ASYMPTOTIC PARTITION FORMULAE. III. PARTITIONS INTO k-th POWERS.[1]

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Introduction.

1.1. In a well-known memoir[2] Hardy and Ramanujan obtained the asymptotic expansion of $p(n)$, the number of unrestricted partitions of n, by applying Cauchy's Theorem to the generating function

$$\prod_{l=1}^{\infty}(1-x^l)^{-1} = 1 + \sum_{n=1}^{\infty} p(n)x^n. \tag{1.11}$$

My object here is to find the asymptotic expansion of $p_k(n)$, the number of partitions of n into k-th powers.[3] The result is to some extent a generalisation of theirs, and part of this paper consists of an application of their method to the new generating function

$$f(x) = \prod_{l=1}^{\infty}(1-x^{l^k})^{-1} = 1 + \sum_{n=1}^{\infty} p_k(n)x^n.$$

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[1] The previous papers of this series are

I. Plane partitions. *Quart. J. of Math.* (Oxford series), 2 (1931), 177—189.

II. Weighted partitions. *Proc. London Math. Soc.* (2), 36 (1933), 117—141.

The present paper may be read independently of these.

[2] *Proc. London Math. Soc.* (2), 17 (1918), 75—115. The reader will find it interesting to compare the methods and results of this paper with my work here.

[3] The problem was suggested to me by Professor Hardy, to whom my thanks are also due for much valuable advice in the course of the investigation.

Before we can use this method, however, we have to solve two subsidiary problems which are much more complicated for general k than in the particular case $k = 1$ considered by Hardy and Ramanujan. It is obvious that $f(x)$ is regular for $|x| < 1$ and that every point of the circumference $|x| = 1$ is an essential singularity of the function. We must know the nature of the singularities of $f(x)$ at the rational points

$$x = e^{\frac{2p\pi i}{q}}$$

or, what is the same thing, the behaviour of $f(x)$ as x approaches such a point. This information is contained in a certain transformation formula.

When $k = 1$, $f(x)$ is, apart from a trivial factor, the reciprocal of an elliptic modular function, and the transformation theory follows at once from the properties of such functions. For general k this is not the case, and we have to develop the transformation theory of $f(x)$ entirely afresh. This work takes up the whole of Part I of this paper. We state the resulting formula as a theorem; it is apparently new and possibly of some interest in itself, apart from the application with which we are here chiefly concerned.

We find that as x approaches 1, $f(x)$ behaves to a first approximation like

$$\frac{x^j}{(2\pi)^{\frac{1}{2}k}}\left(\log\frac{1}{x}\right)^{\frac{1}{2}} \exp\left\{\frac{\Gamma\left(1+\frac{1}{k}\right)\zeta\left(1+\frac{1}{k}\right)}{\left(\log\frac{1}{x}\right)^{\frac{1}{k}}}\right\}, \qquad (1.12)$$

where $\Gamma(t)$ and $\zeta(t)$ are the ordinary Gamma and Riemann-zeta functions. This brings us to our second subsidiary problem, namely the determination of an auxiliary function with an isolated singularity of the type (1.12) at $x = 1$. In a recent note[1], I introduced a generalised Bessel function to enable me to construct such an auxiliary function, and discussed its properties.

We write

$$a = \frac{1}{k}, \quad b = \frac{1}{k+1}, \quad j = j(k) = 0 \qquad (k \text{ even}),$$

$$j = j(k) = \frac{(-1)^{\frac{1}{2}(k+1)}}{(2\pi)^{k+1}}\Gamma(k+1)\zeta(k+1) \quad (k \text{ odd}).^{2}$$

[1] *Journal London Math. Soc.*, 8 (1933), 71—79.

[2] If $k = 1$, then $j = -\frac{1}{24}$. Readers familiar with the memoir of Hardy and Ramanujan referred to above will recollect the appearance of the number $-\frac{1}{24}$.

Then the appropriate generalised Bessel function here is

$$\varphi(z) = \sum_{l=0}^{\infty} \frac{z^l}{\Gamma(l+1)\,\Gamma\left(la - \frac{1}{2}\right)};$$

and our simplest result, found by examining in detail the singularity of $f(x)$ at $x = 1$, is contained in the following theorem.

Theorem 1. *We have*

$$p_k(n) = (n+j)^{-\frac{3}{2}} (2\pi)^{-\frac{1}{2}k} \varphi\{\Gamma(1+a)\zeta(1+a)(n+j)^a\} + O(e^{(\Delta-\alpha)n^b}),$$

where

$$\Delta = (k+1)\{a\Gamma(1+a)\zeta(1+a)\}^{1-b}, \quad \alpha = \alpha(k) > 0.$$

Using the known asymptotic expansion[1] of $\varphi(z)$ we can deduce the expansion of $p_k(n)$ in terms of elementary functions of n.

Theorem 2. *We have*

$$p_k(n) = B_0 (n+j)^{b-\frac{3}{2}} e^{\Delta(n+j)^b} \left\{1 + \sum_{m=1}^{M-1} \frac{(-1)^m b_m}{(n+j)^{mb}} + O\left(\frac{1}{n^{Mb}}\right)\right\},$$

where

$$B_0 = \frac{\Delta}{(2\pi)^{\frac{1}{2}(k+1)}} \frac{k^{\frac{1}{2}}}{(k+1)^{\frac{3}{2}}},$$

and b_m is a function of k and m only, which may be calculated with sufficient labour for any given values of k and m; in particular

$$b_1 = \frac{11k^2 + 11k + 2}{24k\Delta}.$$

Hardy and Ramanujan (loc. cit. p. 111) give the result

$$p_k(n) \sim B_0 n^{b-\frac{3}{2}} e^{\Delta n^b}$$

without proof. This is equivalent to Theorem 2 with $M = 1$, but without any statement as to the order of the error term. Apart from this, Theorem 2 is, I believe, new.

[1] Lemma 33.

1.2. We shall, however, go much further than Theorem 1. We have by Cauchy's Theorem

$$p_k(n) = \frac{1}{2\pi i}\int_{\Gamma} \frac{f(x)\,dx}{x^{n+1}},$$

where Γ is a circle with centre at $x = 0$ and radius less than unity. We shall take the radius of Γ very nearly unity. Then the dominant term of $p_k(n)$ given in Theorem 1 arises from the integral along the portion of Γ in the immediate neighbourhood of $x = 1$. Every point of the circumference of the unit circle is an essential singularity of $f(x)$, as we saw, but the singularity at $x = 1$ is the »heaviest». When we consider the next »heaviest» singularities, and also find a better approximation than (1.12) for $f(x)$ near $x = 1$, we can find for any $\varepsilon > 0$ an expansion for $p_k(n)$ with error only $O(e^{\varepsilon n^b})$.

The numbers k and q are positive integers. The number p is also an integer, satisfying the conditions

$$1 \leq p < q, \quad (p, q) = 1, \tag{1.21}$$

except that, when $q = 1$, $p = 0$ only. We use $\sum_p$ to denote summation over all such values of p for the particular value of q in question. We write q_1 for the least positive integer such that $q \mid q_1^k$. Then $q_1 \mid q$ and we may write $q = q_1 q_2$. We define d_h by

$$p h^k \equiv d_h \pmod q, \quad 0 \leq d_h < q;$$

also

$$\omega_{p,q} = 1 \qquad (k \text{ even}),$$

$$\omega_{p,q} = \exp\left\{\pi i \left(\frac{1}{q^2}\sum_{h=1}^{q} h\, d_h - \frac{1}{4}(q - q_2)\right)\right\} \qquad (k \text{ odd}),$$

$$e_q(l) = \exp\left(\frac{2\pi i l}{q}\right), \quad S_{l,q} = \sum_{h=1}^{q} e_q(l h^k),$$

$$A_{p,q} = \frac{\Gamma(1+a)}{q}\sum_{m=1}^{\infty}\frac{S_{pm,q}}{m^{1+a}},$$

$$C_{p,q} = \left(\frac{q_1}{2\pi}\right)^{\frac{1}{2}k}\omega_{p,q},$$

$$c_{q,0} = 1, \quad A_{p,q,0} = A_{p,q}.$$

For values of $t > 0$, $c_{q,t}$ is a q-th root of unity, depending on k, q and t, and $\Lambda_{p,q,t}$ is a function of k, p, q and t. For any particular values of these parameters, $c_{q,t}$ and $\Lambda_{p,q,t}$ may be calculated, but the process is in general very tedious. The method of calculation will be described later (see § 9.4). Then our full result is

Theorem 3. *For any $\varepsilon > 0$, there exist an integer $q_0 = q_0(k, \varepsilon)$ and integers $T_{q,\varepsilon} = T(k, q, \varepsilon)$, such that*

$$p_k(n) = (n+j)^{-\frac{3}{2}} \sum_{q \leq q_0} \sum_p C_{p,q}\, e_q(-pn) \sum_{t=0}^{T_{q,\varepsilon}} c_{q,t}\, \varphi(\Lambda_{p,q,t}(n+j)^a) + O(e^{\varepsilon n^b}).$$

When $k = 1$, that is, in the case of unrestricted partitions, we have

$$j = -\frac{1}{24}, \quad q_1 = q, \quad T_{q,\varepsilon} = 0, \quad \Lambda_{p,q} = \frac{\pi^2}{6q},$$

$$\varphi(z) = \sum_{l=0}^{\infty} \frac{z^l}{\Gamma(l+1)\Gamma\left(l-\frac{1}{2}\right)} = \frac{z^{\frac{3}{2}}}{\pi^{\frac{1}{2}}} \frac{d}{dz}\left\{\frac{\cosh(2z^{\frac{1}{2}})}{z^{\frac{1}{2}}}\right\}.$$

If we write

$$D = \frac{2\pi}{\sqrt{6}}, \quad \varrho_n = \left(n - \frac{1}{24}\right)^{\frac{1}{2}},$$

we have

$$\varphi(\Lambda_{p,q}\varrho_n^2) = \frac{\varrho_n^3}{2\pi^{\frac{1}{2}}} \frac{d}{dn}\left\{\frac{e^{\frac{D\varrho_n}{q}}}{\varrho_n}\right\} + O(n^{\frac{1}{2}} e^{-2n^{\frac{1}{2}}}),$$

and so the expansion becomes

$$p_1(n) = \frac{q^{\frac{1}{2}}}{2\pi\sqrt{2}} \sum_{q \leq q_0} \frac{d}{dn}\left\{\frac{e^{\frac{D\varrho_n}{q}}}{\varrho_n}\right\} \sum_p \omega_{p,q}\, e_q(-pn) + O(e^{\varepsilon n^{\frac{1}{2}}}).$$

Hardy and Ramanujan obtained the last result. In fact, they went further; taking q_0 an appropriate function of n, they made the error term only $O(n^{-\frac{1}{4}})$. If we attempt to make a similar improvement for $p_k(n)$, we find that we can choose $q_0 = q_0(k, n)$ so that the error term is $O(e^{n^d})$, where $d < b$. This is not so good as the result for $k = 1$, and, in view of the heavy analysis required for this further step, I am content to prove Theorem 3.

Part I. Transformation of the Generating Function.

2. 1. Our first step is to find the transformation formula for $f(x)$ for general k, which will exhibit its behaviour near the rational points on the circumference $|x| = 1$. Our result includes that for $k = 1$, which may also be deduced from the theory of modular functions. For general k no such deduction is possible.

Before stating the transformation formula, we must define certain symbols.

The numbers p, h, s and l are integers. Of these, p satisfies the conditions (1. 22) and

$$1 \leq h \leq q, \quad 1 \leq s \leq k, \quad l \geq 0.$$

If $d_h \neq 0$, we write

$$\mu_{h,s} = \frac{d_h}{q} \text{ (} s \text{ odd)}, \quad \mu_{h,s} = \frac{q - d_h}{q} \text{ (} s \text{ even)}.$$

If $d_h = 0$, we take $\mu_{h,s} = 1$. Hence always $q\mu_{h,s} \geq 1$.

In connection with any particular values of p and q we write

$$X = x e_q(-p) = e^{-y}, \quad Y = q^k y,$$

and we take y real and positive when X is real and $0 < X < 1$. We write also

$$t_s = \left(\frac{2\pi}{Y}\right)^a \exp\left\{a\pi i\left(s - \frac{1}{2}\right)\right\},$$

where Y^a is that k-th root of Y which is real and positive when Y is real and positive,

$$g(h, l, s) = \exp\{2\pi i (l + \mu_{h,s})^a t_s\}\, e_q(-h)$$

$$= \exp\left\{\frac{(2\pi)^{1+a}(l + \mu_{h,s})^a e^{\frac{1}{2}\pi a i (2s+k-1)}}{q y^a} - \frac{2h\pi i}{q}\right\},$$

and finally

$$P_{p,q} = \prod_{h=1}^{q} \prod_{s=1}^{k} \prod_{l=0}^{\infty} \{1 - g(h, l, s)\}^{-1}.$$

We are now in a position to state our transformation theorem.

Theorem 4. *If* $\Re(y) > 0$, *then* $P_{p,q}$ *is convergent and*

$$f(x) = f(e^{-y} e_q(p)) = C_{p,q} y^{\frac{1}{2}} e^{jy} \exp\left(\frac{\Lambda_{p,q}}{y^a}\right) P_{p,q}. \qquad (2.11)$$

In the case $k = 1$, (2. 11) reduces to the modular function transformation used by Hardy and Ramanujan.[1] These authors also stated Theorem 4 in the particular case $k = 2$, $q = 1$, $p = 0$, without proof.

