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Picard potentials and Hill's equation on a torus

by

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1. Introduction

Hill's equation has drawn an enormous amount of consideration due to its ubiquity in applications as well as its structural richness. Of particular importance in the last 20 years is its connection with the KdV hierarchy and hence with integrable systems.

We show in this paper that regarding the independent variable as a complex variable yields a breakthrough for the problem of an efficient characterization of all elliptic finite-gap potentials, a major open problem in the field. Specifically, we show that elliptic finite-gap potentials of Hill's equation are precisely those for which all solutions for all spectral parameters are meromorphic functions in the independent variable, complementing a classical theorem of Picard. The intimate connection between Picard's theorem and elliptic finite-gap solutions of completely integrable systems is established in this paper for the first time.

In addition, we construct the hyperelliptic Riemann surface associated with a finite-gap potential (not necessarily elliptic), i.e., determine its branch and singular points from a comparison of the geometric and algebraic multiplicities of eigenvalues of certain auxiliary operators associated with Hill's equation. These multiplicities are intimately correlated with the pole structure of the diagonal Green's function of the operator $H=d^2/dx^2+q(x)$ in $L^2(\mathbf{R})$. Our construction is new in the present general complex-valued periodic finite-gap case.

Before describing our approach in some detail, we shall give a brief account of the history of the problem involved. This theme dates back to a 1940 paper of Ince [43] who studied what is presently called the Lamé–Ince potential

$$q(x)=-g(g+1)\mathcal{P}(x+\omega_3),\quad g\in\mathbf{N},\ x\in\mathbf{R},\tag{1.1}$$

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in connection with the second-order ordinary differential equation

$$\psi''(E,x)+q(x)\psi(E,x)=E\psi(E,x), \quad E\in\mathbf{C}. \tag{1.2}$$

Here $\mathcal{P}(x):=\mathcal{P}(x;\omega_1,\omega_3)$ denotes the elliptic Weierstrass function with fundamental periods $2\omega_1$ and $2\omega_3$ ($\mathrm{Im}(\omega_3/\omega_1)\neq 0$). In the special case where ω_1 is real and ω_3 is purely imaginary the potential $q(x)$ in (1.1) is real-valued and Ince's striking result [43], in modern spectral theoretic terminology, yields that the spectrum of the unique self-adjoint operator associated with the differential expression $L=d^2/dx^2+q(x)$ in $L^2(\mathbf{R})$ exhibits finitely many bands (and gaps, respectively), i.e.,

$$\sigma(L)=(-\infty,E_{2g}]\cup\bigcup_{m=1}^{g}[E_{2m-1},E_{2(m-1)}], \quad E_{2g}<E_{2g-1}<\ldots<E_0. \tag{1.3}$$

Following the traditional terminology, any real-valued potential q that gives rise to a spectrum of the type (1.3) is called a finite-gap potential (although in view of the general complex-valued case described in (3.34), the less frequently used terms finite-zone or finite-band potential might be more appropriate). The proper extension of this notion to general complex-valued meromorphic potentials q and its connection with stationary solutions of the KdV hierarchy on the basis of elementary algebro-geometric concepts is then obtained as follows. Let $L(t)$ be the second-order differential expression

$$L(t)=\frac{d^2}{dx^2}+q(x,t), \quad (x,t)\in\mathbf{R}^2, \tag{1.4}$$

where q depends on the additional (deformation) parameter t. It is well-known (see, e.g., Wilson [81]) that one can find coefficients $p_j(x,t)$ in

$$P_{2g+1}(t)=\frac{d^{2g+1}}{dx^{2g+1}}+p_{2g}(x,t)\frac{d^{2g}}{dx^{2g}}+\ldots+p_0(x,t), \tag{1.5}$$

in such a way that $P_{2g+1}(t)$ and $L(t)$ are almost commuting, i.e., their commutator $[P_{2g+1},L]$ is a multiplication operator. The coefficients p_j are then certain differential polynomials in q, i.e., polynomials in q and its x-derivatives. The pair (P_{2g+1},L) is called a Lax pair, and the equation

$$\frac{d}{dt}L=[P_{2g+1},L], \quad \text{i.e.,} \quad q_t=[P_{2g+1},L] \tag{1.6}$$

is a nonlinear evolution equation for q. The collection of all such equations for all possible choices of P_{2g+1}, $g\in\mathbf{N}_0$, is then called the KdV hierarchy (see §2 for more details). Due to the commutator structure in (1.6), solutions $q(\,\cdot\,,t)$ of the nonlinear evolution equations of the KdV hierarchy represent isospectral deformations of $L(0)$. Novikov [58], Dubrovin [20], Its and Matveev [46], and McKean and van Moerbeke [54] then showed that a real-valued smooth potential q is a finite-gap potential if and only if it satisfies one of the higher-order stationary (i.e., t-independent) KdV equations. Because of these facts it is common to call any complex-valued meromorphic function q a finite-gap potential if q satisfies one (and hence infinitely many) of the equations of the stationary KdV hierarchy.

The stationary KdV hierarchy, characterized by $q_t=0$ or $[P_{2g+1}, L]=0$, is intimately connected with the question of commutativity of ordinary differential expressions. Thus, if $[P_{2g+1}, L]=0$, a celebrated theorem of Burchnall and Chaundy [14], [15] implies that P_{2g+1} and L satisfy an algebraic relation of the form

$$P_{2g+1}^2 = \prod_{m=0}^{2g} (L-E_m), \quad \{E_m\}_{m=0}^{2g} \subset \mathbf{C}. \tag{1.7}$$

The locations E_m of the (finite) branch points and singular points of the associated hyperelliptic curve

$$F^2 = \prod_{m=0}^{2g} (E-E_m) \tag{1.8}$$

are precisely the band (gap) edges of the spectral bands of L (see (1.3)) whenever $q(x)$ is real-valued and smooth for $x \in \mathbf{R}$ (with appropriate generalizations to the complex-valued case, see §2). It is the (possibly singular) hyperelliptic compact Riemann surface K_g of (arithmetic) genus g, obtained upon one-point compactification of the curve (1.8), which signifies that q in $L=d^2/dx^2+q(x)$ represents a finite-gap potential.

