SOME GENERAL DEVELOPMENTS IN THE THEORY OF FUNCTIONS OF A COMPLEX VARIABLE.

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1. Introduction.

It is well known that, following the classical procedure of Cauchy, the theory of analytic functions of a complex variable $z(= x + iy)$ is effectively developed if one starts with a definition according to which the functions under consideration possess a unique derivative in an *open* set O (in the z-plane) and on the basis of this definition establishes the two fundamental Cauchy contour-integral formulas[1]; with the aid of the latter formulas a great number of essential properties of analytic functions can be established. If in the above definition the open set O is replaced by a set E, which is not necessarily open and which, in fact, may be without interior points, one would obtain, of course, a very general

[1] The conditions with respect to the derivative can be somewhat lightened (GOURSAT, MONTEL, BESIKOVICH, MENCHOFF and a number of others).

class of functions of a complex variable, much more vast than the class of analytic functions. The first highly important development in this direction is due to É. BOREL[1], who made a significant extension of the class of analytic functions by his introduction of the *»Borel monogenic»* functions; these functions, according to definition, possess continuous unique derivatives over sets of a certain description (not necessarily open) and for these functions Borel established existence of two integral formulas analogous to those of the classical Cauchy theory. The next significant advance was made by W. J. TRJITZINSKY[2], who introduced functions called by him *'general monogenic'*. These are defined as follows. Let (K) be a simple rectifiable curve forming the frontier of a bounded domain K. Let E be a closed set contained in $K + (K)$. According to the definition, given in (T), *$f(z) = u(x, y) + \sqrt{-1}\, w(x, y)$ is general monogenic in E if u, w, the first partials of u and w and the second partials of, say, u are finite continuous in E, u is uniform in E, $\Delta u = 0$ (in E)[3] and w is in E harmonic conjugate of u.* In consequence of the developments of (T) it can be asserted that such functions, together with their first derivatives, are representable in EK as follows:

$$(1.1)\qquad f(\alpha) = h(\alpha) + \int\!\!\int_K \log(z - \alpha)\, q(x, y)\, dx\, dy$$

$$(\alpha = a + \sqrt{-1}\, b;\ h(\alpha) \text{ analytic in } K),^4$$

$$(1.1\text{a})\qquad f^{(1)}(\alpha) = h^{(1)}(\alpha) - \int\!\!\int_K \frac{q(x, y)}{z - \alpha}\, dx\, dy,$$

where $q(x, y)$ is continuous in $K + (K)$ and $q(x, y) = 0$ in E.

These are the fundamental formulas of the theory of general monogenic fonctions. In (T) it has been shown that numerous regularity properties (differentiability, uniqueness properties of various descriptions, etc.), of the functions under consideration, will present themselves if

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[1] É. BOREL, *Leçons sur les fonctions monogènes . . .*, Paris, 1917.

[2] W. J. TRJITZINSKY, *Théorie des fonctions d'une variable complexe définies sur des ensembles généraux*, Annales de l'École Norm. Sup., t. 55—Fasc. 2; pp. 119—191; in the sequel this work will be referred to as (T).

[3] $$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

[4] In (1.1) a suitable determination of the logarithm must be used.

(1. i) $q(x, y) \to 0$ sufficiently rapidly as the point (x, y) (in $C(E) = K + (K) - E$) approaches the frontier of $C(E)$; also if,

(1. ii) having covered $C(E)$ by the sum of a denumerable infinity of circular domains, of radii $\gamma_1, \gamma_2, \ldots$, we have $\gamma_\nu \to 0$ (as $\nu \to \infty$) sufficiently rapidly.

The purpose of the present work is to study functions of the form

$$(1.\,2) \qquad \iint \log(z - \alpha)\, d\mu, \qquad \iint \frac{d\mu}{z - \alpha}$$

where $z (= x + iy)$ is the variable of integration and μ is an additive functions of sets[1], *absolutely continuous or not*[2], with respect to which the integrations are performed.

Analytic functions[3], *»Borel monogenic» functions and »general monogenic» functions (of (T)) are all included in* (1. 2).

It is easily seen that functions of the form

$$(1.\,3) \qquad \sum \frac{b_\nu}{\alpha_\nu - \alpha} \qquad \left(\sum_\nu |b_\nu| \text{ convergent}\right)$$

are representable by integrals of the form of the second one in (1. 2) with μ singular. Functions of the form (1. 3) have been studied by Borel, Denjoy, T. Carleman, A. Beurling and a number of others.

In the sequel it will be also shown that *functions $f(\alpha)$ which in a closed set G are representable as limits of uniformly convergent sequences of analytic functions are representable by integrals* (1. 2), at least when G satisfies certain conditions (to be stated precisely in the sequel).

Our investigations will be mainly regarding functions of the form

$$(1.\,4) \qquad \iint \frac{d\mu}{z - \alpha}.$$

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[1] Throughout, any set mentioned will be implied to be Lebesgue measurable. Regarding additive functions of sets and integrals with respect to such functions see S. Saks, *Theory of the Integral*, Warszawa-Lwow, 1937, in the sequal referred to as (S).

[2] An additive function of sets X, $\mu(X)$, will be said to be absolutely continuous on a set G, if for every set $X < G$, with meas. $X = 0$, we have $\mu(X) = 0$. If an additive function $\mu(X)$ is not absolutely continuous on G then $\mu(X)$ is the sum of an absolutely continuous additive function and of a singular additive function (on G); cf. (S; p. 33). An additive function of sets $\vartheta(X)$ $(X < G)$ will be said to be singular on G if for some set E_0, $< G$ and with meas. $E_0 = 0$, we have $\vartheta(X) = \vartheta(E_0 X)$ for all sets $X < G$. Throughout this work, set-functions will be implied to be additive.

[3] The statement with respect to analytic functions is seen to be true in consequence of certain remarks by Borel; cf. Borel, *loc. cit.*

Of importance for our purposes is the notion of symmetric density, forthwith called density. Let $S(\alpha_0, r)$ be the closed circular region with center at α_0 and radius r. *Density, $\varrho(\alpha_0)$, of μ at the point α_0 will be the limit (as $r \to 0$), if it exists, of*

$$(1.5) \qquad \frac{1}{\pi r^2} \int\int_{S(\alpha_0, r)} |d\mu| = \varrho(\alpha_0, r).$$

In consequence of a theorem of Lebesgue, $\varrho(\alpha_0)$ exists (and is finite) for almost all α_0. In most cases the points of interest in the developments of this paper will be those at which the density is zero.

Let α_0 be a point for which $\varrho(\alpha_0)$ exists. Descriptively we shall term the speed with which $\varrho(\alpha_0, r) \to \varrho(\alpha_0)$ (as $r \to 0$) *rarefication of μ*. The faster $\varrho(\alpha_0, r)$ approaches the limit (as $r \to 0$) the greater will be said to be the rarefication of μ at α_0. Similarly, one may talk of »rarefication» of μ in appropriate neighborhoods of sets.

Our investigation will be largely based on the principle according to which *the greater is the rarefication of μ (μ in (1.2)) the more regularity properties will the corresponding functions* (1.2) *possess.* More precisely, various degrees of rarefication of μ will be determined to secure prescribed regularity properties, of various types, of the corresponding functions (1.2).

2. Functions Representable by Integrals (1.2).

Let G be a closed bounded set. Let $O(\delta)$ ($\delta > 0$) be the set of points at distance $< \delta$ from G. Thus, $O(\delta)$ is an open set containing G. *Let*

$$(2.1) \qquad \delta_1 > \delta_2 > \cdots; \quad \delta_\nu > 0 \ (\nu = 1, 2, \ldots); \quad \lim_\nu \delta_\nu = 0.$$

Suppose there is on hand a sequence of functions $\{f_\nu(\alpha)\}$ ($\nu = 1, 2, \ldots$), $f_\nu(\alpha)$ analytic (uniform) in $O(\delta_\nu)$ ($\nu = 1, 2, \ldots$), converging uniformly in G. By hypothesis, then, there exists a function $f(\alpha)$ such that

$$(2.2) \qquad |f(\alpha) - f_\nu(\alpha)| \leqq \varepsilon_\nu \qquad (\alpha \text{ in } G;\ \nu = 1, 2, \ldots),$$

with $\varepsilon_\nu \to 0$ (as $\nu \to \infty$). We shall examine the possibility of representing $f(\alpha)$ by an integral (1.4), with μ an absolutely continuous set-function.

Write

(2. 3) $$a_\nu(\alpha) = f_{n_\nu}(\alpha) - f_{n_{\nu-1}}(\alpha) \qquad (\nu = 1, 2, \ldots; \; f_{n_0}(\alpha) \equiv 0)$$

where $0 = n_0 < n_1 < n_2 < \cdots$. It is noted that $a_\nu(\alpha)$ *is analytic in* $O(\delta_{n_\nu})$ and that, in view of (2. 2),

(2. 3 a) $$|a_\nu(\alpha)| \leqq |f_{n_\nu}(\alpha) - f(\alpha)| + |f(\alpha) - f_{n_{\nu-1}}(\alpha)| \leqq \varepsilon_{n_\nu} + \varepsilon_{n_{\nu-1}} = \eta_\nu$$

$$(\varepsilon_{n_0} = 0; \; \nu = 1, 2, \ldots; \; \alpha \text{ in } G).$$

The sequence $\{n_\nu\}$ will be so chosen that

(2. 4) $$\sum \eta_\nu$$

converges. *Let* $\bar{O}(\delta)$ *designate* $O(\delta)$ + frontier of $O(\delta)$. It is then observed that $\bar{O}(\delta_{n_{\nu+1}})$ is a closed subset of $O(\delta_{n_\nu})$. In $\bar{O}(\delta_{n_{\nu+1}})$ $a_\nu(\alpha)$ is analytic. Hence

(2. 5) $$|a_\nu(\alpha)| \leqq q_\nu \qquad (\alpha \text{ in } \bar{O}(\delta_{n_{\nu+1}}); \; \nu = 1, 2, \ldots).$$

By a known property of analytic functions we may take

(2. 5 a) $$q_\nu = \text{u. b.}\{[\alpha \text{ in } \bar{O}(\delta_{n_{\nu+1}})] \text{ of } |a_\nu(\alpha)|\} = |a_\nu(\alpha_\nu)|,$$

where α_ν is some point on the frontier of $\bar{O}(\delta_{n_{\nu+1}})$.[1]

Let $S^\nu_{\alpha'} = S(\alpha'; \delta_{n_{\nu+1}})$ be the closed circular domain with center at α' (α' in G) and radius $\delta_{n_{\nu+1}}$. In consequence of the definition of $O(\delta_{n_{\nu+1}})$

(2. 6) $$S^\nu_{\alpha'} < \bar{O}(\delta_{n_{\nu+1}}).$$

For α in $S^\nu_{\alpha'}$ consider the function

$$w_\nu(\alpha) = a_\nu(\alpha) - a_\nu(\alpha') \qquad (\alpha' \text{ in } G).$$

In consequence of (2. 6), (2. 5) and (2. 3 a) one accordingly has

(2. 7) $$|w_\nu(\alpha)| \leqq q_\nu + \eta_\nu \qquad (\alpha \text{ in } S^\nu_{\alpha'});$$

moreover, $w_\nu(\alpha') = 0$ and $w_\nu(\alpha)$ is analytic in $S^\nu_{\alpha'}$, i. e. for $|\alpha - \alpha'| \leqq \delta_{n_{\nu+1}}$. By (2. 7) with the aid of the Lemma of Schwartz it is inferred that

(2. 7 a) $$|w_\nu(\alpha)| \leqq \frac{1}{\delta_{n_{\nu+1}}}(q_\nu + \eta_\nu)|\alpha - \alpha'| \qquad (\alpha \text{ in } S^\nu_{\alpha'}).$$

[1] u. b. here denotes 'least upper bound'.

Whence, by (2. 3 a),

$$|a_\nu(\alpha)| = |w_\nu(\alpha) + a_\nu(\alpha')| \leqq |w_\nu(\alpha)| + |a_\nu(\alpha')|$$

$$\leqq \frac{1}{\delta_{n_{\nu+1}}}(q_\nu + \eta_\nu)|\alpha - \alpha'| + \eta_\nu \qquad (\alpha \text{ in } S^\nu_{\alpha'}).$$

Therefore

$$(2.\ 8) \qquad |a_\nu(\alpha)| \leqq 2\,\eta_\nu \qquad (|\alpha - \alpha'| \leqq \tau_\nu;\ \alpha' \text{ in } G),$$

provided one takes

$$(2.\ 8\text{ a}) \qquad \tau_\nu = \frac{\eta_\nu\,\delta_{n_{\nu+1}}}{q_\nu + \eta_\nu} \qquad (< \delta_{\nu_{\nu+1}}).$$

Clearly (2. 8) *will hold in the set* $\bar{O}(\tau_\nu)$, *contained in* $\bar{O}(\delta_{n_{\nu+1}})$. There exists a sequence of integers $\{m_\nu\}$ $(m_\nu \geqq n_{\nu+1};\ m_1 < m_2 < \cdots)$ so that $O(\delta_{m_\nu}) < O(\tau_\nu)$ $(\nu = 1, 2, \ldots)$. Since $a_\nu(\alpha)$ is analytic in $O(\delta_{n_\nu})$ and since $O(\delta_{m_\nu}) < \bar{O}(\tau_\nu) <$ $< \bar{O}(\delta_{n_{\nu+1}})$ (cf. italics subsequent to (2. 8 a)) one accordingly may state the following.

In $\bar{O}(\delta_{m_\nu})$ $a_\nu(\alpha)$ *is analytic and*

$$(2.\ 9) \qquad |a_\nu(\alpha)| \leqq 2\,\eta_\nu \qquad (\alpha \text{ in } \bar{O}(\delta_{m_\nu});\ \nu = 1, 2, \ldots;\ \eta_\nu = \varepsilon_{n_\nu} + \varepsilon_{n_{\nu-1}});$$

moreover $O(\delta_{m_1}) > O(\delta_{m_2}) > \cdots$ *and* $\lim_\nu O(\delta_{m_\nu}) = \lim_\nu \bar{O}(\delta_{m_\nu}) = G$. *Furthermore,*

$$(2.\ 10) \qquad f(\alpha) = \sum_{\nu=1}^{\infty} a_\nu(\alpha) \qquad (\alpha \text{ in } G),$$

where the series converges uniformly and absolutely in G; *the series* $\eta_1 + \eta_2 + \cdots$ *converges for a suitable choice of the sequence* $\{n_\nu\}$.

Now the set $O(\delta_{m_{\nu+1}})$ together with its frontier, that is the set $\bar{O}(\delta_{m_{\nu+1}})$, is contained in the open set $O(\delta_{m_\nu})$. Hence there exists an open set O_ν which contains $O(\delta_{m_{\nu+1}})$, whose frontier consists of a number of polygons,

$$P_\nu,$$

and whose closure $\bar{O}_\nu = O_\nu + P_\nu < O(\delta_{m_\nu})$. Accordingly P_ν will be at a positive distance, say l_ν, from the frontier of $O(\delta_{m_\nu})$. We shall have

$$(2.\ 10\text{ a}) \qquad a_\nu(\alpha) = \frac{1}{2\pi i}\int_{P_\nu} \frac{a_\nu(z)\,dz}{z - \alpha}$$

for α in O_ν, in particular in $O(\delta_{m_{\nu+1}})$. Take $0 < \lambda_\nu < l_\nu$. *Let* $O_{\nu,\varrho}$ *denote an open set containing* O_ν; the frontier of $O_{\nu,\varrho}$ consisting of a number of polygons

(2. 11) $$P_{\nu,\varrho},$$

parallel to the polygons P_ν and at the distance from the latter equal to ϱ, $0 \leqq \varrho \leqq \lambda_\nu$. One has

(2. 11 a) $$a_\nu(\alpha) = \frac{1}{2\pi i}\int_{P_{\nu,\varrho}} \frac{a_\nu(z)\,dz}{z-\alpha} \qquad (\alpha \text{ in } O_\nu)$$

for all ϱ such that $0 \leqq \varrho \leqq \lambda_\nu$. As ϱ varies from 0 to λ_ν the polygons $P_{\nu,\varrho}$ will move, always remaining parallel to P_ν, from $P_{\nu,0} = P_\nu$ to P_{ν,λ_ν}, thus describing the set

(2. 12) $$\Delta_\nu = O_{\nu,\lambda_\nu} - O_\nu + P_{\nu,\lambda_\nu}.$$ [1]

It is clear that $\Delta_\nu < O(\delta_{m_\nu})$ and that Δ_ν has no points in common with $O(\delta_{m_{\nu+1}})$ (cf. italics preceding (2. 10)); thus

(2. 12 a) $$\Delta_\nu < O(\delta_{m_\nu}) - O(\delta_{m_{\nu+1}}).$$

With ds denoting the differential of length along $P_{\nu,\varrho}$, the differential dz involved in (2. 11 a) is seen to be of the form

(2. 13) $$dz = ds\, e^{i\varphi(z)} \qquad (z \text{ on } P_{\nu,\varrho});$$

here $\varphi(z)$ is a real-valued step function, maintaining constant values interior each polygonal side of $P_{\nu,\varrho}$. Let $h_\nu(\varrho)$ $(\geqq 0)$ be a function continuous for

$$0 \leqq \varrho \leqq \lambda_\nu\ .$$

and not identically zero.[2] We introduce (2. 13) in (2. 11 a), multiply both sides by $h_\nu(\varrho)\,d\varrho$ and integrate between the limits $\varrho = 0$, $\varrho = \lambda_\nu$. It is inferred, for α in O_ν,

(2. 14) $$a_\nu(\alpha)\,A_\nu = \frac{1}{2\pi i}\int_{\varrho=0}^{\lambda_\nu}\int_{P_{\nu,\varrho}} \frac{a_\nu(z)}{z-\alpha}\, e^{i\varphi(z)}\, h_\nu(\varrho)\, ds\, d\varrho,$$

where

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[1] The frontier of Δ_ν is $P_{\nu,0} + P_{\nu,\lambda_\nu}$.

[2] It is advantageous to have $h_\nu(\varrho)$ vanish, with suitable rapidity, as $\varrho \to 0$ and as $\varrho \to \lambda_\nu$.

(2. 14 a) $$A_\nu = \int_{\varrho=0}^{\lambda_\nu} h_\nu(\varrho)\, d\varrho \qquad (> 0).$$

On writing, for z in Δ_ν,

(2. 14 b) $$b_\nu(x, y) = \frac{1}{2\pi i}\frac{1}{A_\nu} a_\nu(z)\, e^{i\varphi(z)} h_\nu(\varrho) \qquad (z = x + iy)$$

it is noted that (2. 14) may be written in the form

(2. 15) $$a_\nu(\alpha) = \iint_{\Delta_\nu} \frac{b_\nu(x, y)}{z - \alpha}\, dx\, dy$$

for α in O_ν. In view of (2. 14 b) and (2. 9) and in consequence of a previous remark asserting that $\Delta_\nu < O(\delta_{m_\nu})$, it is observed that

(2. 16) $$|b_\nu(x, y)| \leqq \frac{1}{2\pi}\frac{1}{A_\nu} |a_\nu(z)|\, h_\nu(\varrho) \leqq \frac{\eta_\nu}{\pi A_\nu} h_\nu(\varrho),$$

when z is in Δ_ν. *Let $L_{\nu,\varrho}$ denote the length of $P_{\nu,\varrho}$.* We have

(2. 17) $$L_{\nu,\varrho} \leqq L_\nu^* \qquad [< \infty;\ 0 \leqq \varrho \leqq \lambda_\nu].$$

Whence from (2. 16) with the aid of (2. 14 a) it is inferred that

(2. 18) $$\iint_{\Delta_\nu} |b_\nu(x, y)|\, dx\, dy \leqq \frac{\eta_\nu}{\pi A_\nu} \iint_{\Delta_\nu} h_\nu(\varrho)\, ds\, d\varrho$$

$$= \frac{\eta_\nu}{\pi A_\nu} \int_{\varrho=}^{\lambda_\nu} h_\nu(\varrho)\, d\varrho \int_{P_{\nu,\varrho}} ds = \frac{\eta_\nu}{\pi A_\nu} \int_{\varrho=0}^{\lambda_\nu} h_\nu(\varrho)\, L_{\nu,\varrho}\, d\varrho$$

$$\leqq \frac{L_\nu^* \eta_\nu}{\pi A_\nu} \int_{\varrho=0}^{\lambda_\nu} h_\nu(\varrho)\, d\varrho = \frac{1}{\pi} L_\nu^* \eta_\nu .$$

For L_ν^* we may take the maximum length of $P_{\nu,\varrho}$ $(0 \leqq \varrho \leq \lambda_\nu)$.

In view of (2. 12 a) the sets $\Delta_\nu (\nu = 1, 2, \ldots)$ have no points in common. Form the function $g(x, y)$,

$$g(x, y) = \begin{cases} 0 & \text{(in the complement of } \Delta_1 + \Delta_2 + \cdots), \\ b_\nu(x, y) & \text{(in } \Delta_\nu;\ \nu = 1, 2, \ldots). \end{cases}$$

Let $c(\Delta_\nu) = c_\nu(x, y)$ denote the characteristic function[1] of Δ_ν. We then may write

[1] That is, $c_\nu = 1$ in Δ_ν and $c_\nu = 0$ at other points.

(2. 19) $$g(x, y) = \sum_{\nu=1}^{\infty} c_\nu(x, y)\, b_\nu(x, y).$$

With

(2. 19 a) $$g_n(x, y) = \sum_{\nu=1}^{n} c_\nu(x, y)\, b_\nu(x, y),$$

in view of (2. 15) it follows that

$$\iint \frac{g_n(x, y)\, dx\, dy}{z - \alpha} = \sum_{\nu=1}^{n} \iint \frac{c_\nu(x, y)\, b_\nu(x, y)\, dx\, dy}{z - \alpha}$$

$$= \sum_{\nu=1}^{n} \iint_{\Delta_\nu} \frac{b\ (x, y)\, dx\, dy}{z - \alpha} = a_1(\alpha) + \cdots + a_n(\alpha)$$

for α in G (cf. beginning of this section). Whence in consequence of (2. 10) it is concluded that

(2. 20) $$f(\alpha) = \lim_n \iint \frac{g_n(x, y)\, dx\, dy}{z - \alpha} = \iint \frac{g(x, y)\, dx\, dy}{z - \alpha} \qquad (\alpha \text{ in } G),$$

if the interchange of limiting processes can be justified. We have, by (2. 19),

$$|g_n(x, y)| \leqq \sum_{\nu=1}^{\infty} |c_\nu(x, y)\ b_\nu(x, y)| = \sum_{\nu=1}^{\infty} c_\nu(x, y)\, |b_\nu(x, y)| = g^*(x, y)$$

($n = 1, 2, \ldots$); $g^*(x, y)$ is accordingly a function such that

$$g^*(x, y) = \begin{cases} 0 & (\text{in the complement of } \Delta_1 + \Delta_2 + \cdots), \\ |b_\nu(x, y)| & (\text{in } \Delta_\nu;\ \nu = 1, 2, \ldots); \end{cases}$$

that is, $g^*(x, y) = |g(x, y)|$. This function is summable, if the series

(2. 21) $$S = \sum_\nu L_\nu^* \eta_\nu$$

converges. In fact, in the latter case, with the aid of (2. 18) it is inferred that

$$\iint_X g^*(x, y)\, dx\, dy = \sum_{\nu=1}^{\infty} \iint_{X\Delta_\nu} g^*(x, y)\, dx\, dy = \sum_{\nu=1}^{\infty} \iint_{X\Delta_\nu} |b_\nu(x, y)|\, dx\, dy$$

$$\leqq \sum_{\nu=1}^{\infty} \iint_{\Delta_\nu} |b_\nu(x, y)|\, dx\, dy \leqq \frac{1}{\pi} S$$

for all measurable sets X.

Thus, if (2.21) converges, it may be asserted that *there exists a summable function* $g(x, y)$ *so that*

$$f(\alpha) = \iint \frac{g(x, y)\, dx\, dy}{z - \alpha} = \iint \frac{d\mu}{z - \alpha} \qquad (\alpha \text{ in } G); \tag{2.22}$$

here

$$\mu = \mu(X) = \iint_X g(x, y)\, dx\, dy$$

is an absolutely continuous set-function.

Definition 2.1. *A bounded closed set G will be said to be regular if the following is true. Let $O(\delta)$ be set of points at distance δ (> 0) from G. We take $\delta_1 > \delta_2 > \cdots$ ($\delta_\nu > 0$; $\delta_\nu \to 0$). Let $P(\delta_\nu)$ denote a set of polygons in $O(\delta_\nu) - O(\delta_{\nu+1})$ so chosen that any function analytic in $O(\delta_\nu)$ could be represented in $O(\delta_{\nu+1})$ by the Cauchy contour-integral formula extended over $P(\delta_\nu)$. We designate by $L(P(\delta_\nu))$ the total length of the polygons constituting $P(\delta_\nu)$. The polygons $P(\delta_\nu)$ could be so chosen that the sequence of numbers $L(P(\delta_\nu))$ ($\nu = 1, 2, \ldots$) is bounded.*

It is observed that if G is regular then the polygonal sets P_ν and the numbers λ_ν can be so chosen that the upper bound of the lengths of the P_ν, ($0 \leqq \varrho \leqq \lambda_\nu$; $\nu = 1, 2, \ldots$) is finite. We then shall have $L_\nu^* \leqq b$ ($\nu = 1, 2, \ldots$). In view of the convergence of the series (2.4) the series (2.21) will then also converge and a representation (2.22) will be valid in G.

We are now in position to state the following theorem.

Theorem 2.1. *Let G be a bounded closed set, regular according to the Definition 2.1. Consider the problem formulated at the beginning of this section (cf. the text in connection with (2.1)). Such functions $f(\alpha)$ are representable in G by integrals of the form (2.22) (with $g(x, y)$ summable).*

Note. This theorem may be extended, following the same type of reasoning as just employed, to more general classes of sets G and functions $f(\alpha)$, defined over G.

Consider now functions $f(\alpha)$ of the form (1.3). As remarked in section 1 such functions are representable by integrals (1.4) with μ denoting a singular set-function. We shall examine the possibility of replacing μ in this representation by an absolutely continuous set-function.

It is known[1] that

[1] J. Wolff, Comptes Rendus, t. 173; pp. 1056—57.

(2. 23) $$\frac{1}{\alpha_\nu - \alpha} = \frac{1}{\pi r_\nu^2} \int\int_{S_\nu} \frac{dx\,dy}{z - \alpha} \qquad (S_\nu = S(\alpha_\nu, r_\nu))$$

for α on the frontier and exterior the circular domain $S(\alpha_\nu, r_\nu)$,

$$|z - \alpha_\nu| < r_\nu \qquad (r_\nu > 0).$$

Accordingly,

(2. 24) $$\frac{b_\nu}{z - \alpha} = \int\int_{S_\nu} \frac{\lambda_\nu\,dx\,dy}{z - \alpha} \qquad \left(\lambda_\nu = \frac{b_\nu}{\pi r_\nu^2}\right)$$

for $|\alpha - \alpha_\nu| \geqq r_\nu$. Since by hypothesis $|b_1| + |b_2| + \ldots$ converges, the function

(2. 25) $$f(\alpha) = \sum_{\nu=1}^{\infty} \frac{b_\nu}{\alpha_\nu - \alpha}$$

converges except in a set E_0 of measure zero. Suppose, as can be done without any essential loss of generality, the α_ν ($\nu = 1, 2, \ldots$) are all interior a finite circular domain K. We cover E_0 by a set O consisting of the sum of circular domains

(2. 26) $$|\alpha_\nu - \alpha| < r_\nu \qquad (r_\nu > 0;\ \nu = 1, 2, \ldots).$$

K can be always chosen so that $O < K$. On the other hand, the r_ν can be so selected that meas. O is however small.

Let $f_n(\alpha)$ be the sum of the first n terms of the series (2. 25). In view of (2. 24)

$$f_1(\alpha) = \int\int_K \frac{g_1(z)\,dx\,dy}{z - \alpha} \qquad (\alpha \text{ in } K - S_1),$$

where $g_1(z) = \lambda_1$ in S_1 and and $g_1(z) = 0$ in $K - S_1$. The function $f_2(z)$ is representable as

$$f_2(z) = \int\int_{S_1 - S_1 S_2} \frac{\lambda_1\,dx\,dy}{z - \alpha} + \int\int_{S_2 - S_1 S_2} \frac{\lambda_2\,dx\,dy}{z - \alpha}$$

$$+ \int\int_{S_1 S_2} \frac{(\lambda_1 + \lambda_2)\,dx\,dy}{z - \alpha} = \int\int_K \frac{g_2(z)\,dx\,dy}{z - \alpha} \qquad (\alpha \text{ in } K - (S_1 + S_2))$$

where $g_2(z) = \lambda_1$ (in $S_1 - S_1 S_2$), $= \lambda_2$ (in $S_2 - S_1 S_2$), $= \lambda_1 + \lambda_2$ (in $S_1 S_2$), $= 0$ (elsewhere). It is not difficult to see, then, that

(2. 27) $$f_n(\alpha) = \iint\limits_K \frac{g_n(z)\,dx\,dy}{z-\alpha} \qquad (\text{in } K - (S_1 + S_2 + \cdots + S_n)),$$

where $g_n(z)$ is defined as follows. When z is in $S_1 + S_2 + \cdots + S_n$, a number of the domains S_ν ($\nu = 1, 2, \ldots n$) will contain z. Let

$$S_{\nu_1(z)},\ S_{\nu_2(z)},\ \ldots,\ S_{\nu_{m^1}(z)} \qquad (m^1 = m(z))$$

constitute the totality of such domains. Write

(2. 27 a) $$g_n(z) = \begin{cases} \lambda_{\nu_1(z)} + \lambda_{\nu_2(z)} + \cdots + \lambda_{\nu_{m^1}(z)} & (\text{in } S_1 + S_2 + \cdots + S_n), \\ 0 & (\text{elsewhere}). \end{cases}$$

We have

(2. 28) $$f(\alpha) = \lim_n f_n(\alpha) = \lim_n \iint\limits_K \frac{g_n(z)\,dx\,dy}{z-\alpha} \qquad (\alpha \text{ in } K - O).$$

The function $g_n(z)$ *is simple*[1].

Suppose the $|b_\nu| \to 0$ *(as* $\nu \to \infty$*) sufficiently rapidly so that there exists a sequence* $\{\varrho_\nu\}$ $(\varrho_\nu > 0;\ \nu = 1, 2, \ldots)$ *such that both series*

(2. 29) $$\sum |b_\nu|\, \varrho_\nu^{-2}, \quad \sum \varrho_\nu^2$$

converge. Such sequences $\{\varrho_\nu\}$ *exist, for instance, when*

$$|b_\nu| < a\, \nu^{-2-\varepsilon} \qquad (\varepsilon > 0;\ a > 0;\ \nu = 1, 2, \ldots).$$

Define $g(z)$ *as zero in* $K - O$. For z in O there is a number of domains S_ν ($\nu = 1, 2, \ldots$) containing z; let the totality of such domains (for z fixed in O) be

$$S_{\nu_1(z)}, \quad S_{\nu_2(z)}, \ \ldots.$$

In the set O *we define* $g(z)$ *as*

(2. 30) $$g(z) = \lambda_{\nu_1(z)} + \lambda_{\nu_2(z)} + \ldots.$$

Take

(2. 31) $$r_\nu = h\varrho_\nu \qquad (h > 0)$$

[1] That is, it assumes a finite number of distinct values.

where $\{\varrho_\nu\}$ is a sequence such that the two series (2. 29) converge. Chosing h sufficiently small, meas. O can be made however small[1]. In view of (2. 30) and since $\lambda_\nu = b_\nu / (\pi r_\nu^2)$, it is inferred that inasmuch as the first series (2. 29) converges

(2. 32) $$|g(z)| \leqq \sum_\nu |\lambda_\nu| = h_1 (< \infty) \qquad (z \text{ in } K).$$

The function $g(z)$, being the limit of the sequence of simple (and hence measurable) functions $g_n(z)$, is measurable. In view of (2. 32) $g(z)$ is summable. Also, by (2. 27 a)

$$|g_n(z)| \leqq \sum_{\nu=1}^{\infty} |\lambda_\nu| = h_1 \qquad (\text{in } K;\ n = 1, 2, \ldots);$$

moreover,

$$\left| \frac{g_n(z)}{z - \alpha} \right| \leqq \frac{h_1}{|z - \alpha|},$$

while the function $h_1 |z - \alpha|^{-1}$ is summable. Thus,

$$\lim_n \iint_K \frac{g_n(z)\, dx\, dy}{z - \alpha} = \iint_K \frac{g(z)\, dx\, dy}{z - \alpha} \qquad (\text{in } K - O).$$

Accordingly, in view of (2. 28) it is possible to state the following theorem.

Theorem 2. 2. *Functions $f(\alpha)$ of the form (2. 25), with the $|b_\nu|$ approaching zero sufficiently fast so that the series (2. 29) both converge for some sequence $\{\varrho_\nu\}$, are representable by*

$$f(\alpha) = \iint \frac{d\mu}{z - \alpha} \qquad (\alpha \text{ in } K - O),$$

where $\mu = \mu(X)$ is an absolutely continuous set-function. The open set O is the sum of circular domains $S(\alpha_\nu, h_{\varrho_\nu})$ $(\nu = 1, 2, \ldots)$, with h (> 0) arbitrarily small. Thus, meas. O *can be taken however small*[2].

3. Convergence and Differentiability.

In the integrals (1. 2)

$$\mu = \mu_1 + i\mu_2 \qquad (\mu_1, \mu_2 \text{ real})$$

---

[1] meas. $O \leqq \pi h^2 \sum_\nu \varrho_\nu^2$.

[2] The function μ depends on the choice of O.

is an additive set-function, possibly complex-valued. If one wishes to investigate properties of functions represented by these integrals there is no loss of generality in assuming μ real, for one has

$$(3.1) \qquad \iint g(z,\alpha)\,d\mu = \iint g(z,\alpha)\,d\mu_1 + i\iint g(z,\alpha)\,d\mu_2,$$

where

$$(3.1\text{ a}) \qquad g(z,\alpha) = \log(z-\alpha) \quad \text{or} \quad g(z,\alpha) = (z-\alpha)^{-n} \quad (\text{integer } n > 0)^1.$$

As is well known, a real additive function μ is the difference of two non-negative functions of such kind. Hence *we are justified in confining our attention to integrals of the form*

$$(3.2) \qquad \iint g(z,\alpha)\,d\mu \qquad (\text{cf. } (3.1\text{ a});\ \mu \geqq 0 \text{ or } \mu \leqq 0).$$

Unless stated otherwise, α will be restricted to a bounded simply connected domain K or to some subsets of K, while integrations will be performed over sets $X < K$ and we shall take $\mu \geqq 0$ or $\mu \leqq 0$.