To prove Theorem 4 we need two lemmas.

Lemma 1. *If* C *is any positive number and if*

$$\Re(y) > C|y|^{1+a}, \qquad (2.12)$$

then $P_{p,q}$ *is uniformly convergent with respect to* y.

Lemma 2. *If* y *is real and positive, then* (2. 11) *is true.*

From these two lemmas it follows by the principle of analytic continuation that (2. 11) is true when (2. 12) is satisfied. But for any particular value of y such that $\Re(y) > 0$, we can choose C so that (2. 12) is satisfied. Hence Theorem 4 is an immediate consequence of Lemmas 1 and 2.

2. 2. **Proof of Lemma 1.**

Lemma 3. *If*

$$-\frac{1}{2}\pi < \psi < \frac{1}{2}\pi, \quad \frac{1}{2}\pi(1+a) \leq \vartheta \leq \frac{1}{2}\pi(3-a),$$

then

$$\cos(\vartheta - a\psi) \leq \max\left\{\cos\left(\frac{1}{2}\pi(1+a) - a\psi\right), \cos\left(-\frac{1}{2}\pi(1+a) - a\psi\right)\right\}$$

$$\leq -\frac{2a}{\pi}\cos\psi.$$

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[1] When $k = 1$, my definition of $\omega_{p,q}$ differs in form from that appearing in the transformation used by these authors. The latter definition does not involve the sum

$$\sum_{h=1}^{q-1} h d_h.$$

H. Rademacher (»Zur Theorie der Modulfunktionen», *Journal für Math.*, 167 (1932), 312—336) obtained the modular function transformation for $f(x)$ when $k = 1$ with the form of $\omega_{p,q}$ that I use here. The same author showed (*Journal London Math. Soc.*, 7 (1932), 14—19) by an elementary method that the two definitions are equivalent.

We have

$$\frac{1}{2}\pi < \frac{1}{2}\pi(1+a) - a\psi \leq \vartheta - a\psi \leq \frac{1}{2}\pi(3-a) - a\psi < \frac{3}{2}\pi,$$

and so

$$\begin{aligned}\cos(\vartheta - a\psi) &\leq \max\left\{\cos\left(\frac{1}{2}\pi(1+a) - a\psi\right), \cos\left(\frac{1}{2}\pi(3-a) - a\psi\right)\right\} \\ &= \max\left\{\cos\left(\frac{1}{2}\pi(1+a) - a\psi\right), \cos\left(-\frac{1}{2}\pi(1+a) - a\psi\right)\right\}. \qquad (2.21)\end{aligned}$$

Also

$$\begin{aligned}\cos\left\{\pm\frac{1}{2}\pi(1+a) - a\psi\right\} &= \cos\left\{\frac{1}{2}\pi(1+a) \mp a\psi\right\} \\ &\leq \cos\left\{\frac{1}{2}\pi(1+a) - a|\psi|\right\} = -\sin\left\{a\left(\frac{1}{2}\pi - |\psi|\right)\right\}. \qquad (2.22)\end{aligned}$$

Since

$$0 < a\left(\frac{1}{2}\pi - |\psi|\right) \leq \frac{1}{2}\pi,$$

we have

$$\sin\left\{a\left(\frac{1}{2}\pi - |\psi|\right)\right\} \geq \frac{2a}{\pi}\left(\frac{1}{2}\pi - |\psi|\right) \geq \frac{2a}{\pi}\sin\left(\frac{1}{2}\pi - |\psi|\right) = \frac{2a}{\pi}\cos\psi. \qquad (2.23)$$

The lemma follows from (2.21), (2.22) and (2.23).

Lemma 4. *If* (2.12) *is satisfied and if*

$$C_1 = \frac{4a\,C(2\pi)^a}{q^{1+a}},$$

then

$$|g(h, l, s)| \leq e^{-C_1(l+1)^a}.$$

Let us write $y = |y|e^{i\psi}$, where ψ has its principal value. Since $\Re(y) > 0$, we have $|\psi| < \frac{1}{2}\pi$, and since $1 \leq s \leq k$, the number $\vartheta = \frac{1}{2}\pi a(2s + k - 1)$ satisfies the condition of Lemma 3. Hence, when

$$\Re(y) > C|y|^{1+a},$$

we have

$$\cos\left\{\frac{1}{2}\pi a(2s+k-1)-a\psi\right\}\le-\frac{2a}{\pi}\cos\psi=-\frac{2a\Re(y)}{\pi|y|}<-\frac{2aC|y|^{1}}{\pi},$$

and so

$$|g(h,l,s)|=\exp\left\{\frac{(2\pi)^{1+a}(l+\mu_{h,s})^a\cos\left\{\frac{1}{2}\pi a(2s+k-1)-a\psi\right\}}{q|y|^a}\right\}$$

$$\le\exp\left\{-\frac{2aC(2\pi)^{1+a}(l+\mu_{h,s})^a}{q\pi}\right\}$$

$$=\exp\{-C_1q^a(l+\mu_{h,s})^a\}\le e^{-C_1(l+1)^a},$$

since

$$q(l+\mu_{h,s})\ge l+q\mu_{h,s}\ge l+1.$$

To prove Lemma 1, it is sufficient to show that

$$\sum_{l=0}^{\infty}|g(h,l,s)|$$

is uniformly convergent with respect to y. But this follows from Lemma 4, since

$$\sum_{l=0}^{\infty}e^{-C_1(l+1)^a}$$

is convergent and the terms are independent of y.

Proof of Lemma 2.

3.1. The remainder of Part I is devoted to the proof of Lemma 2. The work is necessarily somewhat lengthy and complicated.

We take y real and positive and write

$$\tau=\tau_h=\frac{h}{q},\quad\nu=\nu_h=\exp(2\pi i\mu_{h,1})=e_q(d_h)=e_q(ph^k),$$

$$\lambda_h=\Gamma(1+a)\sum_{m=1}^{\infty}\frac{e_q(pmh^k)}{m^{1+a}},$$

and, if k is odd,

$$j_h=j_h(q,k)=\frac{(-1)^{\frac{1}{2}(k+1)}}{(2\pi)^{k+1}}\Gamma(k+1)\sum_{m=1}^{\infty}\frac{e_q(mh)}{m^{k+1}},$$

$$\sigma_h=\sigma_{q-h}=(\mu_{q-h,1}-\mu_{h,1})(\tau_{q-h}-\tau_h),$$

while, if k is even,

$$j_h = 0, \quad \sigma_h = 0.$$

We write also

$$S_h = \sum_{l=0}^{\infty} \log \{1 - \nu_h \exp(-Y(l+\tau_h)^k)\},$$

$$S'_h = \sum_{s=1}^{k} \sum_{l=0}^{\infty} \log \{1 - g(h, l, s)\},$$

where the logarithms have their principal values.

Now we have

$$x^{(ql+h)^k} = e_q(ph^k)\, X^{(ql+h)^k} = \nu_h e^{-Y(l+\tau_h)^k},$$

and so

$$-\log f(x) = \sum_{h=1}^{q} \sum_{l=0}^{\infty} \log(1 - x^{(ql+h)^k}) = \sum_{h=1}^{q} S_h,$$

$$f(x) = \exp\left\{-\sum_{h=1}^{q} S_h\right\}.$$

Also, by definition,

$$P_{p,q} = \exp\left\{-\sum_{h=1}^{q} S'_h\right\}.$$

We take ϱ a small number, real, positive and less than $\frac{1}{2}$, and r a large positive integer. We use D to denote a positive number, not always the same at each occurrence, depending at most on k, p, q and y. In Part I, the symbol $O(\;)$ refers to the passage of ϱ to zero and of r to infinity, and the constant implied is of the type D.

3.2. Certain differences arise between the cases k odd and k even, but we shall treat the two cases together. Most of the differences are covered by the notation; thus, when k is odd, the functions j, j_h, σ_h and $\omega_{p,q}$ have various more or less complicated values, while, when k is even, j, j_h and σ_h all vanish and $\omega_{p,q} = 1$. We see that the transformation (2.11) has a slightly simpler form when k is even.

Apart from this, we have to consider three separate cases arising from different values of h. These are:

(i) If $h = q$, we have $\tau_h = 1$, $\mu_h = 1$, $\nu_h = 1$, and

$$S_q = \sum_{l=1}^{\infty} \log(1 - e^{-Yl^k}).$$

(ii) If $h^k \equiv 0 \pmod{q}$, but $h \neq q$, we have $0 < \tau_h < 1$, $\mu_{h,s} = 1$, $\nu_h = 1$ and

$$S_h = \sum_{l=0}^{\infty} \log(1 - e^{-Y(l+\tau)^k}).$$

(iii) If $h^k \not\equiv 0 \pmod{q}$, we have $0 < \tau_h < 1$, and $\nu_h \neq 1$.

In cases (ii) and (iii) we shall treat S_h and S_{q-h} together. Since $h^k \equiv 0 \pmod{q}$ implies that $(q-h)^k \equiv 0 \pmod{q}$, h and $q-h$ appear in the same case. Also, in these two cases,

$$\tau_h + \tau_{q-h} = 1;$$

if k is even,

$$\mu_{h,s} = \mu_{q-h,s}, \quad \nu_h = \nu_{q-h},$$

while if k is odd

$$\mu_{h,s} = \mu_{q-h,s} = 1, \quad \nu_h \nu_{q-h} = 1 \quad \text{(case (ii))},$$

$$\mu_{h,s} + \mu_{q-h,s} = 1, \quad \nu_h \nu_{q-h} = 1 \quad \text{(case (iii))}.$$

If Σ'' denotes summation over those values of h for which $h^k \not\equiv 0 \pmod{q}$, we have, when k is odd,

$$\begin{aligned}
\Sigma'' \sigma_h &= \Sigma''(\mu_{q-h,1} - \mu_{h,1})(\tau_{q-h} - \tau_h) \\
&= \Sigma''(1 - 2\mu_{h,1})(1 - 2\tau_h) \\
&= \Sigma'' 1 - 2\Sigma''\mu_{h,1} - 2\Sigma''\tau_h + 4\Sigma''\tau_h\mu_{h,1} \\
&= \Sigma'' 1 - \Sigma''(\mu_{h,1} + \mu_{q-h,1}) - \Sigma''(\tau_h + \tau_{q-h}) + 4\Sigma''\tau_h\mu_{h,1} \\
&= 4\Sigma''\tau_h\mu_{h,1} - \Sigma'' 1 \\
&= \frac{4}{q^2}\sum_{h=1}^{q-1} h d_h - (q - q_2).
\end{aligned}$$

Then, when k is odd or even, we have

$$\omega_{p,q} = \exp\left(\frac{1}{4}\pi i \Sigma'' \sigma_h\right). \tag{3.21}$$

4.1. *Case* (i): $h = q$.

Lemma 5. *We have*[1]

$$S'_q - S_q = \frac{1}{2}\log Y - \frac{1}{2}k\log 2\pi + \frac{\lambda_q}{Y^a} + jY.$$

We write

$$\chi_1(u) = \frac{kYu^{k-1}\log(1 - e^{2\pi i u})}{e^{Yu^k} - 1}, \quad \chi_2(u) = \frac{kYu^{k-1}\log(1 - e^{-2\pi i u})}{e^{Yu^k} - 1}.$$

We shall define the value of the logarithms precisely at a later stage. The singularities of $\chi_1(u)$ and of $\chi_2(u)$ in the u-plane are at the roots of the equations

$$e^{Yu^k} = 1, \quad e^{2\pi i u} = 1.$$

The functions have simple poles at the points

$$u = l^a t_s \qquad (s = 1, 2, \ldots 2k;\ l = 1, 2, \ldots),$$

and for a given value of s the poles lie along a semi-infinite straight line from the origin inclined at an angle $a\pi\left(s - \frac{1}{2}\right)$ to the positive direction of the real axis. If k is odd, two of these lines coincide with the upper and lower halves of the imaginary axis respectively. The only other singularities of $\chi_1(u)$ and of $\chi_2(u)$ are logarithmic singularities at the points

$$u = l \qquad (l = \ldots -2, -1, 1, 2, \ldots)$$

on the real axis, and a singularity at the origin of a slightly more complicated character.

We shall take ϱ subject to the additional condition

$$\varrho < \frac{1}{2}|t_s|,$$

so that the only singularity of $\chi_1(u)$ or of $\chi_2(u)$ for which $|u| \leq 2\varrho$ is the one at the origin. We write $R = r + \frac{1}{2}$ and choose L an integer such that

[1] If $q = 1$, then $p = 0$, $Y = y$, $h = q$, and

$$S_q = -\log f(x), \quad S'_q = -\log P_{0,1},$$

so that Theorem 4 follows at once from Lemma 5. This enables us to shorten our work considerably if our only object is the proof of Theorem 2 (see § 12.2).

$$\frac{R^k Y}{2\pi} < L < \frac{(2R)^k Y}{2\pi} - \frac{1}{2}.$$

This is always possible for r greater than some number D. Then we write

$$R' = \left\{\frac{2\pi}{Y}\left(L + \frac{1}{2}\right)\right\}^a.$$

We see that $R < R' < 2R$.