While these considerations pertain to general solutions of the stationary KdV hierarchy, we now concentrate on the additional restriction that q be an elliptic function (i.e., meromorphic and doubly periodic) and hence return to our main subject, elliptic finite-gap potentials q for $L=d^2/dx^2+q(x)$, or, equivalently, elliptic solutions of the stationary KdV hierarchy. Ince's remarkable finite-gap result (1.3) remained the only explicit elliptic finite-gap example until the KdV flow $q_t=\frac{1}{4}q_{xxx}+\frac{3}{2}qq_x$ with the initial condition $q(x,0)=-6\mathcal{P}(x)$ was explicitly integrated by Dubrovin and Novikov [22] in 1975 (see also [23], [24], [25], [45]), and found to be of the type

$$q(x,t) = -2\sum_{j=1}^{3} \mathcal{P}(x-x_j(t)) \tag{1.9}$$

for appropriate $\{x_j(t)\}_{1\leqslant j\leqslant 3}$. As observed above, all potentials $q(\cdot,t)$ in (1.9) are isospectral to $q(\cdot,0)=-6\mathcal{P}(\cdot)$.

In 1977, Airault, McKean, and Moser in their seminal paper [2], presented the first systematic study of the isospectral torus $I_{\mathbf{R}}(q_0)$ of real-valued smooth potentials $q_0(x)$ of the type

$$q_0(x) = -2\sum_{j=1}^{M} \mathcal{P}(x-x_j) \tag{1.10}$$

with a finite-gap spectrum of the form (1.3). Among a variety of results they proved that any element q of $I_{\mathbf{R}}(q_0)$ is an elliptic function of the type (1.10) (with different x_j) with M constant throughout $I_{\mathbf{R}}(q_0)$ and that $\dim I_{\mathbf{R}}(q_0) \leqslant M$. In particular, if q_0 evolves according to any equation of the KdV hierarchy it remains an elliptic finite-gap potential. The potential (1.10) is intimately connected with completely integrable many-body systems of the Calogero–Moser-type [17], [57] (see also [18]). This connection with integrable particle systems was subsequently exploited by Krichever [51] in his fundamental construction of elliptic algebro-geometric solutions of the Kadomtsev–Petviashvili equation. In particular, he explicitly determined the underlying algebraic curve Γ and characterized the Baker–Akhiezer function associated with it in terms of elliptic functions as well as the corresponding theta function of Γ. The next breakthrough occurred in 1988 when Verdier [78] published new explicit examples of elliptic finite-gap potentials. Verdier's examples spurred a flurry of activities and inspired Belokolos and Enol'skii [10], Smirnov [68], and subsequently Taimanov [70] and Kostov and Enol'skii [47] to find further such examples by combining the reduction process of Abelian integrals to elliptic integrals (see [6], [7], [8, Chapter 7], [9]) with the aforementioned techniques of Krichever [51], [52]. This development finally culminated in a series of recent results of Treibich and Verdier [74], [75], [76] where it was shown that a general complex-valued potential of the form

$$q(x) = -\sum_{j=1}^{4} d_j \mathcal{P}(x-\omega_j) \tag{1.11}$$

($\omega_2 = \omega_1 + \omega_3$, $\omega_4 = 0$) is a finite-gap potential if and only if $\frac{1}{2}d_j$ are triangular numbers, i.e., if and only if

$$d_j = g_j(g_j+1) \quad \text{for some } g_j \in \mathbf{Z},\ 1 \leqslant j \leqslant 4. \tag{1.12}$$

We shall from now on refer to potentials of the type

$$q(x) = -\sum_{j=1}^{4} g_j(g_j+1)\mathcal{P}(x-\omega_j), \quad g_j \in \mathbf{Z},\ 1 \leqslant j \leqslant 4, \tag{1.13}$$

as Treibich–Verdier potentials. The methods of Treibich and Verdier are based on hyperelliptic tangent covers of the torus $\mathbf{C}/\Lambda$ (Λ being the period lattice generated by $2\omega_1$ and $2\omega_3$). The state of the art of elliptic finite-gap solutions up to 1993 was recently reviewed in a special issue of volume 36 of Acta Applicandae Mathematicae, see, e.g., [11], [26], [53], [69], [71], and [73].

Intrigued by these results and motivated by the fact that a complete characterization of all elliptic finite-gap solutions of the stationary KdV hierarchy is still open, we started to develop our own approach toward a solution of this problem. In contrast to all current approaches in this area, our methods to characterize elliptic finite-gap potentials rely on entirely different ideas based on a powerful theorem of Picard (Theorem 5.1) concerning the existence of solutions which are elliptic of the second kind of (second-order) ordinary differential equations with elliptic coefficients. As we have shown in [36], [37], [38] this approach immediately recovers and extends the results of [10], [47], [68], [70], [74], [75], [76], and [78]. In particular, for reflection-symmetric elliptic finite-gap potentials q (i.e., $q(z)=q(2z_0-z)$ for some $z_0\in\mathbf{C}$) including Lamé–Ince and Treibich–Verdier potentials, our approach reduces the computation of the branch points and singular points of the underlying hyperelliptic curve K_g to certain (constrained) linear algebraic eigenvalue problems.

