a homeomorphism with $|h(x)-x|<\varrho(x)$ for each $x\in\mathbf{R}^p$, then there is a homeomorphism $h^+\colon\mathbf{R}^p\times[0,1]\to\mathbf{R}^p\times[0,1]$ of the form $h^+(x,t)=\big(h_t(x),t\big)$ such that $h_0=h$, $h_1=\mathrm{id}$, and $|h_t(x)-x|<\omega(x)$ for all $x\in\mathbf{R}^p$, $t\in[0,1]$. We may assume that ϱ is invariant with respect to the orthogonal reflection of $\mathbf{R}^p$ in $\mathbf{R}^{p-1}$. Choose a continuous $\delta_0\colon Y\to(0,\infty)$ such that $|f^{-1}(x)-f^{-1}(y)|<\varrho\big(f^{-1}(x)\big)$ if $x,y\in Y$ and $|x-y|<\delta_0(x)$. We show that $\delta\colon Y\to(0,\infty)$, $x\mapsto\delta_0\big(f(x)\big)$, is the desired continuous function.

Thus, let $g\colon Y\to Y$ be a homeomorphism with $|g(x)-f(x)|<\delta(x)$ for each $x\in Y$. Define a homeomorphism $h\colon\mathbf{R}^p\to\mathbf{R}^p$ by letting $h=f^{-1}g$ if $Y=\mathbf{R}^p$ or by letting h

be the extension of $f^{-1}g$ by reflection if $Y=\mathbf{R}^p_+$. Then $|h(x)-x|<\varrho(x)$ for each $x\in\mathbf{R}^p$. Now let h^+ be as above. Writing again $\mathbf{R}^n=\mathbf{R}^p\times\mathbf{R}^{n-p}$ define a homeomorphism $H\colon\mathbf{R}^n\to\mathbf{R}^n$ by $H(x,y)=\bigl(h_t(x),y\bigr)$ where $t=\min(|y|,1)$. Then $H|\mathbf{R}^p=h$, $|H(z)-z|<\eta(z)$ for each $z\in\mathbf{R}^n$, and H respects $\mathbf{R}^{p+1}_+$ if $p<n$. It follows that $g^*=fH$ satisfies the lemma. □

2.10. Lemma. *Let X be a metric space, $Y=X\times\mathbf{R}^1_+$, $f\colon Y\to Y$ a homeomorphism of the form $f(x,t)=\bigl(x,f_x(t)\bigr)$, and $\varepsilon\colon Y\to(0,\infty)$ continuous. Metrize Y by a metric ϱ defined in a standard way. Then there is a* LIP *homeomorphism $g\colon Y\to Y$ of the form $g(x,t)=\bigl(x,g_x(t)\bigr)$ such that $\varrho\bigl(g(y),f(y)\bigr)<\varepsilon(y)$ for each $y\in Y$.*

Proof. Define a continuous $\eta\colon Y\to(0,\infty)$ by

$$\eta(x,t)=\min\{\varepsilon(x,s)/2\,|\,0\le s\le t+1\}.$$

Choose a continuous $\delta\colon Y\to(0,1)$ such that $\varrho\bigl(f(y),f(y')\bigr)<\eta(y)$ if $y,y'\in Y$ and $\varrho(y,y')\le\delta(y)$. By [14; 5.4] we may choose δ to be locally Lipschitz. Define inductively a sequence $\alpha_i\colon X\to\mathbf{R}^1_+$, $i\ge0$, of locally Lipschitz functions by $\alpha_0(x)=0$ and $\alpha_{i+1}(x)=\alpha_i(x)+\delta\bigl(x,\alpha_i(x)\bigr)$; then $\alpha_{i+1}(x)>\alpha_i(x)$, and $\alpha_i(x)\to\infty$ as $i\to\infty$. Define a sequence $\beta_i\colon X\to\mathbf{R}^1_+$, $i\ge0$, of continuous functions by the condition $f\bigl(x,\alpha_i(x)\bigr)=\bigl(x,\beta_i(x)\bigr)$. Then $0=\beta_0(x)<\beta_1(x)<\dots$, $\beta_i(x)\to\infty$ as $i\to\infty$, and $\beta_{i+1}(x)-\beta_i(x)<\eta_i(x)=\eta\bigl(x,\alpha_i(x)\bigr)$. Define $\gamma_0=\beta_0$. For $i\ge1$, by [14; 5.4] choose a locally Lipschitz function $\gamma_i\colon X\to\mathbf{R}^1_+$ such that $\max\{\beta_{i-1}(x),\beta_i(x)-\eta_i(x)\}<\gamma_i(x)<\beta_i(x)$. Then $0=\gamma_0(x)<\gamma_1(x)<\dots$, and $\gamma_i(x)\to\infty$ as $i\to\infty$. Now let $g\colon Y\to Y$ be the bijection of the form $g(x,t)=\bigl(x,g_x(t)\bigr)$ where g_x maps $[\alpha_i(x),\alpha_{i+1}(x)]$ affinely onto $[\gamma_i(x),\gamma_{i+1}(x)]$ for each $i\ge0$. Then g is a LIP homeomorphism by [14; 2.40]. Finally, if $\alpha_i(x)\le t\le\alpha_{i+1}(x)$, then $t\le\alpha_i(x)+1$ and, hence, $\varrho\bigl(g(x,t),f(x,t)\bigr)=|g_x(t)-f_x(t)|<2\eta_i(x)\le\varepsilon(x,t)$. □

3. The main result

In this section we establish the following basic theorem on respectful quasiconformal extension with properties involving the hyperbolic metric.

3.1. Theorem. *Let $1\le p\le n\ge2$, let $X\in\mathcal{X}(n,p)$, let $K\ge1$, and let $\varepsilon>0$. Then there is $L=L(n,K,\varepsilon)\ge1$ with the following property: Let $f\colon\mathbf{R}^{n-1}\to\mathbf{R}^{n-1}$ be a K-*QC *homeomorphism respecting X_0. Then there is a homeomorphism $F\colon\mathbf{R}^n_+\to\mathbf{R}^n_+$ respecting X such that*

1. *$\sigma(F,F_f;H^n)\le\varepsilon$,*
2. *$F|\mathbf{R}^{n-1}=f$,*
3. *$F|H^n$ is L-*BLH*,*
4. *F is L^{2n-2}-*QC*.*

3.2. *Remark.* The absolute case, $X=\emptyset$ or equivalently $X=\mathbf{R}_+^{n,n}=\mathbf{R}_+^n$, of 3.1 is due to Tukia and Väisälä [20; 3.11]. In the proof of their result [19; 7.4] on quasiconformal approximation of solid homeomorphisms they used a variation of their method, and the proof of Theorem 3.1, to be completed in 3.21, is a modification of this proof. The absolute case of 3.1, with (1) omitted, was proved for $n=2$ by Beurling and Ahlfors [3], for $n=3$ by Ahlfors [1], for $n\geq 3$ with small K, using Ahlfors's method, by Sedo and Syčev [16], and for $n\leq 4$ by Carleson [4] (see also [6; 3.12]; the condition (3) is claimed in [20; Introduction]).

From now on, we assume that n, p, X, K, ε, and f are given as in Theorem 3.1.

3.3. Case J

Consider the case of 3.1 where $X=\mathbf{R}_{++}^{n,p}$ with $2\leq p<n$. We call it Case J. We choose a homeomorphism $\eta\colon\mathbf{R}_+^1\to\mathbf{R}_+^1$ depending only on (n,K) such that f^{-1} is η-QS and $\eta(1)>1$. Then we define numbers $\chi=\left(\eta(1)^2-1\right)^{1/2}$ and $\varkappa=\min\{1,1/3\chi\sqrt{n}\}$ and let $C_{\mathrm{J}}\subset\mathbf{R}^{n-1}$ be the set of the points $(x,y,z)\in\mathbf{R}^{n-p}\times\mathbf{R}^{p-2}\times\mathbf{R}_+^1$ with $|x|\leq z/\chi$.