We now define several contours. The contour α_1 is coincident with the real axis from $u = \varrho$ to $u = R$ except near $u = 1, 2, \ldots r$; near such a point α_1 passes round a small semi-circle above the point. The contour β_1 consists of the imaginary axis from $u = i\varrho$ to $u = iR'$; if k is odd, $\chi_1(u)$ has poles on the imaginary axis and β_1 is deformed so as to pass to the *left* of these poles.

The contour γ_1 has three parts γ_1', γ_1'', γ_1'''. On γ_1' we have $u = Re^{i\theta}$ and $0 \leq \theta \leq \frac{1}{4}a\pi$; on γ_1'', $u = ve^{\frac{1}{4}a\pi i}$ and $R \leq v \leq R'$; on γ_1''', $u = R'e^{i\theta}$ and $\frac{1}{4}a\pi \leq \theta \leq \frac{1}{2}\pi$. Finally, δ_1 is the quadrant of the circle $|u| = \varrho$ on which $0 \leq \theta \leq \frac{1}{2}\pi$, and Γ_1 is the closed contour formed by combining $\alpha_1, \beta_1, \gamma_1$ and δ_1. The contours $\alpha_2, \beta_2, \gamma_2, \delta_2$ and Γ_2 are the reflexions of $\alpha_1, \beta_1, \gamma_1, \delta_1$ and Γ_1 in the real axis, except that, if k is odd, β_2 passes to the *right* of the poles on the real axis, so that Γ_2 excludes these poles.

We take the positive directions of α_1, α_2, β_1 and β_2 outwards from the origin, of γ_1, γ_2, δ_1 and δ_2 counterclockwise round the origin and of Γ_1 and Γ_2 counterclockwise. Then we have

$$\Gamma_1 = \alpha_1 - \beta_1 + \gamma_1 - \delta_1, \tag{4.11}$$

$$\Gamma_2 = -\alpha_2 + \beta_2 + \gamma_2 - \delta_2. \tag{4.12}$$

Let us consider the transformation $z = e^{2\pi i u}$. So long as u does not pass outside Γ_1, z will not pass outside a contour in the z-plane consisting of the circumference of the unit circle indented at $z = 1$, this indentation corresponding to the indentations of α_1 at the points $u = 1, 2, \ldots$ Then, if $\log(1 - z)$ be taken real at $z = -1$ $\left(\text{corresponding to } u = \frac{1}{2}\right)$, the logarithm will have its principal value so long as z remains within or on the contour. Hence, if we take

$\log(1 - e^{2\pi i u})$ real at $u = \frac{1}{2}$ on Γ_1, $\log(1 - e^{2\pi i u})$ has its principal value and $\chi_1(u)$ is one-valued at any point on or in the interior of Γ_1.

If we take $\log(1 - e^{-2\pi i u})$ real at $u = \frac{1}{2}$ on Γ_2, a similar argument shows that this logarithm has its principal value at every point on or within Γ_2.

We write

$$I(\Gamma_1) = \int_{\Gamma_1} \chi_1(u)\,du = I(\alpha_1) - I(\beta_1) + I(\gamma_1) - I(\delta_1),$$

$$I(\Gamma_2) = \int_{\Gamma_2} \chi_2(u)\,du = -I(\alpha_2) + I(\beta_2) + I(\gamma_2) - I(\delta_2).$$

A little calculation shows that Γ_1 includes the poles of $\chi_1(u)$ at the points

$$u = l^a t_s \qquad \left(1 \leq s \leq \frac{1}{2}(k+1);\ 1 \leq l \leq L\right),$$

while Γ_2 includes the poles of $\chi_2(u)$ at the points

$$u = l^a t_{k+s} \qquad \left(\frac{1}{2}(k+1) < s \leq k,\ 1 \leq l \leq L\right).$$

Then if k is even, by Cauchy's Theorem,

$$I(\Gamma_1) = 2\pi i \sum_{s=1}^{\frac{1}{2}k} \sum_{l=1}^{L} \log\{1 - \exp(2\pi i l^a t_s)\},$$

$$I(\Gamma_2) = 2\pi i \sum_{s=\frac{1}{2}k+1}^{k} \sum_{l=1}^{L} \log\{1 - \exp(-2\pi i l^a t_{k+s})\},$$

while, if k is odd,

$$I(\Gamma_1) = 2\pi i \sum_{s=1}^{\frac{1}{2}(k+1)} \sum_{l=1}^{L} \log\{1 - \exp(2\pi i l^a t_s)\},$$

$$I(\Gamma_2) = 2\pi i \sum_{s=\frac{1}{2}(k+3)}^{k} \sum_{l=1}^{L} \log\{1 - \exp(-2\pi i l^a t_{k+s})\}.$$

In either case, since $t_{k+s} = -t_s$, we have

$$I(\Gamma_1) + I(\Gamma_2) = 2\pi i \sum_{s=1}^{k} \sum_{l=1}^{L} \log\{1 - \exp(2\pi i l^a t_s)\}$$

$$= 2\pi i \sum_{s=1}^{k} \sum_{l=0}^{L-1} \log\{1 - g(q, l, s)\}. \tag{4. 13}$$

All the above logarithms have their principal values.

4. 2. **Lemma 6.** *If l is any positive integer, and ψ any real number, then*

$$U = \left| e^{2\pi i\left(l+\frac{1}{2}\right)e^{i\psi}} - 1 \right| > D.$$

We have

$$U^2 = 1 - 2e^{-\pi(2l+1)\sin\psi}\cos\{\pi(2l+1)\cos\psi\} + e^{-2\pi(2l+1)\sin\psi}$$
$$\geq \left(1 - e^{-\pi(2l+1)\sin\psi}\right)^2.$$

If

$$\frac{1}{2(2l+1)^{\frac{1}{2}}} \leq \sin\psi \leq 1,$$

then

$$U^2 \geq \left(1 - e^{-\frac{1}{2}\pi(2l+1)^{\frac{1}{2}}}\right)^2 > D$$

If

$$-1 \leq \sin\psi \leq -\frac{1}{2(2l+1)^{\frac{1}{2}}},$$

then

$$U^2 \geq \left(e^{\frac{1}{2}\pi(2l+1)^{\frac{1}{2}}} - 1\right)^2 > D.$$

There remains the case

$$-\frac{1}{2(2l+1)^{\frac{1}{2}}} \leq \sin\psi \leq \frac{1}{2(2l+1)^{\frac{1}{2}}}.$$

If $\cos\psi \geq 0$,

$$1 - \cos\psi = \frac{\sin^2\psi}{1+\cos\psi} \leq \sin^2\psi \leq \frac{1}{4(2l+1)}.$$

If $\cos\psi \leq 0$,

$$1 + \cos\psi = \frac{\sin^2\psi}{1-\cos\psi} \leq \sin^2\psi \leq \frac{1}{4(2l+1)}.$$

Hence, in either case,

$$\cos\psi = \pm 1 + \frac{\xi}{4(2l+1)}, \quad -1 \leq \xi \leq 1,$$

and so

$$\cos\{\pi(2l+1)\cos\psi\} = \cos\left\{\pm\pi(2l+1) + \frac{1}{4}\xi\pi\right\}$$

$$= -\cos\left(\frac{1}{4}\xi\pi\right) < 0.$$

Therefore

$$U^2 = 1 - 2e^{-\pi(2l+1)\sin\psi}\cos\{\pi(2l+1)\cos\psi\} + e^{-2\pi(2l+1)\sin\psi} > 1.$$

Hence, for all values of $\sin\psi$, we have $U^2 > D$, and so $U > D$.

Lemma 7. *If $r > D$, we have*

$$|I(\gamma_1)| < De^{-Dr}, \quad |I(\gamma_2)| < De^{-Dr}. \tag{4.21}$$

(i) Let u lie on γ_1', that is, let $u = Re^{i\theta}$, $0 \leq \theta \leq \frac{1}{4}a\pi$. Then we have

$$\Re(Yu^k) = YR^k\cos k\theta \geq YR^k\cos\frac{1}{4}\pi > DR^k,$$

and, for $R > D$,

$$|e^{Yu^k} - 1| > e^{DR^k} - 1 > De^{DR^k}.$$

By Lemma 6,

$$|1 - e^{2\pi i u}| = |1 - e^{2\pi i(r+\frac{1}{2})e^{i\theta}}| > D.$$

Hence

$$D < |1 - e^{2\pi i u}| \leq 2,$$

and

$$|\log(1 - e^{2\pi i u})| < D.$$

Then

$$\left|\int_{\gamma_1'} \chi_1(u)\,du\right| < DR^k e^{-DR^k} < De^{-Dr}. \tag{4.22}$$

(ii) Let u lie on γ_1'', that is, let $u = ve^{\frac{1}{4}a\pi i}$, $R \leq v \leq R' < 2R$. Then

$$\Re(Yu^k) = Yv^k\cos\frac{1}{4}\pi \geq DR^k.$$

Hence, for $R > D$,

$$|e^{Yu^k} - 1| \geq e^{DR^k} - 1 > De^{DR^k}.$$

Also

$$|e^{2\pi i u}| = e^{-2\pi r \sin \frac{1}{4} a\pi} < e^{-DR},$$

and so

$$|\log(1 - e^{2\pi i u})| < De^{-DR}.$$

Hence

$$\left|\int_{\gamma_1''} \chi_1(u)\,du\right| < DR'^k e^{-DR-DR^k} < De^{-Dr}. \tag{4.23}$$

(iii) Let u lie on γ_1''', that is, let $u = R'e^{i\theta}$ and $\frac{1}{4}a\pi \leq \theta \leq \frac{1}{2}\pi$. Then

$$e^{Yu^k} - 1 = e^{YR'^k e^{ki\theta}} - 1 = e^{2\pi i (L+\frac{1}{2}) e^{i\psi}} - 1,$$

where $\psi = k\theta - \frac{1}{2}\pi$, and so by Lemma 6

$$|e^{Yu^k} - 1| > D.$$

Also

$$|e^{2\pi i u}| \leq e^{-2\pi R' \sin \frac{1}{4} a\pi} < e^{-DR'},$$

and

$$|\log(1 - e^{2\pi i u})| < De^{-DR'}.$$

Hence

$$\left|\int_{\gamma_1'''} \chi_1(u)\,du\right| < DR'^k e^{-DR'} < De^{-Dr}. \tag{4.24}$$

Since $\gamma_1 = \gamma_1' + \gamma_1'' + \gamma_1'''$, the first part of (4.21) follows from (4.22), (4.23) and (4.24). The second part may be proved in the same way.

4.3. **Lemma 8.** *We have*

$$I(\delta_1) = \frac{1}{2}k\pi i\left(\log 2\pi + \log\varrho - \frac{1}{4}\pi\right) + O(\varrho\log\varrho), \tag{4.31}$$

$$I(\delta_2) = \frac{1}{2}k\pi i\left(\log 2\pi + \log\varrho + \frac{1}{4}\pi\right) + O(\varrho\log\varrho). \tag{4.32}$$

These results are a matter of simple calculation and we omit the proof.

Lemma 9. *We have*

$$I(\beta_2) - I(\beta_1) = 2\pi i j Y + O(\varrho^k \log \varrho) + O(e^{-DR}). \tag{4.33}$$

We replace u in $I(\beta_2)$ by $-u$; then β_2 becomes β_1. If k is even, we have at once

$$I(\beta_2) = I(\beta_1), \quad j = 0,$$

so that (4. 33) is true, the right hand side being in fact zero.

If k is odd, we have

$$I(\beta_2) = (-1)^k k Y \int_{\beta_1} \frac{u^{k-1} \log(1 - e^{2\pi i u})}{e^{-Yu^k} - 1} du$$

$$= k Y \int_{\beta_1} \frac{u^{k-1} \log(1 - e^{2\pi i u})}{1 - e^{-Yu^k}} du.$$

Since

$$\frac{1}{1 - e^{-Yu^k}} - \frac{1}{e^{Yu^k} - 1} = 1,$$

we have

$$I(\beta_2) - I(\beta_1) = k Y \int_{\beta_1} u^{k-1} \log(1 - e^{2\pi i u}) du$$

$$= k Y \int_{i\varrho}^{iR'} u^{k-1} \log(1 - e^{2\pi i u}) du,$$

for the integrand has no singularities on the imaginary axis between $u = i\varrho$ and $u = iR'$, and so β_1 may be deformed into this portion of the imaginary axis.

If we put $u = iv$, we have

$$I(\beta_2) - I(\beta_1) = k Y i^k \int_{\varrho}^{R'} v^{k-1} \log(1 - e^{-2\pi v}) dv$$

$$= k Y i^k \int_0^{\infty} v^{k-1} \log(1 - e^{-2\pi v}) dv$$

$$+ O(\varrho^k \log \varrho) + O(e^{-DR'}). \tag{4.34}$$

Also

$$k\int_0^\infty v^{k-1}\log(1-e^{-2\pi v})\,dv = -k\sum_{m=1}^\infty \frac{1}{m}\int_0^\infty v^{k-1}e^{-2\pi m v}\,dv$$

$$= -\frac{\Gamma(k+1)\,\zeta(k+1)}{(2\pi)^k}. \tag{4.35}$$

Then (4.33) follows from (4.34), (4.35) and the definition of j.