Since the main hypothesis in Picard's theorem for a second-order differential equation of the form

$$\psi''(z)+q(z)\psi(z)=E\psi(z), \quad E\in\mathbf{C}, \tag{1.14}$$

with an elliptic potential q assumes the existence of a fundamental system of solutions meromorphic in z for each value of the spectral parameter $E\in\mathbf{C}$, we call any such elliptic function q which gives rise to this property a *Picard potential.* Our main result, a characterization of all elliptic finite-gap solutions of the stationary KdV hierarchy then reads as follows:

THEOREM 1.1. *q is an elliptic finite-gap potential if and only if q is a Picard potential (i.e., if and only if for each $E\in\mathbf{C}$ every solution of $\psi''(z)+q(z)\psi(z)=E\psi(z)$ is meromorphic with respect to z).*

In particular, Theorem 1.1 sheds new light on Picard's theorem since it identifies the elliptic coefficients q for which there exists a meromorphic fundamental system of solutions of (1.14) precisely as the elliptic finite-gap solutions of the stationary KdV hierarchy.

Our proof of Theorem 1.1 in §5 (Theorem 5.7) relies on two main ingredients: A purely Floquet theoretic part to be discussed in §§3 and 4 and an elliptic function part developed in §5.

The Floquet theoretic part is summarized in Theorems 3.2 and 4.1. In particular, Theorem 3.2 illustrates the great variety of possible values of algebraic multiplicities of (anti)periodic, Dirichlet and Neumann eigenvalues in the complex-valued case (as opposed to the self-adjoint case when q is real-valued). Theorem 4.1 on the other hand reconstructs the (possibly singular) hyperelliptic curve K_g associated with a general complex-valued coefficient q (not necessarily elliptic) which gives rise to two linearly independent Floquet solutions of $\psi''+q\psi=E\psi$ for all but finitely many values of E. In addition, we provide a detailed description of the associated Green's function, its singularity structure, and its connection to the hyperelliptic curve associated with q. Both results, Theorems 3.2 and 4.1, are new in the general complex-valued setting.

The elliptic function part in §5 consists of several items. First of all we describe Picard's result in Theorem 5.1. In Proposition 5.6 we prove the key result that all $2\omega_j$-(anti)periodic eigenvalues of q lie in certain half-strips

$$S_j=\{E\in\mathbf{C}:|\operatorname{Im}(|\omega_j|^{-2}\omega_j^2E)|\leqslant C_j,\operatorname{Re}(|\omega_j|^{-2}\omega_j^2E)\leqslant M_j\},\quad j=1,3, \tag{1.15}$$

for suitable constants $C_j>0$, $M_j\in\mathbf{R}$. Without loss of generality we may assume that the fundamental periods $2\omega_1$ and $2\omega_3$ have been chosen in such a way that the angle 2θ between the axes of S_1 and S_3 (i.e., $e^{i\theta}=(\omega_3/\omega_1)|\omega_1/\omega_3|$) satisfies $2\theta\in(0,2\pi)\backslash\{\pi\}$. Then S_1 and S_3 do not intersect outside a sufficiently large disk centered at the origin. A combination of this fact and Picard's Theorem 5.1 then yields a proof of Theorem 1.1 (see the proof of Theorem 5.7).

Finally, we close §5 with a series of remarks that put Theorem 1.1 into proper perspective: Among a variety of points, we stress, in particular, its straightforward applicability based on an elementary Frobenius-type analysis, its property of complementing Picard's original result, and its connection with the Weierstrass theory of reduction of Abelian to elliptic integrals.

2. The KdV hierarchy and hyperelliptic curves

In this section we review basic facts on the stationary KdV hierarchy. Since this material is well-known (see, e.g., [4], [19, Chapter 12], [33], [35]), we confine ourselves to a brief account. Assuming $q\in C^\infty(\mathbf{R})$ or q meromorphic in $\mathbf{C}$ (depending on the particular context in which one is interested) and hence either $x\in\mathbf{R}$ or $x\in\mathbf{C}$, consider the recursion relation

$$\hat f'_{j+1}(x)=\tfrac14\hat f'''_j(x)+q(x)\hat f'_j(x)+\tfrac12 q'(x)\hat f_j(x),\quad 0\leqslant j\leqslant g,\ \hat f_0(x)=1, \tag{2.1}$$

and the associated differential expressions (Lax pair)

$$L=\frac{d^2}{dx^2}+q(x), \tag{2.2}$$

$$\widehat P_{2g+1}=\sum_{j=0}^{g}\left[-\tfrac12\hat f'_j(x)+\hat f_j(x)\frac{d}{dx}\right]L^{g-j},\quad g\in\mathbf{N}_0, \tag{2.3}$$

(here $\mathbf{N}_0 := \mathbf{N} \cup \{0\}$). One can show that

$$[\widehat{P}_{2g+1}, L] = 2\hat{f}'_{g+1} = \tfrac{1}{2}\hat{f}'''_g(x) + 2q(x)\hat{f}'_g(x) + q'(x)\hat{f}_g(x) \tag{2.4}$$

($[\,\cdot\,,\cdot\,]$ the commutator symbol) and explicitly computes from (2.1),

$$\hat{f}_0 = 1, \quad \hat{f}_1 = \tfrac{1}{2}q + c_1, \quad \hat{f}_2 = \tfrac{1}{8}q'' + \tfrac{3}{8}q^2 + \tfrac{1}{2}c_1 q + c_2, \quad \text{etc.}, \tag{2.5}$$

where the c_j are integration constants. Using the convention that the corresponding homogeneous quantities obtained by setting $c_l = 0$ for $l = 1, 2, \ldots$ are denoted by f_j, i.e., $f_j = \hat{f}_j$ ($c_l \equiv 0$), the (homogeneous) stationary KdV hierarchy is then defined as the sequence of equations

$$\mathrm{KdV}_g(q) = 2f'_{g+1} = 0, \quad g \in \mathbf{N}_0. \tag{2.6}$$

Explicitly, this yields

$$\mathrm{KdV}_0(q) = q_x = 0, \quad \mathrm{KdV}_1(q) = \tfrac{1}{4}q''' + \tfrac{3}{2}qq' = 0, \quad \text{etc.} \tag{2.7}$$