3.4. Lemma. *In Case* J, $f|C_{\mathrm{J}}$ *respects* $\mathbf{R}^{n-1,p-1}$.

Proof. Let $X_1=\mathbf{R}^{n-1,p-1}$. Since $X_1\cap C_{\mathrm{J}}=X_0$ and since f respects X_0, it suffices to show that $fC_{\mathrm{J}}\cap(X_1\setminus X_0)=\emptyset$. Thus, suppose, on the contrary, that there is $u\in X_1\setminus X_0$ with $f^{-1}(u)=(x,y,z)\in C_{\mathrm{J}}$. Then $x\neq 0$. Let $z_1=z+|x|/\chi$, $v=(0,y,z_1)\in X_0$, and $w=f(v)\in X_0$. Choose $w_0\in\partial X_0$ with $|w-w_0|=d(w,\partial X_0)$; then $|w-w_0|<|w-u|$ and $v_0=f^{-1}(w_0)\in\partial X_0$. Hence,

$$\begin{aligned}\eta(1)>\eta\left(\frac{|w-w_0|}{|w-u|}\right)&\geq\frac{|v-v_0|}{|v-f^{-1}(u)|}\geq\frac{z_1}{\sqrt{|x|^2+(z_1-z)^2}}\\&\geq\frac{\chi+1/\chi}{\sqrt{1+1/\chi^2}}=\eta(1),\end{aligned}$$

which is a contradiction. □

3.5. Constructions

We define a decomposition $\mathcal{K}=\mathcal{K}_+$ or $\mathcal{K}=\mathcal{K}_{++}$ of H^n into closed n-dimensional rectangular parallelotopes whenever $X=\mathbf{R}_+^{n,p}$ or $X=\mathbf{R}_{++}^{n,p}$, respectively, as follows. In Case J, let $\varkappa$ be as in 3.3; otherwise, let $\varkappa=1$. Let $\mathcal{L}$ be the natural decomposition of $\mathbf{R}^{n-1}\times[\,1,2\,]$ into the closed rectangular parallelotopes which are the translates of the parallelotope $[0,\varkappa]^{n-p}\times[0,1]^p$ with vertices in $(\varkappa\mathbf{Z})^{n-p}\times\mathbf{Z}^{p-1}\times\{1,2\}$. Then let

$$\mathcal{K}_+=\{2^j(Q-\tfrac{1}{2}\varkappa(e_1+\ldots+e_{n-p}))\mid Q\in\mathcal{L},\ j\in\mathbf{Z}\},$$
$$\mathcal{K}_{++}=\{2^j(Q-\tfrac{1}{2}\varkappa(e_1+\ldots+e_{n-p})-\tfrac{1}{2}e_{n-1})\mid Q\in\mathcal{L},\ j\in\mathbf{Z}\}.$$

Thus, except possibly in Case J, the members of $\mathcal{K}$ are cubes. We express $\mathcal{K}$ as a finite disjoint union $\mathcal{K}=\mathcal{K}_1\cup\ldots\cup\mathcal{K}_N$ where each family $\mathcal{K}_i$ is disjoint. In fact, this can be done with $N=2^n$. We set $\mathcal{K}_i^*=\mathcal{K}_1\cup\ldots\cup\mathcal{K}_i$ for $0\le i\le N$.

We define an open parallelotope $P(t)=J^{n-p}(\varkappa t)\times J^p(t)$ and a closed parallelotope $\overline{P}(t)=\operatorname{cl}P(t)$ in $\mathbf{R}^n$ for $t>0$. Suppose that $Q\in\mathcal{K}$. We let z_Q denote the centre and $2\lambda_Q$ the greatest side length of Q. We let $\alpha_Q\colon\mathbf{R}^n\to\mathbf{R}^n$ denote the similarity map $\alpha_Q(x)=z_Q+\lambda_Qx$. For $t>0$ we let $Q(t)=\alpha_QP(t)$ and $\overline{Q}(t)=\alpha_Q\overline{P}(t)$; then $Q=\overline{Q}(1)$. We define a set $X_Q\subset\mathbf{R}^n$ by $X_Q=\emptyset$ if $Q\cap X=\emptyset$, by $X_Q=\mathbf{R}_+^{n-1,p-1}\times\mathbf{R}^1$ if $Q\cap\partial X\ne\emptyset$, and by $X_Q=\mathbf{R}^{n,p}$ otherwise.

In Case J we let $\mathcal{K}_{\mathrm{J}}=\{Q\in\mathcal{K}\mid Q\cap X\ne\emptyset,\ Q\cap\partial X=\emptyset\}$.

The following two lemmas are obvious.

3.6. Lemma. (1) *If $Q,R\in\mathcal{K}$, $Q\cap R\ne\emptyset$, and $Q\ne R$, then* $\operatorname{int}Q\cap\operatorname{int}R=\emptyset$ *and* $\lambda_Q/\lambda_R\in\{\tfrac{1}{2},1,2\}$.

(2) *If $Q\in\mathcal{K}$ and $0<t\le3$, then $2\varkappa t\lambda_Q\sqrt{n}\le d(\overline{Q}(t))\le2t\lambda_Q\sqrt{n}$ and $d(\overline{Q}(t),\mathbf{R}^{n-1})=(3-t)\lambda_Q$.*

(3) *If $Q,R\in\mathcal{K}$ and $Q\cap R=\emptyset$, then $Q(\tfrac{4}{3})\cap R(\tfrac{4}{3})=\emptyset$.* □

3.7. Lemma. (1) *For $Q\in\mathcal{K}$, if $Q\cap X\ne\emptyset$, then $z_Q\in X$, and if $Q\cap X=\emptyset$, then $Q(2)\cap X=\emptyset$. For $Q\in\mathcal{K}_{++}$, if $Q\cap\partial X\ne\emptyset$, then $z_Q\in\partial X$, and if $Q\cap\partial X=\emptyset$, then $Q(2)\cap\partial X=\emptyset$.*

(2) *If $Q\in\mathcal{K}$, then $Q(2)\cap X=Q(2)\cap X_Q=\alpha_Q[P(2)\cap X_Q]$.* □

3.8. Lemma. *In Case* J, *$Q(\tfrac{3}{2})\subset C_{\mathrm{J}}\times\mathbf{R}^1$ for each $Q\in\mathcal{K}_{\mathrm{J}}$.*

Proof. Let $Q\in\mathcal{K}_{\mathrm{J}}$. Consider a point $a=(x,y,z,u)\in\mathbf{R}^{n-p}\times\mathbf{R}^{p-2}\times\mathbf{R}^1\times\mathbf{R}^1$ of $Q(\tfrac{3}{2})$. Since $|x|\le\tfrac{3}{2}\varkappa\lambda_Q\sqrt{n-p}\le\lambda_Q/2\chi\le z/\chi$, we have $a\in C_{\mathrm{J}}\times\mathbf{R}^1$. □

3.9. Lemma. *If $Q\in\mathcal{K}$, then $F_f|Q(\tfrac{3}{2})$ respects X_Q and*

$$F_fQ(\tfrac{3}{2})\cap X=F_fQ(\tfrac{3}{2})\cap X_Q.$$

Proof. For $Q\in\mathcal{K}_{\mathrm{J}}$ in Case J, the first claim follows from 3.4 and 3.8; otherwise the claim is implied by the fact that F_f respects X. Since $Q(\tfrac{3}{2})\cap X=Q(\tfrac{3}{2})\cap X_Q$ by 3.7(2), the second claim follows. □

3.10. Lemma. *There is* $c=c(n)\geq 1$ *such that* $\alpha_Q|\overline{P}(\frac{4}{3})\colon (\overline{P}(\frac{4}{3}),d)\to (H^n,\sigma)$ *is* c-BL *for each* $Q\in\mathcal{K}$.

Proof. As for [19; 6.15] by the aid of 3.6(2). □

3.11. Constructions

For $Q\in\mathcal{K}$ and $z\in\mathbf{R}^n$ we set

$$d_Q^f = d(F_f Q, \mathbf{R}^{n-1}), \quad \beta_Q^f(z) = \frac{z-F_f(z_Q)}{d_Q^f}, \quad F_Q^f = \beta_Q^f F_f \alpha_Q.$$

Thus, β_Q^f is a similarity map $\mathbf{R}^n\to\mathbf{R}^n$, and F_Q^f is an embedding of $\alpha_Q^{-1}\mathbf{R}_+^n\supset\overline{P}(3)$ into $\mathbf{R}^n$. Obviously, β_Q^f respects X_Q. By 3.7(2) and 3.9 it then follows that $F_Q^f|P(\frac{3}{2})$, too, respects X_Q.

For $Y\in\{X_Q\,|\,Q\in\mathcal{K}\}$ we let $\mathcal{G}_Y\subset E_Y(P(\frac{3}{2}),\mathbf{R}^n)$ be the set of all embeddings $F_Q^g|P(\frac{3}{2})$ where $Q\in\mathcal{K}$, $X_Q=Y$, and $g\colon(\mathbf{R}^{n-1},X_0)\to(\mathbf{R}^{n-1},X_0)$ is a K-QC homeomorphism.