Consider the integral

$$(3.3) \qquad \Phi = \iint \frac{d\mu}{|z-\alpha|^n},$$

where n is a positive integer. For the present α will be thought of as fixed in K; we shall find conditions to be satisfied by μ, in the vicinity of α, in order that (3.3) shold converge. Let

$$(3.4) \qquad f_{n,r}(z,\alpha) = \begin{cases} \dfrac{1}{|z-\alpha|^n} & (\text{for } |z-\alpha| > r), \\ \dfrac{1}{r^n} & (\text{for } |z-\alpha| \leqq r). \end{cases}$$

Since in (3.3) $\mu \geqq 0$ and since $f_{n,r}(z,\alpha)$ approaches $|z-\alpha|^{-n}$ increasing monotonically, as $r \to 0$, it is inferred that passage to the limit under the integral sign is possible so as to obtain the relation

$$(3.5) \qquad \lim_r \iint f_{n,r}(z,\alpha)\,d\mu = \iint \frac{d\mu}{|z-\alpha|^n} = \Phi_n.$$

It is of importance to secure finiteness of the limit involved in (3.5). One has

[1] With a suitable determination of the logarithm.

$$\iint f_{n,r}(z,\alpha)\,d\mu = \Phi'_{n,r} + \Phi''_{n,r} \tag{3.6}$$

where

$$\Phi'_{n,r} = \iint\limits_{K-S_r} f_{n,r}(z,\alpha)\,d\mu, \qquad \Phi''_{n,r} = \iint\limits_{S_r} f_{n,r}(z,\alpha)\,d\mu \tag{3.6a}$$

and $S_r = S(\alpha, r)$ *denotes the circular region* (*with center at* α) $|z-\alpha| \leqq r$; r is taken sufficiently small so that $S_r < K$. In view of (3. 4)

$$\Phi'_{n,r} = \iint\limits_{K-S_r} \frac{d\mu}{|z-\alpha|^n}, \qquad \Phi''_{n,r} = \frac{1}{r^n}\mu(S(\alpha,r)). \tag{3.6b}$$

$\Phi'_{n,r}$ increases monotonically as $r \to 0$; we wish to determine a rarefication of »mass» $\mu(\geqq 0)$, in the vicinity of the point α, which would secure finiteness of the limit

$$\lim_{r} \Phi'_{n,r} = \Phi'_n. \tag{3.7}$$

With $r_0(>0)$ such that $S_{r_0} = S(\alpha, r_0) > K$ one has

$$\Phi'_{n,r} = \iint\limits_{K-S_{r_0}} \frac{d\mu}{|z-\alpha|^n} + \iint\limits_{S_{r_0,r}} \frac{d\mu}{|z-\alpha|^n} \qquad (0 < r \leqq r_0), \tag{3.8}$$

where $S_{r_0,r} = S_{r_0} - S_r$. The first integral in the second member above is obviously finite. The second one satisfies the inequality

$$\iint\limits_{S_{r_0,r}} \frac{d\mu}{|z-\alpha|^n} < \sum_{\nu=1}^{\infty} \lambda_\nu(\alpha), \tag{3.9}$$

where

$$\lambda_\nu(\alpha) = \iint\limits_{G_\nu} \frac{d\mu}{|z-\alpha|^n}, \qquad G_\nu = S\left(\alpha, \frac{r_0}{\nu}\right) - S\left(\alpha, \frac{r_0}{\nu+1}\right). \tag{3.9a}$$

In G_ν

$$\frac{r_0}{\nu+1} < |z-\alpha| \leqq \frac{r_0}{\nu}$$

so that

$$\lambda_\nu(\alpha) < \frac{(\nu+1)^n}{r_0^n} \iint\limits_{G_\nu} d\mu = \frac{(\nu+1)^n}{r_0^n}\mu(G_\nu) = \frac{(\nu+1)^n}{r_0^n}(\mu_\nu - \mu_{\nu+1}), \tag{3.9b}$$

where $\mu_\nu = \mu\left(S\left(\alpha, \frac{r_0}{\nu}\right)\right)$. Thus

$$\sum_{\nu=1}^{\infty} \lambda_\nu(\alpha) < r_0^{-n} \sum_{\nu=1}^{\infty} (\nu + 1)^n (\mu_\nu - \mu_{\nu+1}) = r_0^{-n} \sum_{\nu=1}^{\infty} [(\nu + 1)^n - \nu^n] \mu_\nu + r_0^{-n} \mu_1$$
$$< h_n \sum_{\nu=1}^{\infty} \nu^{n-1} \mu_\nu.$$

Whence, in view of (3. 8) and (3. 9) *the limit* Φ'_n *of* (3. 7) *is finite whenever the series*

$$\Gamma_\alpha = \sum_{\nu=1}^{\infty} \nu^{n-1} \mu\left(S\left(\alpha, \frac{r_0}{\nu}\right)\right) \tag{3. 10}$$

converges. The condition of convergence of (3. 10) amounts to the requirement that $\mu(S(\alpha, r))$ should approach zero sufficiently rapidly as $r \to 0$; in other words, this is a condition of rarefication of mass μ in the vicinity of α. If Γ_α converges

$$\lim_\nu \nu^n \mu\left(S\left(\alpha, \frac{r_0}{\nu}\right)\right) = 0.$$

The latter fact, however, would imply that (when $r \to 0$ through values r_0/ν ($\nu = 1, 2, \ldots$))

$$\lim_r \Phi''_{n,r} = 0 \qquad \text{(cf. (3. 6 b))},$$

whenever (3. 10) converges; *the integral* Φ_n *of* (3. 3) *will then converge for the value* α *under consideration.*

When studying real parts of integrals

$$\int\int \log(z - \alpha)\, d\mu \qquad (\mu \geqq 0 \text{ or } \mu \leqq 0) \tag{3. 11}$$

it is sufficient to examine

$$\Phi = \int\int \log \frac{1}{|z - \alpha|}\, d\mu \qquad (\mu \geqq 0). \tag{3. 11 a}$$

As before let α be fixed in K. Let $r_0 (> 0)$ be sufficiently small so that $r_0 \leqq 1$ and so that the closed circular region $S(\alpha, r_0) < K$. The integral (3. 11 a) will converge if

$$\Phi_0 = \int\int_{S(\alpha, r_0)} \log \frac{1}{|z - \alpha|}\, d\mu \tag{3. 11 b}$$

converges. With $0 < r \leqq r_0$ let

$$f_r(z, \alpha) = \begin{cases} \log \frac{1}{|z-\alpha|} & (\text{in } S(\alpha, r_0) - S(\alpha, r)), \\ \log \frac{1}{r} & (\text{in } S(\alpha, r)). \end{cases} \tag{3.12}$$

Since $f_r(z, \alpha) \to \log \frac{1}{|z-\alpha|}$ increasing monotonically, as $r \to 0$, it follows that

$$\lim_r \iint_{S_{r_0}} f_r(z, \alpha)\, d\mu = \Phi_0 \qquad (\text{cf. } (3.11\text{ b});\ S_{r_0} = S(\alpha, r_0)). \tag{3.13}$$

We wish to secure finiteness of the latter integral. Now

$$\Phi_0 = \Phi_r' + \Phi_r'', \tag{3.13 a}$$

$$\Phi_r' = \iint_{S_{r_0} - S_r} f_r\, d\mu, \quad \Phi_r'' = \iint_{S_r} f_r\, d\mu. \tag{3.13 b}$$

On taking account of (3.12) one has

$$\Phi_r' = \iint_{S_{r_0} - S_r} \log \frac{1}{|z-\alpha|}\, d\mu, \quad \Phi_r'' = \mu(S(\alpha, r)) \log \frac{1}{r}. \tag{3.13 c}$$

We observe that Φ_r' increases monotonically as $r \to 0$; thus

$$\lim_r \Phi_r' = \Phi', \tag{3.14}$$

where Φ' may be infinite. Now $S_{r_0} - S_r\ (r > 0)$ is a subset of $S_{r_0} = H_1 + H_2 + \ldots$, where

$$H_\nu = S\left(\alpha, \frac{r_0}{\nu}\right) - S\left(\alpha, \frac{r_0}{\nu + 1}\right); \tag{3.15}$$

moreover, throughout S_{r_0} $\log(1/|z-\alpha|) \geqq 0$. Thus

$$\Phi_r' < \sum_{\nu=1}^{\infty} \tau_\nu, \tag{3.16}$$

where

$$\tau_\nu = \iint_{H_\nu} \log \frac{1}{|z-\alpha|}\, d\mu < \log\left[\frac{\nu+1}{r_0}\right] \iint_{H_\nu} d\mu = \log\left(\frac{\nu+1}{r_0}\right)(\mu_\nu - \mu_{\nu+1})$$

with $\mu_\nu = \mu(S(\alpha, r_0/\nu))$. The series of (3. 16) will converge if

$$\sum_{\nu=1}^{\infty} \log(\nu+1)(\mu_\nu - \mu_{\nu+1}) = \sum_{\nu=1}^{\infty} [\log(\nu+1) - \log \nu]\, \mu_\nu$$

converges; this will take place if and only if

$$T_\alpha = \sum_{\nu=1}^{\infty} \frac{1}{\nu}\, \mu\left(S\left(\alpha, \frac{r_0}{\nu}\right)\right) \tag{3. 17}$$

converges, *in which case the limit* Φ' *in* (3. 14) *will be finite.* The ν-th term of the series (3. 17) may be written in the form

$$\frac{1}{\nu \log(\nu+1)} \eta_\nu \qquad \left[\eta_\nu = \mu\left(S\left(\alpha, \frac{r_0}{\nu}\right)\right) \log(\nu+1)\right]. \tag{3. 18}$$

Now

$$\sum \frac{1}{\nu \log(\nu+1)}$$

diverges. This, together with other considerations, implies

$$\lim_\nu \eta_\nu = 0. \tag{3. 19}$$

In view of the form of η_ν, as given in (3. 18), (3. 19) is seen to imply

$$\lim \Phi''_r = 0 \qquad (\text{cf. } (3.\ 13\,\text{c}); r_0 = r_0/\nu\ (\nu = 1, 2, \ldots)). \tag{3. 19 a}$$

Whence, by (3. 11 b), (3. 13 a) and in view of the statement in connection with (3. 17), it is concluded that *the integral* Φ *in* (3. 11 a) *is finite whenever, for the value* α *under consideration, the series* (3. 17) *converges.*

In studying the imaginary part of an integral (3. 11) it is sufficient to consider integrals of the form

$$\int\int \varphi(z, \alpha)\, d\mu \qquad (\mu \geqq 0) \tag{3. 20}$$

where $\varphi(z, \alpha)$ = angle of $z - \alpha$ and where a suitable determination of $\varphi(z, \alpha)$ is chosen. For α fixed we may, for instance, extend a cut from α to the right, parallel to the axis of reals, and take $0 \leqq \varphi(z, \alpha) < 2\pi$. As a function of z $\varphi(z, \alpha)$ will be continuous except along the cut. It is obvious that (3. 20) will converge whether the set-function μ is absolutely continuous or not. The component (3. 11 a) is of greater interest than (3. 20).

Theorem 3. 1. *With n denoting a positive integer the integral*

$$\int\int \frac{d\mu}{(z-\alpha)^n} \qquad (\mu \geqq 0) \tag{3. 21}$$

will converge for every value of α (in K) for which the series Γ_α of (3. 10) *converges. With a suitable determination of the logarithm the integral*

$$\int\int \log \frac{1}{z-\alpha}\, d\mu \qquad (\mu \geqq 0) \tag{3. 22}$$

will converge whenever the series T_α of (3. 17) *converges.*

We shall now discuss some consequences of this theorem. In consequence of the definition of the average density $\varrho\left(\alpha, \frac{r_0}{\nu}\right)$ one has

$$\mu\left(S\left(\alpha, \frac{r_0}{\nu}\right)\right) = \frac{\pi r_0^2}{\nu^2}\, \varrho\left(\alpha, \frac{r_0}{\nu}\right).$$

Accordingly (3. 10) and (3. 17) may be written as follows:

$$\Gamma_\alpha = \pi r_0^2 \sum_\nu \nu^{n-3}\, \varrho\left(\alpha, \frac{r_0}{\nu}\right) \qquad (n > 0), \tag{3. 23}$$

$$T_\alpha = \pi r_0^2 \sum_\nu \frac{1}{\nu^3}\, \varrho\left(\alpha, \frac{r_0}{\nu}\right). \tag{3. 24}$$

Now, in consequence of a theorem of Lebesgue the density $\varrho(\alpha)$,

$$\varrho(\alpha) = \lim_{r=0} \varrho(\alpha, r) = \lim_\nu \varrho\left(\alpha, \frac{r_0}{\nu}\right), \tag{3. 25}$$

exists and is finite for almost all α. Hence for almost all values of α the series T_α, Γ_α (with $n = 1$) converge.

Corollary 3. 1. *The integrals* (3. 22), (3. 21) *(with $n = 1$) exist at all points of K at which the density $\varrho(\alpha)$ (cf.* (3. 25)*) is finite; this takes place almost everywhere in K. The integrals* (3. 22) *will also converge at those points α for which $\varrho(\alpha) = \infty$, provided* (3. 24) *converges*[1]. *The integral* (3. 21) *(with $n = 1$) will converge for points α with density $\varrho(\alpha) = \infty$, provided* (3. 23) *(with $n = 1$) converges. If for*

[1] That is, provided $\varrho(\alpha, r) \to \infty$ (as $r \to 0$) not too fast.

$n \geqq 2$ and for some a in K the series (3. 23) converges, necessarily[1] $\varrho(a) = 0$ *and the integral* (3. 21) *will converge.*

Note. An integral will be said to exist only if it has a finite value.

We have thus established degrees of rarefication of »mass» μ securing convergence of the integrals (3. 22), (3. 21) for a value α, under consideration.

Conditions will be established under which the integral

$$f(\alpha) = \int\int \frac{d\mu}{(z-\alpha)^n} \qquad \text{(integer } n > 0) \tag{3. 26}$$

is differentiable. First we shall *assume that there exists a set $G < K$, dense in itself, at every point of which Γ_α of* (3. 23) *converges when n is replaced by $n+1$.* By Corollary 3. 1 the integral (3. 26) will converge in G. Let α, for the present, be fixed in G. With β in G, we have

$$\frac{f(\alpha) - f(\beta)}{\alpha - \beta} = \Delta(\alpha, \beta) = \int\int \frac{g(z, \beta)\, d\mu}{(z-\alpha)^n (z-\beta)^n}, \tag{3. 27}$$

$$g(z, \beta) = (z-\beta)^{n-1} + (z-\beta)^{n-2}(z-\alpha) + \cdots + (z-\beta)(z-\alpha)^{n-2} + (z-\alpha)^{n-1}. \tag{3. 27a}$$

There exist sequences $\{\beta_\nu\}$ $(\nu = 1, 2, \ldots;\ \beta_\nu \neq \alpha)$ such that β_ν is in G and $\lim\limits_\nu \beta_\nu = \alpha$. We shall have, for a sequence $\{\beta_\nu\}$,

$$\lim_\nu \Delta(\alpha, \beta_\nu) = \int\int \lim_\nu \frac{g(z, \beta_\nu)\, d\mu}{(z-\alpha)^n (z-\beta_\nu)^n} = \int\int \frac{n\, d\mu}{(z-\alpha)^{n+1}} = f^{(1)}(\alpha) \tag{3. 28}$$

whenever the interchange of integration and of passage to the limit, here involved, is justifiable. It is noted that in the latter case the derivative is unique, that is, it is independent of the choice of the sequence $\{\beta_\nu\}$. It will be convenient to *designate α by β_0.* It is observed that every point z of K has associated with it at least one integer

$$m(z) \geqq 0$$

so that

$$|z - \beta_{m(z)}| \leqq |z - \beta_\nu| \qquad (\nu = 0, 1, \ldots). \tag{3. 29}$$

Define E'_m as the subset of K such that

$$|z - \beta_m| \leqq |z - \beta_\nu| \qquad (\nu = 0, 1, \ldots). \tag{3. 30}$$

[1] The condition of convergence of (3. 23) (with $n \geqq 2$) amounts to the requirement that $\varrho(\alpha, r)$ should approach zero sufficiently rapidly, as $r \to 0$.

The sets E_0', E_1', E_2', ... may have points in common since the integer $m(z)$, referred to above, is not always unique. In view of the statement in connection with (3. 29) every point z of K belongs to at least one set E_j'. Thus

(3. 31) $$K = E_0' + E_1' + E_2' + \cdots.$$

The sets E_ν ($\nu = 0, 1, \ldots$), formed by means of the relations

$$E_0 = E_0', \quad E_1 = E_1' - E_0, \quad E_2 = E_2' - (E_0 + E_1), \ldots,$$
$$E_\nu = E_\nu' - (E_0 + E_1 + \cdots + E_{\nu-1}), \cdots$$

are without common points and

(3, 31 a) $$K = E_0 + E_1 + \cdots,$$

while

(3. 32) $$|z - \beta_m| \leqq |z - \beta_\nu| \qquad (\nu = 0, 1, \ldots)$$

for z in E_m.

Separating the real and imaginary components it is observed that justification of (3. 28) amounts to that of the relations

(3. 33) $$\lim_\nu \int\int f_{\nu,\alpha}(z)\, d\mu = \int\int \lim_\nu f_{\nu,\alpha}(z)\, d\mu,$$

where $f_{\nu,\alpha}(z)$ is the real or imaginary part of

(3. 33 a) $$h_\nu(z) = (z - \alpha)^{-n} (z - \beta_\nu)^{-n} g(z, \beta_\nu) \qquad \text{(cf. (3. 27 a))}.$$

In either case

(3. 34) $$|f_{\nu,\alpha}(z)| \leqq |h_\nu(z)| \leqq \sum_{K=0}^{n-1} \frac{1}{|z - \beta_\nu|^{K+1} |z - \alpha|^{n-K}} = h_\nu^*(z).$$

For z in E_m, in view of (3. 32), (3. 34) and since $\alpha = \beta_0$, one has

(3. 34 a) $$h_\nu^*(z) \leqq \frac{n}{|z - \beta_m|^{n+1}} \qquad (\nu = 1, 2, \ldots).$$

If $w(z)$ is defined by the relations

(3. 35) $$w(z) = \frac{n}{|z - \beta_m|^{n+1}} \qquad (\text{for } z \text{ in } E_m;\ m = 0, 1, \ldots),$$

in view of (3. 31 a) $w(z)$ will be uniquely specified throughout K; moreover, by (3. 34) and (3. 34 a), it will follow that

$$|f_{\nu,\alpha}(z)| \leqq w(z) \qquad (z \text{ in } K;\ \nu = 1, 2, \ldots). \tag{3. 35 a}$$

In this connection it is to be noted that the above developments were given for a value α fixed in G (cf. italics subsequent to (3. 26)) and for a particular sequence $\{\beta_\nu\}$ of the type introduced subsequent to (3. 27 a); thus, the function $w(z)$ involved in (3. 35 a) depends[1] on the sequence $\{\beta_\nu\}$. In virtue of (3. 35 a), the interchange of limiting processes indicated in (3. 33) is certainly permissible if the integral

$$\int\int w(z)\, d\mu = \sum_{m=0}^{\infty} \int\int_{E_m} \frac{n\, d\mu}{|z - \beta_m|^{n+1}} \tag{3. 36}$$

converges. First we observe that, inasmuch as β_m is in G so that Γ_{β_m} of (3. 23; with n replaced by $n + 1$) converges, the integral displayed in the second member of (3. 36) exists and is finite (cf. Corollary 3. 1). Whence it is inferred that the integral in the first member of (3. 36) exists as a finite value, if the series displayed in (3. 36) converges. Let $S\left(\beta_m, \frac{r_0}{\nu}\right)$ be the closed circular region with center at β_m and radius $\frac{r_0}{\nu}$ $(\nu = 1, 2, \ldots)$, *where* r_0 *is the diameter of* K. *On writing* $H_{\nu, m} = E_m S\left(\beta_m, \frac{r_0}{\nu}\right)$ one obtains

$$E_m = \sum_{\nu=1}^{\infty} (H_{\nu, m} - H_{\nu+1, m}).$$

In $H_{\nu,m} - H_{\nu+1,m}$ $r_0/(\nu + 1) < |z - \beta_m|$. Thus

$$\frac{1}{|z - \beta_m|^{n+1}} < \frac{1}{r_0^{n+1}} (\nu + 1)^{n+1} \qquad (\text{in } H_{\nu, m} - H_{\nu+1, m})$$

and, with integration extended over $H_{\nu, m} - H_{\nu+1,m}$, it is concluded that

$$\tau_{\nu, m} = \int\int \frac{d\mu}{|z - \beta_m|^{n+1}} < \frac{1}{r_0^{n+1}} (\nu + 1)^{n+1} \mu (H_{\nu, m} - H_{\nu+1, m}).$$

Whence

[1] Is, in fact, determined by the β_ν.

(3. 37) $$\iint_{E_m} \frac{d\mu}{|z-\beta_m|^{n+1}} = \sum_{\nu=1}^{\infty} \tau_{\nu,m} < h_{n+1} \sum_{\nu=1}^{\infty} (\nu + 1)^n \mu(H_{\nu,m}),$$

where $h_{n+1} = (n+1) r_0^{-n-1}$. Since β_m is in G and G has been defined as the set in which the series Γ_α (with n replaced by $n+1$; cf. (3. 10)) converges, it is observed that the series of the last member of (3. 37) converges. Hence, by (3. 36) and (3. 37)

(3. 38) $$\iint w(z)\, d\mu \leqq h_{n+1} \sum_{m=0}^{\infty} \sum_{\nu=1}^{\infty} (\nu+1)^n \mu(H_{\nu,m}).$$

Since $E_j E_k = 0$ for $j \neq k$, it follows that

$$H_{\nu,j} H_{\nu,k} = 0 \qquad \text{(for } j \neq k\text{)}$$

and

(3. 38 a) $$\sum_{m=0}^{\infty} \mu(H_{\nu,m}) = \mu(T'_\nu)$$

where

(3. 38 b) $$T'_\nu = H_{\nu,0} + H_{\nu,1} + \cdots = E_0 S\left(\beta_0, \frac{r_0}{\nu}\right) + E_1 S\left(\beta_1, \frac{r_0}{\nu}\right) + \cdots$$

$$< \sum_j S\left(\beta_j, \frac{r_0}{\nu}\right) = T_\nu \qquad (\beta_0 = \alpha).$$

By virtue of (3. 38), (3. 38 a) and (3. 38 b) the integral (3. 38) has a finite value if the series

$$W = \sum_{\nu=1}^{\infty} \nu^n \mu(T_\nu)$$

converges.

Theorem 3. 2. *Consider the integral*

(3. 39) $$\iint_K \frac{d\mu}{(z-\alpha)^n} \qquad (\textit{integer } n > 0).$$

Let there be a set dense in itself, $G < K$, such that in G the series Γ_α (with n replaced by $n+1$; cf. (3. 23) *or* (3. 10)) *converges. Consider sequences β_ν $(\nu = 1, 2, \ldots)$, where $\beta_\nu < G$ and*

$$\lim_\nu \beta_\nu = \alpha \qquad (\alpha \textit{ in } G;\ \beta_\nu \neq \alpha).$$

Designate by T_ν the part of K consisting of points at the distance $\leqq r_0/\nu$ (r_0 = diameter of K) from the set of points

$$\alpha, \beta_1, \beta_2, \ldots.$$

If for every sequence $\{\beta_\nu\}$ of the above description, with the same α for limiting point, the series

$$\sum_\nu \nu^n \mu(T_\nu) \tag{3.40}$$

converges, the integral (3. 39) *will possess a unique finite derivative at the point α in question:*

$$\frac{d}{d\alpha} \iint\limits_K \frac{d\mu}{(z-\alpha)^n} = \iint \frac{n\, d\mu}{(z-\alpha)^{n+1}}.$$

Note. The condition of convergence of (3. 40) relates to rarefication »mass» μ ($\geqq 0$) in the vicinity of α.

It is of importance to obtain conditions securing differentiability of integrals (3. 39) in the case when μ is an absolutely continuous set-function. Let $\{\beta_\nu\}$, with $\lim \beta_\nu = \alpha$, be a sequence of the same description as given previously. We designate by B_ν the part common with K of the perpendicular bisector of the segment (α, β_ν). Let K'_ν be the part of K lying to one side of B_ν, containing α and B_ν. We denote $K - K'_\nu$ by K''_ν. It is observed that

$$|z-\alpha| \leqq |z-\beta_\nu| \quad (\text{in } K'_\nu), \quad |z-\beta_\nu| < |z-\alpha| \qquad (\text{in } K''_\nu). \tag{3.41}$$

With sets $X < K$, one may write

$$X = X K'_\nu + X K''_\nu.$$

Thus, if $f_{\nu,\alpha}(z)$ denotes the real (or imaginary part) of $h_\nu(z)$ (cf. (3. 33 a)),

$$\left|\iint\limits_X f_{\nu,\alpha}(z)\, d\mu\right| \leqq \iint\limits_X |f_{\nu,\alpha}(z)|\, d\mu \leqq \iint\limits_X h_\nu^*(z)\, d\mu \tag{3.42}$$

$$= \iint\limits_{X K'_\nu} h_\nu^*(z)\, d\mu + \iint\limits_{X K''_\nu} h_\nu^*(z)\, d\mu,$$

where $h_\nu^*(z)$ is the function of (3. 34). On taking account of the form of $h_\nu^*(z)$ as well as of the inequalities (3. 41), from (3. 42) it is inferred that

$$(3.43)\qquad \iint_X |f_{\nu,\alpha}(z)|\,d\mu < \iint_{X K'_\nu} \frac{n\,d\mu}{|z-\alpha|^{n+1}} + \iint_{X K''_\nu} \frac{n\,d\mu}{|z-\beta_\nu|^{n+1}}$$

$$< \iint_X \frac{n\,d\mu}{|z-\alpha|^{n+1}} + \iint_X \frac{n\,d\mu}{|z-\beta_\nu|^{n+1}}.$$

Now α and β_ν are in $G = G_{n+1}$, *where* G_{n+1} *is the set,* $< K$, *in which the series*

$$(3.44)\qquad \Gamma_\beta^{n+1} = \sum_{\nu=1}^{\infty} \nu^n \mu\left(S\left(\beta, \frac{r_0}{\nu}\right)\right)$$

converges. For any particular β in G_{n+1} the integral

$$(3.45)\qquad \Phi_\beta^{n+1}(X) = \iint_X \frac{d\mu}{|z-\beta|^{n+1}}$$

exists and is finite (cf. Corollary 3.1). With μ assumed to be an absolutely continuous set-function, the set-function (3.45) is absolutely continuous for β in G_{n+1}. Suppose there exists a set $G'_{n+1} < G_{n+1}$, the set G'_{n+1} being dense in itself and such that *absolute continuity of the integral* (3.45) *is uniform with respect to* β, *for* β *in* G'_{n+1}. Uniformity of absolute continuity, here, is to be construed in the following sense. Given $\varepsilon\,(>0)$, however small, there exists $\delta = \delta(\varepsilon)$, *independent of* β and such that $\lim_\varepsilon \delta(\varepsilon) = 0$, so that

$$(3.46)\qquad \Phi_\beta^{n+1}(X) \leqq \varepsilon \qquad \text{(all } \beta \text{ in } G'_{n+1})$$

for all sets $X < K$ with meas. $X \leqq \delta(\varepsilon)$.

Consider the inequality (3.43) where $\{\beta_\nu\}$ is a sequence with the properties

$$\text{(i)}\qquad \beta_\nu < G'_{n+1} \qquad (\nu = 1, 2, \ldots)$$

$$\text{(ii)}\qquad \lim_\nu \beta_\nu = \alpha < G'_{n+1}.$$

If we assign $\varepsilon\,(>0)$, however small, in view of the notation (3.45) and in consequence of the statement made in connection with (3.46), the inequality (3.43) will yield the result:

$$(3.47)\qquad \iint_X |f_{\nu,\alpha}(z)|\,d\mu < n\,\Phi_\alpha^{n+1}(X) + n\,\Phi_{\beta_\nu}^{n+1}(X) \leqq 2n\varepsilon,$$

whenever meas. $X \leqq \delta(\varepsilon)$. This property will persist for all sequences $\{\beta_\nu\}$ satisfying the above conditions (i), (ii). Whence it is observed that the absolute continuity of the integrals of the first member of (3. 42) is uniform with respect to ν. By a known theorem concerning passage to the limit under the integral sign it will follow that the relation (3. 33) is justified (for α in G'_{n+1}); that is, (3. 28) under stated conditions will be justified.

Theorem 3. 3. *Consider the integral*

$$\int\int_K \frac{d\mu}{(z-\alpha)^n} \qquad (\text{integer } n > 0;\ \mu \geqq 0) \tag{3. 48}$$

where μ is an absolutely continuous set-function. Let $G_{n+1} = G < K$ be the set of points β at which the series Γ_β^{n+1} of (3. 44) converges. Suppose there exists a set G'_{n+1}, dense in itself, such that $G'_{n+1} < G_{n+1}$ and such that the absolute continuity[1] *of*

$$\int\int_X \frac{d\mu}{(z-\alpha)^{n+1}} \qquad (X < K)$$

is uniform with respect to β for β in G'_{n+1}. We then have

$$\frac{d}{d\alpha}\int\int_K \frac{d\mu}{(z-\alpha)^n} = \int\int \frac{n\,d\mu}{(z-\alpha)^{n+1}}$$

for all α in G_{n+1}.

Note. In the next section there will be given in some detail results concerning uniformity of absolute continuity.

4. Approximations of Integrals (1. 2) by Analytic Functions.

Suppose μ is an additive set-function. The integral

$$\Phi_\alpha^n(X) = \int\int_X \frac{d\mu}{|z-\alpha|^n} \qquad (X < K;\ \text{integer } n > 0;\ \mu \geqq 0) \tag{4. 1}$$

will have a finite value for α in the set G_n in which the series $\Gamma_\alpha = \Gamma_\alpha^n$ of (3. 10) converges. If we repeat the steps used in establishing (3. 37), replacing E_m, β_m, $n+1$ and $H_{\nu,m}$ by X, α, n and

[1] That is, absolute continuity of the real and imaginary parts.

(4. 2) $$H_\nu = X\,S\left(\alpha,\ \frac{r_0}{\nu}\right),$$

respectively, it is inferred that

(4. 3) $$\Phi_\alpha^n(X) \leqq h_n \sum_{\nu=1}^{\infty} (\nu + 1)^{n-1} \mu\left(X\,S\left(\alpha,\ \frac{r_0}{\nu}\right)\right) < \infty$$

for α in G_n and for sets $X < K$ $(h_n = n\,r_0^{-n})$.

Suppose there exists a function $t(\nu)$, *independent of* α, *such that*

(4. 4) $$\mu\left(K\,S\left(\alpha,\ \frac{r_0}{\nu}\right)\right) \leqq t(\nu) \qquad (\nu = 1, 2, \ldots),$$

for all α *in a set* $G(n)$, *and such that the series*

(4. 4 a) $$S_n = \sum_{\nu=1}^{\infty} (\nu + 1)^{n-1}\, t(\nu)$$

converges. Necessarily $t(\nu) \to 0$, as $\nu \to \infty$; there is no loss of generality, if (for convenience) $t(\nu)$ *is assumed to approach zero monotonically*, as $\nu \to \infty$. Since for $X < K$

$$\mu\left(X\,S\left(\alpha,\ \frac{r_0}{\nu}\right)\right) \leqq \mu\left(K\,S\left(\alpha,\ \frac{r_0}{\nu}\right)\right),$$

in consequence of (4. 3) and of (4. 4) it is inferred that $G(n) < G_n$.

A condition of the type stated in connection with (4. 4) and (4. 4 a) is a statement to the effect that the *degree of rarefication of »mass»*, implied by the convergence of the series Γ_α^n (cf. (3. 10)) *is uniform for* α *in* $G(n)$. It is clear that for $G(n)$ one may take a finite number of any points G_n; of interest, however, are cases when $G(n)$ actually consists of an infinity of points.

Inasmuch as μ is taken absolutely continuous, it can be asserted that, *given* $\varepsilon\,(> 0)$, *there exists* $\delta(\varepsilon)$ $(\delta(\varepsilon) > 0;\ \delta(\varepsilon) \to 0$ *as* $\varepsilon \to 0)$ *so that*

(4. 5) $$\mu(Y) \leqq \delta(\varepsilon) \qquad (Y < K),$$

whenever meas. $Y \leqq \varepsilon$. An integer $m(\varepsilon)$ will be defined by the relation[1]

(4. 5 a) $$m(\varepsilon) = [\delta(\varepsilon)^{-K/n}],$$

[1] $[b]$ denotes the greatest integer $\leqq b$.

where $0 < K < 1$ and K is independent of ε. We shall take $\varepsilon \leqq \varepsilon_0$, where $\varepsilon_0 (>0)$ is sufficiently small so that

$$m(\varepsilon) \geqq 1 \qquad (0 < \varepsilon \leqq \varepsilon_0).$$

In view of the statement in connection with (4. 5) and in consequence of (4. 4) the following will hold. Whenever

$$\text{meas. } X \leqq \varepsilon \qquad (X < K) \tag{4. 6}$$

the inequalities

$$\mu\left(X S\left(\alpha, \frac{r_0}{\nu}\right)\right) \leqq \mu(X) \leqq \delta(\varepsilon) \qquad (\text{all } \alpha), \tag{4. 6 a}$$

$$\mu\left(X S\left(\alpha, \frac{r_0}{\nu}\right)\right) \leqq \mu\left(K S\left(\alpha, \frac{r_0}{\nu}\right)\right) \leqq t(\nu) \qquad (\alpha \text{ in } G(n)) \tag{4. 6 b}$$

will be satisfied for $\nu = 1, 2, \ldots$.