The corresponding inhomogeneous version of $\mathrm{KdV}_g(q) = 0$ is then defined by

$$\hat{f}'_{g+1} = \sum_{j=0}^{g} c_{g-j} f'_{j+1} = 0, \tag{2.8}$$

where $c_0 = 1$ and $c_1, \ldots, c_g$ are arbitrary complex constants.

If one assigns to $q^{(l)} = d^l q/dx^l$ the degree $\deg(q^{(l)}) = l + 2$, $l \in \mathbf{N}_0$, then the homogeneous differential polynomial f_j with respect to q turns out to have degree $2j$, i.e.,

$$\deg(f_j) = 2j, \quad j \in \mathbf{N}_0. \tag{2.9}$$

Next, introduce the polynomial $\widehat{F}_g(E, x)$ in $E \in \mathbf{C}$,

$$\widehat{F}_g(E, x) = \sum_{j=0}^{g} \hat{f}_{g-j}(x) E^j. \tag{2.10}$$

Since $\hat{f}_0(x) = 1$,

$$\widehat{R}_{2g+1}(E, x) = (E - q(x))\widehat{F}_g(E, x)^2 - \tfrac{1}{2}\widehat{F}''_g(E, x)\widehat{F}_g(E, x) + \tfrac{1}{4}\widehat{F}'_g(E, x)^2 \tag{2.11}$$

is a monic polynomial in E of degree $2g + 1$. However, equations (2.1) and (2.8) imply that

$$\tfrac{1}{2}\widehat{F}'''_g - 2(E - q)\widehat{F}'_g + q'\widehat{F}_g = 0, \tag{2.12}$$

and this shows that $\widehat{R}_{2g+1}(E,x)$ is in fact independent of x. Hence it can be written as

$$\widehat{R}_{2g+1}(E)=\prod_{m=0}^{2g}(E-\widehat{E}_m),\quad \{\widehat{E}_m\}_{m=0}^{2g}\subset \mathbf{C}. \tag{2.13}$$

By (2.4) the inhomogeneous KdV equation (2.8) is equivalent to the commutativity of L and $\widehat{P}_{2g+1}$. This shows that

$$[\widehat{P}_{2g+1},L]=0, \tag{2.14}$$

and therefore, if $L\psi=E\psi$, this implies that $\widehat{P}_{2g+1}^2\psi=\widehat{R}_{2g+1}(E)\psi$. Thus $[\widehat{P}_{2g+1},L]=0$ implies

$$\widehat{P}_{2g+1}^2=\widehat{R}_{2g+1}(L)=\prod_{m=0}^{2g}(L-\widehat{E}_m), \tag{2.15}$$

a celebrated theorem by Burchnall and Chaundy [14], [15].

In §4 we will need the converse of the above procedure. It is given by

PROPOSITION 2.1. *Assume that $\widehat{F}_g(E,x)$, given by (2.10) with $\hat{f}_0(x)=1$, is twice continuously differentiable with respect to x, and that*

$$(E-q(x))\widehat{F}_g(E,x)^2-\tfrac{1}{2}\widehat{F}_g''(E,x)\widehat{F}_g(E,x)+\tfrac{1}{4}\widehat{F}_g'(E,x)^2 \tag{2.16}$$

is independent of x. Then $q\in C^\infty(\mathbf{R})$. Also the functions $\hat{f}_j(x)$ are infinitely often differentiable and satisfy the recursion relations (2.1) for $j=0,\dots,g-1$. Moreover, $\hat{f}_g$ satisfies

$$\tfrac{1}{4}\hat{f}_g'''(x)+q(x)\hat{f}_g'(x)+\tfrac{1}{2}q'(x)\hat{f}_g(x)=0, \tag{2.17}$$

i.e., the differential expression $\widehat{P}_{2g+1}$ given in (2.3) commutes with the expression $L=d^2/dx^2+q$.

Proof. The expression given in (2.16) is a monic polynomial in E of degree $2g+1$ whose coefficients are constants. We denote it by

$$\widehat{R}_{2g+1}(E)=\sum_{j=0}^{2g+1}\tilde{c}_{2g+1-j}E^j, \tag{2.18}$$

where $\tilde{c}_0=1$. Comparing the coefficients of E^{2g+1-k} in (2.16) and (2.18) yields

$$q(x)=2\hat{f}_1(x)-\tilde{c}_1 \tag{2.19}$$

for $k=1$, and

$$\begin{aligned}\hat{f}''_{k-1}(x) = 4\hat{f}_k(x) - 4q(x)\hat{f}_{k-1}(x) + 2\hat{f}_{k-1}(x)\hat{f}_1(x) - 2\tilde{c}_1 \\ -\sum_{j=1}^{k-2}\big(\hat{f}''_j(x)\hat{f}_{k-1-j}(x) - \tfrac{1}{2}\hat{f}'_j(x)\hat{f}'_{k-1-j}(x) \\ + 2q(x)\hat{f}_j(x)\hat{f}_{k-1-j}(x) - 2\hat{f}_j(x)\hat{f}_{k-j}(x)\big)\end{aligned} \tag{2.20}$$

for $k=2, ..., g+1$. In (2.20) we have introduced $\hat{f}_{g+1}=0$ for ease of notation.