3.12. Lemma. *The sets* $\mathcal{G}_Y$ *are solid.*

Proof. This can be proved as the implication (1)⇒(3) of [19; 6.17] is proved (cf. also [19; 6.20]) by the aid of 2.3, 3.6(2), and 3.10. □

3.13. Lemma. *Let* $Q\in\mathcal{K}$ *and let* $\varepsilon'>0$. *Then there exists a* LIP *embedding* $\psi\colon P(\frac{4}{3})\to\mathbf{R}^n$ *respecting* X_Q *such that* $d(\psi,F_Q^f;P(\frac{4}{3}))<\varepsilon'$.

Proof. Choose $\delta>0$ such that $\delta\leq\varepsilon' d_Q^f$ and $\delta\leq d(F_fQ(\frac{4}{3}),H^n\setminus F_fQ(\frac{3}{2}))$. We show that there is a LIP homeomorphism $\mu\colon H^n\to H^n$ respecting X such that $d(\mu,F_f;H^n)<\delta$. Then the LIP embedding $\psi=\beta_Q^f\mu\alpha_Q|P(\frac{4}{3})$ satisfies the lemma. In fact, $d(\psi,F_Q^f;P(\frac{4}{3}))<\delta/d_Q^f\leq\varepsilon'$ and, since μ respects X and $\mu Q(\frac{4}{3})\subset F_fQ(\frac{3}{2})$, it follows by 3.7(2) and 3.9 that ψ respects X_Q.

Suppose first that $n\neq 4$, that $p\neq 4$ if $X=\mathbf{R}_+^{n,p}$, and that $p\neq 4,5$ if $X=\mathbf{R}_{++}^{n,p}$. Then the existence of μ follows from 2.8.

Suppose now that $n\neq 5$, that $p\neq 5$ if $X=\mathbf{R}_+^{n,p}$, and that $p\neq 5,6$ if $X=\mathbf{R}_{++}^{n,p}$. Then by 2.8 there is a LIP homeomorphism $g\colon(\mathbf{R}^{n-1},X_0)\to(\mathbf{R}^{n-1},X_0)$ such that $d(g,f)<\delta/3$. From [20; 2.16] (in whose proof the value of $r_0>0$ plays no role) it follows that the homeomorphism $F_g\colon(\mathbf{R}_+^n,X)\to(\mathbf{R}_+^n,X)$ is LIP. Since $d(F_g,F_f)\leq 3d(g,f)$, we can take $\mu=F_g|H^n$.

Only the cases (a) $(n,p)=(5,4)$ and (b) $X=\mathbf{R}_{++}^{n,5}$ remain. In these cases, by 2.8 and 2.9 there is a homeomorphism $g\colon(\mathbf{R}^{n-1},X_0)\to(\mathbf{R}^{n-1},X_0)$ such that

$d(g,f)<\delta/9$ and such that (a) $g|X_0$ or (b) $g|\partial X_0$, respectively, is LIP. Now $d(F_g,F_f)<\delta/3$. Since the homeomorphism $\varphi=F_g(g^{-1}\times\mathrm{id})\colon \mathbf{R}^n_+\to\mathbf{R}^n_+$ has the form $\varphi(x,t)=\big(x,\tau_g(g^{-1}(x),t)\big)$, by 2.10 there is a LIP homeomorphism $\varphi'\colon(\mathbf{R}^n_+,X)\to(\mathbf{R}^n_+,X)$ such that $d(\varphi',\varphi)<\delta/3$. Then $h=\varphi'(g\times\mathrm{id})\colon\mathbf{R}^n_+\to\mathbf{R}^n_+$ is a homeomorphism respecting X such that $d(h,F_g)<\delta/3$ and such that (a) $h|X$ or (b) $h|\partial X_0\times\mathbf{R}^1_+$, respectively, is LIP. Hence, by 2.8 there is a LIP homeomorphism $\mu\colon H^n\to H^n$ respecting X such that $d(\mu,h;H^n)<\delta/3$ implying $d(\mu,F_f;H^n)<\delta$. □

3.14. *Remark.* The application of 2.8 in the proof of 3.13 does not lead to a dependence on the stable homeomorphism theorem. For consider, e.g., the case $\partial X_0\neq\emptyset$. We may assume that the LQS homeomorphisms f and $f|\partial X_0$ are sense-preserving. Then by the proof of [19; 3.12] there are isotopies $f\cong\mathrm{id}$ and $f|\partial X_0\cong\mathrm{id}$, and these can be extended to isotopies $F_f\cong\mathrm{id}$ and $F_f|\,\mathrm{cl}\,(\partial X\cap H^n)\cong\mathrm{id}$. Thus, the homeomorphisms f, $f|\partial X_0$, $F_f|H^n$, and $F_f|\partial X\cap H^n$ are stable as needed if $(n,p)\neq(5,4)$ and $p\neq5$. On the other hand, if $(n,p)=(5,4)$ or $p=5$, it is easy to see that h is stable as needed.

3.15. Lemma. *There is $M=M(n,K)\geq1$ such that*

$$1/M\leq d_Q^f/d_R^f\leq M,\quad d_Q^f\leq Md(F_f\overline{Q}(\tfrac43),\mathbf{R}^{n-1})$$

whenever $Q,R\in\mathcal{K}$ with $Q\cap R\neq\emptyset$.

Proof. The former assertion can be proved as that in [19; 7.5] by the aid of 3.6(2) and 2.3. The latter assertion then follows from the fact that

$$\overline{Q}(\tfrac43)\subset\bigcup\{R\in\mathcal{K}\mid Q\cap R\neq\emptyset\}$$

by 3.6(3). □

3.16. Lemma. *There is $c'=c'(n,K)\geq1$ such that*

$$\beta_Q^f|F_f\overline{Q}(\tfrac43)\colon\big(F_f\overline{Q}(\tfrac43),\sigma\big)\to(\mathbf{R}^n,d)$$

is c'-BL for each $Q\in\mathcal{K}$.

Proof. As for [19; 7.6] by the aid of 3.6(2), 2.3, and 3.15. □

3.17. Constructions

For $0\leq i\leq N$ we set

$$V_i=\bigcup\{Q(1+2^{-i-1})\mid Q\in\mathcal{K}_i^*\},\quad W_i=\bigcup\{Q(1+2^{-i-2})\mid Q\in\mathcal{K}_i^*\};$$

then $V_0=W_0=\emptyset$. If $Q\in\mathcal{K}_i$ and $1<t\leq\frac{4}{3}$, we set

$$V_Q(t)=P(t)\cap\alpha_Q^{-1}V_{i-1},\quad W_Q(t)=P(t)\cap\alpha_Q^{-1}W_{i-1};$$

then, by 3.6(3) if $i\geq 2$ or trivially if $i=1$, $\alpha_Q V_Q(\frac{4}{3})$ is the union of the sets $Q_R=Q(\frac{4}{3})\cap R(1+2^{-i})$ where $R\in\mathcal{K}_{i-1}^*$ and $Q\cap R\neq\emptyset$. Clearly, the set

$$S=\{V_Q(\tfrac{4}{3})\mid Q\in\mathcal{K}\}\cup\{W_Q(\tfrac{4}{3})\mid Q\in\mathcal{K}\}$$

is finite.

We apply 2.7 for $Q\in\mathcal{K}$ with $U=P(\frac{4}{3})$, $U'=P(\frac{5}{4})$, $V=V_Q(\frac{4}{3})$, $W=W_Q(\frac{4}{3})$, $Y=X_Q$, and $\mathcal{F}=\mathcal{F}_{X_Q}=\{g|P(\frac{4}{3})\,|\,g\in\mathcal{G}_{X_Q}\}$; this is possible by 3.12 and 3.13. Since S and $\{X_Q|Q\in\mathcal{K}\}$ are finite, we obtain:

3.18. Lemma. *Let $\varepsilon'>0$ and $L\geq 1$. Then there are positive numbers $\delta=\delta(\varepsilon',n,K)\leq\varepsilon'$ and $L'=L'(\varepsilon',n,K,L)\geq L$ with the following property:*

Let $Q\in\mathcal{K}$, let $g\in\mathcal{F}_{X_Q}$, and let $h\colon V_Q(\frac{4}{3})\to\mathbf{R}^n$ be a locally L-BL embedding respecting X_Q such that $d(h,g;V_Q(\frac{4}{3}))\leq\delta$. Then there is an L'-BL embedding $h'\colon P(\frac{5}{4})\to\mathbf{R}^n$ respecting X_Q such that $d(h',g;P(\frac{5}{4}))\leq\varepsilon'$ and $h'=h$ on $W_Q(\frac{5}{4})$. □