4.4. **Lemma 10.** *We have*

$$I(\alpha_1)-I(\alpha_2) = 2\pi i S_q + 2\pi i \lambda_q Y^{-a}$$
$$+\pi i\log(Y\varrho^k) + O(\varrho\log\varrho) + O(e^{-Dr}), \tag{4.41}$$

where $\log S_q$ *and* $\log(Y\varrho^k)$ *are real.*

We take $\log(1-e^{-Yu^k})$ real on the positive half of the real axis and write

$$\xi_1(u) = \frac{2\pi i\log(1-e^{-Yu^k})}{1-e^{-2\pi i u}},\quad \xi_2(u) = \frac{2\pi i\log(1-e^{-Yu^k})}{e^{2\pi i u}-1}.$$

Then we see that

$$\frac{d}{du}\{\log(1-e^{-Yu^k})\log(1-e^{2\pi i u})\} = \chi_1(u)+\xi_1(u),$$

$$\frac{d}{du}\{\log(1-e^{-Yu^k})\log(1-e^{-2\pi i u})\} = \chi_2(u)+\xi_2(u).$$

Hence

$$I(\alpha_1) = \int_{\alpha_1}\chi(u)\,du$$

$$= -\int_{\alpha_1}\xi_1(u)\,du + \log 2\log(1-e^{-YR^k}) - \log(1-e^{-Y\varrho^k})\log(1-e^{2\pi i\varrho})$$

$$= -\int_{\alpha_1}\xi_1(u)\,du - \left\{\log(2\pi\varrho)-\frac{1}{2}\pi i\right\}\log(Y\varrho^k) + O(\varrho\log\varrho) + O(e^{-YR^k}).$$

Similarly

$$I(\alpha_2) = -\int_{\alpha_2} \xi_2(u)\,du - \left\{\log(2\pi\varrho) + \frac{1}{2}\pi i\right\}\log(Y\varrho^k) + O(e^{-YR^k}) + O(\varrho\log\varrho);$$

and so

$$I(\alpha_1) - I(\alpha_2) = \int_{\alpha_2} \xi_2(u)\,du - \int_{\alpha_1} \xi_1(u)\,du + \pi i\log(Y\varrho^k) + O(\varrho\log\varrho) + O(e^{-YR^k}). \tag{4.42}$$

Now

$$\int_{\alpha_2} \xi_1(u)\,du - \int_{\alpha_1} \xi_1(u)\,du = 2\pi i\sum_{l=1}^{r}\log(1 - e^{-Yl^k})$$

by Cauchy's Theorem. Since the logarithms have their principal value, we have

$$\left|S_q - \sum_{l=1}^{r}\log(1 - e^{-Yl^k})\right| \leq \sum_{l=r+1}^{\infty}\left|\log(1 - e^{-Yl^k})\right| < D\sum_{l=r+1}^{\infty} e^{-Yl^k} < De^{-Dr}.$$

Hence

$$\int_{\alpha_2} \xi_1(u)\,du - \int_{\alpha_1} \xi_1(u)\,du = 2\pi i S_q + O(e^{-Dr}). \tag{4.43}$$

Also

$$\int_{\alpha_2} \{\xi_2(u) - \xi_1(u)\}\,du = \int_{\alpha_2} \log(1 - e^{-Yu^k})\,du = \int_{\varrho}^{R} \log(1 - e^{-Yu^k})\,du = \int_0^{\infty} \log(1 - e^{-Yu^k})\,du + O(\varrho\log\varrho) + O(e^{-Dr}), \tag{4.44}$$

and

$$\int_0^{\infty} \log(1 - e^{-Yu^k})\,du = -\sum_{m=1}^{\infty}\frac{1}{m}\int_0^{\infty} e^{-mYu^k}\,du$$

$$= - Y^{-a}\zeta(1 + a)\int_0^\infty e^{-u^k}du$$

$$= - Y^{-a}\Gamma(1 + a)\zeta(1 + a) = - \lambda_q Y^{-a}. \tag{4.45}$$

Then (4.41) follows from (4.42) to (4.45).

4.5. We can now prove Lemma 5. By (4.11) and (4.12),

$$I(\Gamma_1) + I(\Gamma_2) = I(\alpha_1) - I(\alpha_2) + I(\beta_2) - I(\beta_1) + I(\gamma_1) + I(\gamma_2) - I(\delta_1) - I(\delta_2).$$

If we substitute the results of Lemmas 7, 8, 9 and 10, we have

$$\frac{1}{2\pi i}\{I(\Gamma_1) + I(\Gamma_2)\} = S_q + \lambda_q Y^{-a} + jY + \frac{1}{2}\log Y - \frac{1}{2}k\log 2\pi + O(\varrho\log\varrho) + O(e^{-Dr}).$$

Now let $r \to \infty$ through positive integral values and let $\varrho \to 0$. Then $L \to \infty$, and, by (4.13),

$$S'_q = S_q + \lambda_q Y^{-a} + jY + \frac{1}{2}\log Y - \frac{1}{2}k\log(2\pi).$$

This is Lemma 5.

5.1. *Case* (ii): $h \neq q$, $h^k \equiv 0 \pmod{q}$. Here we have $h \equiv 0 \pmod{q_1}$.

Lemma 11. *If $h \neq q$ and $h^k \equiv 0 \pmod{q}$, we have*

$$S'_h + S'_{q-h} - S_h - S_{q-h} = (\lambda_h + \lambda_{q-h})Y^{-a} + (j_h + j_{q-h})Y - \frac{1}{2}k\{\log(2\sin\tau_h\pi) + \log(2\sin\tau_{q-h}\pi)\}.$$

The result is symmetrical in h and $q-h$; it is therefore sufficient to prove it when $h \leq \frac{1}{2}q$. Then we have

$$0 < \tau = \tau_h \leq \frac{1}{2}.$$

We now take

$$R = r + \frac{1}{2} + \tau, \quad R'' = r + \frac{1}{2} - \tau,$$

and choose L so that

$$\frac{R^k Y}{2\pi} < L < \frac{(2R'')^k Y}{2\pi} - \frac{1}{2},$$

which is clearly possible for $r > D$. We write

$$R' = \left\{\frac{2\pi}{Y}\left(L + \frac{1}{2}\right)\right\}^a.$$

Then

$$R'' < R < R' < 2R'' < 2R.$$

The contour α_1 is coincident with the real axis from ϱ to R, except that it passes above the points

$$u = l + \tau \qquad (l = 0, 1, 2, \ldots r),$$

while α_4 is coincident with the real axis from $-\varrho$ to $-R''$, except that it passes above the points

$$u = -l + \tau \qquad (l = 1, 2, \ldots r).$$

The contours α_2 and α_3 are the reflexions of α_1 and α_4 in the real axis.

We take β_1 and β_2 as before, except for the change in the value of R'; β_3 coincides with β_2, and β_4 with β_1. The contours $\delta_1, \delta_2, \delta_3, \delta_4$ are the quadrants of the circle $|u| = \varrho$ on which, if $u = \varrho e^{i\theta}$,

$$0 \leq \theta \leq \frac{1}{2}\pi, \quad -\frac{1}{2}\pi \leq \theta \leq 0, \quad -\pi \leq \theta \leq -\frac{1}{2}\pi, \quad \frac{1}{2}\pi \leq \theta \leq \pi,$$

respectively. The contours γ_1 and γ_2 are defined as before in terms of the new values of R and R'. The contour γ_3 consists of the three parts $\gamma_3', \gamma_3'', \gamma_3'''$; on γ_3', $u = R'' e^{i\theta}$ and $-\pi \leq \theta \leq -\pi + \frac{1}{4}a\pi$, on γ_3''', $u = R' e^{i\theta}$ and $-\pi + \frac{1}{4}a\pi \leq$ $\leq \theta \leq -\frac{1}{2}\pi$, and γ_3'' is the appropriate segment of the straight line

$$\theta = -\pi + \frac{1}{4}a\pi.$$

Finally, γ_4 is the reflexion of γ_3 in the real axis.

The positive directions of the α and the β contours are outwards from the origin, and of the γ and the δ contours counterclockwise about the origin. Then we write

$$\left.\begin{aligned} \Gamma_1 &= \alpha_1 - \beta_1 + \gamma_1 - \delta_1, \\ \Gamma_2 &= -\alpha_2 + \beta_2 + \gamma_2 - \delta_2, \\ \Gamma_3 &= \alpha_3 - \beta_3 + \gamma_3 - \delta_3, \\ \Gamma_4 &= -\alpha_4 + \beta_4 + \gamma_4 - \delta_4. \end{aligned}\right\} \tag{5.11}$$

We now write

$$\chi_1(u) = \frac{k\,Y u^{k-1} \log\left(1 - e^{2\pi i (u-\tau)}\right)}{e^{Y u^k} - 1},$$

$$\chi_2(u) = \frac{k\,Y u^{k-1} \log\left(1 - e^{-2\pi i (u-\tau)}\right)}{e^{Y u^k} - 1}.$$

If k is even,

$$\chi_3(u) = \chi_2(u), \quad \chi_4(u) = \chi_1(u),$$

while, if k is odd,

$$\chi_3(u) = e^{Y u^k} \chi_2(u), \quad \chi_4(u) = e^{Y u^k} \chi_1(u).$$

We take $\log\left(1 - e^{2\pi i (u-\tau)}\right)$ real at $u = \tau + \frac{1}{2}$ on Γ_1, and $\log\left(1 - e^{-2\pi i (u-\tau)}\right)$ real at $u = \tau + \frac{1}{2}$ on Γ_2. We can then show that $\log\left(1 - e^{2\pi i (u-\tau)}\right)$ has its principal value on or within Γ_1 and Γ_4, and similarly for the other logarithm and the other contours.

The χ-functions have poles at the points

$$t_s l^a \qquad (1 \leq s \leq 2k,\ l \geq 1)$$

and at the origin, and logarithmic singularities at the points

$$u = \tau + l \qquad (l = \ldots -2, -1, 0, 1, 2, \ldots).$$

We write also

$$\xi_1(u) = \frac{2\pi i \log\left(1 - e^{-Y u^k}\right)}{1 - e^{-2\pi i (u-\tau)}},$$

$$\xi_2(u) = \frac{2\pi i \log\left(1 - e^{-Y u^k}\right)}{e^{2\pi i (u-\tau)} - 1},$$

where $\log\left(1 - e^{-Y u^k}\right)$ is real on the positive half of the real axis. If k is even, we take $\xi_3(u)$ and $\xi_4(u)$ of the same form as $\xi_2(u)$ and $\xi_1(u)$ respectively, but the logarithm is now to be taken real on the negative half of the real axis. If k is odd, we take

$$\xi_3(u) = \frac{2\pi i \log(1 - e^{Yu^k})}{e^{2\pi i(u-\tau)} - 1},$$

$$\xi_4(u) = \frac{2\pi i \log(1 - e^{Yu^k})}{1 - e^{-2\pi i(u-\tau)}},$$

where the logarithm is real on the negative half of the real axis.

Further, we write

$$I(\Gamma_1) = \int_{\Gamma_1} \chi_1(u)\,du = I(\alpha_1) - I(\beta_1) + I(\gamma_1) - I(\delta_1),$$

and so on.

5. 2. **Lemma 12.** *For $r > D$, we have*

$$|I(\gamma_1)| < De^{-Dr}, \tag{5.21}$$

and similarly for $I(\gamma_2)$, $I(\gamma_3)$, $I(\gamma_4)$.

The method of proof of these results is clear from that of Lemma 7.

Lemma 13. *We have*

$$\left.\begin{aligned} I(\delta_1) &= \tfrac{1}{2}k\pi i \log(1 - e^{-2\pi i\tau}) + O(\varrho), \\ I(\delta_2) &= \tfrac{1}{2}k\pi i \log(1 - e^{2\pi i\tau}) + O(\varrho), \\ I(\delta_3) &= \tfrac{1}{2}k\pi i \log(1 - e^{2\pi i\tau}) + O(\varrho), \\ I(\delta_4) &= \tfrac{1}{2}k\pi i \log(1 - e^{-2\pi i\tau}) + O(\varrho). \end{aligned}\right\} \tag{5.22}$$

where the logarithms have their principal values.

Since the singularity of every $\chi(u)$ at the origin is now a simple pole, these results are immediate consequences of well-known theorems in the theory of residues.