With the aid of (4. 6 a) it is concluded that

$$\sum_{\nu=1}^{m(\varepsilon)} (\nu + 1)^{n-1} \mu\left(X S\left(\alpha, \frac{r_0}{\nu}\right)\right) \leqq \delta(\varepsilon)\, \zeta_n(\varepsilon) \qquad (\text{all } \alpha), \tag{4. 7}$$

where

$$\zeta_n(\varepsilon) < \frac{1}{n}(m(\varepsilon) + 2)^n \leqq \lambda_n m^n(\varepsilon) \qquad \left(\lambda_n = \frac{1}{n} 3^n\right).$$

By (4. 5 a)

$$\delta(\varepsilon)\, \zeta_n(\varepsilon) < \lambda_n \delta^{1-K}(\varepsilon) \qquad (0 < \varepsilon \leqq \varepsilon_0). \tag{4. 7 a}$$

Thus, since $1 - K > 0$ and $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$, we have

$$\lim_{\varepsilon} \delta(\varepsilon)\, \zeta_n(\varepsilon) = 0. \tag{4. 7 b}$$

On the other hand, by virtue of (4. 6 b)

$$\sum_{\nu > m(\varepsilon)} (\nu + 1)^{n-1} \mu\left(X S\left(\alpha, \frac{r_0}{\nu}\right)\right) \leqq \sum_{\nu > m(\varepsilon)} (\nu + 1)^{n-1} t(\nu) = \zeta_n'(\varepsilon) \tag{4. 8}$$

for α in $G(n)$. On noting that the series, last displayed, is the remainder after $m(\varepsilon)$ terms of the convergent series (4. 4 a) and that $m(\varepsilon)$ (cf. (4. 5 a)) $\to \infty$, as $\varepsilon \to 0$, it is observed that

$$\lim_{\varepsilon} \zeta_n'(\varepsilon) = 0. \tag{4. 8 a}$$

By (4. 3), (4. 7), (4. 7 a) and (4. 8), whenever $X < K$ and meas. $X \leqq \varepsilon$,

(4. 9) $$\Phi_{\alpha}^{n}(X) < h_n \eta_n(\varepsilon) \qquad (\text{all } \alpha \text{ in } G(n)),$$

where

(4. 9 a) $$\eta_n(\varepsilon) = \lambda_n \delta^{1-K}(\varepsilon) + \zeta_n'(\varepsilon) \qquad \left(0 < K < 1;\ \lambda_n = \frac{1}{n} 3^n\right)$$

(cf. (4. 8)) and

(4. 9 b) $$\lim_{\varepsilon=0} \eta_n(\varepsilon) = 0.$$

We are now ready to state the following theorem.

Theorem 4. 1. *Consider the integral*

(4. 10) $$\int\int_X \frac{d\mu}{(z-\alpha)^n} \qquad (\textit{integer } n > 0;\ \mu \geqq 0)$$

where $X < K$ *and* μ *is an absolutely continuous set-function. Suppose there exists a function* $t(v)$ *and a set* $G(n)$ *for which the italicized statement in connection with* (4. 4) *and* (4. 4 a) *holds. This degree of rarefication of »mass»* μ*, in the vicinity of the set* $G(n)$*, will secure the following property.*

The integral (4. 10)*, as function of sets* $X < K$*, is absolutely continuous uniformly with respect to* α*, for* α *in* $G(n)$*. More precisely, under stated conditions, whenever* meas. $X \leqq \varepsilon$*, one has*

(4. 10 a) $$\int\int_X \frac{d\mu}{|z-\alpha|^n} < h_n \eta_n(\varepsilon) \qquad (\alpha \textit{ in } G(n);\ h_n = n r_0^{-n}).$$

Here $\eta_n(\varepsilon)$ *is given by* (4. 9 a) *and* (4. 8) *and satisfies* (4. 9 b)*. The function* $\delta(\varepsilon)$*, involved in* (4. 9 a) *is the function so designated in the italics in connection with* (4. 5).

Note. More generally, instead of defining $m(\varepsilon)$ by (4. 5 a) we may proceed as follows. *Let*

(4. 11) $$m(\varepsilon) = [\sigma(\varepsilon)] \qquad (\sigma(\varepsilon) > 0 \text{ for } \varepsilon > 0)$$

where $\sigma(\varepsilon)$ *is a function such that, as* $\varepsilon \to 0$

(4. 11 a) $$\sigma(\varepsilon) \to \infty, \quad \delta(\varepsilon)\sigma^n(\varepsilon) \to 0.$$

Such functions $\sigma(\varepsilon)$ exists, inasmuch as $\delta(\varepsilon) \to 0$ (as $\varepsilon \to 0$). The function

$$\sigma(\varepsilon) = \delta(\varepsilon)^{-K/n} \qquad (0 < K < 1)$$

is the one already used. In place of (4. 7 a) one has

$$\delta(\varepsilon)\zeta_n(\varepsilon) < \lambda_n \delta(\varepsilon)\sigma^n(\varepsilon) \qquad (0 < \varepsilon \leqq \varepsilon_0)^1.$$

In (4. 10 a) *the function* $\eta_n(\varepsilon)$ *may be taken of the form*

$$\eta_n(\varepsilon) = \lambda_n \delta(\varepsilon)\sigma^n(\varepsilon) + \zeta_n'(\varepsilon) \qquad \text{(cf. (4. 5), (4. 8))}, \tag{4. 12}$$

with $m(\varepsilon)$ *defined by* (4. 11) *and* $\sigma(\varepsilon)$ *satisfying* (4. 11 a).

In view of Theorom 3. 3 the following Corollary to Theorem 4. 1 is inferred.

Corollary 4. 1. *Let* n *be an integer* $\geqq 2$. *Consider the function*

$$f(\alpha) = \int\int_K \frac{d\mu}{(z-\alpha)^{n-1}} \qquad (\mu \geqq 0) \tag{4. 13}$$

where μ *is absolutely continuous. Suppose there exists a function* $t(\nu)$, *independent of* α, *and a set* $G(n)$, *dense in itself, such that*

$$\mu\left(K\, S\left(\alpha, \frac{r_0}{\nu}\right)\right) \leqq t(\nu) \qquad (\nu = 1, 2, \ldots;\ \alpha \text{ in } G(n)),$$

while the series

$$S_n = \sum_{\nu=1}^{\infty} (\nu + 1)^{n-1}\, t(\nu)$$

converges. This rarefication of »mass» μ *implies that* $f(\alpha)$ *has a finite unique derivative at every point of* $G(n)$,

$$f^{(1)}(\alpha) = \int\int_K \frac{(n-1)\, d\mu}{(z-\alpha)^n}.$$

When μ is absolutely continuous, then density $\varrho(z)$ exists and is finite almost everywhere in K; moreover, $\varrho(z)$ will be summable over K. Thus, $d\mu$ may be replaced by $\varrho(z)\,dx\,dy$. Results of the type of those given in Theorem 4. 1 assume a particularly simple form in the important special case of (4. 10) when

$$|\varrho(z)| \leqq b \qquad (\text{in } K) \tag{4. 14}$$

and $n = 1$. We then have

[1] We take $\varepsilon \leqq \varepsilon_0$, where ε_0 is sufficiently small so that $m(\varepsilon) \geqq 1$ for $0 < \varepsilon \leqq \varepsilon_0$.

(4. 15) $$\left|\iint_X \frac{d\mu}{z-\alpha}\right| \leqq b \iint \frac{dx\,dy}{|z-\alpha|} \qquad (X < K).$$

Recalling that for the integral (4. 1) we have previously obtained the inequality (4. 3), it is concluded that

(4. 16) $$\Phi_\alpha(X) = \iint_X \frac{dx\,dy}{|z-\alpha|} \leqq h_1 \sum_{\nu=1}^{\infty} \text{meas. } X S\left(\alpha, \frac{r_0}{\nu}\right),$$

the latter series being convergent for all α in K and for all sets $X < K$. In place of (4. 4) we now derive

(4. 17) $$\text{meas. } K S\left(\alpha, \frac{r_0}{\nu}\right) \leqq \frac{\pi r_0^2}{\nu^2} = t(\nu) \qquad (\nu = 1, 2, \ldots)$$

for all α in the set $G(1) = K$. The series (4. 4 a) now becomes

(4. 17 a) $$S_1 = \sum_{\nu=1}^{\infty} t(\nu)$$

and is seen to converge. With meas. $X \leqq \varepsilon$, the inequality (4. 7) could be written in the form

(4. 18) $$\sum_{\nu=1}^{m(\varepsilon)} \text{meas. } X S\left(\alpha, \frac{r_0}{\nu}\right) \leqq \varepsilon\, m(\varepsilon) \qquad (\text{all } \alpha).$$

We choose integral-valued $m(\varepsilon)$ so that

(4. 19) $$m(\varepsilon) \to \infty, \quad \varepsilon\, m(\varepsilon) \to 0 \qquad (\text{as } \varepsilon \to 0).$$

In view of (4. 17)

(4. 20) $$\sum_{\nu > m(\varepsilon)} \text{meas. } X S\left(\alpha, \frac{r_0}{\nu}\right) \leqq \sum_{\nu > m(\varepsilon)} \text{meas. } K S\left(\alpha, \frac{r_0}{\nu}\right)$$
$$\leqq \pi r_0^2 \sum_{\nu > m(\varepsilon)} \frac{1}{\nu^2} < \frac{\pi r_0^2}{m(\varepsilon)} \qquad (\alpha \text{ in } K).$$

By (4. 18), (4. 20) from (4. 16), (4. 15) it is inferred that, whenever meas. $X \leqq \varepsilon$,

(4. 21) $$\left|\iint_X \frac{d\mu}{z-\alpha}\right| < b\, h_1\, \eta(\varepsilon) \qquad (\alpha \text{ in } K);$$

here

$$\eta(\varepsilon) = \varepsilon\, m(\varepsilon) + \frac{\pi r_0^2}{m(\varepsilon)} \to 0$$

as $\varepsilon \to 0$. It is convenient to take $m(\varepsilon) = \left[\varepsilon^{-\frac{1}{2}}\right]$. One then may write (4. 21) in the form:

(4. 22) $$\left| \int_X\int \frac{d\mu}{z-\alpha} \right| < b_1 \varepsilon^{\frac{1}{2}} \qquad (0 < \varepsilon \leqq \varepsilon_0;\ \alpha \text{ in } K),$$

whenever meas. $X \leqq \varepsilon$ $(X < K)$.

Corollary 4. 2. *Consider the integral*

(4. 23) $$\int_X\int \frac{\varrho(z)\, dx\, dy}{z-\alpha} = \int_X\int \frac{d\mu}{z-\alpha}$$

with $\varrho(z)$ summable in K and $|\varrho(z)|$ uniformly bounded in K. Absolute continuity of this integral considered as function of sets $X < K$, will be uniform with respect to α for α in K. The inequality (4. 22) *will be valid for α in K, whenever* meas. $X \leqq \varepsilon$.

Corollary 4. 3. *Consider the integral*

(4. 24) $$\int_X\int \frac{\varrho(z)\, dx\, dy}{(z-\alpha)^n} = \int_X\int \frac{d\mu}{(z-\alpha)^n} \qquad (X < K)$$

where n is an integer $\geqq 2$, $\varrho(z)$ is summable in K and $|\varrho(z)| \leqq b$ (in K). Suppose there exists a set $G(n) < K$ and a function $t(\nu)$ (> 0) so that meas. $K S\left(\alpha, \frac{r_0}{\nu}\right)$ $\leqq t(\nu)$ $(\nu = 1, 2, \ldots;$ *α in $G(n)$) and so that the series S_n of* (4. 4 a) *converges. We then have*

(4. 25) $$\int_X\int \left| \frac{\varrho(z)\, dx\, dy}{(z-\alpha)^n} \right| < b\, h_n\, \eta_n(\varepsilon) \qquad (\alpha \text{ in } G(n)$$

for sets $X < K$ with meas. $X \leqq \varepsilon$ $(0 < \varepsilon \leqq \varepsilon_0)$; *here*

(4. 25 a) $$\eta_n(\varepsilon) = \lambda_n\, \varepsilon\, \sigma^n(\varepsilon) + \sum_{\nu > m(\varepsilon)} (\nu + 1)^{n-1}\, t(\nu)$$

$$\left(\lambda_n = \frac{1}{n} 3^n;\quad \sigma(\varepsilon) > 0;\quad \sigma^{-1}(\varepsilon),\quad \varepsilon\, \sigma^n(\varepsilon) \to 0 \text{ as } \varepsilon \to 0\right)$$

and $m(\varepsilon) = [\sigma(\varepsilon)]$, while $h_n = n\, r_0^{-n}$.

This Corollary is established by noting that the first member in (4.25) is equal to or is less than $b\,\Phi_\alpha^n(X)$, where

$$\Phi_\alpha^n(X) = \iint\limits_X \frac{dx\,dy}{|z-\alpha|^n} \leqq h_n \sum_{\nu=1}^{\infty} (\nu+1)^{n-1} \text{ meas. } X\,S\left(\alpha, \frac{r_0}{\nu}\right)$$

and by replacing in the developments given in connection with Theorem 4.1 μ and $\delta(\varepsilon)$ by meas. and ε, respectively.

In the investigation of various uniqueness properties of functions of the form (1.4) of fundamental importance is approximation by uniformly convergent sequences of analytic functions, together with the degree of approximation, expressed in terms of rarefication of »mass» μ. We shall first note the following fact.

Suppose $\mu \geqq 0$ is a set-function not necessarily absolutely continuous. Let H be a closed set $< K$. At every point of $K - H$ the function

$$h(\alpha) = \iint\limits_H \frac{d\mu}{z-\alpha} \tag{4.26}$$

will be analytic. If H is not necessarily closed, $h(\alpha)$ will be analytic at every interior point of $K - H$.

To prove this statement it is sufficient to demonstrate that at every point α_0 of the open set $K - H$ $h(\alpha)$ has a unique derivative. Now α_0 is center of a closed circular region $S(\alpha_0, 2\,\delta)$ of positive radius $2\,\delta$ such that $S(\alpha_0, 2\,\delta) < K - H$. With $\{\beta_\nu\}$ $(\nu = 1, 2, \ldots)$ denoting a sequence in $S(\alpha_0, \delta)$ such that

$$\beta_\nu \neq \alpha, \qquad \lim_\nu \beta_\nu = \alpha$$

we consider the expression

$$\frac{h(\alpha_0) - h(\beta_\nu)}{\alpha_0 - \beta_\nu} = \varDelta(\alpha_0, \beta_\nu) = \iint\limits_H \frac{d\mu}{(z-\alpha_0)(z-\beta_\nu)}. \tag{4.27}$$

Justification of passage to the limit under the integral sign, involved in the relation

$$\lim_\nu \varDelta(\alpha_0, \beta_\nu) = \iint\limits_H \frac{d\mu}{(z-\alpha_0)^2} = h^{(1)}(\alpha_0), \tag{4.28}$$

amounts to justification of the equality

(4. 28 a)
$$\lim_{\nu} \iint_{H} h_{\nu, \alpha_0}(z)\, d\mu = \iint_{H} \lim_{\nu} h_{\nu, \alpha_0}(z)\, d\mu,$$

where $h_{\nu, \alpha_0}(z)$ is the real or imaginary part of $(z-\alpha_0)^{-1}(z-\beta_\nu)^{-1}$. Since $|z-\alpha_0| > 2\delta$, $|z-\beta_\nu| > \delta$ (z in H), one has

$$|h_{\nu, \alpha_0}(z)| < \frac{1}{2\delta^2} \qquad (\nu = 1, 2, \ldots;\ z \text{ in } H).$$

On the other hand, $\lim_{\nu} h_{\nu, \alpha_0}(z)$ equals the real or imaginary part (as the case may be) of $(z-\alpha_0)^{-2}$. These considerations are sufficient to justify (4. 28 a), (4. 28); the italicized statement in connection with (4. 26) is thus verified.[1]

Suppose now that $\mu\,(\geqq 0)$ is absolutely continuous. The density of μ, $\varrho(z)$, exists and is finite in $K - K_0$, with

(4. 29)
$$\text{meas. } K_0 = 0.$$

We then may write ($\varrho(z)$ being summable over K)

(4. 30)
$$f_n(\alpha) = \iint_{K} \frac{d\mu}{(z-\alpha)^n} = \iint_{K} \frac{\varrho(z)\, dx\, dy}{(z-\alpha)^n}.$$

Suppose there is a set $G < G(n)$ ($G(n)$ *the set so denoted in Theorem* 4. 1) *in which* $\varrho(z) = 0$.[2] We then may write

(4. 30 a)
$$f_n(\alpha) = \iint_{K-G} \frac{d\mu}{(z-\alpha)^n};$$

this-function will certainly be defined for α in G. Let $X_1, X_2, \ldots$ be a sequence of *closed sets* $< K - G$, such that

(4. 31)
$$X_1 < X_2 < \cdots$$

and

(4. 31 a)
$$\lim_{\nu} \text{meas. } X_\nu = \text{meas. } (K - G).$$

We shall write

[1] If $K-H$ is not connected it might happen that $h(\alpha)$ is equal to distinct analytic functions in various part of $K-H$.

[2] When $n \geqq 2$ necessarily $\varrho(z) = 0$ in $G(n)$ (cf. (4. 4) and note convergence of (4. 4 a)).

$$\text{meas.}\,(K - G - X_\nu) = \varepsilon_\nu; \tag{4. 31 b}$$

thus $\lim \varepsilon_\nu = 0$. *The function*

$$f_{n,\nu}(\alpha) = \iint\limits_{X_\nu} \frac{d\mu}{(z-\alpha)^n} = \Phi_\alpha^n(X_\nu) \tag{4. 32}$$

will be analytic in the open set $K - X_\nu > G$;

$$K - X_1 > K - X_2 > \cdots; \quad \lim_\nu \text{meas.}\,(K - X_\nu) = \text{meas.}\, G. \tag{4. 32 a}$$

In view of (4. 31 a) and of (4. 30 a) and in consequence of absolute continuity it is inferred that

$$\lim_\nu f_{n,\nu}(\alpha) = \lim_\nu \Phi_\alpha^n(X_\nu) = \Phi_\alpha^n(\lim X_\nu) = \Phi_\alpha^n(K - G) = f_n(\alpha) \tag{4. 33}$$

for α in G. To obtain information regarding the degree of approximation we note that

$$f_n(\alpha) - f_{n,\nu}(\alpha) = \iint \frac{d\mu}{(z-\alpha)^n} \qquad (\alpha \text{ in } G)$$

with integration extended over $K - G - X_\nu$. On taking account of (4. 31 b) and of Theorem 4. 1 it is concluded that

$$|f_n(\alpha) - f_{n,\nu}(\alpha)| < h_n \eta_n(\varepsilon_\nu) \qquad (\alpha \text{ in } G;\ \nu = 1, 2, \ldots), \tag{4. 33 a}$$

where $\eta_n(\varepsilon)$ *is specified by the italicized statement in connection with* (4. 12).

Theorem 4. 2. *Let* $\mu (\geqq 0)$ *be an absolutely continuous set-function of density* $\varrho(z)$. *Suppose there exists a set* $G < G(n)$ ($G(n)$ *the set of Theorem* 4. 1) *in which* $\varrho(z) = 0$. *The function* $f_n(\alpha)$ *of* (4. 30) *can be approximated in* G *by a uniformly convergent sequence of analytic functions* $\{f_{n,\nu}(\alpha)\}$ $(\nu = 1, 2, \ldots)$ *of the form* (4. 32), *where the closed sets* $X_\nu < K - G$ *and satisfy* (4. 31), (4. 31 a). *The degree of approximation is given by* (4. 33 a), *where* $\eta_n(\varepsilon)$ *is the function of* (4. 12) *and* $\varepsilon_\nu = \text{meas.}\,(K - G - X_\nu)$.

Note. Similar results can be obtained with the aid of Corollaries 4. 2 and 4. 3. Thus, for instance, consider the function

$$f(\alpha) = \iint\limits_K \frac{\varrho(z)\,dx\,dy}{z-\alpha} = \iint\limits_K \frac{d\mu}{z-\alpha} \tag{4. 34}$$

where $\varrho(z)$ is summable and $|\varrho(z)|$ is uniformly bounded over K. Let G be the set in which $\varrho(z) = 0$. Let $X_1, X_2, \ldots$ be closed sets $< K - G$, such that (4. 31), (4. 31 a) is satisfied. Define $f_\nu(\alpha)$ by (4. 32; with $n = 1$). Then, in consequence of (4. 22),

$$|f(\alpha) - f_\nu(\alpha)| < b_1 \varepsilon_\nu^{1/2} \qquad (\nu = 1, 2, \ldots; \ \alpha \text{ in } G), \tag{4. 35}$$

where $\varepsilon_\nu = \text{meas.}\,(K - G - X_\nu) \to 0$ (as $\nu \to \infty$).

When μ is not an absolutely continuous function of sets $X < K$, with the aid of LEBESGUE's decomposition theorem one may write

$$\iint_X \frac{d\mu}{(z-\alpha)^n} = \iint_X \frac{d\mu_1}{(z-\alpha)^n} + \iint_X \frac{d\vartheta}{(z-\alpha)^n} \qquad (X < K) \tag{4. 36}$$

where μ_1, ϑ are (additive) set-functions, the first absolutely continuous, the latter singular (the integrals are supposed to exist for α in some set $< K$). We are thus brought to the consideration of functions of the form

$$s_n(\alpha) = \iint_K \frac{d\vartheta}{(z-\alpha)^n} \qquad (\vartheta \geqq 0), \tag{4. 37}$$

the set-function ϑ being singular. There exists *a set of measure zero* $K^0 < K$ so that

$$\Phi_\alpha^n(X) = \iint_X \frac{d\vartheta}{(z-\alpha)^n} = \iint_{X K^0} \frac{d\vartheta}{(z-\alpha)^n} \qquad (X < K). \tag{4. 37 a}$$

Suppose there exists a set of positive measure $G_n < K\,(G_n K^0 = 0)$ *such that*

$$\Gamma_\alpha^n = \sum_{\nu=1} \nu^{n-1} \vartheta\left(K S\left(\alpha, \frac{r_0}{\nu}\right)\right) \qquad (r_0 = \text{diameter of } K) \tag{4. 38}$$

converges in G_n. *By Theorem* 3. 1 *the integrals* (4. 37), (4. 37 a) *have finite values for* α *in* G_n.

Since density of additive singular set-functions is zero almost everywhere, in consequence of Corollary 3. 1 it is concluded that the function $s_1(\alpha)$ (cf. (4. 37)) is defined and is finite almost everywhere in K. Hence meas. $G_1 =$ meas. K.

There exists an open set $O < K^0$ of measure as small as desired. Let us take meas. $O <$ meas. G_n. Let $\{H_\nu\}$ $(\nu = 1, 2, \ldots)$ be a sequence of closed sets, such that

$$H_1 < H_2 < \ldots; \quad H_\nu \to O \qquad (\text{as } \nu \to \infty). \tag{4. 39}$$

We then have $K - H_\nu$ open and $K - H_\nu \to K - O$ (as $\nu \to \infty$). The function

$$s_{n,\nu}(\alpha) = \iint\limits_{H_\nu} \ldots = \iint\limits_{H_\nu K^0} \frac{d\vartheta}{(z-\alpha)^n} \tag{4. 40}$$

is analytic in $K - H_\nu$ and, by (4. 39) and (4. 37 a),

$$\lim_\nu s_{n,\nu}(\alpha) = \Phi^n_\alpha(\lim_\nu H_\nu) = \Phi^n_\alpha(O\,K^0) = \Phi^n_\alpha(K^0) = s_n(\alpha) \tag{4. 41}$$

for α in $G_n - O$. Since $G_n K^0 = 0$ and since $O > K^0$, O cannot be contained in G_n; however, the sets G_n and O may have points in common. One has meas. $(G_n - O) > 0$; in fact, by appropriate choice of O it is possible to arrange to have meas. $(G_n - O)$ arbitrarily near meas. G_n.

In consequence of a theorem due to Egoroff[1], given $\varepsilon_1 (> 0)$, however small, there exists a subset $G^n < G_n - O$ with

$$\text{meas. } (G_n - O) - \text{meas. } G^n \leqq \varepsilon_1 \tag{4. 42}$$

such that

$$s_{n,\nu}(\alpha) \to s_n(\alpha) \qquad (\text{as } \nu \to \infty) \tag{4. 42 a}$$

uniformly for α in G^n. By suitable choice of O and of ε_1 it can be arranged to have (4. 42 a) satisfied, as stated, for α in $G^n < G_n$, with meas. G^n arbitrarily close to meas. G_n.

Theorem 4. 3. *Consider a function $s_n(\alpha)$ defined by (4. 37) $\vartheta (\geqq 0)$ denoting a singular set-function. Let $K^0 < K$ be the set of measure zero for which (4. 37 a) holds (for sets $X < K$). Suppose there exists a set $G_n < K$,*

$$G_n K^0 = 0, \qquad \text{meas. } G_n > 0,$$

in which the series Γ^n_α of (4. 38) converges. Given $\varepsilon (> 0)$, however small, a sequence of analytic functions, $s_{n,\nu}(\alpha)$ $(\nu = 1, 2, \ldots)$, and a set $G^n < G_n$, with

[1] Cf. (S; p. 18).

$$\text{meas.}\,(G_n - G^n) \leqq \varepsilon,$$

can be found such that

$$s_{n,\nu}(\alpha) \to s_n(\alpha) \qquad (as\ \nu \to \infty)$$

uniformly for α *in* G^n. *The functions* $s_{n,\nu}(\alpha)$ *may be taken in the form* (4. 40), *the* H_ν *being closed sets satisfying* (4. 39) *and* $O, < K^0$, *denoting a suitable open set of sufficiently small measure.*

We continue to adhere to the notation so far introduced. *Suppose there exists a set* $G(n) < G_n - O$ in which the rarefication of »mass» ϑ, specified in connection with (4. 38), is maintained uniformly; that is, *suppose there exists a function* $\tau(\nu)$, *independent of* α, *such that the series*

$$S_n = \sum_{\nu=1}^{\infty} (\nu + 1)^{n-1} \tau(\nu) \tag{4. 43}$$

converges[1] *while*

$$\vartheta\left(K S\left(\alpha, \frac{r_0}{\nu}\right)\right) \leqq \tau(\nu) \qquad (\nu = 1, 2, \ldots) \tag{4. 43 a}$$

for all α *in* $G(n)$ (*compare with* (4. 4)).

In consequence of (4. 33 a) and since, for sets $X < K$, we have

$$\vartheta\left(X S\left(\alpha, \frac{r_0}{\nu}\right)\right) \leqq \vartheta\left(K S\left(\alpha, \frac{r_0}{\nu}\right)\right)$$

it is inferred that

$$\Phi_\alpha^n(X) = \iint\limits_X \frac{d\vartheta}{|z - \alpha|^n} \leqq h_n \sum_{\nu=1}^{\infty} (\nu + 1)^{n-1} \vartheta\left(X S\left(\alpha, \frac{r_0}{\nu}\right)\right) \tag{4. 44}$$

$$\leqq h_n S_n \qquad (\text{cf. } (4.\ 3) \text{ and } (4.\ 43);\ X < K)$$

when α is in $G(n)$. If $s_{n,\nu}(\alpha)$ is defined by (4. 40), it is to be recalled that (4. 41) will hold in $G_n - O$; since $G(n) < G_n - O$ we shall have à fortiori

$$\lim_{\nu} s_{n,\nu}(\alpha) = s_n(\alpha) \qquad (\alpha \text{ in } G(n)). \tag{4. 45}$$

In order to investigate the character of convergence we form the difference

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[1] ν is replaced by $\nu + 1$ in order to conform with (4. 4 a).

$$s_n(\alpha) - s_{n,\nu}(\alpha) = \int\!\!\cdot\!\!\int_{O-H_\nu} \frac{d\vartheta}{(z-\alpha)^n}.$$

One has

(4. 45 a) $$|s_n(\alpha) - s_{n,\nu}(\alpha)| \leqq \Phi^n_\alpha(O - H_\nu) \qquad \text{(cf. 4. 44)}$$

for α in $G(n)$ and $\nu = 1, 2, \ldots$.

For α in K

(4. 46) $$\vartheta\left((O - H_\nu)\, S\left(\alpha, \frac{r_0}{k}\right)\right) \leqq \vartheta(O - H_\nu) = \xi_\nu \qquad (k = 1, 2, \ldots).$$

On the other hand, by (4. 43 a),

(4. 46 a) $$\vartheta\left((O - H_\nu)\, S\left(\alpha, \frac{r_0}{k}\right)\right) \leqq \vartheta\left(K S\left(\alpha, \frac{r_0}{k}\right)\right) \leqq \tau(\nu) \qquad (k = 1, 2, \ldots)$$

when α is in $G(n)$. Inasmuch as $H_\nu \to O$ (as $\nu \to 0$) it follows that

(4. 46 b) $$\lim_\nu \xi_\nu = 0;$$

moreover, the ξ_ν as well as $\tau(\nu)$ are independent of α. By (4. 45 a), (4. 44; with $X = O - H_\nu$) and (4. 46), (4. 46 a) we have

(4. 47) $$|s_n(\alpha) - s_{n,\nu}(\alpha)| \leqq h_n \left[\sum_{k=1}^{k(\nu)} + \sum_{k>k(\nu)}\right] (k+1)^{n-1}\, \vartheta\left((O - H_\nu)\, S\left(\alpha, \frac{r_0}{k}\right)\right)$$

$$\leqq h_n \left[\sum_{k=1}^{k(\nu)} (k+1)^{n-1} \xi_\nu + \sum_{k>k(\nu)} (k+1)^{n-1} \tau(k)\right] = h_n \zeta'(\nu) \qquad (\alpha \text{ in } G(n)),$$

where the integer $k(\nu)$ ($\geqq 1$) is at our disposal. Now

$$\sum_{k=1}^{k(\nu)} (k+1)^{n-1} < \frac{3^n}{n} k^n(\nu);$$

thus

(4. 47 a) $$\zeta'(\nu) < \frac{3^n}{n} \xi_\nu k^n(\nu) + \sum_{k>k(\nu)} (k+1)^{n-1} \tau(\nu) = \zeta(\nu).$$

The integer $k(\nu)$ ($\nu = 1, 2, \ldots$) will be chosen so that

(4. 48) $$\xi_\nu k^n(\nu) \to 0, \qquad k(\nu) \to \infty \qquad (\text{as } \nu \to \infty).$$

For instance, one may take

(4. 48 a) $$k(\nu) = [\xi_\nu^{-\gamma/n}] \qquad (0 < \gamma < 1)$$

for $\nu \geqq \nu_0$ (ν_0 sufficiently great)[1]. With (4. 48) satisfied, in consequence of convergence of the series S_n (of (4. 43)) it is concluded that

$$\lim_\nu \zeta(\nu) = 0.$$

Theorem 4. 4. *Consider a function* $s_n(\alpha)$ *as given by* (4. 37) *with* $\vartheta(\geqq 0)$ *singular. We note that for a set* K^0 *of measure zero* (4. 37 a) *will hold. Let* $O > K^0$ *be open and of as small measure as desired. Designate by* H_ν, $< O$, *a sequence of closed sets such that*

$$H_1 < H_2 < H_3 < \ldots; \qquad \lim_\nu H_\nu = O.$$

Suppose there exists a set $G(n) < K - O$ *such that there exists a function* $\tau(\nu)$ *for which* (4. 43 a) *is satisfied (in* $G(n)$*), while the series* (4. 43) *converges. The function*

$$s_{n,\nu}(\alpha) = \iint_{H_\nu} \frac{d\vartheta}{(z-\alpha)^n}$$

will be analytic in $K - H_\nu$; $H(K - H_\nu) = K - O > G(n)$. *For* α *in* $G(n)$ $s_{n,\nu}(\alpha) \to s_n(\alpha)$ *(as* $\nu \to \infty$*) uniformly. In fact,*

(4. 49) $$|s_\nu(\alpha) - s_{n,\nu}(\alpha)| < h_\nu \zeta(\nu) \qquad (\alpha \text{ in } G(n);\ \nu = 1, 2, \ldots).$$

Here $\zeta(\nu)$ *is defined by* (4. 47 a), *with* $\varepsilon_\nu = \vartheta(O - H_\nu)$ *and with* $K(\nu)$ *satisfying* (4. 48); $\lim_\nu \zeta(\nu) = 0$.

We shall now proceed to derive approximations by analytic functions following a modification of the above methods.

Let $\mu(\geqq 0)$ be a set function not necessarily absolutely continuous. Suppose there exists a set $G(n) < K$ so that for some function $t(\nu)$ (independent of α),

(4. 50) $$\mu\left(K S\left(\alpha, \frac{r_0}{\nu}\right)\right) \leqq t(\nu) \qquad (\nu = 1, 2, \ldots;\ \alpha \text{ in } G(n)),$$

while

(4. 50 a) $$S_n = \sum_\nu (\nu + 1)^{n-1} t(\nu)$$

[1] For $\nu \leqq \nu_0$ definition of $k(\nu)$ is immaterial.

converges. It is noted that, with $X < K$,

$$\text{(4. 51)} \qquad \Phi_\alpha^n(X) = \iint_X \frac{d\mu}{(z-\alpha)^n} = \left[\iint^{(1)} + \iint^{(2)}\right] \frac{d\mu}{(z-\alpha)^n}.$$

In the last member above the first integration displayed is over $X\,S(\alpha, \delta)$; the second inegration is over $X(K - S(\alpha, \delta))(\delta > 0)$. When z is in the latter set $|z - \alpha| > \delta$; thus

$$\text{(4. 52)} \qquad \left|\iint^{(2)} \frac{d\mu}{(z-\alpha)^n}\right| < \delta^{-n} \iint^{(2)} d\mu < \delta^{-n} \mu(X).$$

On the other hand, in consequence of developments previously given,

$$\text{(4. 53)} \qquad \left|\iint^{(1)} \cdots\right| \leqq h_n \sum_{\nu=1}^{\infty} (\nu + 1)^{n-1} \mu\left(X\,S(\alpha, \delta)\,S\left(\alpha, \frac{r_0}{\nu}\right)\right) = h_n H_\alpha^n(X/\delta)$$

(cf. (4. 3)), with X replaced by $X\,S(\alpha, \delta)$ when α is in $G(n)$. For $\nu = 1, 2, \ldots$

$$\mu\left(X\,S(\alpha, \delta)\,S\left(\alpha, \frac{r_0}{\nu}\right)\right) \leqq \begin{cases} \mu\left(K\,S\left(\alpha, \frac{r_0}{\nu}\right)\right) \leqq t(\nu), \\ \mu(K\,S(\alpha, \delta)) \leqq t(\nu(\delta)), \end{cases}$$

provided α is in $G(n)$; here

$$\text{(4. 54)} \qquad \nu(\delta) = \left[\frac{r_0}{\delta}\right]^1$$

By (4. 53) with the aid of the subsequent inequalities it is inferred that

$$\text{(4. 55)} \qquad H_\alpha^n(X/\delta) \leqq \sum_{\nu=1}^{\nu_\delta} (\nu + 1)^{n-1}\, t(\nu(\delta)) + \sum_{\nu > \nu_\delta} (\nu + 1)^{n-1}\, t(\nu)$$

where ν_δ is at our disposal. We have

$$\text{(4. 55 a)} \qquad 2^{n-1} + \cdots + (1 + \nu_\delta)^{n-1} \leqq K_n \nu_\delta^n.$$

Take ν_δ, integral-valued, so that

$$\text{(4. 56)} \qquad \nu_\delta \to \infty; \qquad \nu_\delta^n\, t(\nu(\delta)) \to 0 \qquad (\text{as } \delta \to 0).$$

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[1] Under (4. 54) $\delta \leqq r_0/\nu(\delta)$ and $S(\alpha, \delta) < S(\alpha, r_0/\nu(\delta))$.