3.19. Constructions

First note that $\frac{5}{4}, \frac{4}{3}, \frac{17}{12}, \frac{3}{2}$ is an arithmetic progression with difference $\frac{1}{12}$. By 3.12, there is a number $q=q(n,K)>0$ such that $|g(x)-g(y)|\geq q$ whenever $Q\in\mathcal{K}$, $g\in\mathcal{G}_{X_Q}$, and $x,y\in\overline{P}(\frac{17}{12})$ with $|x-y|\geq\varkappa/12$. Let c, c', and M be as in 3.10, 3.16, and 3.15, respectively. Define numbers $\delta_N\geq\delta_{N-1}\geq\ldots\geq\delta_0>0$ by $\delta_N=\min\big(q/(M+2),\varepsilon/c'\big)$ and $\delta_{j-1}=\delta(\delta_j,n,K)/M$, where $\delta(\ )$ is as in 3.18. We also define numbers $L_0\leq\ldots\leq L_N$ by $L_0=1$ and $L_j=cc'L'(\delta_j,n,K,\lambda_j)$, where $L'(\ )$ is as in 3.18, where $j\geq 1$, and where $\lambda_j=cc'ML_{j-1}$ (this M is erroneously missing in [19; 7.9]). Observe that the sequences $(\delta_0,\ldots,\delta_N)$ and $(L_0,\ldots,L_N)$ depend only on (n,K,ε). We show by induction that the following lemma is true for every integer $j\in[0,N]$:

3.20$_j$. Lemma. *There is an embedding $F_j\colon V_j\to H^n$ respecting X with the following properties:*

(1) *$d\big(F_j,F_f;Q(1+2^{-j-1})\big)\leq\delta_j d_Q^f$ for every $Q\in\mathcal{K}_j^*$.*

(2) *$F_jQ(1+2^{-j-1})\subset F_fQ(\frac{4}{3})$ for every $Q\in\mathcal{K}_j^*$.*

(3) *F_j is locally L_j-BLH.*

Proof. Since $V_0=\emptyset$, 3.20$_0$ is true. Suppose that 3.20$_{j-1}$ is true. Thus we have an embedding $F_{j-1}\colon V_{j-1}\to H^n$. We define $F_j(x)=F_{j-1}(x)$ for $x\in W_{j-1}$. Let $Q\in\mathcal{K}_j$.

Then $F_Q^f|P(\frac{3}{2})\in\mathcal{G}_{X_Q}$. Define an embedding $h_Q=\beta_Q^f F_{j-1}\alpha_Q|V_Q(\frac{4}{3})$. Consider $R\in\mathcal{K}_{j-1}^*$ with $Q\cap R\neq\emptyset$. By $3.20_{j-1}(1)$ and 3.15 we obtain

$$d(h_Q,F_Q^f;\alpha_Q^{-1}Q_R)=d(F_{j-1},F_f;Q_R)/d_Q^f\leq M\delta_{j-1}=\delta(\delta_j,n,K).$$

Hence, $d\big(h_Q,F_Q^f;V_Q(\frac{4}{3})\big)\leq\delta(\delta_j,n,K)$. For R as above we have

$$h_Q|\alpha_Q^{-1}Q_R=\big(\beta_Q^f(\beta_R^f)^{-1}\big)(\beta_R^f F_{j-1}|Q_R)(\alpha_Q|\alpha_Q^{-1}Q_R).$$

Here $F_{j-1}Q_R\subset F_fR(\frac{4}{3})$ by $3.20_{j-1}(2)$. Hence, $\beta_R^f|F_{j-1}Q_R$ is c'-BL between σ and d by 3.16. Moreover, $\beta_Q^f(\beta_R^f)^{-1}$ is M-BL in d by 3.15. Thus, $h_Q|\alpha_Q^{-1}Q_R$ is locally λ_j-BL in d by $3.20_{j-1}(3)$ and 3.10. Hence, h_Q is locally λ_j-BL. We show h_Q to respect X_Q. Suppose that R is still as above. Let $x\in Q_R$ and $y\in\partial\overline{Q}(\frac{17}{12})$. Since $|\alpha_Q^{-1}(x)-\alpha_Q^{-1}(y)|\geq\varkappa/12$, the choice of q implies $|F_Q^f(\alpha_Q^{-1}(x))-F_Q^f(\alpha_Q^{-1}(y))|\geq q$ yielding $|F_f(x)-F_f(y)|\geq qd_Q^f$. Since

$$|F_{j-1}(x)-F_f(x)|\leq M\delta_{j-1}d_Q^f\leq M\delta_N d_Q^f<qd_Q^f,$$

we conclude that $F_{j-1}Q_R\subset F_fQ(\frac{17}{12})$. Since F_{j-1} respects X, it follows by 3.7(2) and 3.9 that $h_Q|\alpha_Q^{-1}Q_R$ respects X_Q. Thus, indeed, h_Q respects X_Q. Hence, we can apply 3.18 with $\varepsilon'=\delta_j$, $g=F_Q^f|P(\frac{4}{3})$, $h=h_Q$. We obtain an (L_j/cc')-BL embedding $h_Q'\colon P(\frac{5}{4})\to\mathbf{R}^n$ respecting X_Q such that $d\big(h_Q',F_Q^f;P(\frac{5}{4})\big)\leq\delta_j$ and $h_Q'=h_Q$ on $W_Q(\frac{5}{4})$. Setting $F_j=(\beta_Q^f)^{-1}h_Q'\alpha_Q^{-1}$ on $Q(1+2^{-j-1})$ we obtain a well-defined map $F_j\colon V_j\to\mathbf{R}^n$. We show that F_j satisfies the conditions (1), (2), and (3), that F_j is injective, and that F_j respects X.

To prove (1), let $Q\in\mathcal{K}_j^*$. If $Q\in\mathcal{K}_{j-1}^*$, (1) follows from 3.20_{j-1}. If $Q\in\mathcal{K}_j$, we obtain

$$d\big(F_j,F_f;Q(1+2^{-j-1})\big)=d_Q^f d\big(h_Q',F_Q^f;P(1+2^{-j-1})\big)\leq\delta_j d_Q^f.$$

To prove (2), let again $Q\in\mathcal{K}_j^*$. If $Q\in\mathcal{K}_{j-1}^*$, (2) follows from 3.20_{j-1}. Suppose $Q\in\mathcal{K}_j$. Then $d\big(h_Q',F_Q^f;P(\frac{5}{4})\big)\leq\delta_j\leq\delta_N<q$ implying $h_Q'P(1+2^{-j-1})\subset F_Q^fP(\frac{4}{3})$. Hence (2) is true. Observe that (2) implies $F_jV_j\subset H^n$.

If $Q\in\mathcal{K}_{j-1}^*$, then F_j is locally L_j-BLH on $Q(1+2^{-j-1})$ by 3.20_{j-1}. If $Q\in\mathcal{K}_j$, then 3.10, 3.16, and (2) imply that $F_j|Q(1+2^{-j-1})$ is L_j-BLH. Hence, F_j is a locally L_j-BLH immersion.

We now show that F_j is injective. First, $F_j|W_{j-1}$ and $F_j|Q(1+2^{-j-1})$ for each $Q\in\mathcal{K}_j$ are injective. Moreover, if $Q,R\in\mathcal{K}_j^*$ and $Q\cap R=\emptyset$, then (2) and 3.6(3) imply that $F_jQ(1+2^{-j-1})\cap F_jR(1+2^{-j-1})=\emptyset$. Hence, it suffices to show that $F_j(x)\neq F_j(y)$ whenever $j\geq2$, $x\neq y$, $x\in Q(1+2^{-j-1})$, and $y\in R(1+2^{-j-1})$ where $Q\in\mathcal{K}_j$,

$R\in\mathcal{K}^*_{j-1}$, and $Q\cap R\neq\emptyset$. The equality $F_j=(\beta^f_Q)^{-1}h'_Q\alpha_Q^{-1}$ is valid on $Q(\frac{5}{4})\cap V_j$. Hence we may assume that $y\notin Q(\frac{5}{4})$. Since $d\big(F^f_Q(\alpha_Q^{-1}(x)),F^f_Q\partial\overline{P}(\frac{5}{4})\big)\geq q$, we then have $|F_f(x)-F_f(y)|\geq qd^f_Q$. By (1) and 3.15 we obtain

$$\begin{aligned}|F_j(x)-F_j(y)|&\geq|F_f(x)-F_f(y)|-|F_j(x)-F_f(x)|-|F_j(y)-F_f(y)|\\&\geq qd^f_Q-\delta_jd^f_Q-\delta_jd^f_R\geq\big(q-(M+1)\delta_N\big)d^f_Q\geq\delta_Nd^f_Q>0.\end{aligned}$$

It follows that $F_j\colon V_j\to H^n$ is an embedding.