Lemma 14. *We have*

$$I(\beta_4) - I(\beta_1) = 2\pi i j_{q-h} Y + O(\varrho^k) + O(e^{-Dr}), \tag{5.23}$$

and

$$I(\beta_2) - I(\beta_3) = 2\pi i j_h \quad Y + O(\varrho^k) + O(e^{-Dr}). \tag{5.24}$$

If k is even,

$$I(\beta_4) = I(\beta_1), \quad I(\beta_3) = I(\beta_2), \quad j_h = j_{q-h} = 0,$$

and the result is obvious. If k is odd, we have

$$\begin{aligned} I(\beta_4) - I(\beta_1) &= \int_{\beta_4} \chi_4(u)\,du - \int_{\beta_1} \chi_1(u)\,du \\ &= \int_{\beta_1} (e^{Yu^k} - 1)\,\chi_1(u)\,du \\ &= kY \int_{\beta_1} u^{k-1} \log(1 - e^{2\pi i(u-\tau)})\,du \\ &= kYi^k \int_0^\infty v^{k-1} \log(1 - e^{-2\pi(v+i\tau)})\,dv \\ &\qquad + O(\varrho^k) + O(e^{-DR'}). \end{aligned}$$

Since

$$k\int_0^\infty v^{k-1} \log(1 - e^{-2\pi(v+i\tau)})\,dv$$

$$= -k\sum_{m=1}^{\infty} \frac{e^{-2\pi m i\tau}}{m} \int_0^\infty v^{k-1} e^{-2\pi m v}\,dv = (-1)^{\frac{1}{2}(k-1)}\, 2\pi j_{q-h},$$

we have (5.23), and (5.24) may be proved in the same way.

5.3. **Lemma 15.** *We have*

$$\begin{aligned} &I(\alpha_1) - I(\alpha_2) + I(\alpha_3) - I(\alpha_4) \\ &\quad = 2\pi i\{Y^{-a}(\lambda_h + \lambda_{q-h}) + S_h + S_{q-h}\} + O(\varrho \log \varrho) + O(e^{-Dr}). \end{aligned} \tag{5.31}$$

We have

$$\frac{d}{du}\{\log(1 - e^{2\pi i(u-\tau)}) \log(1 - e^{-Yu^k})\} = \chi_1(u) + \xi_1(u),$$

and so on. Hence

$$I(\alpha_1) = -\int\limits_{\alpha_1} \xi_1(u)\,du - \log(1 - e^{-2\pi i\tau}) \log(Y\varrho^k) + E_1,$$

$$I(\alpha_2) = -\int\limits_{\alpha_2} \xi_2(u)\,du - \log(1 - e^{2\pi i\tau}) \log(Y\varrho^k) + E_2,$$

$$I(\alpha_3) = -\int\limits_{\alpha_3} \xi_3(u)\,du - \log(1 - e^{2\pi i\tau}) \log(Y\varrho^k) + E_3,$$

$$I(\alpha_4) = -\int\limits_{\alpha_4} \xi_4(u)\,du - \log(1 - e^{-2\pi i\tau}) \log(Y\varrho^k) + E_4,$$

where E_1, E_2, E_3, E_4 are numbers of the type

$$O(\varrho \log \varrho) + O(e^{-Dr^k}).$$

Then we have

$$I(\alpha_1) - I(\alpha_2) + I(\alpha_3) - I(\alpha_4)$$
$$= \int\limits_{\alpha_2} \xi_2\,du + \int\limits_{\alpha_4} \xi_4\,du - \int\limits_{\alpha_1} \xi_1\,du - \int\limits_{\alpha_3} \xi_3\,du$$
$$+ O(\varrho \log \varrho) + O(e^{-Dr}).$$

Now, as before, we find that

$$\int\limits_{\alpha_2} \xi_2\,du - \int\limits_{\alpha_1} \xi_1\,du$$
$$= 2\pi i\, S_h + 2\pi i\, Y^{-a}\, \Gamma(1+a)\,\zeta(1+a)$$
$$+ O(\varrho \log \varrho) + O(e^{-Dr}).$$

In the same way, whether k is odd or even,

$$\int\limits_{\alpha_3} \xi_3\,du - \int\limits_{\alpha_4} \xi_4\,du = -2\pi i\, S_{q-h} - 2\pi i\, Y^{-a}\, \Gamma(1+a)\,\zeta(1+a)$$
$$+ O(\varrho \log \varrho) + O(e^{-Dr}).$$

Since $h^k \equiv 0 \pmod{q}$, we have

$$e_q(mph^k) = 1, \quad e_q(mp(q-h)^k) = 1,$$
$$\lambda_h = \lambda_{q-h} = \Gamma(1+a)\,\zeta(1+a),$$

and the lemma follows at once.

5.4. By Cauchy's Theorem,

$$\lim_{\varrho\to 0,\, r\to\infty} \frac{1}{2\pi i}\{I(\Gamma_1)+I(\Gamma_2)+I(\Gamma_3)+I(\Gamma_4)\}$$
$$= S_h' + S_{q-h}'. \tag{5.41}$$

But by (5.11) and Lemmas 12, 13, 14 and 15, we see that this limit is equal to

$$S_h + S_{q-h} + (\lambda_h + \lambda_{q-h})\, Y^{-a} + (j_h + j_{q-h})\, Y$$
$$- \frac{1}{2} k \{\log(1 - e^{2\pi i\tau}) + \log(1 - e^{-2\pi i\tau})\}. \tag{5.42}$$

Now

$$\frac{1}{2} k \log\{(1 - e^{2\pi i\tau})(1 - e^{-2\pi i\tau})\} = \frac{1}{2} k \log(4\sin^2 \pi\tau)$$
$$= \frac{1}{2} k \{\log(2\sin \tau_h \pi) + \log(2\sin \tau_{q-h}\pi)\}, \tag{5.43}$$

since $\tau_h = \tau$ and $\tau_{q-h} = 1 - \tau$. Lemma 11 follows from (5.41), (5.42) and (5.43).

5.5. *Case* (iii). We have still to consider the case $h^k \not\equiv 0 \pmod q$. We take now

$$\chi_1(u) = \frac{\nu k\, Y u^{k-1} \log(1 - e^{2\pi i(u-\tau)})}{e^{Yu^k} - \nu},$$

$$\chi_2(u) = \frac{\nu k\, Y u^{k-1} \log(1 - e^{-2\pi i(u-\tau)})}{e^{Yu^k} - \nu}.$$

If k is even, we write

$$\chi_3(u) = \chi_2(u), \quad \chi_4(u) = \chi_1(u),$$

but if k is odd

$$\chi_3(u) = \nu^{-1} e^{Yu^k} \chi_2(u), \quad \chi_4(u) = \nu^{-1} e^{Yu^k} \chi_1(u).$$

These functions have poles at the points

$$u = t_s (l + \mu_{h,s})^a \qquad (1 \leq s \leq 2k;\ l \geq 0)$$

and logarithmic singularities at the points

$$u = l + \tau \qquad (l = \ldots -2, -1, 0, 1, 2, \ldots),$$

but the functions are regular at the origin.

We need not give the new forms of the ξ functions. We remark only that in ξ_1, for example, we take that value of $\log(1 - \nu e^{-Yu^k})$ which tends to zero as $u \to +\infty$ along the real axis.

The α and β contours are the same as before, while the δ contours are unnecessary here; we may put $\varrho = 0$ at once, as the functions are all regular at the origin. The γ contours have to be modified to avoid the singularities of the χ-functions and to enable us to prove a result of the type of Lemmas 7 and 12.

The details of the work will be sufficiently clear from the foregoing, and we content ourselves with stating Lemma 16.

Lemma 16. *If h^k is not a multiple of q, we have*

$$S'_h + S'_{q-h} - S_h - S_{q-h} = Y^{-a}(\lambda_h + \lambda_{q-h}) + Y(j_h + j_{q-h}) + \frac{1}{4}\pi i(\sigma_h + \sigma_{q-h}).$$

The term $\frac{1}{4}\pi i(\sigma_h + \sigma_{q-h})$ arises from

$$\int_{\alpha_1} \{\chi_1(u) + \xi_1(u)\}\, du = -\log(1-\nu)\log(1 - e^{2\pi i\tau}) + O(e^{-DR})$$

and the three similar expressions. If k is even, the sum of the four is zero.

5.6. **Proof of Lemma 2.** We now combine Lemmas 5, 11 and 16. Let Σ' denote summation over those values of h for which $h^k \equiv 0 \pmod q$ and $h \neq q$, and Σ'' summation over those values of h for which $h^k \not\equiv 0 \pmod q$. We have

$$\sum_{h=1}^{q} S'_h - \sum_{h=1}^{q} S_h = Y^{-a}\sum_{h=1}^{q}\lambda_h + Y\sum_{h=1}^{q} j_h + \frac{1}{2}\log Y$$
$$- \frac{1}{2}k\log 2\pi + \frac{1}{4}\pi i\,\Sigma''\sigma_h - \frac{1}{2}k\,\Sigma'\log(2\sin\tau_h\pi),$$

the logarithms on the right hand side being real.

Now

$$\prod_{h=1}^{q_2-1}\left(2\sin\frac{l\pi}{q_2}\right) = q_2,$$

and so

$$\Sigma'\log\left(2\sin\frac{h\pi}{q}\right) = \sum_{l=1}^{q_2-1}\log\left(2\sin\frac{l\pi}{q_2}\right) = \log q_2.$$

Also

$$Y^{-a}\sum_{h=1}^{q}\lambda_h = \frac{\Gamma(1+a)}{qy^a}\sum_{m=1}^{\infty}\frac{S_{pm,q}}{m^{1+a}} = \frac{\varDelta_{p,q}}{y^a}.$$

The value of the sum

$$\sum_{h=1}^{q} e_q(mh)$$

is q or 0 according as m is, or is not, a multiple of q. Hence, if k is odd,

$$q^k\sum_{m=1}^{\infty}\sum_{h=1}^{q}\frac{e_q(mh)}{m^{k+1}} = q^k\sum_{t=1}^{\infty}\frac{q}{(qt)^{k+1}} = \zeta(k+1),$$

and so

$$Y\sum_{h=1}^{q} j_h = \frac{(-1)^{\frac{1}{2}(k+1)}\,y\,\Gamma(k+1)}{(2\pi)^{k+1}}\,q^k\sum_{m=1}^{\infty}\sum_{h=1}^{q}\frac{e_q(mh)}{m^{k+1}}$$
$$= jy.$$

If k is even,

$$Y\sum_{h=1}^{q} j_h = 0 = jy.$$

Also, by (3.21),

$$\log\omega_{p,q} = \frac{1}{4}\pi i\,\Sigma''\sigma_h.$$

Hence we have

$$\log f(x) - \log P_{p,q} = \varDelta_{p,q}y^{-a} + jy + \frac{1}{2}\log y$$
$$+ \log\omega_{p,q} + \frac{1}{2}k(\log q - \log q_2 - \log 2\pi).$$

Taking exponentials and noting that $q = q_1 q_2$, we have (2.11) for y real and positive. This is Lemma 2, and the proof of Theorem 4 is thus complete.

Part II. The Asymptotic Expansion of $p_k(n)$.

6. We now turn to the proof of Theorems 1, 2 and 3. We shall first prove Theorem 3, from which the others may be readily deduced. We first define the meaning of certain further symbols.

We take ε an arbitrary positive number, and δ a positive number whose choice is subsequent to that of k and ε. N is a positive number which is ultimately chosen as a function of k, n and ε. The numbers ε and δ are to be thought of as small, while n and N are to be thought of as large. We write

$$c = 2^{k-1}.$$

A and B are positive numbers, whose values varies from one occurrence to another. In any particular occurrence, A without a suffix is an absolute constant, while B depends at most on k. If, however, A and B depend on other parameters these are indicated explicitly by suffixes, for example,

$$A_\delta = A(\delta), \quad B_{q,\varepsilon} = B(k, q, \varepsilon).$$

The constant implied in the $O(\ )$ notation is henceforth of the type B_ε, except in Lemma 33.

In the complex x-plane, Γ is the circle

$$|x| = e^{-\frac{1}{N}}.$$

We take the Farey dissection of order N^{1-b} of this circle and consider two kinds of arcs: —

(i) Major arcs, $\mathfrak{M}$ or $\mathfrak{M}_{p,q}$, such that $q \leq N^b$,

(ii) Minor arcs, $\mathfrak{m}$ or $\mathfrak{m}_{p,q}$, such that $N^b < q \leq N^{1-b}$.

The restriction y real and positive is now removed and, in connection with every pair of values of p and q, we write

$$H(\Delta) = \exp\left(\frac{\Delta}{y^a}\right), \quad U(\Delta, X) = U(\Delta, e^{-y}) = y^{\frac{1}{2}} e^{jy} H(\Delta),$$

where $y, y^{\frac{1}{2}}, y^a$ are real and positive for X real and $0 < X < 1$. When x lies on Γ, we write

$$y = \frac{1}{N} - i\theta,$$

and for x on $\mathfrak{M}_{p,q}$ or on $\mathfrak{m}_{p,q}$ we have

$$-\theta'_{p,q} \leq \theta \leq \theta_{p,q},$$

where

$$\frac{\pi}{qN^{1-b}} \leq \theta'_{p,q} < \frac{2\pi}{qN^{1-b}}, \quad \frac{\pi}{qN^{1-b}} \leq \theta_{p,q} < \frac{2\pi}{qN^{1-b}}.$$

Upper Bound for $|\log f(x)|$ on the Minor Arcs.

7. 1. If $k = 1$, then $b = \frac{1}{2} = 1 - b$ and there are no minor arcs. Hence we need only consider the case $k \geq 2$ in this section.

We take $\log f(x) = 0$ at $x = 0$. Then $\log f(x)$ is one-valued for every value of x within the unit circle, for $f(x)$ has no zeros within the unit circle.