By hypothesis $\hat{f}_1, ..., \hat{f}_g \in C^2(\mathbf{R})$. Equation (2.19) now yields that $q \in C^2(\mathbf{R})$. Then the equations in (2.20) show recursively that $\hat{f}_k \in C^4(\mathbf{R})$ for $k=1, ..., g$. By induction it follows that q and the functions $\hat{f}_1, ..., \hat{f}_g$ are infinitely often differentiable. Thus we may differentiate (2.19) with respect to x to obtain (2.1) for $j=0$. Also we may differentiate all the equations (2.20) with respect to x. Applying this procedure inductively then proves the validity of (2.1) for $j=1, ..., g-1$ and of (2.17). The final statement then follows from (2.4). □

The nonlinear recursion formalism for $\hat{f}_k$ resulting from (2.20) can be read off from the results of §2.2 in [32].

Equation (2.15) illustrates the intimate connection between the stationary KdV equation $\hat{f}'_{g+1}=0$ in (2.8) and the compact (possibly singular) hyperelliptic curve K_g of (arithmetic) genus g obtained upon one-point compactification of the curve

$$F^2 = \widehat{R}_{2g+1}(E) = \prod_{m=0}^{2g}(E - \widehat{E}_m). \tag{2.21}$$

The above formalism leads to the following standard definition.

Definition 2.2. Any solution q of one of the stationary KdV equations (2.8) is called an (*algebro-geometric*) *finite-gap potential* associated with the KdV hierarchy.

Finite-gap potentials q can be expressed in terms of the Riemann theta function or through τ-functions associated with the curve K_g (see, e.g., [46], [67]).

3. Floquet theory

Next we turn to Floquet theory in connection with a complex-valued nonconstant periodic potential q and assume for the rest of this section that

$$q \in C^0(\mathbf{R}), \quad q(x+\Omega) = q(x), \quad x \in \mathbf{R}, \tag{3.1}$$

for some $\Omega > 0$.

Floquet solutions $\psi(E,x)$ of $Ly=Ey$ are characterized by the property

$$\psi(E,x+\Omega)=\varrho(E)\psi(E,x) \quad \text{for all } x\in\mathbf{R}, \tag{3.2}$$

where $\varrho(E)$ is a so-called Floquet multiplier.

We introduce the fundamental system of solutions $c(E,x,x_0)$ and $s(E,x,x_0)$ of $Ly=Ey$ defined by

$$c(E,x_0,x_0)=s'(E,x_0,x_0)=1, \quad c'(E,x_0,x_0)=s(E,x_0,x_0)=0. \tag{3.3}$$

The functions $c(E,x,x_0)$ and $s(E,x,x_0)$ and their x-derivatives are entire functions of E for every choice of x and x_0. The coefficients of Floquet solutions written as linear combinations of the fundamental solutions $c(E,x,x_0)$ and $s(E,x,x_0)$ and the Floquet multipliers are given through eigenvectors and eigenvalues of the monodromy matrix of L, i.e., through

$$M(E,x_0)=\begin{pmatrix} c(E,x_0+\Omega,x_0) & s(E,x_0+\Omega,x_0) \\ c'(E,x_0+\Omega,x_0) & s'(E,x_0+\Omega,x_0)\end{pmatrix}. \tag{3.4}$$

More precisely, the Floquet multipliers are the eigenvalues of $M(E,x_0)$ and hence are given by

$$\varrho_\pm=\Delta(E)\pm\sqrt{\Delta(E)^2-1}, \tag{3.5}$$

where $\Delta(E)$ is half the trace of $M(E,x_0)$, i.e.,

$$\Delta(E)=\tfrac{1}{2}[c(E,x_0+\Omega,x_0)+s'(E,x_0+\Omega,x_0)]. \tag{3.6}$$

That $\Delta(E)$ is independent of x_0 follows from equations (3.7) and (3.8) below.

The functions $c(E,x,x_0)$ and $s(E,x,x_0)$ satisfy certain Volterra integral equations which imply that

$$\frac{\partial s(E,x,x_0)}{\partial x_0}=-c(E,x,x_0) \quad \text{and} \quad \frac{\partial c(E,x,x_0)}{\partial x_0}=(q(x_0)-E)s(E,x,x_0).$$

Hence

$$\frac{d}{dx_0}c^{(k)}(E,x_0+\Omega,x_0)=c^{(k+1)}(E,x_0+\Omega,x_0)+(q(x_0)-E)s^{(k)}(E,x_0+\Omega,x_0), \tag{3.7}$$

$$\frac{d}{dx_0}s^{(k)}(E,x_0+\Omega,x_0)=s^{(k+1)}(E,x_0+\Omega,x_0)-c^{(k)}(E,x_0+\Omega,x_0) \tag{3.8}$$

for $k=0,1$.

For each $E\in\mathbf{C}$ there exists at least one nontrivial Floquet solution. In fact, since together with $\varrho(E)$, $1/\varrho(E)$ is also a Floquet multiplier, there are two linearly independent Floquet solutions for a given E provided $\varrho(E)^2\neq 1$. Floquet solutions can be expressed in terms of the fundamental system $c(E,x,x_0)$ and $s(E,x,x_0)$ by

$$\psi_\pm(E,x,x_0)=c(E,x,x_0)+\frac{\varrho_\pm(E)-c(E,x_0+\Omega,x_0)}{s(E,x_0+\Omega,x_0)}s(E,x,x_0), \tag{3.9}$$

if $s(E,x_0+\Omega,x_0)\neq 0$, or by

$$\tilde{\psi}_\pm(E,x,x_0)=s(E,x,x_0)+\frac{\varrho_\pm(E)-s'(E,x_0+\Omega,x_0)}{c'(E,x_0+\Omega,x_0)}c(E,x,x_0), \tag{3.10}$$

if $c'(E,x_0+\Omega,x_0)\neq 0$. If both $s(E,x_0+\Omega,x_0)$ and $c'(E,x_0+\Omega,x_0)$ are equal to zero, then $s(E,x,x_0)$ and $c(E,x,x_0)$ are linearly independent Floquet solutions.