We finally show F_j to respect X. First, $F_j|W_{j-1}$ respects X. Consider $Q\in\mathcal{K}_j$. Then $F_j|Q(1+2^{-j-1})$ respects X_Q. Hence, it follows from (2), 3.7(2), and 3.9 that $F_j|Q(1+2^{-j-1})$ respects X. Thus, F_j respects X. □

3.21. *Completion of the proof of* 3.1. We show that 3.1 is true with $L=L_N$ defined in 3.19. Let $F_N\colon H^n\to H^n$ be the map of 3.20_N. We show that the map $F=F_N\cup f\colon\mathbf{R}^n_+\to\mathbf{R}^n_+$ satisfies 3.1. First, F_N is an embedding, and F_N is locally L_N-BLH. Clearly F respects X. To prove the condition (1) of 3.1, let $x\in H^n$. Choose $Q\in\mathcal{K}$ containing x. Then $3.20_N(1)$ yields $|F_N(x)-F_f(x)|\leq\delta_Nd^f_Q\leq\varepsilon d^f_Q/c'$. By $3.20_N(2)$ and 3.16 this implies $\sigma\big(F_N(x),F_f(x)\big)\leq\varepsilon$. Hence (1) is true. Since $F_fH^n=H^n$, it follows from (1) that $F_NH^n=H^n$. From (1) it also follows that $F_N(x)\to f(x_0)$ as $x\to x_0\in\mathbf{R}^{n-1}$. Thus, F is a homeomorphism. Hence, (3) obtains by [19; 6.21]. Finally, (4) follows from (3) and from the analytic definition of quasiconformality [23; 34.2]. □

The following lemma will be needed in the last section.

3.22. Lemma. *Suppose that* $f|B^{n-1}(r)=\mathrm{id}$ *for some* $r>0$ *in Theorem* 3.1. *Then there is* $\omega_n\in(0,1)$ *depending only on* n *such that* F *can be chosen so as to satisfy* $F|C^n(\omega_nr)=\mathrm{id}$.

Proof. We check that the above construction for F works if just one choice is made more carefully. First note that $F_f|C^n(r)=\mathrm{id}$. Thus, if $Q\in\mathcal{K}$ and $Q(2)\subset C^n(r)$, then $d^f_Q=2\lambda_Q$ by 3.6(2), and therefore $F^f_Q|P(2)$ is a restriction of the map $\theta\colon\mathbf{R}^n\to\mathbf{R}^n$, $x\mapsto x/2$. Let $\omega=(1+2\sqrt{2n})^{-1}$. Then $Q\in\mathcal{K}$ and $Q\cap C^n(\omega r)\neq\emptyset$ imply $Q(2)\subset C^n(r)$; this follows by 3.6(2) from the facts $d(Q,\mathbf{R}^{n-1})<\omega r$ and $d\big(C^n(\omega r),\mathbf{R}^n_+\setminus C^n(r)\big)=(1-\omega)r/\sqrt{2}$. Let $\omega_n=\omega^N$.

We show that we can add to 3.20_j the following condition:

(4) $F_j|Q(1+2^{-j-1})=\mathrm{id}$ *for every* $Q\in\mathcal{K}^*_j$ *with* $Q\cap C^n(\omega^jr)\neq\emptyset$.

Let $Q\in\mathcal{K}^*_j$ with $Q\cap C^n(\omega^jr)\neq\emptyset$. If $Q\in\mathcal{K}^*_{j-1}$, then (4) follows from $3.20_{j-1}(4)$. Let $Q\in\mathcal{K}_j$. Consider $R\in\mathcal{K}^*_{j-1}$ with $Q\cap R\neq\emptyset$. Since $Q(2)\subset C^n(\omega^{j-1}r)$, we conclude that $F_{j-1}|Q_R=\mathrm{id}$. Hence, $h_Q=\theta$ on $\alpha_Q^{-1}Q_R$. Thus, $h_Q=\theta|V_Q(\frac{4}{3})$. Now note that θ

is 2-BL with $2 \le M \le \lambda_j \le L_j/cc'$ and that θ respects X_Q. Therefore we can define $h'_Q = \theta | P(\frac{5}{4})$. It follows that $F_j | Q(1+2^{-j-1}) = \mathrm{id}$.

We now have $F | C^n(\omega_n r) = \mathrm{id}$. □

4. Complementary results

In this section we first apply Theorem 3.1 to show in 4.1 that $F|X$ in 3.1 can be prescribed if F is claimed to be quasiconformal only. In 4.3 we consider Euclidean Lipschitz properties that F inherits from f in 3.1. Theorem 4.4 uses another extension method at the limit $K \to 1$. Higher codimensional extension is the topic of 4.5. Theorems 4.6 and 4.8 are corollaries for quasisymmetric homeomorphisms of sphere pairs.

4.1. Theorem. *Let $1 \le p \le n \ge 2$, let $X \in \mathcal{X}(n,p)$, and let*

$$f: (\mathbf{R}^{n-1}, X_0) \to (\mathbf{R}^{n-1}, X_0) \quad \textit{and} \quad g: (X, X_0) \to (X, X_0)$$

be homeomorphisms with $f|X_0 = g|X_0$. Let f and g be K-QC and $\tau = (n, K)$ whenever $p \ge 2$; let $f \cup g: \mathbf{R}^{n-1} \cup X \to \mathbf{R}^{n-1} \cup X$ be η-QS and $\tau = (n, \eta)$ whenever $p = 1$. Then there is a K^*-QC homeomorphism $F: \mathbf{R}^n_+ \to \mathbf{R}^n_+$ with $F|\mathbf{R}^{n-1} = f$ and $F|X = g$ where K^* depends only on τ.

Proof. By 3.1 there is a K_0-QC homeomorphism $F_0: (\mathbf{R}^n_+, X) \to (\mathbf{R}^n_+, X)$ extending f where $K_0 = K_0(\tau)$. By replacing (f, g) by $(F_0^{-1} f, F_0^{-1} g)$ we may assume $f = \mathrm{id}$. Then $g|X_0 = \mathrm{id}$. If $X = \mathbf{R}^{n,p}_{++}$, let $X_1 = \mathbf{R}^{n,p}_+$; then we can extend g by reflection to a K-QC homeomorphism $g_1: X_1 \to X_1$ with $g_1 | \partial X_1 = \mathrm{id}$. Thus, we may assume $X = \mathbf{R}^{n,p}_+$. Then, proceeding by induction on $n-p$, we may assume $p = n-1$. Let $Y = \{x \in \mathbf{R}^n_+ \,|\, x_1 \ge 0\}$ and $Z = \{x \in \mathbf{R}^n_+ \,|\, x_1 \le 0\}$; then $\mathbf{R}^n_+ = Y \cup Z$ and $X = Y \cap Z$. Let $\varphi_Y = (f \cup g)|\partial Y$ and $\varphi_Z = (f \cup g)|\partial Z$. There are L-BL homeomorphisms $\alpha: Y \to \mathbf{R}^n_+$ and $\beta: Z \to \mathbf{R}^n_+$ with L an absolute constant such that $\alpha | Y \cap \mathbf{R}^{n-1} = \mathrm{id}$ and $\beta | Z \cap \mathbf{R}^{n-1} = \mathrm{id}$ and such that $\alpha|X$ and $\beta|X$ are isometric. Then the self-homeomorphisms $g_Y = \alpha \varphi_Y \alpha^{-1}$ and $g_Z = \beta \varphi_Z \beta^{-1}$ of $\mathbf{R}^{n-1}$ are K-QC if $p \ge 2$ by [23; 35.1] and η_1-QS with $\eta_1(t) = L^2 \eta(L^2 t)$ if $p = 1$. By 3.1 there are K_1-QC self-homeomorphisms G_Y and G_Z of $\mathbf{R}^n_+$ extending g_Y and g_Z, respectively, with $K_1 = K_1(\tau)$. Let $K^* = L^{4n-4} K_1$. Then $F_Y = \alpha^{-1} G_Y \alpha: Y \to Y$ and $F_Z = \beta^{-1} G_Z \beta: Z \to Z$ are K^*-QC homeomorphisms extending φ_Y and φ_Z, respectively. Thus, $F = F_Y \cup F_Z: \mathbf{R}^n_+ \to \mathbf{R}^n_+$ is the desired K^*-QC homeomorphism. □

4.2. *Remark.* The assumptions in 4.1 are necessary. For consider first a K-QC homeomorphism $F: \mathbf{R}^n_+ \to \mathbf{R}^n_+$, $n \ge 2$. Then F is η-QS with η depending only on K by [2; 5.23]. Suppose $n \ge 3$. Then $F|\mathbf{R}^{n-1}$ is K-QC by [7; Corollary] as is also

$F|X$ by [7; Theorem 2] if $2 \le p \le n$, $X \in \mathcal{X}(n,p)$, and F respects X. Moreover, the case $p=1$ of 4.1 really differs from the case $p \ge 2$. For example, the homeomorphism $f: \mathbf{R}^{n-1} \to \mathbf{R}^{n-1}$, $x \mapsto |x|x$, is QS (by [23; 16.2] if $n \ge 3$) as is also $g = \mathrm{id}|\mathbf{R}_+^{n,1}$, but $f \cup g$ is not LQS at 0.