Lemma 17. *If $\gamma > 0$, $k \geq 2$ and x lies on $\mathfrak{m}_{p,q}$, then*

$$|\log f(x)| < B_\gamma N^{a-bc+\gamma}. \tag{7. 11}$$

We use $[u]$ to denote the integral part of u and write

$$\beta = \frac{\gamma}{k+3}, \quad \omega_m = \left[\left(\frac{N}{m}\right)^{a+\beta}\right],$$

$$\Psi(v) = \Psi(v, p, q, m) = \sum_{1 \leq l \leq \left[\left(\frac{j}{m}\right)^a\right]} e_q(pl^k m).$$

We have

$$\log f(x) = -\sum_{l=1}^{\infty} \log(1 - x^{l^k}) = \sum_{l=1}^{\infty}\sum_{m=1}^{\infty} \frac{x^{l^k m}}{m}$$

$$= \sum_{m \leq N^b} \sum_{l=1}^{\omega_m} + \sum_{m \leq N^b} \sum_{l=\omega_m+1}^{\infty} + \sum_{m > N^b} \sum_{l=1}^{\infty}$$

$$= Z_1 + Z_2 + Z_3. \tag{7. 12}$$

7. 2. **Lemma 18.** *On the whole circle Γ,*

$$|Z_2| < B_\gamma \log N, \quad |Z_3| < BN^b.$$

If $\zeta > 0$,

$$\sum_{l=1}^{\infty} e^{-\zeta l^k} \leq \int_0^\infty e^{-\zeta u^k} du = \zeta^{-a} \int_0^\infty e^{-u^k} du = B\zeta^{-a}.$$

Also, on Γ, $|x| = e^{-\frac{1}{N}}$, and so

$$|Z_3| \leq \sum_{m > N^b} \frac{1}{m} \sum_{l=1}^{\infty} e^{-\frac{l^k m}{N}} \leq BN^a \sum_{m > N^b} \frac{1}{m^{1+a}}$$

$$< BN^{a(1-b)} = BN^b.$$

Again

$$\sum_{l=\omega_m+1}^{\infty} e^{-\frac{l^k m}{N}} \leq 1 + \int_{\omega_m+1}^{\infty} e^{-\frac{m u^k}{N}} du$$

$$\leq 1 + \left(\frac{N}{m}\right)^a \int_{\left(\frac{N}{m}\right)^\beta}^{\infty} e^{-u^k} du$$

$$\leq 1 + \left(\frac{N}{m}\right)^a e^{-\frac{1}{2}\left(\frac{N}{m}\right)^{k\beta}} \int_0^{\infty} e^{-\frac{1}{2}u^k} du$$

$$\leq 1 + B\left(\frac{N}{m}\right)^a e^{-\frac{1}{2}\left(\frac{N}{m}\right)^{k\beta}} < B_\beta.$$

Hence on Γ

$$|Z_2| \leq B_\beta \sum_{m \leq N^b} \frac{1}{m} < B_\gamma \log N.$$

Lemma 19. *On a minor arc* $\mathfrak{m}_{p,q}$, *if* $k \geq 2$,

$$|Z_1| < B_\gamma N^{a-bc+\gamma}.$$

We write

$$\frac{pm}{q} = \frac{p_m}{q_m}, \quad (p_m, q_m) = 1.$$

Then

$$\Psi(v) = \sum_{1 \leq l \leq \left[\left(\frac{v}{m}\right)^a\right]} e_{q_m}(p_m l^k),$$

where $e_{q_m}(p_m)$ is a primitive q_m-th root of unity. Now $a \geq c$; also

$$N^b < q \leq N^{1-b}, \quad q_m \leq q \leq m q_m.$$

Hence, if $v \leq m\omega_m^k$, we have[1]

[1] Landau, *Vorlesungen über Zahlentheorie*, I, (Leipzig, 1927), Satz 267.

$$|\Psi(v)| < B_\beta \omega_m^\beta q_m^\beta (\omega_m^{1-c} + \omega_m q_m^{-c} + \omega_m^{1-ck} q_m^c)$$
$$< B_\beta N^{(k+2)\beta}\left(N^{a(1-c)} + \frac{N^a}{m^a q_m^c} + N^{a-c} q_m^c\right)$$
$$< B_\beta N^{(k+2)\beta}(N^{a(1-c)} + N^a q^{-c} + N^{a-c} q^c)$$
$$< B_\beta N^{a-bc+(k+2)\beta}.$$

On a minor arc,

$$|1 - X| < B|y| < B\left(\frac{1}{N^2} + \theta^2\right)^{\frac{1}{2}} < \frac{B}{N}\left(1 + \frac{N^{2b}}{q^2}\right)^{\frac{1}{2}} \leq \frac{B}{N},$$
$$1 - |X| > \frac{B}{N}, \quad \frac{|1 - X|}{1 - |X|} < B.$$

Now

$$\sum_{l=1}^{\omega_m} x^{l^k m} = \sum_{l=1}^{\omega_m} e_q(mpl^k) X^{l^k m}$$
$$= \sum_{v=1}^{m\omega_m^k} \{\Psi(v) - \Psi(v-1)\} X^v$$
$$= \sum_{v=1}^{m\omega_m^k - 1} \Psi(v) X^v (1 - X) + \Psi(m\omega_m^k) X^{m\omega_m^k},$$

and so

$$\left|\sum_{l=1}^{\omega_m} x^{l^k m}\right| \leq B_\beta N^{a-bc+(k+2)\beta}\left(1 + |1 - X| \sum_{j=1}^{\infty} |X|^j\right)$$
$$= B_\beta N^{a-bc+(k+2)\beta}\left(1 + \frac{|1 - X|}{1 - |X|}\right)$$
$$< B_\beta N^{a-bc+(k+2)\beta}.$$

Hence

$$|Z_1| < B_\beta N^{a-bc+(k+2)\beta} \sum_{m \leq N^b} \frac{1}{m}$$
$$< B_\beta N^{a-bc+(k+3)\beta} = B_\gamma N^{a-bc+\gamma}.$$

Now $b \leq a - bc$, and so Lemma 17 follows at once from (7.12) and Lemmas 18 and 19.

Upper Bound for $|f(x)|$ on a Major Arc.

8.1. **Lemma 20.** *If $q > 2$, we have*

$$|f(x)| < \exp(BN^a q^{-a} \log\log q)$$

on $\mathfrak{M}_{p,q}$.

Since $q \leq N^b$, on $\mathfrak{M}_{p,q}$,

$$|y|^2 = \frac{1}{N^2} + \theta^2 \leq \frac{1}{N^2} + \frac{4\pi^2}{q^2 N^{2(1-b)}} < \frac{8\pi^2}{q^2 N^{2(1-b)}},$$

and so

$$\frac{|y|^{1+a}}{\Re(y)} < \left(\frac{2\pi\sqrt{2}}{q}\right)^{1+a} N^{1-(1+a)(1-b)} = \left(\frac{2\pi\sqrt{2}}{q}\right)^{1+a}. \qquad (8.11)$$

Hence, if we write $C = \left(\frac{q}{2\pi\sqrt{2}}\right)^{1+a}$ in Lemma 4, we see that, on $\mathfrak{M}_{p,q}$,

$$|g(h, l, s)| \leq e^{-C_1(l+1)^a},$$

where

$$C_1 = \frac{2^{\frac{1}{2}(1-a)} a}{\pi} = B,$$

and so

$$|P_{p,q}| \leq \prod_{h=1}^{q} \prod_{s=1}^{k} \prod_{l=0}^{\infty} \{1 - e^{-B(l+1)^a}\}^{-1}$$

$$= B^{qk} = e^{Bq}.$$

By Theorem 4, on $\mathfrak{M}_{p,q}$,

$$|f(x)| = Bq_1^{\frac{1}{2}k} \left| \omega_{p,q} y^{\frac{1}{2}} e^{jy} \exp\left(\frac{A_{p,q}}{y^a}\right) P_{p,q} \right|$$

$$\leq Bq^{\frac{1}{2}k} \exp\left|\frac{A_{p,q}}{y^a}\right| e^{Bq}$$

$$\leq \exp\{Bq + |A_{p,q}| N^a\},$$

since

$$\frac{1}{|y|} \leq \frac{1}{\Re(y)} = N.$$

Now $3 \leq q \leq N^b$, and so

$$q^{1+a} \leq N^a, \quad q \leq N^a q^{-a} < A N^a q^{-a} \log\log q.$$

Hence to complete the proof of Lemma 20 we have only to prove

Lemma 21. *If* $q > 2$,

$$|A_{p,q}| < B q^{-a} \log\log q.$$

8.2. **Proof of Lemma 21.** We require certain preliminary lemmas.

Lemma 22. *If* $(p, q) = 1$ *and* $(m, q) = r_m$, *then*

$$|S_{pm,q}| \leq B r_m^a q^{1-a}.$$

If we write $q = r_m q_m$ and $pm = r_m p_m$, we have $(p_m, q_m) = 1$ and

$$S_{pm,q} = r_m S_{p_m, q_m}.$$

But[1]

$$|S_{p_m, q_m}| < B q_m^{1-a}.$$

Hence

$$|S_{pm,q}| < B r_m q_m^{1-a} = B r_m^a q^{1-a}.$$

Lemma 23.[2] *If P denotes any prime number and* $q > 2$, *then*

$$\prod_{P|q} \left(1 - \frac{1}{P}\right)^{-1} < B \log\log q.$$

Lemma 24. *We have*

$$\sum_{m=1}^{\infty} \frac{r_m^a}{m^{1+a}} \leq \zeta(1+a) \prod_{P|q} \left(1 - \frac{1}{P}\right)^{-1}.$$

We know that

$$\zeta(1+a) = \sum_{m=1}^{\infty} \frac{1}{m^{1+a}} = \prod_{P} \left(1 + \frac{1}{P^{1+a}} + \frac{1}{P^{2(1+a)}} + \cdots\right);$$

similarly, since any factor of r_m must divide both q and m, we have

[1] Landau, *Vorlesungen* I, Satz 267 ($k = 2$) and Satz 315 ($k \geq 3$). For $k = 1$, the result is trivial.

[2] Landau, *Handbuch der Lehre von der Verteilung der Primzahlen* I, 216—219.

$$\sum_{m=1}^{\infty} \frac{r_m^a}{m^{1+a}} \leq \prod_{P|q}\left(1 + \frac{1}{P} + \frac{1}{P^2} + \cdots\right)\prod_{P\nmid q}\left(1 + \frac{1}{P^{1+a}} + \frac{1}{P^{2(1+a)}} + \cdots\right)$$
$$\leq \zeta(1 + a)\prod_{P|q}\left(1 - \frac{1}{P}\right)^{-1}.$$

Lemma 21 follows at once from the last three lemmas, for

$$|A_{p,q}| \leq \frac{1}{q}\sum_{m=1}^{\infty}\frac{S_{pm,q}}{m^{1+a}} < \frac{B}{q^a}\sum_{m=1}^{\infty}\frac{r_m^a}{m^{1+a}}$$
$$< Bq^{-a}\log\log q.$$

Approximation to $f(x)$ on $\mathfrak{M}_{p,q}$.

9.1. We have now to replace Lemma 20 by a more precise result, which will be used for the smaller values of q. We shall express $f(x)$ on $\mathfrak{M}_{p,q}$ as the sum of a number of functions of the type $U(A, X)$ plus an error term. Our result is

Lemma 25. *If x lies on $\mathfrak{M}_{p,q}$ and $\delta > 0$, we have*

$$\left|f(x) - C_{p,q}\sum_{t=0}^{T_{a,\delta}} c_{q,t}\, U(A_{p,q,t}, X)\right| < B_{q,\delta}\, e^{\delta N^a},$$

where $T_{q,\delta}$ is a function of k, q and δ.

9.2. We first prove a series of lemmas with regard to the behaviour of $|g(h, l, s)|$ and $|H(\zeta e^{i\vartheta})|$ on $\mathfrak{M}_{p,q}$. If $\zeta \geq 0$, we write

$$K(\zeta) = \max\{|H(\zeta e^{\frac{1}{2}\pi i(1+a)})|,\ |H(\zeta e^{-\frac{1}{2}\pi i(1+a)})|\},$$
$$\eta = K\left(\frac{(2\pi)^{1+a}}{k^a q^{1+2a}}\right),\quad C_2 = \exp\left(-\frac{a}{\pi q^a k^a}\right).$$

As before

$$y = |y|e^{i\psi} = \frac{1}{N} - i\theta,$$

where
$$-\frac{1}{2}\pi < \psi < \frac{1}{2}\pi.$$

We write
$$m = qlk + q(s - 1) + h,$$
$$1 \leq s \leq k,\quad 1 \leq h \leq q,\quad l \geq 0.$$

Then there is a one-to-one correspondence between the positive integers m and the sets of integers (h, l, s). Hence g_m is uniquely defined by

$$g_m = g(h, l, s).$$

Lemma 26. *If* $|x| < 1$, $\zeta \geq \zeta' > 0$, *and*

$$\frac{1}{2}\pi(1 + a) \leq \vartheta \leq \frac{1}{2}\pi(3 - a),$$

then

$$|H(\zeta e^{i\vartheta})| \leq K(\zeta') < \exp\left(-\frac{2a\zeta' \cos\psi}{\pi |y|^a}\right).$$

We have

$$|H(\zeta e^{i\vartheta})| = \exp\left\{\frac{\zeta}{|y|^a}\cos(\vartheta - a\psi)\right\},$$

and a corresponding expression for $K(\zeta')$. Then the lemma follows from Lemma 3.