Associated with the second-order differential expression $L=d^2/dx^2+q(x)$ we consider densely defined closed linear operators H, $H_D(x_0)$, $H_N(x_0)$, and H_θ in $L^2(\mathbf{R})$ and $L^2((x_0,x_0+\Omega))$, $x_0\in\mathbf{R}$, respectively. Let

$$Hf=Lf,\quad f\in H^{2,2}(\mathbf{R}), \tag{3.11}$$

$$H_D(x_0)f=Lf,\quad f\in\{g\in H^{2,2}((x_0,x_0+\Omega)):g(x_0)=0=g(x_0+\Omega)\}, \tag{3.12}$$

$$H_N(x_0)f=Lf,\quad f\in\{g\in H^{2,2}((x_0,x_0+\Omega)):g'(x_0)=0=g'(x_0+\Omega)\}, \tag{3.13}$$

and for $\theta\in\mathbf{C}$,

$$H(\theta)f=Lf,\quad f\in\{g\in H^{2,2}((x_0,x_0+\Omega)):g^{(k)}(x_0+\Omega)=e^{i\theta}g^{(k)}(x_0),\, k=0,1\} \tag{3.14}$$

($H^{p,r}(\cdot)$ being the usual Sobolev spaces with r distributional derivatives in $L^p(\cdot)$). Next we denote the purely discrete spectra of $H_D(x_0)$, $H_N(x_0)$, and $H(\theta)$ by

$$\begin{aligned}\sigma(H_D(x_0))&=\{\mu_n(x_0)\}_{n\in\mathbf{N}},\\ \sigma(H_N(x_0))&=\{\nu_n(x_0)\}_{n\in\mathbf{N}_0},\\ \sigma(H(\theta))&=\{E_n(\theta)\}_{n\in\mathbf{N}_0},\end{aligned} \tag{3.15}$$

respectively. Note that, while $H(\theta)$ depends on x_0 its spectrum does not. We agree that in (3.15) as well as in the rest of the paper all point spectra (i.e., sets of eigenvalues) are recorded in such a way that all eigenvalues are consistently repeated according to their algebraic multiplicity unless explicitly stated otherwise.

The eigenvalues of $H_D(x_0)$ ($H_N(x_0)$) are called Dirichlet (Neumann) eigenvalues with respect to the interval $[x_0,x_0+\Omega]$. The eigenvalues of $H(\theta)$ are precisely those values E where the monodromy matrix $M(E,x_0)$ of L has eigenvalues $\varrho=e^{\pm i\theta}$.

The eigenvalues $E_n(0)$ ($E_n(\pi)$) of $H(0)$ ($H(\pi)$) are called the periodic (antiperiodic) eigenvalues associated with q. Note that the (anti)periodic eigenvalues $E_n(0)$ ($E_n(\pi)$) are the zeros of $\Delta(\cdot)-1$ ($\Delta(\cdot)+1$) and that their algebraic multiplicities coincide with the orders of the respective zeros (see, e.g., [39]). In the following we denote the zeros of $\Delta(E)^2-1$ by E_n, $n\in\mathbf{N}_0$. They are repeated according to their multiplicity and are related to the (anti)periodic eigenvalues via

$$E_{4n}=E_{2n}(0), \quad E_{4n+1}=E_{2n}(\pi), \quad E_{4n+2}=E_{2n+1}(\pi), \quad E_{4n+3}=E_{2n+1}(0) \tag{3.16}$$

for $n\in\mathbf{N}_0$. We also introduce

$$d(E)=\operatorname{ord}_E(\Delta(\cdot)^2-1), \tag{3.17}$$

the order of E as a zero of $\Delta(\cdot)^2-1$ ($d(E)=0$ if $\Delta(E)^2-1\neq 0$).

Similarly the Dirichlet eigenvalues $\mu_n(x_0)$ and the Neumann eigenvalues $\nu_n(x_0)$ of $H_D(x_0)$ and $H_N(x_0)$ are the zeros of the functions $s(\cdot, x_0+\Omega, x_0)$ and $c'(\cdot, x_0+\Omega, x_0)$, respectively. Again their algebraic multiplicities coincide precisely with the multiplicities of the respective zeros (see, e.g., [39]). These multiplicities depend in general on x_0. We introduce the notation

$$p(E,x_0)=\operatorname{ord}_E(s(\cdot, x_0+\Omega, x_0)), \tag{3.18}$$

$$r(E,x_0)=\operatorname{ord}_E(c'(\cdot, x_0+\Omega, x_0)), \tag{3.19}$$

and remark that $p(E,x_0)$ and $r(E,x_0)$ are combinations of movable and immovable parts. Specifically, define

$$p_i(E)=\min\{p(E,x_0): x_0\in\mathbf{R}\}, \quad r_i(E)=\min\{r(E,x_0): x_0\in\mathbf{R}\}$$

and $p_m(E,x_0)$ and $r_m(E,x_0)$ by

$$p(E,x_0)=p_i(E)+p_m(E,x_0), \tag{3.20}$$

$$r(E,x_0)=r_i(E)+r_m(E,x_0). \tag{3.21}$$

If $p_i(E)>0$ ($r_i(E)>0$) then E is a Dirichlet (Neumann) eigenvalue no matter what x_0 is and we will call E an immovable Dirichlet (Neumann) eigenvalue. Otherwise, if $p_i(E)=0$ ($r_i(E)=0$) but $p(E,x_0)>0$ ($r(E,x_0)>0$), we call E a movable Dirichlet (Neumann) eigenvalue.