4.3. Preservation of Lipschitz properties in 3.1

Consider the behaviour in the Euclidean metric of the homeomorphism F as given in Theorem 3.1. We show that F inherits various Lipschitz properties from f. Of course, $F|H^n$ is LIP. For $z=(x,t) \in H^n$ write $\delta(z) = d(z, \mathbf{R}^{n-1}) = t$. Then $L_d(z,F) = L_\sigma(z,F)\delta(F(z))/\delta(z)$ and $L_d(F(z), F^{-1}) = L_\sigma(F(z), F^{-1})\delta(z)/\delta(F(z))$. Here $L_\sigma(z,F) \le L$ and $L_\sigma(F(z), F^{-1}) \le L$ by 3.1(3), $e^{-\varepsilon} \le \delta(F(z))/\delta(F_f(z)) \le e^{\varepsilon}$ by 3.1(1), and $\delta(F_f(z)) = \tau_f(x,t)$. Suppose that $f|B^{n-1}(r)$ is λ-Lipschitz. If $z \in C^n(r)$, then $\tau_f(x,t) \le \lambda t$ and thus $L_d(z,F) \le \lambda^* = e^{\varepsilon} L \lambda$. Hence, $F|C^n(r)$ is λ^*-Lipschitz. If $f^{-1}|fB^{n-1}(r)$ is λ-Lipschitz and $z \in C^n(r)$, then $\tau_f(x,t) \ge t/\lambda$ and thus $L_d(F(z), F^{-1}) \le \lambda^*$. It follows, e.g., that if f is λ-BL, then F is λ^*-BL, and that if f is LIP, then F is LIP. Assume now $p \ge 2$. Choose a homeomorphism $\eta: \mathbf{R}_+^1 \to \mathbf{R}_+^1$ depending only on (n,K) such that f is η-QS. If $z \in X$, define $\tau_f^0(z) = \max\{|f(x) - f(y)| \,|\, y \in X_0,\ |x-y| = t\}$. Then $\tau_f^0(z) \le \tau_f(z) \le \eta(1)\tau_f^0(z)$. It follows that if $f|X_0$ is λ-BL, then $F|X$ is $\eta(1)e^{\varepsilon}L\lambda$-BL, and that if $f|X_0$ is LIP, then $F|X$ is LIP. In the case $p=1$, note that if f is as in 4.2, then $|F(0,t)| \le e^{\varepsilon}\tau_f(0,t) = e^{\varepsilon}t^2$ for $t>0$, and therefore $F|X$ is not LIP.

For K-QC homeomorphisms with K sufficiently close to 1 there is an elementary and explicit extension method due to Tukia and Väisälä [21; 5.4], who considered more generally s-QS embeddings $f: \mathbf{R}^k \to \mathbf{R}^n$, $k<n$, with small s. In the next theorem we check that in our special case this method is respectful. The BLH condition of the theorem was known to Tukia and Väisälä [26; 4.5].

4.4. Theorem. *Let $1 \le p \le n \ge 2$ and $X \in \mathcal{X}(n,p)$. Then there is $K_0 = K_0(n) > 1$ with the following property: Let $f: (\mathbf{R}^{n-1}, X_0) \to (\mathbf{R}^{n-1}, X_0)$ be a K-QC homeomorphism with $K \le K_0$. Then there is a homeomorphism $F: (\mathbf{R}_+^n, X) \to (\mathbf{R}_+^n, X)$ which satisfies the conditions (2)–(4) of 3.1 with $L = L(n,K) \to 1$ as $K \to 1$ and for which $F|H^n$ is piecewise-affine.*

Proof. Replace p by m and let $X' = \mathbf{R}_+^{n,m}$. It suffices to modify the proof of [21; 5.4] in the special case $f\mathbf{R}^p = \mathbf{R}^p$ with $p = n-1$ as follows. First, make sure that the vertices of the cubes in $\mathcal{J}(n-1)$ of side length 1 are in $\mathbf{Z}^{n-1}$ and that each of $X' \cap H^n$ and $\partial X \cap H^n$ is the underlying space of a subcomplex of the triangulation W of $\mathbf{R}^n \setminus \mathbf{R}^{n-1}$. Let each $(n-1)$-frame w_Q be ordered as $(w_Q^{n-m+1}, \dots, w_Q^{n-1}, w_Q^1, \dots, w_Q^{n-m})$ when forming the orthonormal $(n-1)$-frame $v_Q = G(w_Q)$. Observe that in [21] no use has been made of the stated sense-preserving nature of various embeddings. Thus, it is not necessary to assume the n-frames u_Q to be positively oriented, and we can define $u_Q^n = e_n$ for each $Q \in \mathcal{J}(n-1)$. It follows that $h_Q \mathbf{R}_+^n = \mathbf{R}_+^n$ for each Q, that $h_Q X = X$ if $a_Q \in \partial X_0$, and that $h_Q X' = X'$ if $a_Q \in X_0$. Then $g\mathbf{R}_+^n = \mathbf{R}_+^n$. For each vertex b of W in X we choose the cube $Q = Q(b)$ in such a way that $a_Q \in X_0$, with $a_Q \in \partial X_0$ whenever $b \in \partial X$. If $X = X'$, it follows that $gX_0 = X_0$ and $gX \subset X$, which imply that $gX = X$. In the case $X \neq X'$ we have that $g[\partial X] \subset \partial X$, yielding $g[\partial X] = \partial X$, and $gX \subset X'$, from which we conclude that $gX = X$.

Finally, from [21; (5.7), (5.8), and 3.5] it easily follows that $L_\sigma(x,g) \leq L_1$ and $l_\sigma(x,g) \geq 1/L_1$ for each $x \in H^n$, where

$$L_1 = L_1(n,q) = \bigl(1 + 18n^2(n+1)M(n)q\bigr)/(1 - 9n^2 q) \to 1$$

as $q \to 0$, which implies that $g|H^n$ is L_1-BLH. Thus, $F = g|\mathbf{R}_+^n$ satisfies the theorem. □

4.5. Extension from dimension $n-1$ to $n+k$

Let $1 \leq p \leq n \geq 2$ and $k \geq 1$. For simplicity we consider only the case $X_0 = \mathbf{R}^{n-1,p-1}$; cf. 5.5 for the case $X_0 = \mathbf{R}_+^{n-1,p-1}$. Suppose that f is a K-QC self-homeomorphism of $(\mathbf{R}^{n-1}, \mathbf{R}^{n-1,p-1})$. Let F be the K_0^*-QC self-homeomorphism of $(\mathbf{R}_+^n, \mathbf{R}_+^{n,p})$ extending f with $K_0^* = K_0^*(n,K)$ whose existence is guaranteed by 3.1. Then we can extend F by reflection to a K_0^*-QC self-homeomorphism F_0 of $(\mathbf{R}^n, \mathbf{R}^{n,p})$. Repeating this process, we can extend f to a K_k^*-QC self-homeomorphism F_k of $(\mathbf{R}^{n+k}, \mathbf{R}^{n+k,p+k})$ with $K_k^* = K_k^*(n+k,K)$. By 4.4 we may assume that $K_k^* \to 1$ as $K \to 1$. By 3.1 and 4.4 we can choose $F|H^n$ to be L-BLH with $L = L(n,K) \to 1$ as $K \to 1$. Then F_k can also be obtained by rotating F around $\mathbf{R}^{n-1}$. More precisely, let $z \in \mathbf{R}^{n+k}$ and write $z = (x, te)$ where $x \in \mathbf{R}^{n-1}$, $e \in S^k$, and $t \geq 0$. Then define $F_k(x,te) = (x', t'e)$ where $(x',t') = F(x,t)$. Now it is easy to show that $F_k|\mathbf{R}^{n+k} \setminus \mathbf{R}^{n-1}$ is L-BLH; cf. [20; 3.13]. Then F_k is $L^{2(n+k-1)}$-QC. Still further, the same method of [21; 5.4] we used for 4.4 gives directly a number $K_k = K_k(n+k) > 1$ and in the case $K \leq K_k$ an extension F_k such that $F_k|\mathbf{R}^{n+k} \setminus \mathbf{R}^{n-1}$ is piecewise-affine and L_k-BLH with $L_k = L_k(n+k,K) \to 1$ as $K \to 1$.