One may take

(4. 56a) $$\nu_\delta = [\, t(\nu(\delta))^{-\gamma/n}] \qquad (0 < \gamma < 1).$$

Thus, for α in $G(n)$,

(4. 57) $$H_\alpha^n (X/\delta) \leqq k_n \nu_\delta^n t(\nu(\delta)) + \sum_{\nu > \nu_\delta} (\nu + 1)^{n-1} t(\nu) = \theta'_n(\delta).$$

Since (4. 50a) converges, in view of (4. 56) it is concluded that

(4. 57a) $$\theta'_n(\delta) \to 0 \qquad (\text{as } \delta \to 0).$$

By (4. 51), (4. 52), (4. 53) and (4. 57) we have, when α is in $G(n)$,

(4. 58) $$| \Phi_\alpha^n (X) | < \delta^{-n} \mu(X) + h_n \theta'_n(\delta).$$

Suppose $\theta_n(\delta)$ is a suitable continuous function of δ ($0 < \delta \leqq \delta_0$; δ_0 sufficiently small) such that

(4 .58a) $$\theta'_n(\delta) \leqq \theta_n(\delta) \qquad (0 < \delta \leqq \delta_0);$$

moreover, we chose $\theta_n(\delta)$ so that $\lim\limits_{\delta} \theta_n(\delta) = 0$. By (4. 58), for α in $G(n)$,

(4. 59) $$| \Phi_\alpha^n (X) | < \delta^{-n} \mu(X) + h_n \theta_n(\delta) = F_n(X, \delta) \qquad (0 < \delta \leqq \delta_0).$$

If $\mu(X) \leqq \varepsilon_0$, where ε_0 (> 0) is sufficiently small, there exist values δ such that

(4. 60) $$\delta^{-n} \mu(X) = h_n \theta_n(\delta);$$

let $\delta(\mu(X))$ be the greatest of these values. Now, in (4. 59) the first member is independent of δ, which is accordingly at our disposal. Thus, with the above choice of δ,

(4. 61) $$| \Phi_\alpha^n (X) | < F_n(\mu(X)) = 2\,\delta^{-n} \mu(X) = 2\, h_n \theta_n(\delta)$$
$$(\alpha \text{ in } G(n);\ \delta = \delta(\mu(X));\quad X < K).$$

In view of (4. 60) $\delta(\mu(X)) \to 0$, *as* $\mu(X) \to 0$. Thus

$$\theta_n(\delta(\mu(X))) \to 0 \qquad (\text{as } \mu(X) \to 0)$$

and, in consequence of (4. 61), *the same will be true of* $F_n(\mu(X))$.

When $t(\nu) \to 0$ (as $\nu \to \infty$) monotonically[1] and sufficienly rapidly so that, for some λ'_n, one has

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[1] The monotone character of $t(\nu)$ is merely for convenience and is otherwise not essential.

(4. 62) $$(\nu+2)^{n-1}t(\nu+1)+(\nu+3)^{n-1}t(\nu+2)+\cdots\leqq\lambda'_n\nu^n t(\nu)$$

in (4. 57) we may take $\theta'_n(\delta)$ of the form

$$\theta'_n(\delta)=K_n\nu^n t(\nu(\delta))+\lambda'_n\nu^n t(\nu),$$

where $\nu=\nu_\delta$ satisfies (4. 56). Now

$$\frac{r_0}{\delta}-1<\nu(\delta)\leqq\frac{r_0}{\delta}.$$

Thus, defining $t(\nu)$ for ν positive and not integral-valued so that $t(\nu)$ *is continuous monotone for all* $\nu>0$, we have

(4. 62 a) $$t(\nu(\delta))\leqq t\left(\frac{r_0-\delta}{\delta}\right)\leqq t\left(\frac{r'}{\delta}\right)\qquad(r'=r_0-\delta_0),$$

provided $0<\delta\leqq\delta_0$ where δ_0 is suitably small. Let us take $\nu_\delta=\nu(\delta)$[1]. Then, by (4. 62 a), it is inferred that

(4. 63) $$\theta'_n(\delta)=\lambda_n\nu^n(\delta)\,t(\nu(\delta))\leqq\lambda_n r_0^n\delta^{-n}t\left(\frac{r'}{\delta}\right)=\theta_n(\delta)$$

(r' from (4. 62 a); $\lambda_n=k_n+\lambda'_n$ (cf. (4. 62), (4. 55 a)); $0<\delta\leqq\delta_0$).

Since, by hypothesis $\nu^n t(\nu)\to 0$, as $\nu\to\infty$, it is clear that

$$\lim_{\nu}\theta_n(\delta)=0;$$

moreover, $\theta_n(\delta)$, as given by (4. 63), is continuous in δ. Furthermore, (4. 60) now assumes the form

(4. 64) $$\mu(X)=h'_n t\left(\frac{r'}{\delta}\right)\qquad(h'_n=h_n\lambda_n r_0^n).$$

With $\mu(X)\leqq\varepsilon_0$ ($\varepsilon_0>0$ sufficiently small), this equation has a unique solution $\delta=\delta(\mu(X))$ and

(4. 65) $$F_n(\mu(X))=2\,\delta^{-n}\mu(X)=2\,h'_n\delta^{-n}t\left(\frac{r'}{\delta}\right)\qquad(\delta=\delta(\mu(X)))$$

for $0<\delta\leqq\delta_0$ and $\mu(X)\leqq\varepsilon_0$

[1] This ν_δ will satisfy (4. 56).

When $t(\nu)$ is continuous monotone, such that (4. 62) holds the following is observed. The faster $t(\nu) \to 0$ (as $\nu \to \infty$) the greater will $\delta(\mu(X))$ (cf. (4. 64)) be; that is, the slower will

$$\delta(\mu(X)) \to 0 \qquad (\text{as } \mu(X) \to 0)$$

and, by (4. 65), the faster will $F_n(\mu(X)) \to 0$ (as $\mu(X) \to 0$).

We have accordingly established the following result regarding »continuity» of $\Phi^n_\alpha(X)$ (cf. (4. 51)), considered as a function of sets $X < K$.

Theorem 4. 5. *Let $\mu(\geqq 0)$ be a set-function, not necessarily absolutely continuous. Suppose there exists a set $G(n) < K$ such that for some monotone continuous function $t(\nu)$, satisfying the statement in connection with (4. 62), inequalities (4. 50) hold for α in $G(n)$. With $\Phi^n_\alpha(X)$ defined by (4. 51) (integer $n \geqq 1$), one has*

$$(4.\ 66) \qquad |\Phi^n_\alpha(X)| < F_n(\mu(X)) = 2\, h'_n\, \delta^{-n}\, t\left(\frac{r'}{\delta}\right)$$

$$(\alpha \text{ in } G(n);\quad h'_n = h_n \lambda_n r_0^n \quad (cf.\ (4.\ 63));\quad r' = r_0 - \delta_0)$$

for sets $X < K$. In (4. 66) $\delta = \delta(\mu(X))$ is solution of the equation $\mu(X) = h'_n\, t(r'\,\delta^{-1})$. The faster $t(\nu) \to 0$ (as $\nu \to \infty$), the faster will $F_n(\mu(X)) \to 0$ (as $\mu(X) \to 0$).

Note. Inequality (4. 66) gives degree of »continuity» (with respect to sets $X < K$) of integrals $\Phi^n_\alpha(X)$. Under stated conditions, this continuity is uniform for α in $G(n)$. The last sentence of the theorem may be interpreting as signifying that the greater is the rarefication of »mass» μ in the neighborhood of the set $G(n)$ the greater is the degree of continuity (point-set continuity) of $\Phi^n_\alpha(X)$ for α in $G(n)$. In (4. 66) this dependence has been made explicit.

Under the conditions of the above theorem the density $\varrho(z)$ of the »mass» μ will be zero in $G(n)$ when $n \geqq 2$. In any case, suppose a set $G(n)$ exists in which $\varrho(z) = 0$, while (4. 50) holds for α in $G(n)$. Let O denote a suitable open set

$$(4.\ 67) \qquad K > O > K - G(n)$$

of measure as close as desired to that of $K - G(n)$; we have

$$(4.\ 68) \qquad f_n(\alpha) = \Phi^n_\alpha(K) = \Phi^n_\alpha(O) \qquad (\text{cf. } (4.\ 51)).$$

Let $H_1 < H_2 < \ldots, H_\nu < O$, be closed sets such that $\lim H_\nu = O$. The function

$$(4.\ 68\,\text{a}) \qquad f_{n,\nu}(\alpha) = \Phi^n_\alpha(H_\nu)$$

is analytic in $K - H_\nu$. We have

$$\lim_\nu f_{n,\nu}(\alpha) = \Phi^n_\alpha(\lim_\nu H_\nu) = \Phi^n_\alpha(O) = f_n(\alpha)$$

for α in $G^n = G(n) - O$[1]. If meas. $G(n) > 0$ then the set O can be so chosen that meas. $G^n > 0$; in fact, one may arrange to have meas. G^n arbitrarily close to that of $G(n)$. To investigate the degree of approximation of $f_n(\alpha)$ by $f_{n,\nu}(\alpha)$ we write

$$|f_n(\alpha) - f_{n,\nu}(\alpha)| = |\Phi^n_\alpha(O) - \Phi^n_\alpha(H_\nu)| = |\Phi^n_\alpha(O - H_\nu)|.$$

By (4. 66) (with $X = O - H_\nu$)

$$|f_\nu(\alpha) - f_{n,\nu}(\alpha)| < F_n(\mu(O - H_\nu)) \tag{4. 69}$$

for α in G^n.

When a closed set $G(n)$ exists such that for some $t(\nu)$ (4. 50a) converges, (4. 50) holds, while $\varrho(z) = 0$ (in $G(n)$), then the set O introduced in (4. 67) can be taken as $O = K - G(n)$; furthermore, we then have $G^n = G(n)$ and inequalities (4. 69) will continue to hold.

Theorem 4. 6. *Let $\mu\,(\geqq 0)$ be a set-function not necessarily absolutely continuous. Suppose there exists a set $G(n)$ such that for some function $t(\nu)$ the conditions of Theorem 4. 5 are satisfied. Consider the function $f_n(\alpha)$, as defined by (4. 68). $f_n(\alpha)$ may be approximated by analytic functions as follows.*

Let open O, satisfying (4. 67), be of measure as near as desired to that of $K - G(n)$. When $G(n)$ is a closed set we take $O = K - G(n)$. Let $H_1 < H_2 < \ldots$ be closed subsets of O; $\lim_\nu H_\nu = O$. The functions $f_{n,\nu}(\alpha) = \Phi^n_\alpha(H_\nu)$, analytic in $K - H_\nu$ $(\nu = 1, 2, \ldots)$, converge uniformly to $f_n(\alpha)$ for α in $G^n = G(n) - O$. In fact, for α in G^n,

$$|f_n(\alpha) - f_{n,\nu}(\alpha)| < F_n(\mu(O - H_\nu)) \qquad (\nu = 1, 2, \ldots), \tag{4. 70}$$

where $F_n(\mu)$ is the function defined in Theorem 4. 5. When μ is absolutely continuous, Theorem 4. 2 will hold, as stated, with inequalities (4. 33 a) replaced by

$$|f_n(\alpha) - f_{n,\nu}(\alpha)| < F_n(\mu(K - G - X)) \qquad (\nu = 1, 2, \ldots;\ \alpha \text{ in } G); \tag{4. 70 a}$$

here G, X_ν are sets so denoted in Theorem 4. 2.

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[1] In fact, for every α for which the integral $\Phi^n_\alpha(O)$ exists.

5. Approximations by Rational Functions.

In a paper by J. Wolff[1] are found the following results regarding representations of analytic functions by series of the form

(5. 1) $$\sum \frac{A_K}{z - \alpha_K},$$

where

(5. 1 a) $$S(A) = \sum |A_K|$$

converges.

Suppose $f(z)$ is analytic in an open bounded set D. Let D_1 be any open set whose closure $\bar{D}_1 = D_1 +$ frontier of $D_1 < D$. Form an open set $\varDelta > D_1$, with frontier consisting of a finite number of polygons P, such that $\bar{\varDelta} = \varDelta + P < D$. Let $\delta (> 0)$ be the distance from P to the frontier of D. Construct polygons P_n parallel to P, exterior to $\varDelta$ at distance $\delta 2^{-n}$ from P. Designate by $\varDelta_n$ the open set having P_n for frontier and containing D_1. The function $f(z)$ is representable in D_1 by a series (5. 1) with $S(A)$ (of (5. 1 a)) $< 2ML$. Here M is the upper bound of $|f(z)|$ on the P_n $(n = 1, 2, \ldots)$ and L is the upper bound of the lengths of the P_n $(n = 1, 2, \ldots)$. The α_k are on the P_n $(n = 1, 2, \ldots)$, a finite number of α_k being situated on each P_n.

The above result of Wolff can be applied to obtain representations for functions which are limits of analytic functions. In this connection we shall investigate functions $f(\alpha)$ of the type referred to in Theorem 2. 1. We thus have a closed bounded set G and a sequence of functions $f_\nu(\alpha)$ $(\nu = 1, 2, \ldots)$, where $f_\nu(\alpha)$ is analytic (uniform) in the open set $O(\delta_\nu) > G$. Here $O(\delta)$ is the set of points at distance $< \delta$ from G. We have $O(\delta_\nu) \to G$ (as $\nu \to \infty$), inasmuch as we take $\delta_1 > \delta_2 > \cdots (\delta_\nu > 0)$, $\lim \delta_\nu = 0$. Consider $f(\alpha)$ such that in G $|f(\alpha) - f_\nu(\alpha)| \leqq \varepsilon_\nu (\nu = 1, 2, \ldots; \; \varepsilon_\nu \to 0$ as $\nu \to \infty)$. Under these circumstances a sequence $\{m_\nu\}$ can be found, $m_1 < m_2 < \cdots$, as well as functions $a_\nu(\alpha)$ such that the following is true (cf. (2. 9), (2. 10)).

In $\bar{O}(\delta_{m_\nu})$ $a_\nu(\alpha)$ is analytic (uniform),

(5. 2) $$|a_\nu(\alpha)| \leqq 2\eta_\nu \qquad (\text{in } \bar{O}(\delta_{m_\nu});\ \nu = 1, 2, \ldots);$$

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[1] J. Wolff, *Sur les séries* $\sum \frac{A_K}{z - \alpha_K}$, Comptes Rendus, t. 173; pp. 1327—28.

$$O(\delta_{m_1}) > O(\delta_{m_2}) > \cdots; \qquad O(\delta_{m_\nu}) \to G;$$

(5. 2 a) $$f(\alpha) = \sum_{\nu=1}^{\infty} a_\nu(\alpha) \qquad \text{(in } G\text{)}$$

and the series

(5. 2 b) $$\sum \eta_\nu$$

converges[1].

Subsequent to (2. 10) we introduced an open set $O_\nu > O(\delta_{m_{n+1}})$ with frontier consisting of a set P_ν of polygons, while $O(\delta_{m_\nu}) > \bar{O}_\nu = O_\nu + P_\nu$. With $l_\nu (> 0)$ denoting the distance from P_ν to the frontier of $O(\delta_{m_\nu})$ we take $0 < \lambda_\nu < l_\nu$ and from an open set $O_{\nu,\varrho} > O_\nu$ $(0 \leqq \varrho \leqq \lambda_\nu)$, the frontier of $O_{\nu,\varrho}$ being a set $P_{\nu,\varrho}$ of polygons parallel to P_ν and at the distance ϱ from P_ν.

Let $L_{\nu,\varrho}$ denote the sum of lengths of the polygons constituting $P_{\nu,\varrho}$ and let

(5. 3) $$L_\nu = \text{upper bound } L_{\nu,\varrho} \qquad (0 \leqq \varrho \leqq \lambda_\nu).$$

The polygonal sets $P_{\nu,\varrho}$, formed for $0 \leqq \varrho \leqq \lambda_\nu$, are in $O(\delta_{m_\nu}) - O(\delta_{m_{\nu+1}}) < O(\delta_{m_\nu})$; hence by (5. 2)

(5. 4) $$|a_\nu(\alpha)| \leqq 2\,\eta_\nu \qquad (\text{on } P_{\nu,\varrho};\ 0 \leqq \varrho \leqq \lambda_\nu).$$

Applying the results of Wolff, referred to above, to the function $a_\nu(\alpha)$ it is inferred that

(5. 5) $$a_\nu(\alpha) = \sum \frac{A_{\nu,n}}{\alpha_{\nu,n} - \alpha} \qquad (\alpha \text{ in } O(\delta_{m_{\nu+1}}))$$

where, in view of (5. 4) and (5. 3),

(5. 5 a) $$\sum_n |A_{\nu,n}| < 4\,\eta_\nu\, L_\nu;$$

moreover, it is observed that the $\alpha_{\nu,n}$ are on the polygons $P_{\nu,\varrho}$ $(0 < \varrho \leqq \lambda_\nu)$, the limiting points of the $\alpha_{\nu,n}$ $(n = 1, 2, \ldots)$ being all on $P_{\nu,0} = P_\nu$.

Let the set G be regular according to Definition 2. 1. Then

(5. 6) $$L_\nu \leqq L < \infty \qquad (\nu = 1, 2, \ldots).$$

In consequence of (5. 2 a) and (5. 5)

(5. 7) $$f(\alpha) = \sum_\nu \sum_n \frac{A_{\nu,n}}{\alpha_{\nu,n} - \alpha} = \sum_{K=1}^{\infty} \frac{a_K}{\alpha_K - \alpha}$$

[1] $\bar{E}$ denotes E + limiting points of E.

for α in G. Here $\{a_K\} = \{A_{\nu,n}\}$, $\{\alpha_K\} = \{\alpha_{\nu,n}\}$[1], the α_K are not in G and, in view of (5. 5 a),

(5. 7 a) $$\sum_K |a_K| = \sum_\nu \sum_n |A_{\nu,n}| < 4 \sum_\nu \eta_\nu L_\nu = S.$$

In consequence of (5. 6) and since the series (5. 2 b) converges; it is seen that S converges.

Theorem 5. 1. *Let G be bounded, closed, regular (cf. Definition 2. 1). Suppose $f(\alpha)$ is in G the limit of a uniformly convergent sequence of analytic functions, as stated in the beginning of section 2. For α in G we then have the representation*

(5. 8) $$f(\alpha) = \sum_K \frac{a_K}{\alpha_K - \alpha} \qquad (\alpha_K \text{ not in } G),$$

where

(5. 8 a) $$\sum_K |a_K|$$

converges.

Note. This theorem can be extended to certain more general classes of functions. However, we shall not undertake such an extension at this time.

We now turn our attention to Theorem 4. 2. With the aid of this theorem the following result can be proved.

Theorem 5. 2. *Let $\mu(\geqq 0)$ be absolutely continuous. Suppose there exists a set $H < K$ in which the density of μ is zero and which is such that for a function $t(\nu)$, independent of α,*

(5. 9) $$S_n = \sum_\nu (\nu + 1)^{n-1} t(\nu) \qquad (\text{integer } n \geqq 1)$$

converges, while

$$\mu\left(K S\left(\alpha, \frac{r_0}{\nu}\right)\right) \leqq t(\nu) \cdot \qquad (\nu = 1, 2, \ldots; \alpha \text{ in } G).$$

Consider the function

(5. 10) $$f_n(\alpha) = \iint_K \frac{d\mu}{(z-\alpha)^n} = \Phi_\alpha^n(K) = \Phi_\alpha^n(K - G).$$

In a set $G' < G$ of measure arbitrarily close to that of G $f_n(\alpha)$ can be approximated uniformly by finite sums

[1] That is, the indicated sequences (one simple, another double) constitute merely a rearrangement one of another.

(5. 11) $$r(\alpha) = \sum_{j=1}^{K} \frac{a_j}{\alpha - \alpha_j} \qquad (\alpha_j \text{ not in } G').$$

When closed sets $X_\nu < K_1 - G$, *where* $K_1 = \overline{K}$, *can be found such that* meas. $X_\nu \to$ meas. $(K_1 - G)$ *(as* $\nu \to \infty$*) while*

(5. 12) $$K_1 - X_\nu > \overline{K_1 - X_{\nu+1}} \qquad (\nu = 1, 2, \ldots)^1,$$

the set G' *can be replaced by* G.

To prove this theorem we note that, according to Theorem 4. 2,

(5. 13) $$|f_n(\alpha) - f_{n,\nu}(\alpha)| < h_n \eta_n(\varepsilon_\nu) \qquad (\nu = 1, 2, \ldots; \alpha \text{ in } G),$$

where $\eta_n(\varepsilon)$ is given by (4. 12), (cf. (4. 11 a), (4. 11), (4. 8),

$$\varepsilon_\nu = \text{meas.}\,(K - G - X_\nu)$$

and

$$f_{n,\nu}(\alpha) = \Phi^n_\alpha(X_\nu) \qquad (\text{closed } X_\nu < K - G),$$

with $X_1 < X_2 < \cdots$ and $\varepsilon_\nu \to 0$ (as $\nu \to \infty$); moreover, $\eta_n(\varepsilon_\nu) \to 0$ (as $\nu \to \infty$). The function $f_{n,\nu}(\alpha)$ is analytic in the open set $K - X_\nu > G$. Let D_ν consist of the points of $K - X_\nu$ at distance $> \xi 2^{-\nu}$ ($\xi > 0$) from X_ν. The set D_ν will be open and $\bar{D}_\nu < K - X_\nu$. By choosing ξ sufficiently small the measure of

$$G' = \prod_\nu G \bar{D}_\nu \qquad (G' < G)$$

can be made as near as desired to that of G. Thus, if meas. $G > 0$ one may always arrange to have meas. $G' > 0$. Assign $\varepsilon (> 0)$, however small; by Wolff's theorem there exists a sum $r_{\nu,\varepsilon}(\alpha)$ of form (5. 11) such that

(5. 13 a) $$|f_{n,\nu}(\alpha) - r_{\nu,\varepsilon}(\alpha)| \leqq \frac{\varepsilon}{2}$$

for α in $\bar{D}_\nu$. By (5. 13) and (5. 13 a)

(5. 14) $$|f_n(\alpha) - r_{\nu,\varepsilon}(\alpha)| \leqq |f_n(\alpha) - f_{n,\nu}(\alpha)| + |f_{n,\nu}(\alpha) - r_{\nu,\varepsilon}(\alpha)|$$
$$< h_n \eta_n(\varepsilon_\nu) + \frac{\varepsilon}{2} \leqq \varepsilon \qquad (\alpha \text{ in } G\bar{D}_\nu),$$

provided that $\nu = \nu(\varepsilon)$ is taken sufficiently great.[2] When (5. 12) is satisfied we may replace D_ν by $K_1 - X_{\nu+1}$; (5. 14) will then hold in G.

[1] $\overline{E}$ = closure of E.

[2] It is to be noted that $\eta_n(\varepsilon_\nu) \to 0$ (as $\nu \to \infty$).

In view of (5. 14), given $\varepsilon > 0$ (however small), there exist a sum $r_\varepsilon(\alpha)$ of the form (5. 11) such that

$$|f_n(\alpha) - r_\varepsilon(\alpha)| < \varepsilon \qquad (\alpha \text{ in } G' < G).$$

Here G' is the product of the $G\bar{D}_\nu$ $(\nu = 1, 2, \ldots)$, unless (5. 12) is satisfied when one may take $G' = G$. This completes the proof of the theorem.

Theorem 5. 3. *Let $\vartheta (\geqq 0)$ be a singular set-function. We have $\vartheta(X) = \vartheta(X K^0)$ for sets $X < K$; here $K^0 < K$ and* meas. $K^0 = 0$. *Designate by $O > K^0$ an open set of as small measure as desired. Let $G(n) < K - O$ be a set such that for a function $\tau(\nu)$, independent of α,*

$$S_n = \sum_\nu (\nu + 1)^{n-1} \tau(\nu) \qquad (integer\ n \geqq 1) \tag{5. 15}$$

converges, while

$$\vartheta\left(K S\left(\alpha, \frac{r_0}{\nu}\right)\right) \leqq \tau(\nu) \qquad (\nu = 1, 2, \ldots;\ \alpha\ in\ G(n)).$$

Consider the function

$$s_n(\alpha) = \iint_K \frac{d\vartheta}{(z - \alpha)^n} = \Phi_\alpha^n(K) = \Phi_\alpha^n(K^0). \tag{5. 16}$$

In the set $G(n)$ $s_n(\alpha)$ can be approximated uniformly by finite sums of the form (5. 11) *(α_j not in $G(n)$).*

To prove the above we recall Theorem 4. 4. The closed subsets H_ν $(\nu = 1, 2, \ldots)$ of O, referred to in that theorem, will be defined as follows. H_ν is to be the part of O at distance $\geqq \frac{1}{\nu}$ from the frontier of O. Applying Wolff's expansion to the function $s_{n,\nu}(\alpha) = \Phi_\alpha^n(H_\nu)$, analytic in $K - H_\nu$, it is observed that, inasmuch as

$$K_1 - H_\nu > \overline{K_1 - H_{\nu+1}} \qquad (K_1 = \bar{K})$$

there exists a sum $r_{\nu,\varepsilon}(\alpha)$ of the form (5. 11) such that

$$|s_{n,\nu}(\alpha) - r_{\nu,\varepsilon}(\alpha)| \leqq \frac{\varepsilon}{2} \tag{5. 17}$$

for α in $K - H_{\nu+1}$; in particular, (5. 17) will hold in

$$\Pi(K - H_{\nu+1}) = K - O$$

Now, in view of Theorem 4. 4,

$$|s_n(\alpha) - s_{n,\nu}(\alpha)| < h_n \zeta(\nu) \qquad (\nu = 1, 2, \ldots)$$

for α in $G(n)$; here $\zeta(\nu) \to 0$ as $\nu \to \infty$ Thus, by (5. 17),

$$|s_n(\alpha) - r_{\nu,\varepsilon}(\alpha)| \leqq |s_n(\alpha) - s_{n,\nu}(\alpha)| + |s_{n,\nu}(\alpha) - r_{\nu,\varepsilon}(\alpha)|$$

$$< h_n \zeta(\nu) + \frac{\varepsilon}{2} \leqq \varepsilon \qquad (\nu = \nu(\varepsilon))$$

when α is in $(K - O)\, G(n) = G(n)$.[1] Such an inequality is obtained for every $\varepsilon (> 0)$, however small. This establishes the Theorem.

Using the above methods or suitable modifications of them, numerous further results along the lines of Wolff's expansions can be obtained.[2] The methods so far used appear to be adequate for the treatment, in the essential particulars, of problems of this type.

6. **Continuity.**

The real or imaginary parts of the integrals studied in this work as functions of α are semi-continuous in one sense or other. In any perfect subset of a set where a function, under consideration, can be uniformly approximated by analytic functions such a function will be, of course, continuous. Such approximations were involved in Theorems 2. 1, 4. 2, 4. 3, 4. 4, 5. 1, 5. 2, 5. 3. The purpose of this section is to investigate the »degree of continuity» of the classes of functions considered in this paper. First, limits of convergent sequences of analytic functions will be considered. For this purpose it will be convenient to obtain a function

$$(6.\ 1) \qquad w = h(\varepsilon, z)$$

effecting a conformal transformation of a domain $D(\varepsilon)$, in the z-plane, on the interior of the unit circle in the w-plane. In this connection $D(\varepsilon)$ is taken to be the domain containing the real interval $(0, 2a)$ $(a > 0)$ and bounded by two circular arcs, C_u and C_l, extending from $z = 0$ to $z = 2a$ above and below the axis of reals, respectively. C_u and C_l are to be arcs of the circles

$$(6.\ 2) \qquad |z - (a - ib)| = \sqrt{a^2 + b^2} = R, \quad |z - (a + ib)| = R \qquad (b > 0),$$

[1] Since $K - O > G(n)$.

[2] One may go to the definition of the integrals involved and make use of uniformity of absolute continuity of the integrals (for α within certain sets, supposed to exist).

respectively, while b is to be chosen so that *the line $x = a$ intersects C_u and C_l at points which are at the distance 2ε from each other;* that is,

$$b = \frac{1}{2\varepsilon}(a^2 - \varepsilon^2), \quad R = \frac{1}{2\varepsilon}(a^2 + \varepsilon^2)$$

and $R - b = \varepsilon$. The tangents at $z = 0$ to C_u and C_l make an angle τ (which we take positive) with the positive direction of the real axis,

(6. 2 a) $$\operatorname{tg} \tau = \frac{2 a \varepsilon}{a^2 - \varepsilon^2}.$$

The function

(6. 3) $$z_1 = \frac{z}{2a - z}$$

will transform $D(\varepsilon)$ on the interior of the angular region $W(\varepsilon)$, in the z_1-plane

(6. 3 a) $$-\tau < \text{angle } z_1 < \tau.$$

The z-interval $(0, 2a)$ will go into the positive axis of reals in the z_1-plane, the z-points $0, a, 2a$ going into the z_1-points $0, 1, \infty$, respectively. The further transformation

(6. 4) $$z_2 = z_1^\sigma \qquad \left(\sigma = \frac{\pi}{2\tau}\right)$$

will map $W(\varepsilon)$ on the z_2-half plane[1]

(6. 4 a) $$-\frac{\pi}{2} < \text{angle } z_2 < \frac{\pi}{2}$$

the z_1-points $0, 1, \infty$ going into the z_2-points $0, 1, \infty$, respectively. The half plane (6. 4 a) is finally mapped on the interior of the w-unit-circle by means of the transformation

(6. 5) $$w = h(\varepsilon, z) = \frac{z_2 - 1}{z_2 + 1}.$$

The z_2-points $0, 1, \infty$ will go into the w-points $-1, 0, 1$, respectively. By (6. 5), (6. 4) and (6. 3) *we have*

(6. 6) $$w = h(\varepsilon, z) = \frac{z^\sigma - (2a - z)^\sigma}{z^\sigma + (2a - z)^\sigma},$$

where $\sigma = \pi/(2\tau)$ and τ is defined by (6. 2 a).

[1] We take the determination for which $z_1^\sigma > 0$ when $z_1 > 0$.