Lemma 27. *If* $|x| < 1$,

$$|g_m| \leq \eta^{m^a}.$$

We put

$$\vartheta = \frac{1}{2}\pi a(k + 2s - 1),$$

$$\zeta = \frac{(2\pi)^{1+a}(l + \mu_{h,s})^a}{q},$$

$$\zeta' = \frac{(2\pi)^{1+a} m^a}{q^{1+2a} k^a}$$

in Lemma 26. We have

$$q^2 k(l + \mu_{h,s}) \geq qk(l + q\mu_{h,s}) \geq qk(l + 1) \geq m,$$

$$(l + \mu_{h,s})^a \geq \frac{m^a}{q^{2a} k^a},$$

$$\zeta > \zeta' > 0,$$

and

$$1 \leq s \leq k, \quad \frac{1}{2}\pi(1 + a) \leq \vartheta \leq \frac{1}{2}\pi(3 - a).$$

The conditions of Lemma 26 are therefore satisfied. But

$$|g_m| = |g(h, l, s)| = |H(\zeta e^{i\vartheta})|, \quad K(\zeta') = \eta^{m^a},$$

and so the lemma follows.

Lemma 28. *On* $\mathfrak{M}_{p,q}$, $0 \leq \eta \leq C_2$.

On $\mathfrak{M}_{p,q}$, by (8. 11),

$$\frac{|y|^a}{\cos\psi} = \frac{|y|^{1+a}}{\Re(y)} < \left(\frac{2\pi\sqrt{2}}{q}\right)^{1+a}.$$

Hence, by Lemma 26,

$$K\left(\frac{(2\pi)^{1+a}}{q^{1+a}}\right) \leq \exp\left(-\frac{2a}{\pi}\left(\frac{2\pi}{q}\right)^{1+a}\frac{\cos\psi}{|y|^a}\right)$$

$$\leq \exp\left(-\frac{2^{\frac{1}{2}(1-a)}a}{\pi}\right) \leq \exp\left(-\frac{a}{\pi}\right),$$

and so

$$\eta \leq \exp\left(-\frac{a}{\pi(qk)^a}\right) = C_2.$$

Lemma 29. *On* $\mathfrak{M}_{p,q}$, *if* $\zeta > 0$ *and* $V > 0$, *then*

$$|H(\zeta e^{i\vartheta})|\, K(V) \leq \exp\left\{\left(\frac{\zeta}{1+a}\right)^{1+a}\left(\frac{\pi N}{2V}\right)^a\right\}.$$

We have always

$$|H(\zeta e^{i\vartheta})| \leq \exp\left(\frac{\zeta}{|y|^a}\right),$$

and by Lemma 26

$$K(V) \leq \exp\left(-\frac{2aV\cos\psi}{\pi|y|^a}\right).$$

On $\mathfrak{M}_{p,q}$, $|y|^{-1} = N\cos\psi$, and so

$$|H(\zeta e^{i\vartheta})|\, K(V) \leq \exp\left\{N^a(\cos\psi)^a\left(\zeta - \frac{2Va}{\pi}\cos\psi\right)\right\}$$

$$= \exp\{N^a w((\cos\psi)^a)\},$$

where

$$w(u) = u\left(\zeta - \frac{2Va}{\pi}u^k\right).$$

A simple maximum argument shows that, when $u \geq 0$,

$$w(u) \leq \left(\frac{\pi}{2V}\right)^a \left(\frac{\zeta}{1+a}\right)^{1+a}.$$

Since $(\cos\psi)^a > 0$, the lemma follows at once.

9.3. **Lemma 30.** *For any $W > 0$, there exist an integer $T = B_W$ and functions $G_1, G_2, \ldots$, each a product of functions of the form $g(h, l, s)$, such that*

$$\left| P_{p,q} - 1 - \sum_{t=1}^{T} G_t \right| \leq B_{q,W}\, \eta^W \tag{9.31}$$

on $\mathfrak{M}_{p,q}$.

We can expand $P_{p,q}$ formally thus:

$$P_{p,q} = \prod_{m=1}^{\infty} (1 - g_m)^{-1} = \prod_{m=1}^{\infty} (1 + g_m + g_m^2 + \cdots)$$

$$= 1 + \sum_{t=1}^{\infty} G_t = S,$$

where G_t is a product of functions of the form g_m. If

$$G_t = g_{m_1} g_{m_2} g_{m_3} \cdots g_{m_r},$$

we write

$$v(t) = m_1^a + m_2^a + \cdots + m_r^a;$$

repetitions of the numbers m may of course occur. It is clear that for any positive A there are only a finite number of terms G_t such that $v(t) < A$. We may then suppose S arranged so that

$$v(t+1) \geq v(t)$$

for all $t \geq 1$.

By Lemmas 27 and 28,

$$|g_m| \leq \eta^{m^a}, \quad 0 \leq \eta \leq C_2 < 1$$

on $\mathfrak{M}_{p,q}$, and so

$$\sum_{m=1}^{\infty} g_m$$

and $P_{p,q}$ are both absolutely convergent. The above expansion and rearrangement are then justified and we have

$$P_{p,q} = S.$$

If we write S_2 for the result of replacing every g_m in S by $C_2^{m^a}$, and P_2 for the corresponding infinite product, the same argument shows that

$$1 + \sum_{t=1}^{\infty} C_2^{v(t)} = S_2 = P_2;$$

and clearly

$$P_2 = \prod_{m=1}^{\infty} (1 - C_2^{m^a})^{-1} = B_q.$$

If T is the least value of t such that $v(t) \geq W$, we have $v(t) \geq W$ for all $t \geq T$. Then $T = B_W$, and on $\mathfrak{M}_{p,q}$

$$\begin{aligned} \left| P_{p,q} - 1 - \sum_{t=1}^{T} G_t \right| &= \left| \sum_{t=T+1}^{\infty} G_t \right| \\ &\leq \sum_{t=T+1}^{\infty} \eta^{v(t)} = \eta^W \sum_{t=T+1}^{\infty} \eta^{v(t)-W} \\ &\leq \eta^W \sum_{t=T+1}^{\infty} C_2^{v(t)-W} < \eta^W C_2^{-W} S_2 \\ &= B_{q,W}\,\eta^W. \end{aligned}$$

This is Lemma 30.

Multiplying both sides of (9. 31) by

$$C_{p,q}\, y^{\frac{1}{2}} e^{jy} H(\varLambda_{p,q}),$$

and using Theorem 4, we have on $\mathfrak{M}_{p,q}$

$$\begin{aligned} &\left| f(x) - C_{p,q}\, y^{\frac{1}{2}} e^{jy} H(\varLambda_{p,q}) \left(1 + \sum_{t=1}^{T} G_t \right) \right| \\ &\leq B_{q,W}\, \eta^W \,|\, H(\varLambda_{p,q}) \,| \\ &= B_{q,W} \,|\, H(\varLambda_{p,q}) \,|\, K\left(\frac{(2\pi)^{1+a}\, W}{k^a q^{1+2a}} \right). \end{aligned} \tag{9. 32}$$

If we choose V and W so that

$$\left(\frac{\zeta}{1+a} \right)^{1+a} \left(\frac{\pi}{2V} \right)^a = \delta, \quad W = \frac{q^{1+2a} k^a V}{(2\pi)^{1+a}},$$

the expression on the right of (9. 32) becomes

$$B_{q,V}\,|H(\varDelta_{p,q})|\,K(V) < B_{q,\delta}\, e^{\delta N^a}$$

by Lemma 29.

On the other hand, by definition, $g(h, l, s)$ is the product of a function $H(\varDelta)$ and the number $e_q(-h)$. Since

$$H(\varDelta_1)\,H(\varDelta_2)\cdots = H(\varDelta_1 + \varDelta_2 + \cdots),$$

we have

$$y^{\frac{1}{2}}\, e^{jy}\, H(\varDelta_{p,q})\, G_t = c_{q,t}\, y^{\frac{1}{2}}\, e^{jy}\, H(\varDelta_{p,q,t}) = c_{q,t}\, U(\varDelta_{p,q,t}, X),$$

where $c_{q,t}$ is a q-th root of unity.

9.4. *The calculation of* $c_{q,t}$ *and* $\varDelta_{p,q,t}$. It is clear that, for any particular values of the parameters k, p and q, the functions

$$G_1,\ G_2,\ \ldots$$

may be calculated in succession. Hence the values of $c_{q,t}$ and $\varDelta_{p,q,t}$ may be calculated. The amount of work may be shortened by various considerations. Thus, if

$$\frac{1}{2}\pi(1+a) \leq \arg \varDelta_{p,q,t} \leq \frac{1}{2}\pi(3-a),$$

the corresponding terms may be omitted in Lemma 25 and hence in Theorem 3. Again it is not difficult to prove that, when

$$\frac{1}{2}\pi(1+a) \leq \arg \varDelta_{p,q} \leq \frac{1}{2}\pi(3-a),$$

then

$$\frac{1}{2}\pi(1+a) \leq \arg \varDelta_{p,q,t} \leq \frac{1}{2}\pi(3-a) \qquad (9.41)$$

for all $t \geq 0$. Hence in this case

$$|f(x)| < B_{q,\delta}$$

on $\mathfrak{M}_{p,q}$, and the terms corresponding to p, q may be omitted in Theorem 3. The same result follows directly from Lemma 33 and the fact that, for sufficiently large n,

$$|\varphi(\varDelta_{p,q,t}(n+j)^a)| < 1,$$

when (9.41) is satisfied.

In addition, the work of expanding $P_{p,q}$ may be shortened by various combinations of the factors $\{1 - g(h, l, s)\}$. Thus, if

$$\Omega = \exp\left\{\frac{(2\pi)^{1+a}(l+1)^a e^{\frac{1}{2}\pi a i(2s+k-1)}}{q y^a}\right\},$$

we have

$$g(rq_1, l, s) = \Omega e_q(-rq_1) = \Omega e_{q_2}(-r),$$

and

$$\prod_{r=1}^{q_2}\{1 - g(rq_1, l, s)\} = \prod_{r=1}^{q_2}\{1 - \Omega e_{q_2}(-r)\}$$
$$= 1 - \Omega^{q_2}.$$

It seems, however, that no simple formulae can be found for $\varDelta_{p,q,t}$ and $c_{q,t}$ in general.

When $k = 1$, the calculations are simple, since $T_{q,\varepsilon} = 0$ and

$$\varDelta_{p,q,0} = \varDelta_{p,q} = \frac{\pi^2}{6q}.$$

The Generalised Bessel Function and the Auxiliary Function.

10. 1. We now introduce the integral function

$$\varphi(z) = \sum_{l=0}^{\infty} \frac{z^l}{\Gamma(l+1)\,\Gamma\left(la - \frac{1}{2}\right)},$$

in terms of which $p_k(n)$ will ultimately be expanded. This is a particular case of the generalised Bessel function introduced in a recent paper.[1]

We shall use this function to construct an auxiliary function with a singularity of the type of

$$U(\varDelta, x) = x^{-j}\left\{\log\frac{1}{x}\right\}^{\frac{1}{2}} \exp\left\{\frac{\varDelta}{\left(\log\frac{1}{x}\right)^a}\right\}$$

at $x = 1$. We take x^{-j}, $\left(\log\frac{1}{x}\right)^{\frac{1}{2}}$ and $\left(\log\frac{1}{x}\right)^a$ real and positive on the interval

[1] *Jour. London Math. Soc.*, 8 (1933), 71—79.

(0, 1) of the real axis in the x-plane. Let M be the least positive integer such that $M + j > 1$. When $m \geq M$, we take $(m+j)^a$, $(m+j)^{-\frac{3}{2}}$ real and positive. For $|x| < 1$, we write

$$F_\Delta(x) = \sum_{m=M}^{\infty} (m+j)^{-\frac{3}{2}} \varphi(\Delta (n+j)^a)\, x^m,$$

$$Q_\Delta(x) = F_\Delta(x) - U(\Delta, x).$$

Then $F_\Delta(x)$ is defined for $|x| < 1$, and $Q_\Delta(x)$ and $U(\Delta, x)$ are defined precisely for x real and $0 < x < 1$. For other values of x, these functions are defined by analytic continuation.

The u-plane is cut from $-\infty$ to 0 and from 1 to ∞ along the real axis. Then $U(\Delta, u)$ is one-valued and regular in and on the boundary of the region so defined except at $u = 0$ and $u = 1$. Γ_5 is a contour in the u-plane, and consists of the real axis from $-\infty$ to $-\frac{1}{4}$, the small circle $|u| = \frac{1}{4}$ taken in a counterclockwise direction round the origin, and the real axis from $-\frac{1}{4}$ to $-\infty$. Γ_6 consists of the real axis from $+\infty$ to $\frac{5}{4}$, the small circle $|u - 1| = \frac{1}{4}$ taken in a clockwise direction round $u = 1$, and the real axis from $\frac{5}{4}$ to $+\infty$.