The previous theorems imply analogous results, Theorems 4.6 and 4.8, on extending QS self-homeomorphisms to ball pairs from the bounding sphere pairs. The absolute case $p = n$ of the first theorem and a generalization of the absolute case of the second theorem to embeddings $f\colon S^k \to \mathbf{R}^n$, $k < n$, are due to Tukia and Väisälä ([20; 3.15.4], [22; 2.18] and, respectively, [21; 5.23], [26; 4.6]).

4.6. Theorem. *Let* $1\le p\le n\ge 2$ *and let* $f\colon(S^{n-1},S^{p-1})\to(S^{n-1},S^{p-1})$ *be an* η-QS *homeomorphism. Then* f *can be extended to an* η^*-QS *homeomorphism* $F\colon(\bar{B}^n,\bar{B}^p)\to(\bar{B}^n,\bar{B}^p)$ *where* η^* *depends only on* n *and* η. *Moreover, one of the following two conditions can be added:*

(1) *We can choose* $F|B^n$ *to be* L-BLH *with* $L=L(n,\eta)$.

(2) *If* $g\colon\bar{B}^p\to\bar{B}^p$ *is an* η-QS *homeomorphism with* $f|S^{p-1}=g|S^{p-1}$ *and if* $f\cup g$ *is* η-QS *whenever* $p=1$, *then we can choose* $F|\bar{B}^p=g$.

Proof. We may assume that $f(e_1)=e_1$. Let $X=\mathbf{R}_+^{n,p}$. Choose a Möbius homeomorphism $\varphi\colon\dot{\mathbf{R}}^n\to\dot{\mathbf{R}}^n$ such that $\varphi H^n=B^n$, that $\varphi(\infty)=e_1$, and that $\varphi\dot{\mathbf{R}}^{n,p}=\dot{\mathbf{R}}^p$. Then $\varphi X=\bar{B}^p\setminus\{e_1\}$. By [25; 3.2, (1.8), and 3.10], the homeomorphism f_1: $(\mathbf{R}^{n-1},X_0)\to(\mathbf{R}^{n-1},X_0)$ defined by $\varphi^{-1}f\varphi$ is θ-QS with θ depending only on η as also in (2) are the homeomorphism $g_1\colon X\to X$ defined by $\varphi^{-1}g\varphi$ and $f_1\cup g_1$ whenever $p=1$. By 3.1 we can extend f_1 to a K-QC homeomorphism $F_1\colon(\mathbf{R}_+^n,X)\to(\mathbf{R}_+^n,X)$ with $K=K(n,\eta)$. Moreover, we may assume in (1) by 3.1 that $F_1|H^n$ is L-BLH with $L=L(n,\eta)$ and in (2) by 4.1 that $F_1|X=g_1$. Extend F_1 to a homeomorphism $F_1\colon\dot{\mathbf{R}}_+^n\to\dot{\mathbf{R}}_+^n$. Then $F=\varphi F_1\varphi^{-1}\colon(\bar{B}^n,\bar{B}^p)\to(\bar{B}^n,\bar{B}^p)$ is a K-QC homeomorphism extending f such that $F|B^n$ is L-BLH in (1) and that $F|\bar{B}^p=g$ in (2). Now $|F(0)|\le a(n,\eta)<1$ by [22; 2.17]. Hence, there is a K_0-QC homeomorphism $h\colon\mathbf{R}^n\to\mathbf{R}^n$ such that $h(F(0))=0$, $h|\mathbf{R}^n\setminus B^n=\mathrm{id}$, and $K_0=K_0(n,\eta)$. Let $F_2\colon\mathbf{R}^n\to\mathbf{R}^n$ be the K_0K-QC homeomorphism obtained from hF by reflection; then $F_3=h^{-1}F_2\colon\mathbf{R}^n\to\mathbf{R}^n$ is K_0^2K-QC and $F=F_3|\bar{B}^n$. Hence, F is η^*-QS with η^* depending only on (n,η). □

4.7. Lemma. *Let* $n\ge 2$, *and let* $f\colon\bar{B}^n\to\bar{B}^n$ *be a* K-QC *homeomorphism such that* $f|S^{n-1}$ *is* s-QS *with* $s\le\frac{1}{4}$. *Then* $|f(0)|\le a=a(n,K,s)<1$ *with* $a\to 0$ *as* $K\to 1$ *and* $s\to 0$.

Proof. By [26; 2.3], $f|S^{n-1}$ is η-QS with a universal homeomorphism η. Hence, $|f(0)|\le b(n,K)<1$ by [22; 2.17]. Suppose that the second assertion of the lemma is not true. Then for some $\varepsilon>0$ there are K_j-QC homeomorphisms $f_j\colon\bar{B}^n\to\bar{B}^n$ with $f_j|S^{n-1}$ being s_j-QS such that $K_j\to 1$, $\frac{1}{4}\ge s_j\to 0$, and $|f_j(0)|\ge\varepsilon$. Applying [18; 3.5–3.7] and passing to a subsequence, we may assume that $(f_j|S^{n-1})$ converges uniformly to a homeomorphism $h\colon S^{n-1}\to S^{n-1}$, which must be an isometry. Extending each f_j by reflection to a K_j-QC homeomorphism $g_j\colon\dot{\mathbf{R}}^n\to\dot{\mathbf{R}}^n$, applying [23; 19.4(2), 20.5, 21.5, and 37.3], and passing again to a subsequence, we may assume that (f_j) converges uniformly to a Möbius homeomorphism $g\colon\bar{B}^n\to\bar{B}^n$. Since $g|S^{n-1}=h$, it follows that g is an isometry, which contradicts the inequality $|g(0)|\ge\varepsilon$. □

4.8. Theorem. *Let $1 \le p \le n \ge 2$. Then there is $s_0 = s_0(n) > 0$ with the following property: Let $f: (S^{n-1}, S^{p-1}) \to (S^{n-1}, S^{p-1})$ be an s-QS homeomorphism with $s \le s_0$. Then f can be extended to an s^*-QS homeomorphism $F: (\bar{B}^n, \bar{B}^p) \to (\bar{B}^n, \bar{B}^p)$ such that $F|B^n$ is L-BLH where $s^* = s^*(n,s) \to 0$ and $L = L(n,s) \to 1$ as $s \to 0$.*

Proof. We may assume that $f(e_1) = e_1$. Define X, φ, f_1 as in the proof of 4.6. From [25; 3.8 and 3.10] we see that there is an absolute constant $s_1 > 0$ such that if $s \le s_1$, then f_1 is K-QC with $K = K(n,s) \to 1$ as $s \to 0$. Thus, choosing $s_0 = s_0(n) > 0$ with $s_0 \le \min(s_1, \frac{1}{4})$ small enough and assuming $s \le s_0$, by 4.4 we can extend f_1 to a homeomorphism $F_1: (\mathbf{R}^n_+, X) \to (\mathbf{R}^n_+, X)$ such that $F_1|H^n$ is L-BLH with $L = L(n,s) \to 1$ as $s \to 0$. Define F and choose h as in the proof of 4.6 but now by 4.7 with $K_0 = K_0(n,s) \to 1$ as $s \to 0$. Then F satisfies the theorem. □

5. Extension of locally quasisymmetric homeomorphisms

In this section we use mainly the term LQS rather than the term LQC. We prove LQS versions of Theorems 3.1 and 4.1 and of the higher codimensional extension 4.5.

In the following lemma ω_n is the number of 3.22.