The asymptotic behavior of $c'(E,x_0+\Omega,x_0)$, $s(E,x_0+\Omega,x_0)$, and $\Delta(E)$ as $|E|$ tends to infinity, i.e.,

$$s(E,x_0+\Omega,x_0)=(-E)^{-1/2}\sin[(-E)^{1/2}\Omega]+O(|E|^{-1}e^{|\operatorname{Im}(-E)^{1/2}|\Omega}), \tag{3.22}$$

$$c'(E,x_0+\Omega,x_0)=(-E)^{-1/2}\sin[(-E)^{1/2}\Omega]+O(|E|^{-1}e^{|\operatorname{Im}(-E)^{1/2}|\Omega}), \tag{3.23}$$

$$\Delta(E)=\cos[(-E)^{1/2}\Omega]+O(|E|^{-1/2}e^{|\operatorname{Im}(-E)^{1/2}|\Omega}), \tag{3.24}$$

obtained by a standard iteration of Volterra integral equations together with Rouché's theorem, then proves the following facts:

(1) The zeros $\mu_n(x_0)$ of $s(E,x_0+\Omega,x_0)$ and the zeros $\nu_n(x_0)$ of $c'(\,\cdot\,,x_0+\Omega,x_0)$ are simple for $n\in\mathbf{N}$ sufficiently large.

(2) The zeros E_n of $\Delta(E)^2-1$ are at most double for $n\in\mathbf{N}$ large enough.

(3) $\mu_n(x_0)$, $\nu_n(x_0)$, and E_n can be arranged such that they have the following asymptotic behavior as n tends to infinity (we assume in the following that they are actually arranged in this way):

$$\mu_n(x_0)=-\frac{n^2\pi^2}{\Omega^2}+O(1), \tag{3.25}$$

$$\nu_n(x_0)=-\frac{n^2\pi^2}{\Omega^2}+O(1), \tag{3.26}$$

$$E_{2n-1},\, E_{2n}=-\frac{n^2\pi^2}{\Omega^2}+O(1). \tag{3.27}$$

The Hadamard factorizations of $s(E,x_0+\Omega,x_0)$, $c'(E,x_0+\Omega,x_0)$, and $\Delta(E)^2-1$ therefore read

$$s(E,x_0+\Omega,x_0)=c_1(x_0)\prod_{n=1}^{\infty}\left(1-\frac{E}{\mu_n(x_0)}\right), \tag{3.28}$$

$$c'(E,x_0+\Omega,x_0)=c_2(x_0)\prod_{n=0}^{\infty}\left(1-\frac{E}{\nu_n(x_0)}\right), \tag{3.29}$$

$$\Delta(E)^2-1=c_3^2\prod_{n=0}^{\infty}\left(1-\frac{E}{E_n}\right) \tag{3.30}$$

for suitable E-independent and nonvanishing $c_1(x_0)$, $c_2(x_0)$, and c_3. Equations (3.28)–(3.30) assume that none of the eigenvalues is equal to zero. If this were to happen these equations have to be replaced by obvious modifications.

For more details on algebraic versus geometric multiplicities of eigenvalue problems of the type of $H_D(x_0)$ and $H(\theta)$ see, e.g., [39].

We then have the following

PROPOSITION 3.1. (i) *The number λ is an immovable Dirichlet eigenvalue if and only if it is also an immovable Neumann eigenvalue. In particular, in this case there are two linearly independent (anti)periodic solutions of $Ly=Ey$.*

(ii) *If λ is an (anti)periodic eigenvalue and also both a Dirichlet and a Neumann eigenvalue with respect to the interval $[x_0,x_0+\Omega]$, then λ is an immovable Dirichlet and Neumann eigenvalue.*

Proof. (i) Assume that λ is an immovable Dirichlet eigenvalue. Then we have $s(\lambda, x_0+\Omega, x_0)=0$ for every $x_0 \in \mathbf{R}$. By (3.8), $s'(\lambda, x_0+\Omega, x_0)$ is continuous as a function of x_0. On the other hand $s'(\lambda, x_0+\Omega, x_0)$ is always equal to one of the two Floquet multipliers. Therefore, we infer that $s'(\lambda, x_0+\Omega, x_0)$ is in fact equal to some constant ϱ. In particular we have $s(\lambda, x+\Omega, x_0)=\varrho s(\lambda, x, x_0)$ for all $x_0 \in \mathbf{R}$.

Now consider the function $s(\lambda, x, \tilde{x}_0)$. Then we obtain

$$\begin{aligned} s(\lambda, x+\Omega, \tilde{x}_0) &= \alpha s(\lambda, x+\Omega, x_0)+\beta c(\lambda, x+\Omega, x_0) \\ &= \alpha\varrho s(\lambda, x, x_0)+\beta c(\lambda, x+\Omega, x_0) \end{aligned} \tag{3.31}$$

and

$$s(\lambda, x+\Omega, \tilde{x}_0) = \varrho s(\lambda, x, \tilde{x}_0) = \varrho(\alpha s(\lambda, x, x_0)+\beta c(\lambda, x, x_0)). \tag{3.32}$$

Since the left-hand sides are equal, comparing the right-hand sides yields

$$c(\lambda, x+\Omega, x_0) = \varrho c(\lambda, x, x_0) \tag{3.33}$$

if $\tilde{x}_0$ is chosen such that $\beta=s(\lambda, \tilde{x}_0, x_0)\neq 0$. Hence $c(\lambda, x, x_0)$ is a solution satisfying Neumann boundary conditions, i.e., λ is among the Neumann eigenvalues $\{\nu_n(x_0)\}_{n\in\mathbf{N}_0}$. Since x_0 is arbitrary we find that λ is, in fact, an immovable Neumann eigenvalue. Similarly, using that q is nonconstant, one shows that λ is an immovable Dirichlet eigenvalue if it is an immovable Neumann eigenvalue. Moreover, $c(\lambda, x, x_0)$ and $s(\lambda, x, x_0)$ are linearly independent Floquet solutions of $Ly=\lambda y$ with the same Floquet multiplier ϱ implying $\varrho^2=1$. Thus $c(\lambda, x, x_0)$ and $s(\lambda, x, x_0)$ are indeed both periodic or both antiperiodic.