Let G be a closed bounded set containing the closed interval $I\,(0 \leqq \alpha \leqq 2a)$. Suppose $f_\nu(\alpha)$ is analytic (uniform) in $O(\delta_\nu)$ (notation of section 2), where $\delta_\nu > 0$ and

(6. 7) $$\delta_1 > \delta_2 > \cdots, \quad \delta_\nu \to 0 \qquad (\text{as } \nu \to \infty).$$

Let the sequence $\{f_\nu(\alpha)\}$ converge uniformly, for α in G, to a limiting function $f(\alpha)$. Thus,

(6. 8) $$|f(\alpha) - f_\nu(\alpha)| \leqq \varepsilon_\nu \quad (\alpha \text{ in } G;\ \nu = 1, 2, \ldots;\ \lim \varepsilon_\nu = 0).$$

Inasmuch as the $f_\nu(\alpha)$ and $f(\alpha)$ are functions satisfying the conditions stated at the beginning of section 2, the statement in connection with (2. 9) and (2. 10) will hold, the series

(6. 9) $$\sum_\nu \eta_\nu \qquad (\eta_\nu = \varepsilon_{n_\nu} + \varepsilon_{n_{\nu-1}})$$

being convergent. To secure convergence of (6. 9) one needs merely to make a suitable choice of the sequence (n_ν). If we write

(6. 10) $$g_\nu(\alpha) = a_1(\alpha) + a_2(\alpha) + \cdots + a_\nu(\alpha),$$

in view of of (2. 10) it is observed that

(6. 10 a) $$g_\nu(\alpha) \to f(\alpha) \qquad (\text{as } \nu \to \infty)$$

uniformly in G. It is to be recalled that

$$a_\nu(\alpha) = f_{n_\nu}(\alpha) - f_{n_{\nu-1}}(\alpha) \qquad (f_{n_0}(\alpha) \equiv 0;\ n_0 = 0)$$

(cf. (2. 3)) is analytic in $\bar{O}(\delta_{m_\nu})\,(\nu = 1, 2, \ldots)$ and

(6. 10 b) $$|a_\nu(\alpha)| \leqq 2\,\eta_\nu \qquad (\alpha \text{ in } \bar{O}(\delta_{m_\nu}));$$

here $\{m_\nu\}$ is a subsequence of $\{n_\nu\}$ specified subsequent to (2. 8 a). We have $m_\nu \geqq n_\nu\,(\nu = 1, 2, \ldots)$ and $O(\delta_{m_\nu}) < O(\delta_\nu)$. By (6. 10)

(6. 10 c) $$g_\nu(\alpha) = f_{n_\nu}(\alpha) \qquad (\nu = 1, 2, \ldots).$$

Thus, in view of (6. 10) and (6. 10 b)

$$|f_{n_\nu}(\alpha)| \leqq |a_1(\alpha)| + \cdots + |a_\nu(\alpha)| \leqq 2\,(\eta_1 + \eta_2 + \cdots + \eta_\nu)$$

for α in $\bar{O}(\delta_{m_\nu})$. Since the series (6. 9) converges *we accordingly have*

(6. 11) $$|f_{n_\nu}(a)| \leqq S \qquad (a \text{ in } \bar{O}(\delta_{m_\nu})),$$

this being true for $\nu = 1, 2, \ldots$; *moreover,*

(6. 11 a) $$|f(a) - f_{n_\nu}(a)| \leqq \varepsilon_{n_\nu} \qquad (a \text{ in } G;\ \nu = 1, 2, \ldots).$$

Since the interval I, $0 \leqq a \leqq 2a$, is in G and since $\bar{O}(\delta_{m_\nu})$ contains all points at the distance $\leqq \delta_{m_\nu}$ from G, it is clear that

(6. 12) $$\bar{O}(\delta_{m_\nu}) > \bar{D}(\delta_{m_\nu}) \qquad (\nu = 1, 2, \ldots).$$

Suppose $f(a) = 0$; then by (6. 11 a)

(6. 13) $$|f_{n_\nu}(a)| \leqq \varepsilon_{n_\nu} \qquad (\nu = 1, 2, \ldots).$$

If we apply the transformation

(6. 14) $$w = h(\delta_{m_\nu}, a) \qquad (\text{cf. (6. 6) with } \varepsilon = \delta_{m_\nu})$$

the region $D(\delta_{m_\nu})$ will be mapped on the interior of the unit circle in the w-plane. The function

(6. 15) $$F_\nu(w) = f_{n_\nu}(a)$$

will be analytic for $|w| < 1$ and, in view of (6. 13),

(6. 15 a) $$|F_\nu(0)| = |f_{n_\nu}(a)| \leqq \varepsilon_{n_\nu} \qquad (\nu = 1, 2, \ldots);$$

moreover, inasmuch as (6. 11) and (6. 12) are satisfied, one has

(6. 15 b) $$|F_\nu(w)| \leqq S \qquad (|w| < 1).$$

Thus

(6. 15 c) $$|F_\nu(w) - F_\nu(0)| \leqq S + \varepsilon_{n_\nu} \qquad (|w| < 1).$$

Whence, applying the lemma of Schwarz to the function

$$G_\nu(w) = \frac{F_\nu(w) - F_\nu(0)}{S + \varepsilon_{n_\nu}},$$

it is inferred that

$$|G_\nu(w)| \leqq |w| \qquad (|w| < 1);$$

that is,

(6. 16) $$|F_\nu(w) - F_\nu(0)| \leqq (S + \varepsilon_{n_\nu})|w| \qquad (|w| < 1).$$

Let q be the upper bound of $S + \varepsilon_{n_\nu}$ ($\nu = 1, 2, \ldots$). In view of (6. 16) and (6. 15 a) it is then concluded that

$$(6.\ 17) \qquad |F_\nu(w)| \leqq |F_\nu(w) - F_\nu(0)| + |F_\nu(0)| \leqq q|w| + \varepsilon_{n_\nu} \qquad (|w| < 1)$$

for $\nu = 1, 2, \ldots$. Going back to the α-plane, with the aid of the transformation (6. 14) by (6. 15) and (6. 17) we obtain

$$(6.\ 17\text{ a}) \qquad |f_{n_\nu}(\alpha)| \leqq q\,|h(\delta_{m_\nu}, \alpha)| + \varepsilon_{n_\nu}$$

for α in $D(\delta_{m_\nu})$. Such an inequality will hold for $\nu = 1, 2, \ldots$. In particular (5. 17 a) will hold for $0 \leqq \alpha \leqq 2a$. *By virtue of* (6. 17 a) *and* (6. 11 a)

$$(6.\ 18) \qquad |f(\alpha)| \leqq |f_{n_\nu}(\alpha)| + |f(\alpha) - f_{n_\nu}(\alpha)| \leqq q\,|h(\delta_{m_\nu}, \alpha)| + 2\varepsilon_{n_\nu} = w_\nu(\alpha)$$

for $0 \leqq \alpha \leqq 2a$ *and for* $\nu = 1, 2, \ldots$ (*cf.* (6. 6) *with* $\varepsilon = \delta_{m_\nu}$).

With $f(\alpha)$ taken equal to zero at $\alpha = a$, the »*degree of continuity*» (along I) at $\alpha = a$ of the function $f(\alpha)$ will be characterized by the speed with which $f(\alpha) \to 0$ as $\alpha \to a$ (along I). One may also measure the »degree of continuity» as follows. Given $\varepsilon (> 0)$, suppose $l(\varepsilon) (> 0)$ can be found so that $|f(\alpha) - f(a)|$ ($|f(\alpha)|$ in the present case) is $\leqq \varepsilon$ for $a - l(\varepsilon) \leqq \alpha \leqq a + l(\varepsilon)$. The slower $l(\varepsilon) \to 0$, as $\varepsilon \to 0$, the greater will be the »degree of continuity» of $f(\alpha)$ at a (along I).

We shall write

$$(6,\ 19) \qquad \alpha = a(1 + \beta), \quad \zeta = \frac{1 - \beta}{1 + \beta}.$$

Then, by (6. 6), one has

$$(6.\ 20) \qquad h(\delta_{m_\nu}, \alpha) = \frac{1 - \zeta^\sigma}{1 + \zeta^\sigma}; \quad \sigma = \sigma(\nu) = \frac{\pi}{2\tau(\nu)}; \quad \operatorname{tg} \tau(\nu) = \frac{2a\delta_{m_\nu}}{a^2 - \delta_{m_\nu}^2}.$$

Let us take, for instance, $a < \alpha < 2a$; then $0 < \zeta < 1$ and $\zeta \to 1$ when $\alpha \to a$. Let $\varepsilon (> 0)$ be assigned ($< 2q$). Designate by $\nu' = \nu(\varepsilon)$ an integer (taken as small as possible) such that

$$(6.\ 21) \qquad \varepsilon_{n_{\nu'}} \leqq \frac{\varepsilon}{4}.$$

Consider now the function (6. 20) for $\nu = \nu'$. If one takes ζ so that

(6. 22) $$1 > \zeta \geqq q(\varepsilon)^{1/\sigma}, \quad q(\varepsilon) = \frac{2q - \varepsilon}{2q + \varepsilon}, \quad \sigma = \sigma(\nu'),$$

the inequality

(6. 22 a) $$q\frac{1 - \zeta^{\sigma}}{1 + \zeta^{\sigma}} \leqq \frac{\varepsilon}{2} \qquad (\sigma = \sigma(\nu'))$$

will be satisfied. By (6. 22 a) and (6. 21) one then will have

(6. 23) $$w_{\nu'}(\alpha) \leqq \varepsilon,$$

where $w_{\nu'}(\alpha)$ is the function so denoted in (6. 18).

With the aid of (6. 19) it is inferred that the condition (6. 22) is equivalent to

$$\frac{1 - h(\varepsilon)}{1 + h(\varepsilon)} \geqq \beta > 0 \qquad (h(\varepsilon) = q(\varepsilon)^{1/\sigma};\ \sigma = \sigma(\nu')),$$

that is, to

(6. 24) $$a < \alpha \leqq \frac{2a}{1 + h(\varepsilon)} = a + l'(\varepsilon) \qquad (\text{cf. } (6.\ 22),\ (6.\ 20)).$$

Thus, by virtue of (6. 18) and (6. 23),

$$|f(\alpha)| \leqq \varepsilon \qquad (a < \alpha \leqq a + l'(\varepsilon)).$$

It can be shown that the same inequality will hold for $a - l'(\varepsilon) \leqq \alpha < a$. *Whence*, in consequence of (6. 24),

(6. 25) $$|f(\alpha)| \leqq \varepsilon \qquad (a - l'(\varepsilon) \leqq \alpha \leqq a + l'(\varepsilon))$$

where

(6. 25 a) $$l'(\varepsilon) = a\frac{1 - h(\varepsilon)}{1 + h(\varepsilon)} \qquad (h(\varepsilon) = q(\varepsilon)^{1/\sigma};\ \sigma = \sigma(\nu');\ q(\varepsilon) \text{ from } (6.\ 22)),$$

$\sigma(\nu')$ *being defined by* (6. 20) *while* ν' *is the integer specified in connection with* (6. 21). When $\varepsilon \to 0$, $h(\varepsilon) \to 1$ and $l(\varepsilon) \to 0$. Now, since $1 < 1 + h(\varepsilon) < 2$,

(6. 25 b) $$a(1 - h(\varepsilon)) > l'(\varepsilon) > \frac{a}{2}(1 - h(\varepsilon)) = l''(\varepsilon).$$

By (6. 20)

$$\frac{1}{\sigma} \geqq b_0\,\delta_{m_{\nu'}} \qquad (b_0 \text{ independent of } \varepsilon).$$

Thus, in view of (6. 22) and (6. 25 a)

$$h(\varepsilon) < \left(1 - \frac{\varepsilon}{2q}\right)^{1/\sigma} \leqq \left(1 - \frac{\varepsilon}{2q}\right)^{b_0\,\delta_{m_{\nu'}}} \qquad (0 < \varepsilon \leqq \varepsilon_0 < 2q)$$

and, by virtue of (6. 25 b),

(6. 26) $$l'(\varepsilon) > l''(\varepsilon) > \lambda\, \delta_{m_\nu}\, \varepsilon = l(\varepsilon) \qquad (\nu = \nu(\varepsilon),\ \lambda \text{ independent of } \varepsilon)$$

for $0 < \varepsilon \leqq \varepsilon_0$. In view of (6. 26) and (6. 25) it is inferred that (6. 25) *holds with* $l'(\varepsilon)$ *replaced by* $\lambda\, \delta_{m_\nu}\, \varepsilon$ $(\nu = \nu(\varepsilon))$.

By (2. 5) (with 2. 3) we have $|f_{n_\nu}(\alpha) - f_{n_{\nu-1}}(\alpha)| \leqq q_\nu$ (in $\bar{O}(\delta_{n_{\nu+1}})$). On taking account of the statements subsequent to (2. 8 a) it is observed that for m_ν one may take the least integer $(\geqq n_{\nu+1})$ such that

(6. 27) $$\delta_{m_\nu} \leqq \frac{\eta_\nu\, \delta_{n_{\nu+1}}}{q_\nu + \eta_\nu} \qquad (\eta_\nu = \varepsilon_{n_\nu} + \varepsilon_{n_{\nu-1}}).$$

We are now able to formulate the following theorem.

Theorem 6. 1. *Let G be a closed bounded set. Let $\{f_\nu(\alpha)\}$ $(\nu = 1, 2, \ldots)$ be a sequence of functions, $f_\nu(\alpha)$ being analytic (uniform) in $O(\delta_\nu)$ (notation of section 2); $\delta_1 > \delta_2 > \cdots$, $\delta_\nu > 0$, $\lim_\nu \delta_\nu = 0$. Suppose that in G this sequence converges uniformly to a limiting function $f(\alpha)$; thus,*

$$|f(\alpha) - f_\nu(\alpha)| \leqq \varepsilon_\nu \qquad (\alpha \textit{ in } G;\ \nu = 1, 2, \ldots;\ \lim_\nu \varepsilon_\nu = 0).$$

The degree of continuity of $f(\alpha)$ in the neighborhood of a point α_0, interior an interval I belonging to G, can be specified as follows.

Let $\{n_\nu\}$ $(\nu = 1, 2, \ldots)$ be a sequence such that $\varepsilon_{n_1} + \varepsilon_{n_2} + \cdots$ converges.[1] *One has $|f_{n_\nu}(\alpha) - f_{n_{\nu-1}}(\alpha)| \leqq q_\nu$ (in $\bar{O}(\delta_{n_{\nu+1}})$) for $\nu = 1, 2, \ldots$. We form a subsequence $\{m_\nu\}$ $(\nu = 1, 2, \ldots)$ of $\{n_\nu\}$, with $m_\nu (\geqq n_{\nu+1})$ designating the least integer such that (6. 27) holds. Given $\varepsilon (> 0)$, however small, one has*

(6. 28) $$|f(\alpha) - f(\alpha_0)| \leqq \varepsilon$$

for all α, on I, such that

(6. 28 a) $$|\alpha - \alpha_0| \leqq \lambda\, \varepsilon\, \delta_{m(\varepsilon)} = l(\varepsilon) \qquad (\lambda > 0, \textit{ independent of } \varepsilon).$$

Here $m(\varepsilon) = m_{\nu'}$, where ν' is the least of the integers $\nu (= 1, 2, \ldots)$ for which $\varepsilon_{n_\nu} \leqq \varepsilon/4$.[2]

[1] Such a sequence clearly exists since $\lim_\nu \varepsilon_\nu = 0$.

[2] There are infinitely many integers ν for which the latter inequality holds.

Note. If one could manage to carry out the technical steps possibly better results could be obtained, if one replaces $D(\delta)$ by the set of points at distance $< \delta$ from the interval I, under consideration, or by a rectangle enclosing I. We then would have to employ mapping functions distinct from that of (6. 6). If rectangles are used one is brought to the consideration of elliptic functions. The function (6. 6) is preferable on account of its simplicity.

An examination of Theorem 6. 1 makes it apparent that the »degree of continuity» of the limiting function $f(\alpha)$ depends essentially on sequences $\{\varepsilon_\nu\}$ (which determine sequences $\{n_\nu\}$), $\{q_\nu\}$ and $\{\delta_\nu\}$. If we consider the function $l(\varepsilon)$ of (6. 28 a) the following is observed. If $q_\nu \to \infty$ (as $\nu \to \infty$), the slower $q_\nu \to \infty$ the slower will the $m_\nu \to \infty$ [1] and the slower will $\delta_{m(\varepsilon)}$ (with $m(\varepsilon) = m_{\nu'}$) $\to 0$, as $\varepsilon \to 0$; the same will be true for $l(\varepsilon)$. Thus, the slower $q_\nu \to \infty$ (as $\nu \to \infty$), the greater will be the degree of continuity of $f(\alpha)$.[2] *The slower* $\delta_{n_\nu} \to 0$, *as* $\nu \to \infty$, *the slower will the* $m_\nu \to \infty$ *and*, again, *the greater will be the degree of continuity of* $f(\alpha)$. In a similar way one may examine how the speed with which $\varepsilon_\nu \to 0$, as $\nu \to \infty$, is reflected in the degree of continuity of $f(\alpha)$.

With $f_\nu(\alpha)$ $(\nu = 1, 2, \ldots)$ and $f(\alpha)$ satisfying the conditions stated in connection with (6. 7), (6. 8), it is of interest to consider *the important special case when the sequence of upper bounds of* $|f_\nu(\alpha)|$ (*in* $\bar{O}(\delta_{\nu+1})$) *is bounded*. In this case, there exists a finite number M, independent of α, such that

$$|f_\nu(\alpha)| \leqq M \qquad (\alpha \text{ in } \bar{O}(\delta_{\nu+1})) \tag{6. 29}$$

for $\nu = 1, 2, \ldots$. We then have

$$|f_{n_\nu}(\alpha) - f_{n_{\nu-1}}(\alpha)| \leqq 2M \qquad (\alpha \text{ in } \bar{O}(\delta_{\nu+1})).$$

Since $\bar{O}(\delta_{n_{\nu+1}}) < \bar{O}(\delta_{\nu+1})$, the latter inequality will hold in $\bar{O}(\delta_{n_{\nu+1}})$, as well. The numbers q_ν, referred to in the theorem, may accordingly be replaced by $2M$ and the sequence $\{m_\nu\}$ may be defined as a subsequence of $\{n_\nu\}$ such that m_ν is the least integer satisfying the inequality

$$\delta_{m_\nu} \leqq \frac{\eta_\nu \delta_{n_{\nu+1}}}{2M + \eta_\nu} \qquad (m_\nu \geqq n_{\nu+1})$$

$(\nu = 1, 2, \ldots)$; m_ν may be defined as the least integer such that

$$\delta_{m_\nu} \leqq b\, \eta_\nu \delta_{n_{\nu+1}} \qquad (\eta_\nu = \varepsilon_{n_\nu} + \varepsilon_{n_{\nu-1}}), \tag{6. 30}$$

[1] With a suitable choice of m_ν to satisfy (6. 27).

[2] One would expect this intuitively.

where b is the reciprocal of the greatest of the values $2M + \eta_\nu$ $(\nu = 1, 2, \ldots)$.[1] We have $m_1 < m_2 < \cdots$. By definition of m_ν, just given,

$$\delta_{m_\nu} \leqq b\,\eta_\nu\,\delta_{n_{\nu+1}} < \delta_{m_{\nu-1}}.$$

Increasing ν by unity it is inferred that

$$\delta_{m_{\nu+1}} \leqq b\,\eta_{\nu+1}\,\delta_{n_{\nu+2}} < \delta_{m_\nu}. \tag{6.30a}$$

In view of (6. 30 a) it is concluded that

$$l(\varepsilon) > b\,\lambda\,\varepsilon\,\eta_{\nu+1}\,\delta_{n_{\nu+2}} \qquad (\nu = \nu';\ \text{cf. } (6.\ 28\ \text{a})),$$

where ν' is defined as stated subsequent to (6. 28 a). Now

$$\eta_{\nu+1} = \varepsilon_{n_{\nu+1}} + \varepsilon_{n_\nu} > \varepsilon_{n_\nu};$$

thus,

$$l(\varepsilon) > b\,\lambda\,\varepsilon\,\varepsilon_{n_\nu}\,\delta_{n_{\nu+2}} \qquad (\nu = \nu').$$

Accordingly we are able to formulate the following Corollary.

Corollary 6. 1. *If the sequence* $\{f_\nu(\alpha)\}$ $(\nu = 1, 2, \ldots)$, *referred to in Theorem* 6. 1, *possesses the additional property* (6. 29) *the degree of continuity of the limiting function* $f(\alpha)$ *can be specified without the aid of the subsequence* $\{m_\nu\}$ *of* $\{n_\nu\}$. *In fact, with* α_0 *denoting an interior point of an interval* $I < G$, *the following can be stated. Given* $\varepsilon\,(> 0)$, *however small, we have*

$$|f(\alpha) - f(\alpha_0)| \leqq \varepsilon \tag{6.31}$$

for all α, *on* I, *such that*

$$|\alpha - \alpha_0| \leqq \lambda_0\,\varepsilon\,\varepsilon_{n_\nu}\,\delta_{n_{\nu+2}} = l(\varepsilon) \qquad (\lambda_0 > 0,\ \textit{independent of}\ \varepsilon). \tag{6.31a}$$

In (6. 31 a) $\nu = \nu'$, *where* ν' *is the least of the integers* $\nu\,(= 1, 2, \ldots)$ *for which* $\varepsilon_{n_\nu} \leqq \varepsilon/4$.

Note. Except for a constant factor, $l(\varepsilon)$ in (6. 31 a) is of the order of

$$l_0(\varepsilon) = \varepsilon^2\,\delta_{n_{\nu+2}} \qquad (\nu = \nu' = \nu(\varepsilon)).$$

[1] In consequence of the definition of b one has $b\,\eta_\nu\,\delta_{n_{\nu+1}} < \delta_{n_{\nu+1}}$. Hence $m_\nu > n_{\nu+1} > n_{\nu'}$.

The faster $\varepsilon_{n_\nu} \to 0$, as $\nu \to \infty$, the smaller will $\nu' = \nu(\varepsilon)$ be for a given ε and the greater will be $\delta_{n_{\nu+2}}$ (with $\nu = \nu(\varepsilon)$). Thus, *the faster* $\varepsilon_{n_\nu} \to 0$, *as* $\nu \to \infty$, the slower will $l(\varepsilon)$ (of (6. 31 a)) $\to 0$ (as $\varepsilon \to 0$) and *the greater will be the degree of continuity of* $f(\alpha)$ *at* α_0 (on I). It is also clear that the slower $\delta_{n_\nu} \to 0$ (as $\nu \to \infty$) the greater will be the degree of continuity. It is to be noted that the indicated dependence between the rates of decrease of the sequences $\{\varepsilon_{n_\nu}\}$ and $\{\delta_{n_\nu}\}$, on one hand, and of the degree of continuity of $f(\alpha)$, on the other hand, has been made quite explicit for the case under consideration.

We shall now investigate the dependence of the degree of continuity of functions of the form

$$f_n(\alpha) = \iint\limits_K \frac{d\mu}{(z-\alpha)^n} \qquad \text{(integer } n \geqq 1;\ \mu \geqq 0) \tag{6. 32}$$

on the rarefication of mass μ. Offhand, such a dependence is to be expected. Let us assume first that μ is an absolutely continuous set-function and that there exists a set $G(n) < K$, dense in itself, such that for some function $t(\nu)$ (independent of α) $\mu\left(K S\left(\alpha, \frac{r_0}{\nu}\right)\right) \leqq t(\nu)$ ($\nu = 1, 2, \ldots$; α in $G(n)$), while the series S_n of (4. 4 a) converges. In consequence of Theorem 4. 1 one then will have for sets $X < K$

$$\Phi_\alpha^n(X) = \iint\limits_X \frac{d\mu}{|z-\alpha|^n} < h_n \eta_n(\varepsilon) \qquad (\alpha \text{ in } G(n)), \tag{6. 33}$$

whenever meas. $X \leqq \varepsilon$; $\eta_n(\varepsilon)$ may be defined by (4. 12). As previously we let r_0 designate the diameter of K. With α, α_0 in $G(n)$ and $|\alpha - \alpha_0| \leqq \delta$, one has

$$f_n(\alpha_0) - f_n(\alpha) = \iint\limits_K \left(\frac{1}{(z-\alpha_0)^n} - \frac{1}{(z-\alpha)^n}\right) d\mu \tag{6. 34}$$

$$= \left(\iint\limits_{S_0} + \iint\limits_{K_0}\right)\left(\frac{1}{(z-\alpha_0)^n} - \frac{1}{(z-\alpha)^n}\right) d\mu,$$

where

$$S_0 = K S(\alpha_0, 2\delta), \quad K_0 = K - S_0.$$

Since meas. $S_0 \leqq 4\pi\delta^2$, by virtue of the statement in connection with (6. 33) it is inferred that

(6. 35) $$\left|\iint_{S_0}[\ldots]\,d\mu\right| \leqq \Phi^n_{\alpha_0}(S_0) + \Phi^n_{\alpha}(S_0) < 2\,h_n\,\eta_n(4\,\pi\,\delta^2).$$

On the other hand, it is observed that

$$r_0 > |z-\alpha_0| > 2\,\delta, \qquad r_0 > |z-\alpha| > \delta \qquad (z \text{ in } K_0)$$

so that the integrand displayed in (6. 34) will satisfy the inequality

$$\left|\frac{1}{(z-\alpha_0)^n} - \frac{1}{(z-\alpha)^n}\right| = |\alpha-\alpha_0|\left|\frac{(z-\alpha)^{n-1}+\cdots+(z-\alpha_0)^{n-1}}{(z-\alpha_0)^n(z-\alpha)^n}\right|$$

$$< |\alpha-\alpha_0|\, n\, r_0^{n-1}\, 2^{-n}\,\delta^{-2n} \qquad (z \text{ in } K_0);$$

accordingly,

(6. 36) $$\left|\iint_{K_0}[\ldots]\,d\mu\right| < |\alpha-\alpha_0|\,\sigma_n\,\delta^{-2n} \qquad (\sigma_n = n\,2^{-n}\,r_0^{n-1}\,\mu(K))$$

Thus, in view of (6. 35) and (6. 36), from (6. 34) we infer that

(6. 37) $$|f_n(\alpha_0) - f_n(\alpha)| < 2\,h_n\,\eta_n(4\,\pi\,\delta^2) + \sigma_n\,\delta^{-2n}\,|\alpha-\alpha_0|$$

for α, α_0 in $G(n)$ and $|\alpha-\alpha_0| \leqq \delta$.

Let us assign ε (> 0), however small. There exists a $\delta(\varepsilon)(> 0)$ so that

(6. 38) $$\eta_n(4\,\pi\,\delta^2) \leqq \frac{\varepsilon}{4\,h_n} \qquad (0 < \delta \leqq \delta(\varepsilon));$$

such values δ exist since $\eta_n(u) \to 0$, as $u \to 0$. One may define $\delta(\varepsilon)$ as the greatest value for which (6. 38) holds, as stated. Let

(6. 38 a) $$l(\varepsilon) = \frac{1}{2\,\sigma_n}\,\varepsilon\,\delta^{2n}(\varepsilon).$$

Then

(6. 39) $$\sigma_n\,\delta^{-2n}(\varepsilon)\,|\alpha-\alpha_0| \leqq \frac{\varepsilon}{2} \qquad (\text{for } |\alpha-\alpha_0| \leqq l(\varepsilon)).$$

Whence, by (6. 37), (6. 38) and (6. 39)

(6. 40) $$|f_n(\alpha_0) - f_n(\alpha)| < \varepsilon \qquad (\alpha_0, \alpha \text{ in } G(n)),$$

whenever

(6. 40 a) $$|\alpha-\alpha_0| \leqq l(\varepsilon) \qquad (\text{cf. (6. 38 a), (6. 36)})$$

this being so for all ε such that $0 < \varepsilon \leqq \varepsilon_0$.

Thus the following theorem has been proved.

Theorem 6. 2. *Consider a function* $f_n(\alpha)$ *defined by the integral* (6. 32) *with* $\mu(\geqq 0)$ *an absolutely continuous set-function. Suppose there exists a set* $G(n) < K$, *dense in itself, so that for some* $t(\nu)$ *the series* S_n *of* (4. 4) *converges, while* $\mu\left(K S\left(\alpha, \frac{r_0}{\nu}\right)\right) \leqq t(\nu)$ $(\nu = 1, 2, \ldots)$ *in* $G(n)$. *The degree of continuity of* $f_n(\alpha)$ *in* $G(n)$ *can be specified as follows. With* $\varepsilon\,(0 < \varepsilon \leqq \varepsilon_0)$, *however small, one has* $|f_n(\alpha_0) - f_n(\alpha)| < \varepsilon$, *whenever* α_0, α *are in* $G(n)$, *while* $|\alpha - \alpha_0| \leqq l(\varepsilon)$. *Here* $l(\varepsilon) = \varepsilon\delta^{2n}(\varepsilon)/(2\sigma_n)$ (σ_n *from* (6. 36)), *where* $\delta(\varepsilon) \to 0$ (*as* $\varepsilon \to 0$). *One may determine* $\delta(\varepsilon)$ *as follows.*

There exists a function $d(\xi)$ (*cf.* (4. 5)) *so that* $\mu(X) \leqq d(\xi)$, *when* meas. $X \leqq \xi$ ($d(\xi) \to 0$ *as* $\xi \to 0$). *Choose* $\sigma(\xi)\,(> 0)$ *so that*

$$\sigma(\varepsilon) \to \infty, \quad d(\xi)\sigma^n(\varepsilon) \to 0 \quad (as\ \xi \to 0).$$

Let[1] $m(\xi) = [\sigma(\xi)]$ *and write*

$$\zeta_n'(\xi) = \sum_{\nu > m(\xi)} (\nu + 1)^{n-1} t(\nu), \quad \eta_n(\xi) = \lambda_n d(\xi)\sigma^n(\xi) + \zeta_n'(\xi)$$

$(\lambda_n = \frac{1}{n} 3^n)$. *With the aid of* $\eta_n(\xi)$ *we define* $\delta(\varepsilon)$ *as the greatest value such that*

$$\eta_n(4\pi\delta^2) \leqq \varepsilon/(4h_n) \qquad (0 < \delta \leqq \delta(\varepsilon)).$$

Note. With the aid of the above theorem it is easy to show that the faster $d(\xi) \to 0$ (as $\xi \to 0$) and the faster the series S_n converges[2], the slower will $l(\varepsilon)$ (if suitably defined) $\to 0$ (as $\varepsilon \to 0$); that is, the greater will be the degree of continuity of $f_n(\alpha)$.[3]

Consider now a function

$$f(\alpha) = \iint_K \frac{\varrho(z)\,dx\,dy}{z - \alpha}, \tag{6. 41}$$

where $\varrho(z)$ is summable over K and $|\varrho(z)|$ is uniformly bounded in K. Corollary 4. 2 will be applicable, giving the inequality

$$\iint_X \left|\frac{\varrho(z)\,dx\,dy}{z - \alpha}\right| < b_1 \xi^{1/2} \qquad (\alpha \text{ in } K).$$

[1] I. e., $m(\xi)$ is the greatest integer $\leq \sigma(\xi)$.

[2] That is, the greater is the rarefication of »mass» μ, particularly in the neighborhood of the set $G(n)$.

[3] Note that the faster $d(\xi) \to 0$, the faster can $\sigma(\xi)$ be allowed to approach ∞ (subject to the condition $d(\xi)\sigma^n(\xi) \to 0$ (as $\xi \to 0$)); the smaller will $\zeta_n'(\xi)$ be.

whenever meas. $X \leqq \xi (> 0)$. Thus, for the case of $f(\alpha)$, as given by (6. 41), Theorem 6. 2 may be applied with $G(n)$ replaced by K and $h_n \eta_n(\xi)$ replaced by $b_1 \xi^{1/2}$. Hence the following result can be stated.

Corollary 6. 2. *Let* $f(\alpha)$ *be a function of the form* (6. 41), *where* $|\varrho(z)| \leqq b$ (*in* K) *and* $\varrho(z)$ *is summable over* K. *Given* $\varepsilon (0 < \varepsilon \leqq \varepsilon_0)$, *however small, we have* $|f_n(\alpha_0) - f_n(\alpha)| < \varepsilon$, *whenever* α_0, α *are in* K *and*

$$|\alpha - \alpha_0| \leqq l(\varepsilon) = b' \varepsilon^3 \qquad (b' > 0, \textit{ independent of } \varepsilon).$$

Consider now functions of the form

$$s_n(\alpha) = \iint\limits_K \frac{d\vartheta}{(z-\alpha)^n} = \Phi_\alpha^n(K) \qquad (\vartheta \geqq 0, \text{ integer } n > 0), \tag{6. 42}$$

where ϑ is a singular set-function. We have, for sets $X < K$,

$$\Phi_\alpha^n(X) = \Phi_\alpha^n(K^0 X) \qquad (\text{meas. } K^0 = 0).$$

Let $G(n) < K$ be a set such that, for a function $\tau(\nu)$, the series

$$S_n = \sum_\nu (\nu + 1)^{n-1} \tau(\nu) \tag{6. 43}$$

converges, while

$$\vartheta\left(K S\left(\alpha, \frac{r_0}{\nu}\right)\right) \leqq \tau(\nu) \qquad (\nu = 1, 2, \ldots; \alpha \text{ in } G(n)). \tag{6. 43 a}$$

Suppose α_0 is a limiting point of $G(n)$ and let α be any other point in $G(n)$ such that

$$|\alpha - \alpha_0| \leqq \delta \qquad (\delta > 0). \tag{6. 44}$$

We form the difference

$$s_n(\alpha_0) - s_n(\alpha) = \left(\iint^{(1)} + \iint^{(2)}\right)\left(\frac{1}{(z-\alpha_0)^n} - \frac{1}{(z-\alpha)^n}\right) d\vartheta, \tag{6. 45}$$

where

$$\iint^{(1)} = \iint \qquad (\text{over } S(\alpha_0, 2\delta)), \tag{6. 45 a}$$

$$\iint^{(2)} = \iint \qquad (\text{over } K - S(\alpha_0, 2\delta) = K^{(2)}).$$

Under (6. 44), $|z-\alpha_0|^{-1} < 1/(2\delta)$, $|z-\alpha|^{-1} < 1/\delta$ for z in $K^{(2)}$ and

(6. 46) $$\left|\frac{1}{(z-\alpha_0)^n} - \frac{1}{(z-\alpha)^n}\right| < |\alpha - \alpha_0|\, r'_n\, \delta^{-2n} \qquad (r'_n = n r_0^{n-1} 2^{-n};\ z \text{ in } K^{(2)}).$$

Thus

(6. 47) $$\left|\int\int^{(2)} \cdots\right| < |\alpha - \alpha_0|\, r_n\, \delta^{-2n} \qquad (r_n = r'_n\, \vartheta(K);\ \text{cf. } (6.\ 46)).$$

Since, with integration extended over $S(\alpha_0, 2\delta)$,

$$\left|\int\int^{(1)} \cdots\right| \leqq \int\int \frac{d\vartheta}{|z-\alpha_0|^n} + \int\int \frac{d\vartheta}{|z-\alpha|^n},$$

in consequence of (4. 44) it is concluded that

(6. 48) $$\left|\int\int^{(1)}\right| \leqq h_n \sum_{\nu=1}^{\infty} (\nu+1)^{n-1}\, \theta_\nu(\alpha, \alpha_0)$$

where

(6. 48 a) $$\theta_\nu(\alpha, \alpha_0) = \vartheta\left(S(\alpha_0, 2\delta)\, S\left(\alpha_0, \frac{r_0}{\nu}\right)\right) + \vartheta\left(S(\alpha_0, 2\delta)\, S\left(\alpha, \frac{r_0}{\nu}\right)\right).$$

Since $\vartheta \geqq 0$ and since α_0 is in $G(n)$, by (6. 43 a) one has

(6. 49) $$\vartheta\left(S(\alpha_0, 2\delta)\, S\left(\alpha_0, \frac{r_0}{\nu}\right)\right) \leqq \begin{cases} \vartheta(S(\alpha_0, 2\delta)), \\ \vartheta\left(S\left(\alpha_0, \frac{r_0}{\nu}\right)\right) \leqq \tau(\nu) \end{cases}$$

for $\nu = 1, 2, \ldots$. Similarly, on noting that $S(\alpha_0, 2\delta)\, S\left(\alpha, \frac{r_0}{\nu}\right)$ is a subset of $S(\alpha_0, 2\delta)$ and of $S\left(\alpha, \frac{r_0}{\nu}\right)$, it is concluded that

(6. 49 a) $$\vartheta\left(S(\alpha_0, 2\delta)\, S\left(\alpha, \frac{r_0}{\nu}\right)\right) \leqq \begin{cases} \vartheta(S(\alpha_0, 2\delta)), \\ \vartheta\left(S\left(\alpha, \frac{r_0}{\nu}\right)\right) \leqq \tau(\nu) \end{cases}$$

$(\nu = 1, 2, \ldots)$, inasmuch as $\alpha < G(n)$. We note that

$$S(\alpha_0, 2\delta) < S\left(\alpha_0, \frac{r_0}{\nu'}\right)$$

where

(6. 50) $$\nu' = \nu(\varepsilon) = \left[\frac{r^0}{r\delta}\right];^1$$

accordingly

$$\vartheta(S(\alpha_0, 2\delta)) \leqq \vartheta\left(S\left(\alpha_0, \frac{r_0}{\nu'}\right)\right) \leqq \tau(\nu(\delta)).$$

Whence, by (6. 49) and (6. 49 a), the $\theta_\nu(\alpha, \alpha_0)$ of (6. 48) satisfy the inequalities

(6. 51) $$\theta_\nu(\alpha, \alpha_0) \leqq \begin{cases} 2\tau(\nu(\delta)), \\ 2\tau(\nu) \end{cases}$$

for $\nu = 1, 2, \ldots$, provided $|\alpha - \alpha_0| \leqq \delta$. In consequence of (6. 51) and (6. 48)

(6. 52) $$\left|\int\int^{(1)} \cdots\right| \leqq h_n\left(\sum_{\nu=1}^{\nu_\delta} \cdots + \sum_{\nu>\nu_\delta} \cdots\right)$$

$$\leqq 2h\left(\omega(\delta) + \sum_{\nu=1}^{\nu_\delta} (\nu + 1)^{n-1}\tau(\nu(\delta))\right),$$

where

(6. 52 a) $$\omega(\delta) = \sum_{\nu>\nu_\delta} (\nu + 1)^{n-1}\tau(\nu)$$

and ν_δ is an integer at our disposal. Now $2^{n-1} + \cdots + (1 + \nu_\delta)^{n-1} \leqq K_n \nu_\delta^n$. Thus, for α, α_0 in $G(n)$ and $|\alpha - \alpha_0| \leqq \delta$,

(6. 53) $$\left|\int\int^{(1)} \cdots\right| \leqq 2h_n\zeta(\delta),$$

with

(6. 53 a) $$\zeta(\delta) = K_n\nu_\delta^n\tau(\nu(\delta)) + \omega(\delta) \qquad \text{(cf. (6. 50), (6. 52 a))}.$$

To secure the relation $\lim \zeta(\delta) = 0$ we choose ν_δ so that

(6. 54) $$\nu_\delta^n\tau(\nu(\delta)) \to 0, \quad \nu_\delta \to \infty \quad (\text{as } \delta \to 0)^2.$$

One may take

(6. 54 a) $$\nu_\delta = [\tau(\nu(\delta))^{-\gamma/n}] \qquad (0 < \gamma < 1).$$

---

[1] $[b]$ = greatest integer $\leqq b$.