We proved in the paper referred to above that

$$F_\Delta(x) = \frac{x^M}{2\pi i} \int_{\Gamma_5} \frac{U(\Delta, u)\, du}{u^M (u - x)},$$

$$Q_\Delta(x) = \frac{x^M}{2\pi i} \int_{\Gamma_6} \frac{U(\Delta, u)\, du}{u^M (u - x)}.$$

We take N sufficiently large to ensure that $e^{-\frac{1}{N}} > \frac{1}{2}$.

Lemma 31. *If* $x = \exp\left(-\frac{1}{N} + i\theta\right)$, *we have* $|F_\Delta(x)| < B_\Delta$ *when* $\frac{1}{2}\pi \leq \theta \leq \frac{3}{2}\pi$, *and* $|Q_\Delta(x)| < B_\Delta$ *when* $-\frac{1}{2}\pi \leq \theta \leq \frac{1}{2}\pi$.

On Γ_5 and Γ_6 we have

$$\left|\log\frac{1}{u}\right| > A, \quad \exp\left(\left|\frac{A}{\left(\log\frac{1}{u}\right)^a}\right|\right) < B_A,$$

$$\left|\log\frac{1}{u}\right|^{\frac{1}{2}} < B\,|\,u\,|^{M+j-1},$$

and so

$$|\,U(A, u)\,| < B_A\,|\,u\,|^{M-1}.$$

If $\frac{1}{2}\pi \leq \theta \leq \frac{3}{2}\pi$ and u lies on Γ_6, we have

$$|\,u-x\,| > A\,|\,u\,|, \quad |\,x\,|^M < B,$$

and so, by (10. 11),

$$|\,F_A(x)\,| < B_A \int\limits_{\Gamma} \left|\frac{du}{u^2}\right| < B_A.$$

Similarly, if $-\frac{1}{2}\pi \leq \theta \leq \frac{1}{2}\pi$, we have $|\,Q_A(x)\,| < B_A$.

10. 2. **Lemma 32.** *If*

$$E(A) = (n+j)^{-\frac{3}{2}}\varphi\{A(n+j)^a\}\, e_q(-pn) - \frac{1}{2\pi i}\int\limits_{\mathfrak{M}_{p,q}} \frac{U(A, X)}{x^{n+1}}\,dx,$$

then

$$|\,E(A)\,| < B_A \exp\left\{\frac{n}{N} + \frac{q^a|\,A\,|}{\pi^a} N^b\right\}.$$

We have

$$E(A) = \frac{1}{2\pi i}\left\{\int\limits_{-\frac{1}{2}\pi}^{\frac{3}{2}\pi} F_A(X)X^{-n}\,d\theta - \int\limits_{-\theta'_{p,q}}^{\theta_{p,q}} U(A, X)X^{-n}\,d\theta\right\}$$

$$= \frac{1}{2\pi i}\left\{\int\limits_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} F_A(X)X^{-n}\,d\theta + \int\limits_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} Q_A(X)X^{-n}\,d\theta\right.$$

$$\left. + \int\limits_{\theta_{p,q}}^{\frac{1}{2}\pi} U(A, X)X^{-n}\,d\theta + \int\limits_{-\frac{1}{2}\pi}^{-\theta'_{p,q}} U(A, X)X^{-n}\,d\theta\right\}$$

$$= \frac{1}{2\pi i}(E' + E'' + E''' + E''''). \tag{10. 21}$$

On Γ,

$$|X|^{-n} = e^{\frac{n}{N}}, \quad \log\frac{1}{X} = \frac{1}{N} - i\theta,$$

and, when $|\theta| < \Delta$,

$$|U(\Delta, X)| < B\left|\frac{1}{N} - i\theta\right|^{\frac{1}{2}} \exp\left(\left|\frac{\Delta}{\left(\frac{1}{N} - i\theta\right)^a}\right|\right)$$

$$< B \exp|\Delta\theta^{-a}|.$$

On $\mathfrak{M}_{p,q}$,

$$-\theta'_{p,q} \leq \theta \leq \theta_{p,q},$$

where

$$\frac{\pi}{qN^{1-b}} \leq \theta'_{p,q} < \frac{2\pi}{qN^{1-b}}; \quad \frac{\pi}{qN^{1-b}} \leq \theta_{p,q} < \frac{2\pi}{qN^{1-b}}.$$

Let us write

$$\theta_0 = \frac{\pi}{qN^{1-b}}.$$

Then

$$|E'''| \leq Be^{\frac{n}{N}} \int_{\theta_0}^{\frac{1}{2}\pi} e^{|\Delta|\theta^{-a}}\, d\theta < B \exp\left(\frac{n}{N} + \frac{|\Delta|}{\theta_0^a}\right)$$

$$= B \exp\left(\frac{n}{N} + \frac{q^a|\Delta|}{\pi^a}N^b\right), \tag{10.22}$$

and similarly

$$|E''''| < B \exp\left(\frac{n}{N} + \frac{q^a|\Delta|}{\pi^a}N^b\right). \tag{10.23}$$

But, by Lemma 31,

$$|E'| < B_\Delta \exp\left(\frac{n}{N}\right), \quad |E''| < B_\Delta \exp\left(\frac{n}{N}\right), \tag{10.24}$$

and the lemma follows at once from (10.21) to (10.24).

10.3. **Lemma 33.** (*Asymptotic expansion of $\varphi(z)$*). *If $\gamma > 0$ and $|\arg z| \leq \leq \pi - \gamma$, then*

$$\varphi(z) = (az)^{1-b} e^{(k+1)(az)^{1-b}} \left\{\sum_{m=0}^{M-1} \frac{(-1)^m a_m}{(az)^{m(1-b)}} + O\left(\frac{1}{|z|^{M(1-b)}}\right)\right\},$$

where the constant in the $O(\ )$ *term is* $B_{M,\gamma}$, *and* a_m *is the coefficient of* v^{2m} *in the expansion of*

$$\frac{\Gamma\left(m+\frac{1}{2}\right)}{2\pi}\left(\frac{2k}{k+1}\right)^{m+\frac{1}{2}}(1-v)^{\frac{1}{2}}\left\{1+\frac{a+2}{3}v+\frac{(a+2)(a+3)}{3\cdot 4}v^2+\cdots\right\}^{-m-\frac{1}{2}}$$

in powers of v. *In particular,*

$$a_0=\left(\frac{k}{2\pi(k+1)}\right)^{\frac{1}{2}},\quad a_1=\frac{(11k^2+11k+2)a_0}{24k(k+1)}.$$

This is Theorem 5 of my paper referred to above. A proof will, I hope, be published shortly in the *Proceedings of the London Math. Soc.*

Final Lemmas and Proof of Theorem 3.

11.1. We write

$$J_{p,q}=(n+j)^{-\frac{3}{2}}C_{p,q}\sum_{t=1}^{T_{q,\delta}}c_{q,t}\,\Phi(\Lambda_{p,q,t}(n+j)^a),$$

and

$$I_{p,q}=\frac{1}{2\pi i}\int\frac{f(x)\,dx}{x^{n+1}},$$

the integral being taken over the major arc $\mathfrak{M}_{p,q}$ $(q\le N^b)$ or the minor arc $\mathfrak{m}_{p,q}$ $(N^b<q\le N^{1-b})$.

By Lemma 20, we have

$$|f(x)|<\exp\{Bq^{-a}N^a\log\log q\}$$

on $\mathfrak{M}_{p,q}$, provided that $q>2$. Hence there exists an integer $q_0=q_0(k,\delta)=B_\delta$ such that

$$|f(x)|<e^{\delta N^a} \tag{11.11}$$

on $\mathfrak{M}_{p,q}$ when $q>q_0$. Taking this value of q_0 we write

$$J=\sum_{q=1}^{q_0}\sum_p J_{p,q},\quad I_1=\sum_{q=1}^{q_0}\sum_p I_{p,q},$$

$$I_2=\sum_{q_0<q\le N^b}I_{p,q},\quad I_3=\sum_{N^b<q\le N^{1-b}}\sum_p I_{p,q}.$$

Then

$$p_k(n) = \frac{1}{2\pi i}\int_{\Gamma} \frac{f(x)\,dx}{x^{n+1}} = I_1 + I_2 + I_3,$$

$$p_k(n) - J = (I_1 - J) + I_2 + I_3. \tag{11.12}$$

11.2. **Lemma 34.** *We have*

$$|I_1 - J| < B_\delta \exp\left(\delta N^a + \frac{n}{N}\right).$$

If $q \leq q_0$ and $T = T_{q,\delta}$, then

$$\begin{aligned} I_{p,q} - J_{p,q} &= \frac{1}{2\pi i}\int_{\mathfrak{M}_{p,q}} \left\{f(x) - C_{p,q}\sum_{t=0}^{T} c_{q,t}\, U(\varDelta_{p,q,t}, X)\right\}\frac{dx}{x^{n+1}} \\ &- C_{p,q}\sum_{t=1}^{T} c_{q,t}\, E(\varDelta_{p,q,t}) \\ &= J'_{p,q} - J''_{p,q}. \end{aligned}$$

By Lemma 25

$$|J'_{p,q}| < B_\delta \exp\left(\frac{n}{N} + \delta N^a\right).$$

Also, by Lemma 32,

$$|E(\varDelta_{p,q,t})| < B_{p,q,t} \exp\left(\frac{n}{N} + B_{p,q,t} N^b\right).$$

But $t \leq T = B_{q,\delta}$ and $p < q \leq q_0 = B_\delta$, so that $B_{p,q,t} < B_\delta$. Hence

$$\begin{aligned} |E(\varDelta_{p,q,t})| &< B_\delta \exp\left(\frac{n}{N} + B_\delta N^b\right) \\ &< B_\delta \exp\left(\frac{n}{N} + \delta N^a\right), \end{aligned}$$

and so

$$|I_{p,q} - J_{p,q}| < B_\delta \exp\left(\frac{n}{N} + \delta N^a\right).$$

Hence

$$|I_1 - J| \leq \sum_{q \leq q_0} \sum_p |I_{p,q} - J_{p,q}|$$

$$< B_\delta q_0^2 \exp\left(\frac{n}{N} + \delta N^a\right)$$

$$= B_\delta \exp\left(\frac{n}{N} + \delta N^a\right).$$

Lemma 35. *We have*

$$|I_2| + |I_3| < B_\delta \exp\left(\frac{n}{N} + \delta N^a\right).$$

If we put $\gamma = \frac{1}{2} bc$ in Lemma 17, we have

$$|\log f(x)| < BN^{a - \frac{1}{2}bc}, \quad |f(x)| < B_\delta e^{\delta N^a}$$

on $\mathfrak{m}_{p,q}$. Also, by (11. 11),

$$|f(x)| < e^{\delta N^a}$$

on $\mathfrak{M}_{p,q}$, if $q_0 < q \leq N^b$. Hence

$$|I_2 + I_3| \leq \sum_{q_0 \leq q \leq N^{1-b}} \sum_p |I_{p,q}|$$

$$< B_\delta e^{\delta N^a} \int_\Gamma |x|^{-n} d\theta$$

$$= B_\delta \exp\left(\frac{n}{N} + \delta N^a\right).$$

11. 3. By (11. 12) and Lemmas 34 and 35,

$$|p_k(n) - J| < B_\delta \exp\left(\frac{n}{N} + \delta N^a\right). \tag{11. 31}$$

We now take

$$\delta = \left(\frac{1}{2}\varepsilon\right)^{1+a}, \quad N = \frac{2n^{1-b}}{\varepsilon}.$$

Then

$$\frac{n}{N} + \delta N^a = \varepsilon n^b, \quad B_\delta = B_\varepsilon,$$

and (11. 31) leads at once to Theorem 3.

Theorems 1 and 2.

12. 1. Theorem 1 is clearly a particular case of Theorem 3. We apply Lemma 33 to all the φ-functions, except that for which $p=0$, $q=1$ and $t=0$. We have

$$\left|(n+j)^{-\frac{3}{2}}\varphi\left(\varDelta_{p,q,t}(n+j)^{a}\right)\right| < B_{q,t}\exp\left\{\Re\left(\varDelta_{p,q,t}\,t^{1-b}\right)n^{b}\right\},$$

and it is not difficult to prove that there is a positive number $\alpha=\alpha(k)>0$ such that

$$\Re\left(\varDelta_{p,q,t}\,t^{1-b}\right) < \varDelta_{0,1,0}-\alpha$$

for all values of p, q and t except $p=0$, $q=1$, and $t=0$.

Theorem 1 is then proved and Theorem 2 is an immediate corollary of Theorem 1 and Lemma 33.

12. 2. It is possible to prove Theorem 1 directly with substantially less analysis than that required to prove Theorem 3. Most of the proof is the same, but the full Theorem 4 is only needed for the case $p=0$ and $q=1$, that is, for $\mathfrak{M}_{0,1}$. In this case

$$\log f(x) = -S_q,$$

and the whole of section 5 may be omitted. The proof of Lemma 20 by means of Theorem 4 is replaced by a proof somewhat similar to that of Lemma 17. By this means we can prove

Lemma 36. *On* $\mathfrak{M}_{p,q}$,

$$\left|\log f(x) - \frac{\varDelta_{p,q}}{y^{a}}\right| < BN^{b}.$$

From this we can find a suitable upper bound for the contribution of the integral along $\mathfrak{M}_{p,q}$ $(q>1)$ to the value of $p_k(n)$, and so prove Theorem 1 directly.