5.1. Lemma. *Let $1 \le p \le n \ge 2$, let $X \in \mathcal{X}(n,p)$, let $f: (\mathbf{R}^{n-1}, X_0) \to (\mathbf{R}^{n-1}, X_0)$ be an LQS homeomorphism, and let $A \subset \mathbf{R}^n_+$ be compact. Then there is $r_0 > 0$ with the following property: For every $r \ge r_0$ there is a QS homeomorphism $\varphi: (\mathbf{R}^n_+, X) \to (\mathbf{R}^n_+, X)$ such that $\varphi = f$ on $B^{n-1}(r)$, that $\varphi|H^n$ is BLH, and that $\varphi B^n_+(r_0) \supset A$. Moreover, if $f|B^{n-1}(s) = \mathrm{id}$ for some $s > 0$ and if $r \ge s$, then φ can be chosen so as to satisfy $\varphi|C^n(\omega_n s) = \mathrm{id}$.*

Proof. We may replace f by $f - f(0)$ and A by $A - f(0)$ arriving thus always at the situation $f(0) = 0$. Choose $r_0 > 0$ with $F_f B^n_+(r_0) \supset A$. Consider $r \ge r_0$. If it is assumed $f|B^{n-1}(s) = \mathrm{id}$, assume $r \ge s$. Let $C = C^n(2r)$. Then $B^n_+(r_0) \subset C$.

We wish first to extend $f_1 = f|B^{n-1}(2r)$ to a QC self-homeomorphism g of $(\mathbf{R}^{n-1}, X_0)$. By [23; 34.7] or [9; Theorem 4], f_1 is QC. If $n=2$, the existence of g follows from [9; Theorem 5]. Suppose that $n \ge 3$. Let $\alpha: \dot{\mathbf{R}}^{n-1} \to \dot{\mathbf{R}}^{n-1}$ be the inversion in $S^{n-2}(2r)$; then $\alpha \dot{X}_0 = \dot{X}_0$. The LQC embedding $h_0 = \alpha f \alpha: \bar{B}^{n-1}(2r) \setminus \{0\} \to \mathbf{R}^{n-1}$ respects X_0. Thus, by the relative (or respectful) Schoenflies theorem [6; 2.4] (for a slightly corrected and completed proof of which see [12]), there is a QC embedding $h: \bar{B}^{n-1}(2r) \to \mathbf{R}^{n-1}$ extending $h_0|S^{n-2}(2r)$ and, if $p \ne 1$, respecting X_0. By composing h with a suitable QC homeomorphism $\mathbf{R}^{n-1} \to \mathbf{R}^{n-1}$ which respects X_0 if $p \ne 1$ we may assume that $h(0) = 0$ always. Then $g = f_1 \cup (\alpha h \alpha | \mathbf{R}^{n-1} \setminus B^{n-1}(2r))$ is the desired homeomorphism.

We have that $F_g = F_f$ on C. Consider $\varepsilon > 0$. By 3.1 and 3.22 there is a QS homeomorphism $\varphi: (\mathbf{R}^n_+, X) \to (\mathbf{R}^n_+, X)$ extending g such that $\sigma(\varphi, F_g; H^n) < \varepsilon$, such that $\varphi | H^n$ is BLH, and that $\varphi | C^n(\omega_n s) = \mathrm{id}$ if $f | B^{n-1}(s) = \mathrm{id}$. Then $\sigma(\varphi, F_f; C \cap H^n) < \varepsilon$. By choosing ε small enough we have that $d(\varphi, F_f; B^n_+(r_0))$ is so small that $\varphi B^n_+(r_0) \supset A$. Then φ satisfies the lemma. □

5.2. Theorem. *Let* $1 \le p \le n \ge 2$, *let* $X \in \mathcal{X}(n,p)$, *and let* $f: (\mathbf{R}^{n-1}, X_0) \to (\mathbf{R}^{n-1}, X_0)$ *be an* LQS *homeomorphism. Then* f *can be extended to an* LQS *homeomorphism* $F: (\mathbf{R}^n_+, X) \to (\mathbf{R}^n_+, X)$ *which is* LIPH *on* H^n.

Proof. We construct inductively numbers $r_j \ge j$ and QS self-homeomorphisms φ_j of $(\mathbf{R}^n_+, X)$ for $j \ge 1$ such that, setting $s_j = 2r_j/\omega_n$ $(> r_j)$, we have that $\varphi_j = f$ on $B^{n-1}(s_j)$, that $\varphi_j | H^n$ is BLH, that $\varphi_j B^n_+(r_j) \supset B^n_+(j)$, that $r_{j+1} > r_j$, and that $\varphi_{j+1} = \varphi_j$ on $C_j = C^n(\omega_n s_j) \supset B^n_+(r_j)$. We obtain r_1 and φ_1 from 5.1.

Suppose that we have constructed r_j and φ_j. Define an LQS homeomorphism $g = \varphi_j^{-1} f: (\mathbf{R}^{n-1}, X_0) \to (\mathbf{R}^{n-1}, X_0)$. Then $g | B^{n-1}(s_j) = \mathrm{id}$. Thus, by 5.1 there are a number $r_{j+1} \ge \max(j+1, s_j)$ and a QS homeomorphism $\varphi: (\mathbf{R}^n_+, X) \to (\mathbf{R}^n_+, X)$ such that $\varphi = g$ on $B^{n-1}(s_{j+1})$, that $\varphi | H^n$ is BLH, that $\varphi B^n_+(r_{j+1}) \supset \varphi_j^{-1} B^n_+(j+1)$, and that $\varphi | C_j = \mathrm{id}$. Then $\varphi_{j+1} = \varphi_j \varphi: (\mathbf{R}^n_+, X) \to (\mathbf{R}^n_+, X)$ is the desired homeomorphism.

By setting $F = \varphi_j$ on $B^n_+(r_j)$ for each j we obtain the desired LQS homeomorphism $F: (\mathbf{R}^n_+, X) \to (\mathbf{R}^n_+, X)$ extending f and LIPH on H^n. □

5.3. *Remarks.* 1. The absolute case $X = \mathbf{R}^n_+$ of 5.2 without the LIPH property was proved in [10; 9.2]. The above proof for it is a simplification of that in [10].

2. In 5.2, if $p \ge 2$ and $f | X_0$ is LIP, then F can be chosen such that $F | X$ is LIP. This follows from the construction of F, where in the proof of 5.1 note that [12; Theorem 3] produces h with $h | h^{-1} X_0$ LIP whenever $h_0 | h_0^{-1} X_0$ is LIP and that $\varphi | X$ is LIP by 4.3 whenever $g | X_0$ is LIP.

5.4. Theorem. *Let* $1 \le p \le n \ge 2$, *let* $X \in \mathcal{X}(n,p)$, *and let* $f: (\mathbf{R}^{n-1}, X_0) \to (\mathbf{R}^{n-1}, X_0)$ *and* $g: (X, X_0) \to (X, X_0)$ *be* LQS *homeomorphisms with* $f | X_0 = g | X_0$ *and such that* $f \cup g$ *is* LQS *at* 0 *if* $p = 1$. *Then there is an* LQS *homeomorphism* $F: \mathbf{R}^n_+ \to \mathbf{R}^n_+$ *extending* f *and* g.

Proof. The proof is similar to that of 4.1; only resort to 5.2. □

5.5. Extension from dimension $n-1$ to $n+k$

We consider only the case $X_0 = \mathbf{R}^{n-1,p-1}_+$; cf. 4.5 for the case $X_0 = \mathbf{R}^{n-1,p-1}$. However, for notational reasons we change X_0. Thus, define

$$\widehat{\mathbf{R}}^{n,p}_+ = \{x \in \mathbf{R}^{n,p} \mid x_{n-p+1} \ge 0\}$$

for $1 \le p \le n$ as then $(\mathbf{R}^{n+k}, \widehat{\mathbf{R}}_+^{n+k,p+k}) = (\mathbf{R}^n, \widehat{\mathbf{R}}_+^{n,p}) \times \mathbf{R}^k$ for $k \ge 1$. Now suppose that f is an LQS self-homeomorphism of $(\mathbf{R}^{n-1}, \widehat{\mathbf{R}}_+^{n-1,p-1})$ with $2 \le p \le n$. Let F be the LQS self-homeomorphism of $(\mathbf{R}^{n-1}, \widehat{\mathbf{R}}_+^{n-1,p-1}) \times \mathbf{R}_+^1$ extending f which is given by 5.2. Then we can extend F by reflection to an LQS self-homeomorphism F_0 of $(\mathbf{R}^n, \widehat{\mathbf{R}}_+^{n,p})$. Repeating this process, we can extend f to an LQS self-homeomorphism F_k of $(\mathbf{R}^{n+k}, \widehat{\mathbf{R}}_+^{n+k,p+k})$ for each $k \ge 0$. Alternatively, as $F|H^n$ can be chosen LIPH, we can also obtain F_k for each $k \ge 1$ by rotating F around $\mathbf{R}^{n-1}$ as in 4.5; now $F_k|\mathbf{R}^{n+k} \setminus \mathbf{R}^{n-1}$ is LIPH and F_k thus LQS.

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