[2] Note that $\omega(\delta)$, being the remainder after ν_δ terms of the convergent series (6. 43), will $\to 0$ when $\nu_\delta \to \infty$.

It is of advantage to have (6. 54) satisfied in such a way that $\zeta(\delta) \to 0$ (as $\delta \to 0$) as fast as possible.

By virtue of (6. 45), (6. 47) and (6. 53)

(6. 55) $$|s_n(\alpha_0) - s_n(\alpha)| \leqq \left|\int\int^{(1)}\cdots\right| + \left|\int\int^{(2)}\cdots\right| < |\alpha - \alpha_0| r_n \delta^{-2n} + 2 h_n \zeta(\delta)$$

$$(\alpha, \alpha_0 \text{ in } G(n);\ |\alpha - \alpha_0| \leqq \delta).$$

Assign $\varepsilon (> 0)$, however small. Let $\delta = \delta(\varepsilon)$ be a value (which it is advantageous to take as great as possible) so that

(6. 56) $$\zeta(\delta) \leqq \frac{\varepsilon}{4 h_n}.$$

In view of (6. 55) we then have

(6. 57) $$|s_n(\alpha_0) - s_n(\alpha)| < \varepsilon \qquad (\alpha, \alpha_0 \text{ in } G(n)),$$

whenever

(6. 57 a) $$|\alpha - \alpha_0| \leqq l(\varepsilon) = \frac{1}{2 r_n} \varepsilon\, \delta^{2n}(\varepsilon).$$

For $\varepsilon \leqq \varepsilon_0$ $l(\varepsilon) \leqq \delta$.

Theorem 6. 3. *Consider a function $s_n(\alpha)$, as given by (6. 42) with ϑ a singular set-function. Let $G(n) < K$ be such that for some $\tau(\nu)$ (independent of α) S_n of (6. 43) converges while (6. 43 a) holds. The degree of continuity of $s_n(\alpha)$, for α in $G(n)$, depends on the »rarefication» of ϑ, as follows. Given $\varepsilon (> 0)$, however small ($\varepsilon \leqq \varepsilon_0$), we have $|s_n(\alpha_0) - s_n(\alpha)| < \varepsilon$ (α, α_0 in $G(n)$) whenever $|\alpha - \alpha_0| \leqq l(\varepsilon)$; $l(\varepsilon)$ may be defined by the following succession of steps. Define $\nu(\delta)$ by (6. 50) and then define ν_δ (integral-valued) so that (6. 54) holds. We take $\omega(\delta)$ of the form (6. 52 a) and $\zeta(\delta)$ of the form (6. 53 a)[1]. Let $\delta = \delta(\varepsilon)$ be the greatest number such that $\zeta(\delta) \leqq \varepsilon/(4 h_n)$. One then may write*

$$l(\varepsilon) = \frac{1}{2 r_n} \varepsilon\, \delta^{2n}(\varepsilon) \qquad (r_n \textit{ from } (6.\ 47),\ (6.\ 46)).$$

7. Functions Determined by Values on an Arc.

Let $\mu (\geqq 0)$ be a set-function not necessarily absolutely continuous. Suppose there is a set $G = G(1)$ closed, such that density $\varrho(z)$ of μ is zero in G, and

[1] K_n is introduced subsequent to (6. 52 a).

such that conditions of Theorem (4. 6) hold for this set. Vanishing of density in G implies that

$$\frac{\nu^2}{\pi r_0^2}\mu\left(KS\left(\alpha, \frac{r_0}{\nu}\right)\right) \to 0 \qquad (\text{as } \nu \to \infty;\ \alpha \text{ in } G).$$

Accordingly, what is assumed is the following. *There exists a continuous monotone function* $t(\nu)$ *such that*

(7. 1)
$$t(\nu) = \frac{o(\nu)}{\nu^2}, \qquad \lim_{\nu} o(\nu) = 0,$$

while

(7. 1 a)
$$t(\nu + 1) + t(\nu + 2) + \cdots \leqq \lambda' \frac{o(\nu)}{\nu} \qquad (\nu = 1, 2, \ldots)$$

and

(7. 1 b)
$$\mu\left(KS\left(\alpha, \frac{r_0}{\nu}\right)\right) \leqq t(\nu) \qquad (\alpha \text{ in } G;\ \nu = 1, 2, \ldots).$$

G being closed, in Theorem 4. 6 *we take* $O = K - G$. *The closed sets* H_ν *will be selected as the parts of* O *at distance* $\geqq r_0/\nu$ *from the frontier* O. We have

(7. 2)
$$H_1 < H_2 < \cdots; \qquad \lim_{\nu} H_\nu = O.$$

Function

(7. 2 a)
$$f_\nu(\alpha) = \iint\limits_{H_\nu} \frac{d\mu}{z - \alpha}$$

is analytic in $K - H_\nu$,

(7. 2 b)
$$K - H_1 > K - H_2 > \cdots; \quad K - H_\nu \to G \qquad (\text{as } \nu \to \infty),$$

and it approximates the function

(7. 3)
$$f(\alpha) = \iint\limits_{K} \frac{d\mu}{z - \alpha} = \iint\limits_{O} \frac{d\mu}{z - \alpha}$$

as follows:

(7. 3')
$$|f(\alpha) - f_\nu(\alpha)| = \left| \iint\limits_{O - H_\nu} \frac{d\mu}{z - \alpha} \right| < F_1(\mu(O - H_\nu)) = r(\nu)$$

$$(\alpha \text{ in } G; \quad \nu = 1, 2, \ldots),$$

where $F_1(\mu(X))$ *is defined as stated in Theorem* 4. 6. One may also replace the last member in (7. 3') by certain other expressions which can be easily inferred from the developments of section 4.

Let $\widehat{AB}$ be an arc in G and let α_0 be a point on $\widehat{AB}$ so that for a point ζ_0 (in G) the segment (α_0, ζ_0) is in G, while this segment is limit from both sides of segments (ζ_0, β) (β on $\widehat{AB}$) lying in G. We designate by N_ν the set of points at distance $< r_0/\nu$ from the segment (α_0, ζ_0). It is clear that

$$(7.\ 4) \qquad N_\nu < K - H_\nu.$$

Suppose that for $\nu \geqq \nu_0$ frontier of N_ν intersects $\widehat{AB}$ in unique points A'_ν, B'_ν (A'_ν on $\widehat{A\alpha_0}$, B'_ν on $\widehat{\alpha_0 B}$. Designate by Γ'_ν and Γ''_ν the closed regions (which, except for B'_ν and A'_ν, are in N_ν) bounded by the contours $B'_\nu\,\zeta_0$, $\alpha_0 B'_\nu$ and $A'_\nu\,\zeta_0$, $\alpha_0 A'_\nu$, respectively. *There exists a segment* $\zeta_0\, B_\nu$ (B_ν *on* $\widehat{\alpha_0 B'_\nu}$) *for which the angle*[1] β, ζ_0, α_0 *is maximum, under the condition that the segments* (ζ_0, β) (β *on* $\widehat{\alpha_0 \beta'_\nu}$) *be in* G. Similarly is defined a point A_ν on $A'_\nu \alpha_0$. We designate by Γ_ν the domain, $< N_\nu$, bounded by the contour

$$\Gamma_\nu^{(1)} = B_\nu\, \zeta_0\, A_\nu\, B_\nu.$$

On writing

$$(7.\ 5) \qquad \text{angle } B_\nu\, \zeta_0\, A_\nu = \pi K_\nu \text{ [2]}$$

and on noting that $\Gamma_\nu < N_\nu$ it is inferred that

$$(7.\ 5\text{ a}) \qquad 0 < K_\nu < \frac{a'}{\nu}.$$

We introduce now the function

$$(7.\ 6) \qquad q_\nu(\alpha) = \exp. \{[(\alpha - \zeta_0) \exp. (-\sqrt{-1}\, \varphi_\nu)]^{1/K_\nu}\}$$

($\varphi_\nu =$ angle of the bisector $\zeta_0\, D_\nu$ of the angle $B_\nu\, \zeta_0\, A_\nu$).

A function of this type (but independent of ν) has been previously used by T. Carleman in his important investigations of series of the form

$$\sum_n \frac{b_n}{z - a_n}.$$

Subsequently, this function (in the form (7. 6)) has been employed by W. J. Trjitzinsky in his investigation of »general monogenic» functions[3]. The essential feature in our investigations is that this function varies with K_ν ($\nu = \nu_0, \nu_0 + 1, \ldots$).

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[1] This is an angle forming part of Γ'_ν.

[2] This is the angle forming part of $\Gamma'_\nu + \Gamma''_\nu$.

[3] W. J. Trjitzinsky, *loc. cit.*, section 8.

We have

$$\varphi_{\alpha,\nu} = \left|\text{angle}\left[(\alpha - \zeta_0)\exp.\left(-\sqrt{-1}\,\varphi_\nu\right)\right]^{1/K_\nu}\right| = \frac{\pi}{2} - \frac{1}{K_\nu}\,\text{angle}\,(\alpha_0, \zeta_0 A_\nu) \tag{7.6a}$$

for α on (ζ_0, α_0), provided D_ν is on $\widehat{\alpha_0 B_\nu}$, and

$$\varphi_{\alpha,\nu} = \frac{\pi}{2} - \frac{1}{K_\nu}\,\text{angle}\,(\alpha_0, \zeta_0 B_\nu) \qquad \alpha \text{ on } (\zeta_0, \alpha_0)) \tag{7.6b}$$

when D_ν is on $\widehat{\alpha_0 A_\nu}$.

Let $\omega(\nu)$ (> 0) be defined as a continuous function $(\nu \geqq \nu_0)$ which for integral values $\nu (\geqq \nu_0)$ satisfies

$$\omega(\nu) \leqq \text{least}\left[\text{angle}\,(\alpha_0, \zeta_0 A_\nu);\quad \text{angle}\,(\alpha_0, \zeta_0 B_\nu)\right]. \tag{7.7}$$

In view of (7.5) we then have

$$\frac{a'}{\nu} > K_\nu \geqq \frac{2}{\pi}\,\omega(\nu) \qquad (\nu = \nu_0, \nu_0 + 1, \ldots). \tag{7.8}$$

By (7.7) and (7.6a)

$$\varphi_{\alpha,\nu} \leqq \frac{\pi}{2} - \frac{\omega(\nu)}{K_\nu} \qquad (\alpha \text{ on } (\zeta_0, \alpha_0))$$

and

$$\cos\varphi_{\alpha,\nu} \geqq \cos\left[\frac{\pi}{2} - \frac{\omega(\nu)}{K_\nu}\right] > \frac{2}{\pi}\,\frac{\omega(\nu)}{K_\nu} \tag{7.9}$$

$$(\alpha \text{ on } (\zeta_0, \alpha_0);\ \nu = \nu_0, \nu_0 + 1, \ldots).$$[1]

Consequently, by virtue of (7.6), (7.6a) and (7.9),

$$|q_\nu(\alpha)| > e^{|\alpha - \zeta_0|^{1/K_\nu}\lambda(\nu)} \qquad (\alpha \text{ on } (\zeta_0, \alpha_0)) \tag{7.10}$$

where

$$\lambda(\nu) = \frac{2}{\pi}\,\frac{\omega(\nu)}{K_\nu}. \tag{7.10a}$$

Suppose the function $f(\alpha)$ of (7.3) is zero on $\widehat{AB}$. Then by (7.3')

$$|f_\nu(\alpha)| < r(\nu) \qquad (\alpha \text{ on } \widehat{AB};\ \nu \geqq \nu_0). \tag{7.11}$$

On the other hand, whether $f(\alpha)$ is zero on $\widehat{AB}$ or not, one has

[1] ν_0 sufficiently great.

(7. 12) $$|f_\nu(\alpha)| < h \qquad (\alpha \text{ on } (\zeta_0 A_\nu),\ (\zeta_0 B_\nu)).$$

In fact, when α is on the stated segments α is in G; by Theorem 4. 5

$$|f_\nu(\alpha)| = |\Phi'_\alpha(H_\nu)| < F_1(\mu(H_\nu)) \leqq h \qquad (\nu \geqq \nu_0;\ \alpha \text{ in } G),$$

where h is independent of α and ν.

Form the function

(7. 13) $$T_\nu(\alpha) = f_\nu(\alpha)\, q_\nu^\sigma(\alpha) \qquad (\text{cf. } (7.\,6))$$

with $\sigma(>0)$ at our disposal; $T_\nu(\alpha)$ is analytic in Γ_ν (domain introduced preceding (7. 5)), since

$$\Gamma_\nu < N_\nu < K - H_\nu.$$

By (7. 11) and (7. 6) from (7. 13) it is inferred that

(7. 14) $$|T_\nu(\alpha)| < r(\nu)\exp.(\sigma R^{1/K_\nu}) \qquad [R = \max.\,|\zeta_0 - \beta|\ (\beta \text{ on } \widehat{AB});$$

$$\nu = \nu_0, \nu_0 + 1, \ldots;\ \alpha \text{ on } \widehat{AB}].$$

Now $|q_\nu(\alpha)| \leqq 1$ for α on $(\zeta_0 A_\nu)$, $(\zeta_0 B_\nu)$. Hence, by virtue of (7. 12),

(7. 14 a) $$|T_\nu(\alpha)| < h \qquad (\text{on } (\zeta_0 A_\nu),\ (\zeta_0 B_\nu)).$$

In consequence of the maximum property of analytic functions it is observed that inequalities (7. 14), (7. 14 a) imply

(7. 15) $$|T_\nu(\alpha)| < h + r(\nu)\exp.(\sigma R^{1/K_\nu}) \qquad (\alpha \text{ on } (\zeta_0, \alpha_0));$$

this will hold for $\nu = \nu_0, \nu_0 + 1, \ldots$, inasmuch as (ζ_0, α_0) lies in Γ_ν $(\nu = \nu_0, \nu_0 + 1, \ldots)$. By [7. 15), (7. 13) and (7. 10)

(7. 16) $$|f_\nu(\alpha)| = |T_\nu(\alpha)|\,|q_\nu^{-\sigma}| < \exp.\{-\sigma|\alpha - \zeta_0|^{1/K_\nu}\lambda(\nu)\}\cdot$$

$$\cdot[h + r(\nu)\exp.(\sigma R^{1/K_\nu})] \qquad (\alpha \text{ on } (\zeta_0, \alpha_0)).$$

Restrict α to a sub-interval (ζ^0, α_0) of (ζ_0, α_0) so that

(7. 17) $$\frac{R}{|\alpha - \zeta_0|} \leqq g' \qquad (g' > 1;\ \alpha \text{ on } (\zeta^0, \alpha_0)).$$

Then, with $g = R/g'$,

$$\exp.\,|\alpha - \zeta_0| \geqq \exp.\,g \qquad (\alpha \text{ on } (\zeta^0, \alpha_0)).$$

Whence (7. 16) is seen to imply

$$|f_\nu(\alpha)| < [h + r(\nu)\ \exp.\ (\sigma R^{1/K_\nu})]\ \exp.\ (-\sigma g^{1/K_\nu}\lambda(\nu)) \tag{7. 18}$$

$$(\alpha \text{ on } (\zeta^0, \alpha_0);\ \nu = \nu_0, \nu_0 + 1, \ldots).$$

Designate by $\varphi(\nu)$ (> 1) any function which approaches infinity (when $\nu \to \infty$), however slowly. Define $\sigma = \sigma_\nu$ by the equation

$$\exp.\ (\sigma g^{1/K_\nu}\lambda(\nu)) = \varphi(\nu). \tag{7. 19}$$

With this choice of σ it is observed that the second member in (7. 18) will $\to 0$ (as $\nu \to \infty$), provided

$$r(\nu)\ \exp.\ (\sigma_\nu R^{1/K_\nu}) \leqq B \qquad (\nu = \nu_0, \nu_0 + 1, \ldots), \tag{7. 20}$$

where B is some number independent of ν; that is, (7. 20) would imply that

$$\lim_\nu f_\nu(\alpha) = f(\alpha) = 0 \qquad (\alpha \text{ on } (\zeta^0, \alpha_0)). \tag{7. 21}$$

In view of (7. 19) it is noted that (7. 20) may be written in the form

$$r(\nu) \leqq B\,\varphi(\nu)^{-\lambda'(\nu)}, \quad \lambda'(\nu) = \frac{1}{\lambda(\nu)}(g')^{1/K_\nu}. \tag{7. 22}$$

Condition (7. 22), securing (7. 21), amounts to a requirement that $r(\nu) = F_1(\mu(O - H_\nu))$ (cf. (4. 66) for definition of F_1) should approach zero sufficiently rapidly as $\nu \to \infty$. In view of the definition of F_1 this is seen to be a condition requiring a sufficiently high degree of rarefication of »mass» μ in the neighborhood of the set G. We shall now proceed to replace (7. 22) by a more explicit condition. In view of (7. 22) we shall have (7. 21) whenever

$$r(\nu) \leqq B\,\varphi(\nu)^{-\lambda_0(\nu)} \qquad (\lambda_0(\nu) \geqq \lambda'(\nu);\ \nu \geqq \nu_0). \tag{7. 22 a}$$

Now, by (7. 22), (7. 10 a) and (7. 8)

$$\lambda'(\nu) = \frac{\pi}{2}\frac{K_\nu}{\omega(\nu)}(g')^{1/K_\nu} < \frac{a_0}{\nu\,\omega(\nu)}g_0^{1/\omega(\nu)} = \lambda_0(\nu) \tag{7. 23}$$

$$(\nu \geqq \nu_0;\ a_0 = \pi a'/2;\ g_0 = (g')^{\pi/2};\ \omega(\nu) \text{ from } (7.\ 7)).$$

In an extensive variety of cases the function $\omega(\nu)$ of (7. 7) may be taken of the form

$$\omega(\nu) = \frac{b'}{\nu} \qquad (b' > 0, \text{ independent of } \nu); \tag{7. 24}$$

the function $\lambda_0(\nu)$ of (7. 23) is then of the form

$$\lambda_0(\nu) = b\, b_1^{\nu} \qquad (b_1 > 1;\ b > 0).^1 \tag{7. 24 a}$$

Turning our attention to $r(\nu)$ of (7. 3'), in consequence of Theorem 4. 5 we have

$$r(\nu) = 2\, h_1' \frac{1}{\delta}\, t\left(\frac{r'}{\delta}\right) \qquad (r' = r_0 - \delta_0;\ \delta_0 (> 0) \text{ small}) \tag{7. 25}$$

where $\delta = \delta_\nu$ satisfies the equation

$$\mu(O - H_\nu) = \mu_\nu = h_1'\, t\left(\frac{r'}{\delta}\right) \qquad (\text{cf. } (7.\ 1),\ (7.\ 1\ \text{a})). \tag{7. 25 a}$$

Let $\nu = t_{-1}(u)$ be the inverse of the function $u = t(\nu)$. Then

$$\frac{1}{\delta_\nu} = \frac{1}{r'}\, t_{-1}\left(\frac{\mu_\nu}{h_1'}\right)$$

and, in view of (7. 25 a), it is observed that (7. 22 a) holds (for some B) if

$$\mu_\nu\, t_{-1}\left(\frac{\mu_\nu}{h_1'}\right) \leqq B'\, \varphi(\nu)^{-\lambda_0(\nu)} \qquad (\nu = \nu_0,\ \nu_0 + 1, \ldots). \tag{7. 26}$$

The faster $u = t(\nu) \to 0$ (as $\nu \to \infty$), the slower will $t_{-1}(u) \to \infty$ as $u \to 0$. The first member in (7. 26) will approach zero (as $\nu \to \infty$) whenever $t(\nu)$ vanishes sufficiently rapidly (as $\nu \to \infty$); for example, this will be the case when $t(\nu) = \exp.(-\nu)$. In any case, $\mu_\nu\, t_{-1}(\mu_\nu / h_1')$ cannot tend to zero (as $\nu \to \infty$) as fast or faster than μ_ν. In fact, to be able to satisfy (7. 26) at all, one should have

$$\mu_\nu \leqq B'\, \varphi(\nu)^{-\lambda_0(\nu)}\, \psi(\nu) \qquad (\nu \geqq \nu_0), \tag{7. 27}$$

where $\psi(\nu) (> 0)$ is some function such that $\lim_{\nu} \psi(\nu) = 0$. If (7. 27) holds, (7. 26) will be satisfied provided

$$\psi(\nu)\, t_{-1}\left(\frac{\mu_\nu}{h_1'}\right) \leqq 1;$$

that is, (7. 26) will follow from

$$t_{-1}\left(\frac{\mu_\nu}{h_1'}\right) \leqq \frac{1}{\psi(\nu)} \qquad (\nu \geqq \nu_0). \tag{7. 27 a}$$

Inasmuch as $t(\nu)$ is a monotone function $t_{-1}(u)$ *is monotone increasing* (*as* $u \to 0$);

[1] b, b_1 are expressible in terms of g_0, b', a_0 in an obvious manner.

moreover, $\mu_\nu = \mu(O - H_\nu) \to 0$ monotonically as $\nu \to \infty$. Hence (7. 27 a) will be satisfied if

$$\frac{\mu_\nu}{h_1'} \geqq t\left(\frac{1}{\psi(\nu)}\right) \qquad (\nu \geqq \nu_0).$$

Whence we observe that (7. 26) *and, consequently,* (7. 21) *will hold if*

$$(7.\ 28) \qquad h_1'\, t\left(\frac{1}{\psi(\nu)}\right) \leqq \mu(O - H_\nu) \leqq B'\,\varphi(\nu)^{-\lambda_0(\nu)}\,\psi(\nu) \qquad (\nu = \nu_0, \nu_0 + 1, \ldots),$$

where $\psi(\nu)\,(> 0)$ *is some function such that* $\lim_\nu \psi(\nu) = 0$ ($\lambda_0(\nu)$ *from* (7. 23) *or* (7. 24 a), *as the case may be*).

Examples of set-functions $\mu\,(\geqq 0)$ *can be given so that* (7. 28) *holds as stated.*

It is observed that (7. 28) constitutes a condition regarding rarefication of »mass» μ in the vicinity of the set G. In particular, (7. 28) implies that $t(\nu)$ should approach zero (as $\nu \to \infty$) sufficiently rapidly.

Theorem 7. 1. *Let* $\mu\,(\geqq 0)$ *be not necessarily absolutely continuous. Suppose there exists a closed set* G *in which the density of* μ *is zero, while there exists a continuous monotone function* $t(\nu)$ *such that the statement in connection with* (7. 1), (7. 1 a), (7. 1 b) *holds. Define sets* O *and* H_ν *as stated subsequent to* (7. 1 b). *Let* $\widehat{AB}$ *be an arc in* G *and let* α_0 *be a point on the arc so that, for a point* ζ_0 (*in* G), *the segment* (α_0, ζ_0) *is in* G *and is limit, on both sides, of segments* (ζ_0, β) (β *on* $\widehat{AB}$) *lying in* G. *Let* A_ν *and* B_ν *be points* (*situated on the arc*) *referred to in the italics subsequent* (7. 4). *Let* $\omega(\nu)$ *be the least of the angles* $\alpha_0, \zeta_0 A_\nu$ *and* $\alpha_0, \zeta_0 B_\nu$.

Whenever the function

$$f(\alpha) = \iint_K \frac{d\mu}{z - \alpha}$$

vanishes on $\widehat{AB}$ *it will necessarily also vanish on the sub-interval* (ζ^0, α_0) *satisfying* (7. 17), *of* (ζ_0, α_0), *provided*

$$h_1'\, t\left(\frac{1}{\psi(\nu)}\right) \leqq \mu(O - H_\nu) \leqq B'\,\varphi(\nu)^{-\lambda_0(\nu)}\,\psi(\nu) \qquad (\nu \geqq \nu_0;\ h_1' > 0).$$

Here $\varphi(\nu)\ (> 1) \to \infty$ (*as* $\nu \to \infty$), *however slowly,*

$$\lambda_0(\nu) = \frac{a_0}{\nu\,\omega(\nu)}\, g_0^{1/\omega(\nu)} \qquad (g_0 = (g')^{\pi/2} > 1)$$

and $\lim \psi(\nu) = 0$. *When* $\omega(\nu)$ *is of the form* b'/ν *we may take* $\lambda_0(\nu) = b_0 b_1^\nu$ (*suitable* b, b_1; $b > 0$, $b_1 > 1$).

Note. If $f(\alpha) = 0$ on $\widehat{AB}$ then $f(\alpha)$ will also vanish on certain polygonal lines situated in G, provided that the sides of the lines may be taken in succession in the role of the arc $\widehat{AB}$ of the theorem in such a manner that the conditions of the theorem hold with respect to the next side of the polygon. The set $F(\widehat{AB})$, consisting of all such polygonal lines, whether originating from $\widehat{AB}$ or from any side of the polygonal lines, referred to, is connected in a certain sense. The class of functions $\{f(\alpha)\}$, for which the inequalities of the theorem hold (as stated, with reference to all sides of the polygonal lines in question) possess the property that the members of the class are uniquely determined throughout $F(\widehat{AB})$ by their values on $\widehat{AB}$. Throughout $F(\widehat{AB})$ the functions of the class are quasi analytically continuable in the indicated sense.

With the aid of the developments regarding continuity (cf. section 6) it is possible to obtain theorems of the same type as 7. 1 but with the arc $\widehat{AB}$ replaced by *sets non-dense* (in fact, one may take *non-dense denumerable sets*) *on* $\widehat{AB}$.[1] The methods to be used in establishing such theorems are substantially those of this section and of section 7 of (T).[2] In these pages we shall not go into the detailed development of the indicated procedure.

The conditions obtained in the theorem may be etablished in an essentially different form. For this purpose let us recall again the definition of the sets H_ν, referred to in (7. 2), and consider the difference

$$f(\alpha) - f_\nu(\alpha) = \iint\limits_{O-H_\nu} \frac{d\mu}{z-\alpha} = \sum_{i=1}^{\infty} A_{\nu,i}(\alpha); \tag{7. 29}$$

here

$$A_{\nu,i}(\alpha) = \iint\limits_{H_{\nu,i}} \frac{d\mu}{z-\alpha} \qquad (H_{\nu,i} = H_{\nu+i} - H_{\nu+i-1}). \tag{7. 29 a}$$

One has

$$\frac{r_0}{\nu+i-1} \leqq |z-\alpha| \qquad (\alpha \text{ in } G)$$

for z in $H_{\nu,i}$. Thus

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[1] With $\widehat{AB}$ taken rectifiable and such that length of $\widehat{A_\nu B_\nu} \to 0$, as $\nu \to \infty$.

[2] The latter section deals with the stated problem for the case when the functions under consideration are general monogenic, according to TRJITZINSKY.

$$(7.30)\qquad |A_{\nu,i}(\alpha)| \leqq \frac{\nu+i}{r_0} \iint\limits_{H_{\nu,i}} d\mu = \frac{\nu+i}{r_0}\mu(H_{\nu+i} - H_{\nu+i-1})$$

for α in G. Whence, in consequence of (7.29),

$$(7.31)\qquad |f(\alpha) - f_\nu(\alpha)| \leqq \frac{1}{r_0}\sum_{i=1}^{\infty}(\nu+i)\mu(H_{\nu+i} - H_{\nu+i-1})$$

$$(\nu = \nu_0,\ \nu_0+1,\ \ldots;\ \alpha \text{ in } G).$$

Since $H_{i+1} - H_i < O - H_i$, it is observed that (7.31) implies

$$(7.31\text{ a})\qquad |f(\alpha) - f_\nu(\alpha)| \leqq \frac{1}{r_0}\sum_{i \geqq \nu}(i+1)\mu(H_{i+1} - H_i) = r'(\nu)$$

$$\leqq \frac{1}{r_0}\sum_{i \geqq \nu}(i+1)\mu(O - H_i) = r^*(\nu) \qquad (\alpha \text{ in } G).$$

We now may repeat the argument subsequent to (7.3) replacing $r(\nu)$ of the last member of (7.3′) by the function $r^*(\nu)$ from (7.31 a), thus obtaining the condition

$$(7.32)\qquad r^*(\nu) \leqq B\,\varphi(\nu)^{-\lambda_0(\nu)} \qquad \text{(cf. (7.23))}$$

where $\varphi(\nu)$ is the function referred to in (7.22 a). This requires that $\mu(O - H_\nu)$ should approach zero (as $\nu \to \infty$) rather rapidly. Thus, to begin, it is justifiable to assume that

$$(7.33)\qquad r^*(\nu) \leqq r^0\,\nu\,\mu(O - H_\nu) \qquad (\nu \geqq \nu_0).$$

A condition of the form (7.32) is then satisfied if

$$(7.33\text{ a})\qquad \mu(O - H_\nu) < B'\frac{1}{\nu}\varphi(\nu)^{-\lambda_0(\nu)} \qquad (\nu = \nu_0,\ \nu_0+1,\ \ldots).$$

In (7.33), (7.33 a) one may replace $r^*(\nu)$ by $r'(\nu)$ (cf. (7.31 a)) and $O - H_\nu$ by $H_{\nu+1} - H_\nu$.

Corollary 7.1. *Under conditions of Theorem 7.1 the functions in question will possess the stated uniqueness property also when inequalities (7.28) are replaced by the condition (7.33 a).*

8. Quasi-analyticity in the Ordinary Sense.

Let $\mu\,(\geqq 0)$ denote a set-function, not necessarily absolutely continuous, such that there exists a closed set $G < K$ satisfying the following conditions. We have

(8. 1) $$G < G(n) \qquad (n = 1, 2, \ldots),$$

where $G(n)$ is a set satisfying the conditions of Theorem 4. 5. More precisely, *we assume that there exists a monotone continuous functions* $t(\nu)$ *such that*

(8. 2) $$(\nu + 2)^{n-1}\, t(\nu + 1) + (\nu + 3)^{n-1}\, t(\nu + 2) + \cdots \leqq \lambda' \nu^n t(\nu)$$
$$(\nu = 1, 2, \ldots; n = 1, 2, \ldots),$$

while

(8. 2 a) $$\mu\left(K S\left(\alpha, \frac{r_0}{\nu}\right)\right) \leqq t(\nu) \qquad (\nu = 1, 2, \ldots; \alpha \text{ in } G).$$

It is not difficult to see that, under (8. 2), (8. 2 a), the function

(8. 3) $$f(\alpha) = \iint\limits_K \frac{d\mu}{z - \alpha} = \iint\limits_O \frac{d\mu}{z - \alpha} = \Phi'_\alpha(O) \qquad (O = K - G)^{1}$$

is indefinitely differentiable in G; in fact,

(8. 3 a) $$f^{(n)}(\alpha) = n! \iint\limits_O \frac{d\mu}{(z - \alpha)^{n+1}} = n!\, \Phi_\alpha^{n+1}(O)$$
$$(n = 1, 2, \ldots; \alpha \text{ in } G).$$

As in the preceding section let H_ν denote the part of O at distance $\geqq r_0/\nu$ from the frontier of O; thus $\lim\limits_\nu H_\nu = O$. The sets $K - H_\nu$ will be open and

$$K - H_1 > K - H_2 > \cdots \to G.$$

In $K - H_\nu$ the function

(8. 4) $$f_\nu(\alpha) = \iint\limits_{H_\nu} \frac{d\mu}{z - \alpha}$$

will be analytic. For $n = 0, 1, \ldots$ and for α in G we have

(8. 5) $$f^{(n)}(\alpha) - f_\nu^{(n)}(\alpha) = n! \iint\limits_{O - H_\nu} \frac{d\mu}{(z - \alpha)^{n+1}} = n!\, \Phi_\alpha^{n+1}(O - H_\nu)$$
$$(\alpha \text{ in } G;\ \nu = 1, 2, \ldots).$$

---

[1] In G density of μ will be zero.

Thus, in view of Theorem 4. 5 (in particular, of (4 66)),

(8. 5 a) $$|f^{(n)}(\alpha) - f_\nu^{(n)}(\alpha)| < n!\, F_{n+1}(\mu(O - H_\nu)) = r_n(\nu)$$

$$(\alpha \text{ in } G;\ \nu = \nu_0,\ \nu_0 + 1, \ldots;\ n = 0, 1, \ldots)$$

inasmuch as (8. 1) holds. We shall now investigate the form of $r_n(\nu)$. By Theorem 4. 5 (with n replaced by $n + 1$ and X replaced by $O - H_\nu$)

(8. 6) $$r_n(\nu) = n!\, 2\, h_{n+1}^1 \delta^{-n-1} t\left(\frac{r'}{\delta}\right) \qquad (r' = r_0 - \delta_0;\ \delta_0 (> 0) \text{ small}),$$

where $\delta = \delta_{n,\nu}$ satisfies

(8. 6 a) $$\mu(O - H_\nu) = \mu_\nu = h'_{n+1} t\left(\frac{r'}{\delta}\right)$$

and

(8. 6 b) $$h'_{n+1} = h_{n+1} \lambda_{n+1} r_0^{n+1}, \quad \lambda_{n+1} = K_{n+1} + \lambda'$$

(K_{n+1} from (4. 55 a); λ' from (8. 2)).

From (8. 6 a) we have

(8. 6 c) $$\frac{1}{\delta_{n,\nu}} = \frac{1}{r'} t_{-1}\left(\frac{\mu_\nu}{h'_{n+1}}\right),$$

where t_{-1} is the inverse of the function t. Substituting (8. 6 a) and (8. 6 c) in (8. 6) one obtains

(8. 7) $$r_n(\nu) = m_n \mu_\nu t_{-1}^{n+1}\left(\frac{\mu_\nu}{h'_{n+1}}\right) \qquad (m_n = 2\, n!\, (r')^{-n-1}).$$

Inequalities (8. 5 a) (*with* (8. 7)) *are useful in the study of indefinitely differentiable functions* $f(\alpha)$ *of the form* (8. 3).

Let $S(R, r)$ ($R > 1$, $r < 1$) be the closed region of the following description. From the origin O we draw tangents OA'', OB'' to the circle $|z - 1| = r$; the points A'', B'' on the circumference $|z| = R$. Designate by A', B' the points of tangency with $|z - 1| = r$ of the lines OA'', OB'', respectively. $S(R, r)$ is the connected region containing O and bounded by the greater arc $A''B''$ of $|z| = R$, by the smaller arc $A'B'$ of $|z - 1| = r$ and by the segments $A'A''$, $B'B''$.

The following important result of E. Borel[1] will be needed.

»If one writes the following Mittag-Leffler expansion of $1/(1 - z)$:

(8. 8) $$\frac{1}{1 - z} = \sum_{n=0}^{\infty} G_n(z),$$

---

[1] É. Borel, *Sur les séries de polynomes et de fractions rationnelles*, Acta mathematica, vol. 24, pp. 301—381; particularly see pp. 354—358.

where

(8. 8 a)
$$G_0(z) = g_0(z) = 1, \quad G_n(z) = g_n(z) - g_{n-1}(z)$$
$$= \sum_{i=0}^{\alpha_n} g_{n,\,i}\, z^i \qquad (\alpha_n = n^4 + \cdots + n^{4n})$$

with

(8. 8 b)
$$g_n(z) = \sum_{\lambda_1=0}^{n^4} \sum_{\lambda_2=0}^{n^8} \cdots \sum_{\lambda_n=0}^{n^{4n}} \frac{(\lambda_1 + \cdots + \lambda_n)!}{\lambda_1! \ldots \lambda_n!} \left(\frac{z}{n}\right)^{\lambda_1+\cdots+\lambda_n},$$

it can be asserted that

(8. 9)
$$\sum_{n=0}^{\infty} | G_n(z) | < M(R, r) = R^{[8\,R\,r^{-1}]^{32\,R\,r^{-1}}+2}$$

for z in $S(R, r)$, convergence of the series being uniform.»

The above result of Borel will be used in order to obtain conditions regarding »rarefication» of mass μ under which Mittag-Leffler (for short, M.-L.) development of $f(\alpha)$ will be possible around a point α_0 in G, along lines in G. This, incidentally, would establish quasi-analyticity in the ordinary sense (i. e. unique determination by values of the function and of all of its derivatives at a point).

On writing
$$A_\nu(\alpha) = f_\nu(\alpha) - f_{\nu-1}(\alpha) \qquad (\nu = 1, 2, \ldots; f_0(\alpha) \equiv 0)$$
that is,

(8. 10)
$$A_\nu(\alpha) = \iint\limits_{H_\nu - H_{\nu-1}} \frac{d\mu}{z-\alpha} \qquad (\nu = 1, 2, \ldots; H_0 = 0),$$

one has

(8. 10 a)
$$f(\alpha) = \sum_{\nu=1}^{\infty} A_\nu(\alpha) \qquad (\alpha \text{ in } G).$$

Let α_0 be fixed point in G and suppose there exists a segment (α_0, ζ_0) in G. When α is on (α_0, ζ_0) and z is in $H_\nu - H_{\nu-1}$ (in fact, if z is in H_ν), recalling the definition of H_ν in consequence of certain developments previously given by Trjitzinsky[1], it is concluded that the point

(8. 11)
$$u = \frac{\alpha - \alpha_0}{z - \alpha_0}$$

[1] Cf. Trjitzinsky, *loc. cit.*, section 10; in particular the text in connection with (3), ..., (12).

will lie in $S(R_\nu, r_\nu)$ where, with σ_1, σ_2 denoting suitable constants independent of ν,

$$R_\nu = \sigma_1 \nu, \; r_\nu = \frac{\sigma_2}{\nu} \qquad (\sigma_1, \sigma_2 > 0). \tag{8. 12}$$

We then have

$$\frac{1}{z-\alpha} = \frac{1}{z-\alpha_0}\,\frac{1}{1-u} = \frac{1}{z-\alpha_0}\sum_{n=0}^{\infty} G_n(u)$$

where, in view of Borel's result,

$$\sum_{n=0}^{\infty} |G_n(u)| < M\left(\sigma_1 \nu, \frac{\sigma_2}{\nu}\right) = M_\nu \qquad (\alpha \text{ on } (\alpha_0, \zeta_0),\ z \text{ in } H_\nu). \tag{8. 12 a}$$

Consequently

$$A_\nu(\alpha) = \iint\limits_{H_\nu - H_{\nu-1}} \frac{1}{z-\alpha_0}\sum_{n=0}^{\infty} G_n(u)\,d\mu = \sum_{n=0}^{\infty} H_{n,\nu}(\alpha-\alpha_0), \tag{8. 13}$$

with

$$H_{n,\nu}(\alpha-\alpha_0) = \iint\limits_{H_\nu - H_{\nu-1}} G_n(u)\frac{d\mu}{z-\alpha_0},$$

inasmuch as the second member of (8. 13) is uniformly convergent; (8. 13) constitutes a M.-L. expansion of $A_\nu(\alpha)$ around the point α_0, along the segment (α_0, ζ_0). Since for α_0 and z in the indicated sets

$$|z-\alpha_0| \geqq \frac{r_0}{\nu},$$

we have

$$|H_{n,\nu}(\alpha-\alpha_0)| \leqq \frac{\nu}{r_0}\iint\limits_{H_\nu - H_{\nu-1}} |G_n(u)|\,d\mu$$

and, by (8. 12 a),

$$\sum_n |H_{n,\nu}(\alpha-\alpha_0)| \leqq \frac{\nu}{r_0}\iint\limits_{H_\nu - H_{\nu-1}} \sum_n |G_n(u)|\,d\mu \tag{8. 14}$$

$$< \frac{\nu}{r_0} M_\nu\, \mu(H_\nu - H_{\nu-1}) \qquad (\alpha \text{ on } (\alpha_0, \zeta_0)).$$

If the series

$$S = \sum \nu M_\nu\, \mu(H_\nu - H_{\nu-1}) \qquad (\text{cf. } (8. 12 a)) \tag{8. 15}$$

converges, the double series

$$\sum_{n,\nu} H_{n,\nu}(\alpha - \alpha_0)$$

will be absolutely convergent on (α_0, ζ_0). We then may rearrange terms, obtaining by (8. 10 a) and (8. 13),

(8. 16)
$$f(\alpha) = \sum_{\nu=1}^{\infty} \sum_{n=0}^{\infty} H_{n,\nu}(\alpha - \alpha_0) = \sum_{n=0}^{\infty} \sum_{\nu=1}^{\infty} \cdots$$
$$= \sum_{n=0}^{\infty} H_n(\alpha - \alpha_0) \qquad (\alpha \text{ on } (\alpha_0, \zeta_0))$$

where

(8. 16 a)
$$H_n(\alpha - \alpha_0) = \iint_O G_n(u) \frac{d\mu}{z - \alpha_0} \qquad (\text{from } (8.\ 11))$$

inasmuch as

$$\sum (H_\nu - H_{\nu-1}) = O.$$

By (8. 8 a) and (8. 11) from (8. 16 a) we obtain

$$H_n(\alpha - \alpha_0) = \sum_{i=0}^{\alpha_n} g_{n,i} \left[\iint_O \frac{d\mu}{(z - \alpha_0)^{i+1}} \right] (\alpha - \alpha_0)^i = \sum_{i=0}^{\alpha_n} g_{n,i} \frac{f^{(i)}(\alpha_0)}{i\,!} (\alpha - \alpha_0)^i.$$

Consequently (8. 16) is a M.-L. development of $f(\alpha)$ around α_0 (along (α_0, ζ_0)). Thus, the following fact has been established. *If the series* (8. 15) *converges,* $f(\alpha)$ *may be expanded in a M.-L. series around* α_0, *along* (α_0, ζ_0) *in* G; *we then have* $f(\alpha) = 0$ *on* (α_0, ζ_0), *whenever*

$$0 = f(\alpha_0) = f^{(1)}(\alpha_0) = f^{(2)}(\alpha_0) = \cdots.$$

Using the definition of M_ν given in (8. 12 a), in view of (8. 9) it is concluded that, for a suitable $\sigma (> 0)$,

$$M_\nu < (\sigma_1 \nu)^{(\sigma_1{}^2)\sigma\nu^2} \qquad (\nu \geqq \nu_0).$$

Thus, (8. 15) will converge if

(8. 17)
$$\mu(H_\nu - H_{\nu-1}) \leqq \frac{s_\nu}{\nu}(\sigma_1 \nu)^{-(\sigma\nu^2)\sigma\nu^2} = K(\nu) \qquad (\nu \geqq \nu_0),$$

while

(8. 17 a)
$$\sum s_n \qquad (s_\nu > 0)$$

converges. Now

$$H_\nu - H_{\nu-1} < O - H_{\nu-1}.$$

Hence (8. 17) will hold if

(8. 18) $$\mu(O - H_{\nu-1}) \leqq K(\nu) \qquad (\nu \geqq \nu_0;\ \text{cf. } (8.\ 17)).$$

It is observed that if one takes σ sufficiently great the factor s_ν/ν in $K(\nu)$ may be deleted.

A modified method will be now applied. Suppose $f^{(\nu)}(\alpha_0) = 0\ (\nu = 0, 1, \ldots)$ for a fixed point α_0 in G. As before suppose there exists a segment (α_0, ζ_0) in G. For α in (α_0, ζ_0) in view of (8. 5 a) we shall have

(8. 19) $$|f_\nu^{(n)}(\alpha_0)| < r_n(\nu) \qquad (\nu \geqq \nu_0;\ n = 0, 1, \cdots).$$

A M.-L. expansion of $f_\nu(\alpha)$ can be given by (8. 13) if one replaces the set $H_\nu - H_{\nu-1}$ by H_ν; thus,

(8. 20) $$f_\nu(\alpha) = \iint\limits_{H_\nu} \frac{1}{z - \alpha_0} \sum_{n=0}^{\infty} G_n(u)\, d\mu = \sum_{n=0}^{\infty} H^{n,\nu}(\alpha - \alpha_0)$$

where

(8. 20 a) $$H^{n,\nu}(\alpha - \alpha_0) = \iint\limits_{H_\nu} G_n(u) \frac{d\mu}{z - \alpha_0}.$$

By (8. 8 a)

$$G_0(u) + \cdots + G_n(u) = g_n(u)$$

and, in view of (8. 20),

(8. 21) $$f_\nu(\alpha) = \lim_n P_{n,\nu}(\alpha - \alpha_0),\ P_{n,\nu}(\alpha - \alpha_0) = \iint\limits_{H_\nu} \frac{1}{z - \alpha_0} g_n(u)\, d\mu$$

$$= \sum_{i=0}^{\alpha_n} h_{n,i} \left[\iint\limits_{H_\nu} \frac{d\mu}{(z - \alpha_0)^{i+1}} \right] (\alpha - \alpha_0)^i = \sum_{i=0}^{\alpha_n} h_{n,i} \frac{f_\nu^{(i)}(\alpha_0)}{i!} (\alpha - \alpha_0)^i;$$

here

(8. 21 a) $$h_{n,i} = \text{coefficient of } z^i \text{ in } g_n(z) \qquad (\text{cf. } (8.\ 8\ \text{b})).$$

Formula (8. 21) will certainly hold on (α_0, ζ_0), inasmuch as this segment is in G and every point of it is an interior point of the set $K - H_\nu$ in which $f_\nu(\alpha)$ is analytic.

The Mittag-Leffler coefficients $h_{n,i}$ are really summability factors. One has $\lim\limits_{n} h_{n,i} = 1$ and

$$0 < h_{n,i} < h \qquad (h \text{ independent of } n, i);$$

thus, by (8. 21) and (8. 19),

$$(8.\ 22) \qquad |P_{n,\nu}(\alpha - \alpha_0)| < h \sum_{i=0}^{\alpha_n} r_i(\nu) \frac{\tau^i}{i!}$$

$$(\tau = |\alpha_0 - \zeta_0|;\ \alpha \text{ on } (\alpha_0, \zeta_0) \text{ in } G).$$

The $r_i(\nu)$ are given by (8. 7). In this connection it is noted that in view of (4. 55 a) (cf. (8. 6 b)) one may take

$$K_{n+1} = \frac{1}{n+1} 3^{n+1}.$$

Thus, in (8. 6 b) one may put

$$\lambda_{n+1} = \frac{1}{n+1} g^{n+1} \qquad (\text{suitable } g > 3).$$

Taking the expression for h_{n+1} from (4. 10 a), in consequence of (8. 6 b) it is inferred that one may take

$$(8.\ 23) \qquad h'_{n+1} = g^{n+1}.$$

Whence

$$(8.\ 24) \qquad r_i(\nu) = 2\, i!\, (r')^{-i-1} \mu_\nu t_{-1}^{i+1} \left(\frac{\mu_\nu}{g^{i+1}}\right)$$

so that by virtue of (8. 22) one has

$$(8.\ 25) \qquad |P_{n,\nu}(\alpha - \alpha_0)| < \varrho_{n,\nu} = \frac{2h}{r'} \sum_{i=0}^{\alpha_n} \mu_\nu t_{-1}^{i+1} \left(\frac{\mu_\nu}{g^{i+1}}\right) \left(\frac{\tau}{r'}\right)^i$$

for α on (α_0, ζ_0)[1]

We shall now seek a value $N(\nu)$ such that

$$(8.\ 26) \qquad \sum_{n > N(\nu)} |G_n(u)| < \frac{1}{\nu} \qquad (u \text{ in } S(R_\nu, r_\nu)),$$

where R_ν, r_ν are given by (8. 12). Borel has solved this problem (with R_ν, r_ν replaced by any numbers R ($\geqq 2$), $r > 0$) in the case when the second member in

[1] Inasmuch as $t_{-1}(u) \to \infty$, as $u \to 0$, we have $\lim\limits_{n} \varrho_{n,\nu} = \infty$.

(8. 26) is equal to unity, by utilizing certain results from a memoir of MITTAG-LEFFLER[1]. Following Borel's method[2], with obvious modifications, we obtain

(8. 26a) $$N(\nu) = a_1 \nu^2 \qquad (a_1 > 0, \text{ independent of } \nu).$$

In view of (8. 20), (8. 20a), (8. 21)

(8. 27) $$f_\nu(\alpha) - P_{n,\nu}(\alpha - \alpha_0) = \iint\limits_{H_\nu} \frac{1}{z-\alpha_0} \sum_{m>n} G_m(u)\, d\mu \qquad (u \text{ from } (8.\ 11)).$$

Since in (8. 27) u represents a point in $S(R_\nu, r_\nu)$, in consequence of (8. 26) it is inferred that

(8. 27a) $$|f_\nu(\alpha) - P_{n,\nu}(\alpha - \alpha_0)| \leqq \iint\limits_{H_\nu} \sum_{m>n} |G_m(u)| \frac{d\mu}{|z-\alpha_0|} < \frac{1}{\nu} \iint\limits_{H_\nu} \frac{d\mu}{|z-\alpha_0|}$$

$$< \frac{1}{\nu} \iint\limits_{O} \frac{d\mu}{|z-\alpha_0|} = \frac{K'}{\nu} \qquad (n \geqq a_1 \nu^2)$$

for α on (α_0, ζ_0).

By virtue of (8. 25) and (8. 27a) one has

(8. 28) $$|f_\nu(\alpha)| \leqq |P_{n',\nu}(\alpha - \alpha_0)| + |f_\nu(\alpha) - P_{n',\nu}(\alpha - \alpha_0)|$$

$$< \varrho_{n',\nu} + \frac{K'}{\nu} \qquad (n' = \text{least integer} \geqq a_1 \nu^2).$$

Thus

(8. 28a) $$\lim_\nu f_\nu(\alpha) = f(\alpha) = 0 \qquad (\text{on } (\alpha_0, \zeta_0) \text{ in } G),$$

provided

(8. 29) $$\lim_\nu \varrho_{n',\nu} = 0 \qquad (n' \text{ from } (8.\ 28), (8.\ 26\text{a})).$$

In view of (8. 25) condition (8. 29) will be satisfied if

$$\sum_{i=0}^{\alpha_{n'}} \mu_\nu\, t_{-1}^{i+1}(\mu_\nu g^{-1-\alpha_{n'}})\, \tau_1^{i+1} \to 0 \qquad \left(\text{as } \nu \to \infty;\ \tau_1 = \frac{\tau}{r'}\right)^3,$$

that is, if

$$\mu_\nu [\tau_1 t_{-1}(\mu_\nu g^{-1-\alpha_{n'}})]^{1+\alpha_{n'}} \to 0 \qquad (\text{as } \nu \to \infty).$$

[1] MITTAG-LEFFLER, *Sur la représentation analytique d'une branche uniforme . . .*, Acta mathematica, vol. 23, pp, 43—80.

[2] BOREL, *loc. cit.*, pp. 356—358.

[3] This is so because $g > 1$ and $t_{-1}(\eta)$ increases monotonically as $\eta\ (> 0) \to 0$.

If a (>0, independent of ν) is suitably chosen the latter condition will certainly hold whenever

(8. 30) $\varrho(\nu) = \mu_\nu [\tau_1 t_{-1}(\mu_\nu g^{-c(\nu)})]^{c(\nu)} \to 0$ $[c(\nu) = (a\nu^2)^{4a\nu^2};\ \mu_\nu = \mu(O - H_\nu)]$.

The relation (8. 30) secures quasi-analyticity, just as the inequality (8. 18). The two conditions, though both relating to rarefication in the vicinity of the set G, are of a substantially different form.

There exist set-functions μ ($\geqq 0$), *not identically zero, for which* (8. 30) *is satisfied.*

One may write (8. 30) in the form

$$t_{-1}(\mu_\nu g^{-c(\nu)}) \leqq \frac{1}{\tau_1} [o(\nu)\mu_\nu^{-1}]^{1/c(\nu)} \qquad (o(\nu) \to 0).$$

Now, $\mu_\nu g^{-c(\nu)} \to 0$ (as $\nu \to \infty$) and, thus, the first member will approach infinity; hence the second member must approach infinity. Whence

$$\left(\frac{\mu_\nu}{o(\nu)}\right)^{1/c(\nu)} \leqq o_1(\nu) \qquad (o_1(\nu) \to 0).$$

On the other hand, the preceding inequality may be written as

$$\mu_\nu g^{-c(\nu)} \geqq t\left([o(\nu)\mu_\nu^{-1}]^{1/c(\nu)} \frac{1}{\tau_1}\right).$$

Combining the latter two inequalities one obtains

(8. 30 a) $$g^{c(\nu)} t\left([o(\nu)\mu_\nu^{-1}]^{1/c(\nu)} \frac{1}{\tau_1}\right) \leqq \mu_\nu \leqq o(\nu)[o_1(\nu)]^{c(\nu)}$$

$[c(\nu) = (a\nu^2)^{4a\nu^2};\quad \mu_\nu = \mu(O - H_\nu);\quad \nu \geqq \nu_0;\quad o(\nu) \to 0,\ o_1(\nu) \to 0$ (as $\nu \to \infty$)$]$.

Theorem 8. 1. *Let* $\mu(\geqq 0)$ *be a set-function, not necessarily absolutely continuous. Suppose there exists a monotone continuous function* $t(\nu)$ *such that* (8. 2) *holds, while in a closed set* G *one has* (8. 2 a). *Consider functions*

$$f(\alpha) = \iint\limits_K \frac{d\mu}{z - \alpha} = \iint\limits_O \frac{d\mu}{z - \alpha} \qquad (O = K - G).$$

Let H_ν *denote the part of* O *at distance* $\geqq r_0/\nu$ *from the frontier of* O. *If* α_0 *denotes a point in* G *and the segment* (α_0, ζ_0) *is in* G *then* $f(\alpha)$ *can be expressed,* *on* (α_0, ζ_0), *in terms of the values of the derivatives at* α_0,

$$f^{(\nu)}(\alpha_0) \qquad (\nu = 0, 1, \ldots),$$

with the aid of a convergent Mittag-Leffler *expansion, provided that for* $\nu \geqq \nu_0$

$$\mu(H_\nu - H_{\nu-1}) \leqq \frac{s_\nu}{\nu} (\sigma_1 \nu)^{-(\sigma\nu^2)^{\sigma\nu^2}} \tag{8. 31}$$

$$\left(\sum s_\nu \text{ convergent; } s_\nu > 0; \text{ suitable } \sigma_1 (> 0),\ \sigma(> 0)\right).$$

This condition is also satisfied whenever (8. 18) *holds and it implies quasi-analyticity in the ordinary sense of the corresponding class of functions* $f(\alpha)$.

The latter property is also implied by the inequalities (8. 30 a), *where* $g(> 3)$, $a\ (> 0)$, $\tau_1\ (> 0)$ *are suitable constants and* $o(\nu)(> 0)$, $o_1(\nu)(> 0)$ *are functions approaching zero as* $\nu \to \infty$.

A number of developments along the lines of this section have been previously given in a significant paper by R. Caccioppoli[1]; the results obtained by the latter are essentially different from ours.

9. Determination by Values on Sets of Positive Linear Measure.

We designate by G a closed bounded set, in the α-plane, containing a closed interval I $(0 \leqq \alpha \leqq 2a)$. Let $O(\delta)(\delta > 0)$ denote the set of points at distance $< \delta$ from G. If $\delta_1 > \delta_2 > \cdots$ $(\delta_\nu > 0, \lim\limits_\nu \delta_\nu = 0)$, one has

$$O(\delta_1) > O(\delta_2) > \cdots; \quad O(\delta_\nu) \to G \qquad (\text{as } \nu \to \infty).$$

Suppose $f_\nu(\alpha)$ *is analytic (uniform) in* $O(\delta_\nu)$ *and the sequence* $\{f_\nu(\alpha)\}$ *converges uniformly in* G; *designating by* $f(\alpha)$ *the limiting function, we have*

$$|f(\alpha) - f_\nu(\alpha)| \leqq \varepsilon_\nu \qquad (\text{in } G;\ \lim_\nu \varepsilon_\nu = 0). \tag{9. 1}$$

Let us suppose also that

$$|f_\nu(\alpha)| \leqq S \qquad (\alpha \text{ in } \bar{O}(\delta_{\nu+1});\ \nu = 1, 2, \ldots), \tag{9. 2}$$

where S *is independent of* ν.

Let $D(\varepsilon)$, in the α-plane be the domain so designate at the beginning of section 6. The function

[1] R. Caccioppoli, *Le funzioni monogene generalizate definite mediante integrali doppi di Cauchy*, Rendiconti del Seminaro Mat. della R.U. di Padova (1934); pp. 1—26.

(9. 3) $$w = h(\varepsilon, \alpha) = \frac{\alpha^\sigma - (2a - \alpha)^\sigma}{\alpha^\sigma + (2a - \alpha)^\sigma} \qquad \left(\sigma = \frac{\pi}{2\tau}\right),$$

where $\operatorname{tg} \tau = 2a\varepsilon/(a^2 - \varepsilon^2)$, will map $D(\varepsilon)$ conformally on the interior of the unit circle in the w-plane. The interval $(0, 2a)$ and the points $0, a, 2a$ (in the α-plane) will go into the interval $(-1, 1)$ and the points $-1, 0, 1$ (in the w-plane) respectively.

We shall now examine the conditions under which the class of functions $f(\alpha)$, under consideration, has the *property* (P) consisting in unique determination of the members of the class when the functional values are known on a set Γ of positive linear measure, situated on an interval in G. It will be supposed that Γ is on the interval $(a, 2a)$; this entails no loss of generality. It is convenient to formulate the problem as follows.

With Γ denoting a set of the above description, we wish to find conditions under which vanishing of $f(\alpha)$ on Γ implies vanishing of $f(\alpha)$ on $\Gamma' > \Gamma$, where

(9. 4) $$\text{meas. } \Gamma' > \text{meas. } \Gamma.$$

Thus, suppose that

(9. 5) $$f(\alpha) = 0 \qquad (\alpha \text{ on } \Gamma).$$

Inasmuch as $f(\alpha)$ is continuous, *Γ is to be taken closed.* Whence, on writing

(9. 6) $$I_1 = (a, 2a), \qquad O = I_1 - \Gamma,$$

it is observed that

(9. 6 a) $$O = \sum_{i=1}^{\infty} I(\alpha_i', \alpha_i'') \qquad (I(\alpha_i', \alpha_i'') = (\alpha_i', \alpha_i''); \ \alpha_i' < \alpha_i'')$$

where the non-overlapping intervals (α_i', α_i'') are open[1] and are all in I_1. In so far as meas. $\Gamma > 0$, one has

(9. 6 b) $$\text{meas. } O = \sum_i (\alpha_i'' - \alpha_i') < a.$$

By (9. 1)

(9. 7) $$|f(\alpha) - f_\nu(\alpha)| = |f_\nu(\alpha)| \leqq \varepsilon_\nu \qquad (\alpha \text{ in } \Gamma;\ \nu = 1, 2, \ldots).$$

The function

(9. 8) $$w = h(\delta_{\nu+1}, \alpha) \qquad (\text{cf. (9. 3) with } \varepsilon = \delta_{\nu+1})$$

[1] Except, of course, that in some cases there are extreme semiopen intervals.

transforms $D(\delta_{\nu+1})$ on the interior of the unit w-circle. Now $f_\nu(a)$ is analytic in $O(\delta_\nu) > O(\delta_{\nu+1}) > D(\delta_{\nu+1})$ because $\delta_{\nu+1} < \delta_\nu$ and $(0, 2a)$ is in G; thus, application of the transformation (9. 8) to $f_\nu(a)$ will yield the function

$$f_\nu(a) = F_\nu(w), \tag{9. 8 a}$$

analytic in w for $|w| < 1$. In consequence of the inequality (9. 2) we also have

$$|F_\nu(w)| \leqq S \qquad (|w| < 1;\ \nu = 1, 2, \ldots). \tag{9. 8 b}$$

Γ is on $(a, 2a)$ and whence is carried over by (9. 8) into a set $\Gamma_\nu^{(w)}$ situated on the interval $0 \leqq w \leqq 1$. Similarly O is transformed by (9. 8) into an open set $O_\nu^{(w)}$ of the form

$$O_\nu^{(w)} = \sum_{i=1}^{\infty} I(w_i'(\nu),\ w_i''(\nu)) \qquad (I(w_i'(\nu),\ w_i''(\nu)) = (w_i'(\nu),\ w_i''(\nu))), \tag{9. 9}$$

where the $w_i'(\nu)$, $w_i''(\nu)$ are points in the w-plane corresponding to the points a_i', a_i'' of the a-plane, respectively. Consideration of the form of the function (9. 3) leads one to the conclusion that the following is true for the points just referred to (if one keeps ν fixed). If a_1, a_2 are points of the set $\{a_i', a_i''\}$, such that $a_1 < a_2$, then the corresponding w-points, w_1 and w_2[1], *will also satisfy the inequality* $w_1 < w_2$.

We have

$$\Gamma_\nu^{(w)} + O_\nu^{(w)} = (0, 1). \tag{9. 9 a}$$

In consequence of (9. 7) and (9. 8 a)

$$|F_\nu(w)| \leqq \varepsilon_\nu \qquad (w \text{ in } \Gamma_\nu^{(w)};\ \nu = 1, 2, \ldots). \tag{9. 10}$$

The set $\Gamma_\nu^{(w)}$ is in the interval $0 \leqq w \leqq 1$. One has

$$\text{meas. } \Gamma_\nu^{(w)} > 0; \tag{9. 11}$$

we shall now apply a theorem due to A. Beurling[2], which stated in a restricted form, requisite for our purposes, is as follows.

»Let $F(w)$ be analytic for $|w| < 1$ and suppose

$$|F(w)| \leqq S \qquad (|w| < 1). \tag{9. 12}$$

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[1] Belonging, of course, to the set $\{w_i'(\nu),\ w_i''(\nu)\}$ $(i = 1, 2 \ldots)$.

[2] A. Beurling, *Études sur un problème de majoration,* Thèse, Upsala, 1933 (pp. 1--109).

Designate by E a set of values $0 \leqq r \leqq 1$ for which

(9. 12a) $$\mu(r) = \min. |F| \{ \text{on} (|w| = r) \ (r < 1)\} \leqq \varepsilon \qquad (0 < \varepsilon < S).$$

Then for $|w| \leqq r (0 \leqq r < 1)$ one has

(9. 13) $$|F(w)| \leqq \varepsilon \left(\frac{1}{\varepsilon} S\right)^{p(r)},$$

where

(9. 13a) $$p(r) = \frac{4}{\pi} \operatorname{arc\,tg} \sqrt{\frac{\theta + r}{1 + \theta r}} \qquad (\theta = 1 - \text{meas.}\, E)».$$

We apply this theorem to $F(w) = F_\nu(w)$. In view of (9. 8b) S of (9. 12) will be the number so denoted in (9. 8b). In consequence of (9. 10) one may take

(9. 14) $$E = \Gamma_\nu^{(w)}, \ \varepsilon = \varepsilon_\nu.$$

The theorem will then yield the following result:

(9. 15) $$|F_\nu(w)| \leqq \varepsilon_\nu^{1-p_\nu(r)} S^{p_\nu(r)} = \lambda_\nu(r) \qquad (\text{for } |w| \leqq r < 1),$$

where

(9. 15a) $$p_\nu(r) = \frac{4}{\pi} \operatorname{arc\,tg} \sqrt{\frac{\theta_\nu + r}{1 + r\theta_\nu}} \qquad (\theta_\nu = 1 - \text{meas.}\, \Gamma_\nu^{(w)}),$$

(9. 15b) $$\theta_\nu = \text{meas.}\, O_\nu^{(w)} = \sum_i (w_i''(\nu) - w_i'(\nu)).$$

In particular, (9. 15) will hold on the interval $(-r, r)$ $(0 < r < 1)$.

Going back to the variable α, on taking note of (9. 8a), from (9. 15) we obtain

(9. 16) $$|f_\nu(\alpha)| \leqq \lambda_\nu(r) \qquad (\alpha \text{ on } I(r)),$$

where $I(r)$ is the α-interval corresponding to the w-interval $(-r, r)$. It is convenient to arrange to have $I(r)$ independent of ν; let us say

(9. 16a) $$I(r) = (\eta, 2a - \eta) \qquad (0 < \eta < a),$$

where η is independent of ν and is however small. In (9. 16) we then have

(9. 16b) $$r = r_\nu = -h(\delta_{\nu+1}, \eta) = h(\delta_{\nu+1}, 2a - \eta) = \frac{(2a - \eta)^\sigma - \eta^\sigma}{(2a + \eta)^\sigma - \eta^\sigma},$$

as can be seen by considering (9. 8), (9. 3); here

(9. 16c) $$\sigma = \sigma(\nu) = \frac{\pi}{2\tau(\nu)}, \quad \operatorname{tg} \tau(\nu) = \frac{2a\delta_{\nu+1}}{a^2 - \delta_{\nu+1}^2}.$$

In view of (9. 16) and since $\lim_\nu f_\nu(a) = f(a)$ *it is inferred that, if* (9. 5) *holds, we have*

(9. 17) $$f(a) = 0 \qquad (0 < \eta \leqq a \leqq 2a - \eta),$$

provided

(9. 18) $$\lim_\nu \lambda_\nu(r_\nu) = 0 \qquad \text{(cf. (9. 15), (9. 16b))}.$$

We shall now proceed to find conditions securing (9. 18). It is noted first that, in view of (9. 16c) and inasmuch as $\lim_\nu \delta_{\nu+1} = 0$,

(9. 19) $$\sigma(\nu) = \sigma_0(\nu)\,\delta_{\nu+1}^{-1} \qquad (\sigma' \leqq \sigma_0(\nu) \leqq \sigma''; \ \nu \geqq \nu_0)$$

where $\sigma'(>0)$, $\sigma''(>0)$ are independent of ν. On the other hand, by (9. 16b)

(9. 20) $$r_\nu = \frac{\zeta^{\sigma(\nu)} - 1}{\zeta^{\sigma(\nu)} + 1} \qquad \left(\zeta = \frac{2a}{\eta} - 1 > 1\right).$$

For η small ζ is arbitrarily great. One has

(9. 20a) $$0 < r_\nu < 1, \quad r_\nu \to 1 \qquad (\text{as } \nu \to \infty).$$

Consider now θ_ν of (9. 15b). It is of importance to observe that (9. 6b) holds. A constituent interval (a_i', a_i'') of O is transformed by (9. 8) into a constituent interval $(w_i'(\nu), w_i''(\nu))$ of $O_\nu^{(w)}$. We shall compare the length of the latter interval with that of the first. In consequence of (9. 8) with the aid of a mean value theorem it is inferred that

(9. 21) $$0 < w_i''(\nu) - w_i'(\nu) = h(\delta_{\nu+1}, a_i'') - h(\delta_{\nu+1}, a_i') = h^{(1)}(\delta_{\nu+1}, \bar{a}_i)(a_i'' - a_i'),$$

where

(9. 21a) $$a \leqq a_i' < \bar{a}_i = \bar{a}_{i,\nu} < a_i'' \leqq 2a.$$

Now

(9. 22) $$h^{(1)}(\delta_{\nu+1}, a) = \frac{4a\sigma(\nu)\,a^{\sigma(\nu)-1}(2a - a)^{\sigma(\nu)-1}}{[a^{\sigma(\nu)} + (2a - a)^{\sigma(\nu)}]^2} > 0$$

for $0 < \alpha < 2a$; this function attains its maximum (for $0 < \alpha < 2a$) at $\alpha = a$; from $\alpha = a$ to $\alpha = 2a$ it is monotone diminishing. Hence on account of (9. 21 a) it follows that

$$h^{(1)}(\delta_{\nu+1}, \bar{\alpha}_i) < h^{(1)}(\delta_{\nu+1}, \alpha_i')$$

and, by (9. 21),

$$w_i''(\nu) - w_i'(\nu) < h^{(1)}(\delta_{\nu+1}, \alpha_i')(\alpha_i'' - \alpha_i'). \tag{9. 23}$$

Now, in view of (9. 22),

$$h^{(1)}(\delta_{\nu+1}, \alpha) = \frac{4a\,\sigma(\nu)}{\alpha^2}\,\frac{\gamma^{\sigma(\nu)-1}}{[1+\gamma^{\sigma(\nu)}]^2} \leqq \frac{4\,\sigma(\nu)}{a}\gamma^{\sigma(\nu)-1} \tag{9. 23 a}$$

$$\left(a \leqq \alpha \leqq 2a;\ \gamma = -1 + \frac{2a}{\alpha};\ 0 \leqq \gamma \leqq 1\right).$$

For $a + \zeta' \leqq \alpha \leqq 2a\,(0 < \zeta' < a)$ we have

$$\gamma \leqq \gamma_0 = \frac{a - \zeta'}{a + \zeta'} < 1 \qquad (\gamma_0 \text{ independent of } \nu)$$

and

$$h^{(1)}(\delta_{\nu+1}, \alpha) \leqq \frac{4\,\sigma(\nu)}{a}\gamma_0^{\sigma(\nu)} \qquad (a + \zeta' \leqq \alpha \leqq 2a). \tag{9. 23 b}$$

If Γ contains the interval $(a, a + \zeta')$, in place of (9. 21 a) we shall have

$$a + \zeta' \leqq \alpha_i' < \bar{\alpha}_i = \bar{\alpha}_{i,\nu} < \alpha_i'' \leqq 2a$$

and, by (9. 23), (9. 23 b),

$$\omega_i''(\nu) - \omega_i'(\nu) < \frac{4\,\sigma(\nu)}{a}\gamma_0^{\sigma(\nu)}(\alpha_i'' - \alpha_i');$$

thus, in this case,

$$\theta_\nu = \sum_i (w_i''(\nu) - w_i'(\nu)) < \frac{4\,\sigma(\nu)}{a}\gamma_0^{\sigma(\nu)} \text{meas. } O \qquad (\text{cf. (9. 6 b)}) \tag{9. 24}$$

and, inasmuch as $\gamma_0 < 1$ and $\sigma(\nu) \to \infty$ (as $\nu \to \infty$), we shall have

$$\lim_\nu \theta_\nu = 0. \tag{9. 24 a}$$

Suppose now that Γ contains an interval, say $(a, a + \zeta')$ (with $0 < \zeta' < a$)[1], or deleting this condition note that $\theta_\nu < 1$ (cf. (9. 11))[2]. We shall now obtain an explicit form of the condition (9. 18). By (9. 15 a)

[1] There is no essential loss of generality in this choice of the interval.

[2] Obviously $\theta_\nu \leqq 1$ in any case. It is sufficient for our purposes to have $\theta_\nu < 1$ merely for an infinite subsequence of values ν.

(9. 24 b) $$1 - p_\nu(r_\nu) = \frac{2}{\pi} \text{ arc sin } \frac{1 - \varrho_\nu}{1 + \varrho_\nu} \qquad \left(\varrho_\nu = \frac{\theta_\nu + r_\nu}{1 + r_\nu \theta_\nu}\right).$$

Using the definition of $\lambda_\nu(r)$, given in (9. 15), we write

(9. 25) $$\lambda_1(\nu) = \log\left(\frac{1}{\lambda_\nu(r_\nu)}\right) = (1 - p_\nu(r_\nu)) \log\left(\frac{1}{\varepsilon_{n_\nu}}\right) - p_\nu(r_\nu) \log S.$$

By (9. 20 a) and (9. 15 a)

$$\lim_\nu p_\nu(r_\nu) = 1$$

in any case. Whence, by virtue of (9. 25), it is concluded that (9. 18) will hold if $\lambda_1(\nu) \to \infty$ (as $\nu \to \infty$); that is, if

$$\lambda_2(\nu) = \text{arc sin}\left(\frac{1 - \varrho_\nu}{1 + \varrho_\nu}\right) \log\left(\frac{1}{\varepsilon_{n_\nu}}\right) \to \infty \qquad (\text{as } \nu \to \infty).$$

In so far as $(1 - \varrho_\nu)/(1 + \varrho_\nu) \to 0$ (as $\nu \to \infty$), the above condition is equivalent to the relation

$$\lim_\nu \frac{(1 - \theta_\nu)(1 - r_\nu)}{(1 + \theta_\nu)(1 + r_\nu)} \log\left(\frac{1}{\varepsilon_{n_\nu}}\right) = \infty;$$

that is, to

(9. 26) $$\lim_\nu (1 - \theta_\nu)(1 - r_\nu) \log\left(\frac{1}{\varepsilon_{n_\nu}}\right) = \infty.$$

Here, in view of (9. 24 a), the factor $(1 - \theta_\nu)$ may be replaced by unity, whenever Γ contains an interval, as stated. On taking account of (9. 20) it is observed that (9. 26) may be replaced by the formula

$$\lim_\nu (1 - \theta_\nu) \zeta^{-\sigma(\nu)} \log\left(\frac{1}{\varepsilon_{n_\nu}}\right) = \infty,$$

which may be written as

(9. 27) $$\varepsilon_{n_\nu} \leqq \exp.\left(\frac{-1}{1 - \theta_\nu} \psi(\nu) \zeta^{\sigma(\nu)}\right) \qquad (\nu \geqq \nu_0)$$

$$\left(\psi(\nu) (> 0) \to \infty, \text{ however slowly; } \zeta = -1 + \frac{2a}{\eta} > 1\right).$$

To obtain (9. 17) it is sufficient to have (9. 27) satisfied merely for an infinite subsequence of values ν. By (9. 19) it is sufficient to have

(9. 28) $$\varepsilon_\nu \leqq \exp.\left(\frac{-1}{1-\theta_\nu}\psi(\nu)\,A^{1/\delta_\nu+1}\right) \qquad (\nu \geqq \nu_0;\ A = \zeta^{\nu''} > 1).$$

Theorem 9. 1. *With the notation introduced at the beginning of this section, consider functions* $f(\alpha)$, *defined over a closed set* G, *which are limits of uniformly convergent sequences* $\{f_\nu(\alpha)\}$ *of analytic functions, as stated in connection with* (9. 1) *and* (9. 2). *If* $f(\alpha) = 0$ *on a closed set* Γ, *of positive linear measure, situated on an interval, let us say* $(a, 2a)$ $(a > 0)$ (*the interval* $(0, 2a)$ *lying in* G), *then* $f(\alpha)$ *will necessarily vanish also for*

$$0 < \eta \leqq \alpha \leqq 2a - \eta \qquad (\eta \text{ however small}),$$

provided (9. 28) *holds. In* (9. 28) $\zeta = -1 + 2a/\eta > 1$, $\psi(\nu) > 0$ *and*

$$\psi(\nu) \to \infty \qquad (\text{as } \nu \to \infty),$$

however slowly; $1 - \theta_\nu$ $(0 \leqq \theta_\nu < 1)$ *denotes the measure of the closed set* $\Gamma_\nu^{(w)}$ *obtained by applying the transformation* (9. 8) *to the set* Γ.

In the case when Γ *contains an interval* $(a, a + \zeta')$ $(\zeta' > 0)$ *we have* $\theta_\nu \to 0$ (*as* $\nu \to \infty$) *and one may replace* θ_ν *in* (9. 28) *by zero. We have the inequalities* $\theta_\nu < 1$ $(\nu = \nu_0, \nu_0 + 1, \ldots)$ *in any case.* (This may be established, for instance, with the aid of a known theorem on conformal transformations, applied to linear sets of positive measure.)

Note. This theorem gives a condition under which there is on hand a class of functions possessing the uniqueness property (P), referred to preceding (9. 4). The above developments suggest a method of analytic continuation.

With G *closed,* $G < \overline{K}$, *write* $O = K - G$. *As in section 7 let us define* H_ν *as the part of* O *at distance* $\geqq r_0/\nu$ *from the frontier of* O. *We shall have* (7. 2). *Let* $\mu\,(\geqq 0)$ *be a set-function not necessarily absolutely continuous and consider functions of the form*

(9. 29) $$f(\alpha) = \iint_K \frac{d\mu}{z - \alpha} = \iint_O \frac{d\mu}{z - \alpha} = \Phi_\alpha(O),$$

the density of μ *being zero in* G; that is, functions considered in section 7. On writing

(9. 29 a) $$f_\nu(\alpha) = \Phi_\alpha(H_\nu),$$

we obtain

$$f(\alpha) - f_\nu(\alpha) = \Phi_\alpha(O - H_\nu)$$

and, by (7. 31 a),

(9. 30) $$|f(\alpha) - f_\nu(\alpha)| \leqq r'(\nu) = \varepsilon_\nu \qquad (\nu \geqq \nu_0;\ \alpha \text{ in } G).$$

On the other hand,

(9. 31) $$|f_\nu(\alpha)| \leqq \iint\limits_{H_\nu} \frac{d\mu}{|z-\alpha|} < \frac{\nu^2}{r_0} \iint\limits_{H_\nu} d\mu < r_1 \nu^2 \qquad \left(r_1 = \frac{1}{r_0}\mu(K)\right)$$

for α in $\overline{K - H_{\nu+1}}$, inasmuch as $|z - \alpha| \geqq r_0/\nu - r_0/(\nu+1)$ (z in H_ν).[1]

We may now repeat the reasoning made earlier in this section, with ε_ν of (9. 1) replaced by $r'(\nu)$ of (9. 30). However, S of (9. 2) will now depend on ν,

(9. 32) $$S = S_\nu = r_1 \nu^2,$$

and we shall write

(9. 33) $$O(\delta_\nu) = K - H_\nu, \quad \delta_\nu = \frac{r_0}{\nu}.$$

The notation (9. 33) is consistent with that introduced preceding (9. 1), if the convention is made that the set G, now under consideration, contains the frontier of K.[2] Suppose that there exists an interval — which without any loss of generality may be taken as (0, $2a$) ($a > 0$) — in the part of G exclusive of the frontier of K. It is then inferred that, if $f(\alpha) = 0$ on Γ,[3] necessarily

(9. 34) $$f(\alpha) = 0 \qquad (0 < \eta \leqq \alpha \leqq 2a - \eta),$$

provided

(9. 35) $$\lim_\nu \lambda_\nu(r_\nu) = 0 \qquad \text{(cf. statement with (9. 17), (9. 18)).}$$

Here, in virtue of (9. 15) and (9. 32),

$$\lambda_\nu(r) = \varepsilon_\nu^{1-p_\nu(r)}\, r_1^{p_\nu(r)}\, \nu^{2p_\nu(r)}$$

and $r(\nu)$ $\left(\text{cf. (9. 16 b), (9. 16 c) with } \delta_{\nu+1} = \dfrac{r_0}{\nu+1}\right)$ is of the form

(9. 36) $$r_\nu = \frac{\zeta^{\sigma(\nu)} - 1}{\zeta^{\sigma(\nu)} + 1} \qquad \left(\zeta = \frac{2a}{\eta} - 1 > 1\right),$$

[1] The statement in connection with (7. 2 a), (7. 2 b) will apply.

[2] This is not a very essential point.

[3] Γ a set of the type involved in (9. 5).

where

(9. 36 a) $$\sigma(\nu) = \sigma_0(\nu)\,\nu \qquad (\sigma' \leqq \sigma_0(\nu) \leqq \sigma'';\ \nu \geqq \nu_0);$$

moreover (cf. (9. 15 a), (9. 24 b))

$$p_\nu(r_\nu) = \frac{4}{\pi} \text{ arc tg } \sqrt{\varrho_\nu}, \quad 1 - p_\nu(r_\nu) = \frac{2}{\pi} \text{ arc sin } \left(\frac{1-\varrho_\nu}{1+\varrho_\nu}\right) \quad \left(\varrho_\nu = \frac{\theta_\nu + r_\nu}{1 + r_\nu \theta_\nu}\right)$$

with θ_ν having the same significance as before. Thus, if $\theta_\nu < 1$ $(\nu \geqq \nu_0)$, (9. 35) will be satisfied if

$$\log\left(\frac{1}{\lambda_\nu(r_\nu)}\right) = (1 - p_\nu(r_\nu)) \log\left(\frac{1}{\varepsilon_\nu}\right) + p_\nu(r_\nu) \log\left(\frac{1}{r_1}\right) - 2\, p_\nu(r_\nu) \log \nu \to \infty$$

(as $\nu \to \infty$). Now $p_\nu(r_\nu) \to 1$ and $1 - p_\nu(r_\nu)$ is of the order of

$$\frac{2}{\pi}\,\frac{1-\varrho_\nu}{1+\varrho_\nu} = \frac{2}{\pi}\,\frac{(1-\theta_\nu)(1-r_\nu)}{(1+\theta_\nu)(1+r_\nu)}.$$

Whence it is observed that it is sufficient to have

$$\frac{1}{\pi}\,\frac{(1-\theta_\nu)(1-r_\nu)}{1+\theta_\nu} \log\left(\frac{1}{\varepsilon_\nu}\right) - 2 \log \nu \to \infty.$$

Furthermore, it is observed that by (9. 36) and (9. 36 a)

(9. 37) $$\frac{1-r_\nu}{1+\theta_\nu} \geqq \frac{1}{2}(1 - r_\nu) = \frac{1}{\zeta^{\sigma(\nu)} + 1} \geqq \lambda_1 \zeta^{-\sigma(\nu)} \geqq \lambda_1 \zeta^{-\sigma''\nu} \qquad (\lambda_1 > 0).$$

Hence (9. 35) will hold if

$$\frac{1}{2\pi} \lambda_1 (1 - \theta_\nu) \zeta^{-\sigma''\nu} \log\left(\frac{1}{\varepsilon_\nu}\right) - \log \nu \to \infty.$$

The latter condition can be expressed by saying that the first member above is $\geqq \psi(\nu)$ (> 0) where $\psi(\nu) \to \infty$, however slowly; that is,

$$\log\left(\frac{1}{\varepsilon_\nu}\right) \geqq \frac{2\pi}{\lambda_1}\,\frac{1}{1-\theta_\nu} \zeta^{\sigma''\nu} (\psi(\nu) + \log \nu) \qquad (\nu \geqq \nu_0)$$

or

$$r'(\nu) = \varepsilon_\nu \leqq \exp.\left\{-\lambda_0 \frac{1}{1-\theta_\nu} A^\nu(\psi(\nu) + \log \nu)\right\} \tag{9.38}$$

$$\left(\lambda_0 = \frac{2\pi}{\lambda_1} \text{ (cf. (9. 37)); } \sigma'' \text{ from (9. 36 a); } A = \zeta^{\sigma''} > 1;\ \nu \geqq \nu_0\right).$$

In view of the definition of $r'(\nu)$, given in (7. 31 a), $\mu(H_{\nu+1} - H_\nu)$ should approach zero (as $\nu \to \infty$) rather rapidly. Thus we may assume first that

$$\frac{1}{r_0} \sum_{i \geqq \nu} (i+1)\mu(H_{i+1} - H_i) \leqq \frac{1}{K_0} \nu \mu(H_{\nu+1} - H_\nu) \quad (K_0 > 0;\ \nu \geqq \nu_0). \tag{9.39}$$

Accordingly it is observed that (9. 38) will take place if

$$\mu(H_{\nu+1} - H_\nu) \leqq \frac{K_0}{\nu} \exp.\left\{-\frac{\lambda_0}{1-\theta_\nu} A^\nu(\psi(\nu) + \log \nu)\right\} = h(\nu) \tag{9.40}$$

for $\nu \geqq \nu_0$.

Theorem 9. 2. *Let G and O be sets and let $f(\alpha)$ be a function as described in the italics preceding to and in connection with* (9. 29). *Suppose G contains an interval Δ. Without any loss of generality one may take $\Delta = (0, 2a)$ $(a > 0)$. Let Γ be a closed set of positive linear measure, situated on the interval $(a, 2a)$.*

If $f(\alpha) = 0$ on Γ, then necessarily $f(\alpha) = 0$ for $0 < \eta \leqq \alpha \leqq 2a - \eta$ (η however small), provided the rarefication of »mass» is such as specified by (9. 40). *In* (9. 40) $K_0 > 0$, $\lambda_0 = \frac{2\pi}{\lambda_1}$, $\psi(\nu)$ $(> 0) \to +\infty$ *however slowly; $\lambda_1 > 0$ is from* (9. 37);

$$A = \zeta^{\sigma''} > 1 \qquad \left(\zeta = \frac{2a}{\eta} - 1;\ \sigma'' \textit{ from } (9.\ 36\ a)\right).$$

The θ_ν in (9. 40) *have the same significance as in Theorem* 9. 1 *and these numbers have the same bearing in connection with the stated result as indicated at the end of Theorem* 9. 1.

Note. This Theorem furnishes a condition of rarefication of »mass» under which there is on hand a class of functions

$$f(\alpha) = \iint_K \frac{d\mu}{z - \alpha}$$

with the uniqueness property (P) (referred to earlier). It is clear that the condition (9. 40) will insure this property on every interval $\Delta < G$, provided that certain obvious conditions are satisfied by the constants involved in (9. 40).

10. **Applications to Functions of the Form** (1. 3).

The results of the preceding sections can be conveniently applied to functions represented by series (1. 3). Thus, we consider functions

$$S(\alpha) = \sum \frac{b_\nu}{\alpha_\nu - \alpha} \qquad \left(\sum |b_\nu| \text{ convergent}\right), \tag{10. 1}$$

where the b_ν may be complex, $b_\nu = b'_\nu + \sqrt{-1}\, b''_\nu$. We thus have

$$S(\alpha) = S_1(\alpha) + \sqrt{-1}\, S_2(\alpha), \quad S_1 = \sum \frac{b'_\nu}{\alpha_\nu - \alpha}, \quad S_2 = \sum \frac{b''_\nu}{\alpha_\nu - \alpha}; \tag{10. 2}$$

$$S_1 = S_{1,1} - S_{1,2}, \quad S_{1,1} = \sum \frac{a_\nu^{1,1}}{\beta_{\nu,1} - \alpha}, \quad S_{1,2} = \sum \frac{a_\nu^{1,2}}{\beta_{\nu,2} - \alpha};$$

$$S_2 = S_{2,1} - S_{2,2}, \quad S_{2,1} = \sum \frac{a_\nu^{2,1}}{\gamma_{\nu,1} - \alpha}, \quad S_{2,2} = \sum \frac{a_\nu^{2,2}}{\gamma_{\nu,2} - \alpha},$$

where

$$a_\nu^{1,1} \geqq 0, \quad a_\nu^{1,2} \geqq 0, \quad a_\nu^{2,1} \geqq 0, \quad a_\nu^{2,2} \geqq 0 \tag{10. 2 a}$$

and the series

$$\sum_\nu a_\nu^{i,j} \qquad (i, j = 1, 2) \tag{10. 2 b}$$

converge. Conversely, inasmuch as $|b_\nu| \leqq |b'_\nu| + |b''_\nu|$, it is inferred without difficulty that convergence of the four series (10. 2 b) implies convergence of

$$\sum |b_\nu|. \tag{10. 3}$$

Consider functions of the form

$$g(\alpha) = \sum \frac{a_\nu}{\beta_\nu - \alpha} \qquad \left(a_\nu > 0;\ \sum a_\nu \text{ convergent}\right), \tag{10. 4}$$

where as a matter of convenience, entailing no essential loss of generality, the β_ν are assumed to be in K (the bounded domain so designated throughout this work). As a singular integral $g(\alpha)$ of (10. 4) may be expressed as follows:

$$g(\alpha) = \iint_K \frac{d\vartheta}{z - \alpha} \qquad (z \text{ variable of integration}). \tag{10. 5}$$

Here $\vartheta = \vartheta(X)$ $(X < K)$ is a singular set-function such that, for sets $X < K$, we have

(10. 5 a) $$\vartheta(X) = \vartheta(X E_0) = a_{\nu_1} + a_{\nu_2} + \cdots \qquad (\geqq 0),$$

with E_0 denoting the set of points $\{\beta_\nu\}$ $(\nu = 1, 2, \ldots)$ and $(\beta_{\nu_1}, \beta_{\nu_2}, \ldots)$[1] denoting the totality of points in $X E_0$.

Definition 10. 1. *With X denoting a set $< K$, the expression*

(10. 6) $$\mathfrak{S}(X)\, a_\nu$$

is to be formed as follows. Let $\beta_{\nu_1}, \beta_{\nu_2}, \ldots$ denote the totality of all points β_ν (from the series (10. 4)) which are in X; the expression (10. 6) will stand for the sum

(10. 6 a) $$a_{\nu_1} + a_{\nu_2} + \cdots \qquad (\textit{the } a_\nu \textit{ from } (10.\,4)).$$

The following relationship is immediate:

(10. 7) $$\vartheta(X) = \mathfrak{S}(X)\, a_\nu.$$

Also, with the series (10. 1) in view, one may write

(10. 7 a) $$\mathfrak{S}(X)\,|b_\nu| = |b_{\nu_1}| + |b_{\nu_2}| + \cdots,$$

where $\nu_1, \nu_2, \ldots$ are subscripts of all the points α_ν which are in X. The following inequalities will hold:

(10. 8) $$\mathfrak{S}(X)\, a_\nu^{i,j} \leqq \mathfrak{S}(X)\,|b_\nu| \qquad (i, j = 1, 2).$$

It will be sufficient to demonstrate this for $i = j = 1$. We observe that (10. 7 a) will hold as stated. In the series $S_{1,1}$ of (10. 2) the $\beta_{\nu,1}$ represent certain points which constitute a subset of the points α_ν. Of the $\beta_{\nu,1}$ certain ones, say

(10. 8 a) $$\beta_{m_1, 1},\ \beta_{m_2, 1}, \ldots,$$

will be in X; the points (10. 8 a) will constitute a subset of the points $\alpha_{\nu_1}, \alpha_{\nu_2}, \ldots$. Whence the numbers

$$a_{m_1}^{1,1},\quad a_{m_2}^{1,1}, \ldots$$

will form a subset of the real parts of the numbers

$$b_{\nu_1},\quad b_{\nu_2}, \ldots$$

[1] Of course, sequences $(\nu_1, \nu_2, \ldots)$ depend on sets X.

and, accordingly,

$$\mathfrak{S}(X)\,a_\nu^{1,1} = a_{m_1}^{1,1} + a_{m_2}^{1,1} + \cdots \leqq |\Re\, b_{\nu_1}| + |\Re\, b_{\nu_2}| + \cdots$$
$$\leqq |b_{\nu_1}| + |b_{\nu_2}| + \cdots = \mathfrak{S}(X)\,|b_\nu|,$$

which is the desired inequality.

Conversely, since $|b_\nu| \leqq |b_\nu^1| + |b_\nu^{1,1}|$,

(10. 9) $$\mathfrak{S}(X)\,|b_\nu| \leqq \mathfrak{S}(X)\,|b_\nu^1| + \mathfrak{S}(X)\,|b_\nu^{1,1}|.$$

Now

$$\mathfrak{S}(X)\,|b_\nu^1| = \mathfrak{S}(X)\,a_\nu^{1,1} + \mathfrak{S}(X)\,a_\nu^{1,2}$$

and there is a similar relation for the last term in (10. 9). Thus

(10. 10) $$\mathfrak{S}(X)\,|b_\nu| \leqq \mathfrak{S}(X)\,a_\nu^{1,1} + \mathfrak{S}(X)\,a_\nu^{1,2} + \mathfrak{S}(X)\,a_\nu^{2,1} + \mathfrak{S}(X)\,a_\nu^{2,2}.$$

In consequence of Theorem 3. 1 the following result may be stated.

(I.) *The series* (10. 1) *converges for every value α, in K, for which the series*

(10. 11) $$\sum_{m=1}^{\infty} \mathfrak{S}\left(K\,S\left(\alpha,\ \frac{r_0}{m}\right)\right)|b_\nu|$$

converges[1].

In fact, if (10. 11) converges then in view of (10. 8) the four series

(10. 11 a) $$\sum_m \mathfrak{S}\left(K\,S\left(\alpha,\ \frac{r_0}{m}\right)\right)a_\nu^{i,j} \qquad (i, j = 1, 2)$$

will also converge. Now, by (10. 2),

$$S(\alpha) = (S_{1,1} - S_{1,2}) + \sqrt{-1}\,(S_{2,1} - S_{2,2})$$

and, by virtue of the statement made in connection with (10. 5)

(10. 12) $$S_{i,j}(\alpha) = \iint_K \frac{d\,\vartheta^{i,j}}{z-\alpha} \qquad (\vartheta^{i,j} \geqq 0,\ \text{singular set-function}),$$

where $\vartheta^{i,j}$ is formed as indicated in (10. 5 a). In view of (10. 7) the series (10. 11 a) may be written in the form

[1] Here $S(\alpha, \varrho)$ denotes a circle of center α and radius ϱ.

(10. 13) $$\sum_m \vartheta^{i,j}\left(KS\left(\alpha, \frac{r_0}{m}\right)\right) \qquad (i, j = 1, 2).$$

Applying Theorem 3. 1 it is accordingly inferred that the series $S_{i,j}(\alpha)$ $(i, j=1, 2)$ all converge for every α for which (10. 11) converges; the same will be true for $S(\alpha)$. Thus the truth of the statement in connection with (10. 11) has been verified.

By (10. 10)

(10. 14) $$\sum_m \mathfrak{S}\left(S\left(\alpha, \frac{r_0}{m}\right)\right)|b_\nu| \leqq \Gamma_{1,1}(\alpha) + \Gamma_{1,2}(\alpha) + \Gamma_{2,1}(\alpha) + \Gamma_{2,2}(\alpha)$$

where

(10. 14 a) $$\Gamma_{i,j} = \sum_m \mathfrak{S}\left(S\left(\alpha, \frac{r_0}{m}\right)\right) a_\nu^{i,j} \qquad (i, j = 1, 2)^1$$

provided the four series $\Gamma_{i,j}$ converge; in the latter case, of course, the series (10. 11) will converge. In view of the formula (10. 7)

(10. 15) $$\Gamma_{i,j} = \sum_m \vartheta^{i,j}\left(S\left(\alpha, \frac{r_0}{m}\right)\right) = \sum_m \frac{\pi r_0^2}{m^2}\varrho^{i,j}\left(\alpha, \frac{r_0}{m}\right).$$

Here $\varrho^{i,j}(\alpha, r_0/m)$ is an expression which, for $m \to \infty$, will approach $\varrho^{i,j}(\alpha)$, the (symmetric) density at α of $\vartheta^{i,j}$, almost everywhere; moreover, inasmuch as the $\vartheta^{i,j}$ are singular set-functions, the densities $\varrho^{i,j}(\alpha)$ will be zero almost everywhere. This is a consequence of a fundamental result of LEBESGUE. Thus, there exists a set E_0 of zero measure so that the series $\Gamma_{i,j}$ $(i, j = 1, 2)$ all converge in $K - E_0$. In view of (10. 14) the series (10. 11) will also converge in $K - E_0$. Whence we have proved anew the following known classical result.

(II.) *Series* (10. 1) *converge almost everywhere.*

We have obtained more than what is implied by the above statement; that is, there is on hand some information as to the location of the points of convergence; in consequence of (I.) the series (10. 1) converges at α provided

(10. 16) $$\mathfrak{S}\left(S\left(\alpha, \frac{r_0}{m}\right)\right)|b_\nu| \to 0 \qquad (\text{as } m \to \infty)$$

sufficiently fast. It is of interest to note that (10. 1) will converge even at

[1] For m sufficiently great $KS\left(\alpha, \frac{r_0}{m}\right) = S\left(\alpha, \frac{r_0}{m}\right)$ if α is in K.

points α at which the four densities $\varrho^{i,j}(\alpha)$ of the set-functions $\vartheta^{i,j}$ are infinite, provided that

$$\varrho^{i,j}\left(\alpha, \frac{r_0}{m}\right) \to \infty \qquad \text{(as } m \to \infty\text{)}$$

not too fast; a precise statement regarding this fact is an immediate consequence of (I.).

With the aid of Theorem (3. 2) one may study the problem of diffirentiability of series (10. 1). We shall not go into the formulation of the results of this type.

Regarding the degree of continuity the following can be established with the aid of Theorem 6. 3.

(III.) *Suppose there is a set* $G < K$ *so that for some function* $\tau(\nu)$ (> 0), *independent of* α,

$$\mathfrak{S}\left(S\left(\alpha, \frac{r_0}{m}\right)\right) | b_\nu | \leqq \tau(m) \qquad (m = 1, 2, \ldots; \ \alpha \ in\ G), \tag{10. 17}$$

while the series

$$\sum \tau(m) \tag{10. 17 a}$$

converges. Then, given ε $(> 0$, *however small), we shall have*

$$| S(\alpha_0) - S(\alpha) | < \varepsilon \qquad (\alpha, \alpha_0 \ in\ G) \tag{10. 18}$$

whenever

$$| \alpha - \alpha_0 | \leqq l\left(\frac{\varepsilon}{4}\right); \tag{10. 18 a}$$

$l(\varepsilon)$ $(\to 0$, *as* $\varepsilon \to 0)$ *is the function so denoted in Theorem* 6. 3 *and essentially it depends only on* $\tau(\nu)$.[1]

In fact, in view of (10. 8) and (10. 7), it is observed that (10. 17) implies

$$\vartheta^{i,j}\left(K\,S\left(\alpha, \frac{r_0}{m}\right)\right) \leqq \tau(m) \qquad (m = 1, 2, \ldots; \ \alpha \text{ in } G) \tag{10. 19}$$

for $i, j = 1, 2$. These inequalities enable application of Theorem 6. 3 to each of the four functions (10. 12). Thus

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[1] We may take $l(\varepsilon) = \varepsilon\,\delta^2(\varepsilon)/(2\,r_1)$, where $\delta(\varepsilon)$ is defined as follows. Let ν' be the greatest integer $\leqq r^0/(2\,\delta)$ and ν_δ be the greatest integer $\leqq \tau(\nu') - \gamma$ $(0 < \gamma < 1)$. Form the function $\zeta(\delta) = K_1 \nu_\delta \tau(\nu') + \tau(\nu_\delta + 1) + \tau(\nu_\delta + 2) + \cdots$; $\delta(\varepsilon)$ is such that $\zeta(\delta(\varepsilon)) \leqq \varepsilon/(4\,h_1)$. Other ways of defining $l(\varepsilon)$ can be inferred from the considerations of section 6.

(10. 20) $$|S_{i,j}(\alpha_0) - S_{i,j}(\alpha)| < \frac{\varepsilon}{4} \qquad (\alpha,\ \alpha_0 \text{ in } G)$$

for $|\alpha - \alpha_0| \leqq l(\varepsilon/4)$. Now

$$S(\alpha_0) - S(\alpha) = (w_{1,1} - w_{1,2}) + \sqrt{-1}\,(w_{2,1} - w_{2,2}),$$

where

$$w_{i,j} = S_{i,j}(\alpha_0) - S_{i,j}(\alpha).$$

Consequently, by virtue of (10. 20) it is concluded that

$$|S(\alpha_0) - S(\alpha)| \leqq |w_{1,1}| + |w_{1,2}| + |w_{2,1}| + |w_{2,2}| < \varepsilon$$

for α, α_0 in G, provided (10. 18 a) is satisfied.

Sections 7, 8, 9, regarding uniqueness properties, may be applied with the purpose of investigating such properties for functions of the form (10. 1). We shall confine ourselves to the application of section 9. Of interest for our present purpose is Theorem 9. 2. The following result is deduced.

(IV.) *Let G be a closed set, $G < K$. Write $O = K - G$ and define H_ν as the part of O at distance $\geqq r_0/\nu$ from the frontier of O. Suppose G contains an interval Δ, say $\Delta = (0,\ 2a)$ $(a > 0)$. Designate by Γ a closed set, linear meas. $\Gamma > 0$, situated on $(a,\ 2a)$. Noting that the θ_m of Theorem 9. 2 satisfy the inequalities $\theta_m < 1$, it is observed that the class of functions (10. 1) for which*

(10. 21) $$\mathfrak{S}(H_{m+1} - H_m)|b_\nu| \leqq h(m) \qquad (h(m) \text{ from } (9.40);\ m \geqq m_0)$$

will possess the property that its members will be uniquely determined on

$$(0 < \eta \leqq \alpha \leqq 2a - \eta)$$

by their functional values on Γ.

In fact, in consequence of (10. 8) and (10. 7), it is deduced that (10. 21) implies

(10. 22) $$\vartheta^{i,j}(H_{m+1} - H_m) \leqq h(m) \qquad (m \geqq m_0)$$

for $i, j = 1, 2$. Accordingly the conditions of Theorem 9. 2 will be satisfied for each of the four functions (10. 12); whence the $S_{i,j}(\alpha)$ $(i, j = 1, 2)$ will have the desired property and the same will be true for $S(\alpha)$.

Following the indicated lines, many further applications of the results of this work to series of the form (10. 1) can be carried out.