(ii) Suppose for brevity that λ is a periodic eigenvalue. Then $M(E, x_0)$ is the identity matrix and therefore every solution of $L\psi=\lambda\psi$ is periodic. Thus we obtain for any $x_1 \in \mathbf{R}$ that $s(\lambda, x_1+\Omega, x_1)=s(\lambda, x_1, x_1)=0$ and $c'(\lambda, x_1+\Omega, x_1)=c'(\lambda, x_1, x_1)=0$ implying that λ is an immovable Dirichlet and Neumann eigenvalue. □

It was shown by Rofe–Beketov [66] that the spectrum of H is equal to the conditional stability set of L, i.e., the set of all spectral parameters E for which a nontrivial bounded solution of $L\psi=E\psi$ exists. Hence

$$\sigma(H) = \bigcup_{\theta\in[0,2\pi]} \sigma(H(\theta)) = \bigcup_{n\in\mathbf{N}_0} \sigma_n, \quad \text{where } \sigma_n = \bigcup_{\theta\in[0,\pi]} E_n(\theta). \tag{3.34}$$

We note that in the general case where q is complex-valued some of the spectral arcs σ_n may cross each other, see, e.g., [39] and [62] for explicit examples.

The Green's function $G(E, x, x')$ of H, i.e., the integral kernel of the resolvent of H

$$G(E, x, x') = (H-E)^{-1}(x, x'), \quad E\in\mathbf{C}\backslash\sigma(H),\ x, x'\in\mathbf{R}, \tag{3.35}$$

is explicitly given by

$$G(E,x,x')=W(f_-(E,x),f_+(E,x))^{-1}\begin{cases} f_+(E,x)f_-(E,x'), & x\geqslant x', \\ f_-(E,x)f_+(E,x'), & x\leqslant x'. \end{cases} \tag{3.36}$$

Here $f_\pm(E,\cdot)$ solve $Lf=Ef$ and are chosen such that

$$f_\pm(E,\cdot)\in L^2((R,\pm\infty)), \quad E\in\mathbf{C}\backslash\sigma(H),\ R\in\mathbf{R}, \tag{3.37}$$

with $W(f,g)=fg'-f'g$ the Wronskian of f and g.

Equation (3.36) implies that the diagonal Green's function is twice continuously differentiable and satisfies the nonlinear second-order differential equation

$$4(E-q(x))G(E,x,x)^2-2G(E,x,x)G''(E,x,x)+G'(E,x,x)^2=1 \tag{3.38}$$

(the primes denoting derivatives with respect to x).

It follows from (3.34) that $|\varrho(E)|\neq 1$ unless $E\in\sigma(H)$. Therefore, if $E\notin\sigma(H)$ there is precisely one Floquet solution in $L^2((-\infty,R))$ and one in $L^2((R,\infty))$. Letting $\varrho_\pm(E)=e^{\pm i\theta}$ with $\mathrm{Im}(\theta)>0$ we obtain $|\varrho_+(E)|<1<|\varrho_-(E)|$. Hence $f_+(E,x)=\psi_+(E,x,x_0)$ and $f_-(E,x)=\psi_-(E,x,x_0)$. Since $\psi_\pm(E,x_0,x_0)=1$, equations (3.5) and (3.9) imply

$$W(f_-(E,\cdot),f_+(E,\cdot))=\frac{e^{i\theta}-e^{-i\theta}}{s(E,x_0+\Omega,x_0)}=-2\frac{[\Delta(E)^2-1]^{1/2}}{s(E,x_0+\Omega,x_0)}. \tag{3.39}$$

The sign of the square root was chosen such that $[\Delta(E)^2-1]^{1/2}$ is asymptotically equal to $\frac{1}{2}\varrho_-(E)$ for large positive E. Equation (3.39) implies (see also [34])

$$G(E,x_0,x_0)=-\frac{s(E,x_0+\Omega,x_0)}{2[\Delta(E)^2-1]^{1/2}}. \tag{3.40}$$

Closely related to $G(E,x,x)$ is the function

$$H(E,x,x')=\frac{\partial^2 G(E,x,x')}{\partial x\,\partial x'}. \tag{3.41}$$

To evaluate it for $x=x'=x_0$ we use (3.10) and obtain

$$H(E,x_0,x_0)=\frac{c'(E,x_0+\Omega,x_0)}{2[\Delta(E)^2-1]^{1/2}}. \tag{3.42}$$

When $q\in C^1(\mathbf{R})$ the function H satisfies the nonlinear second-order differential equation

$$\begin{aligned} &4(E-q(x))^2H(E,x,x)^2-2q'(x)H(E,x,x)H'(E,x,x) \\ &-2(E-q(x))H(E,x,x)H''(E,x,x)+(E-q(x))H'(E,x,x)^2=(E-q(x))^3. \end{aligned} \tag{3.43}$$

We emphasize that both (3.38) and (3.43) hold universally for any $q\in C^0(\mathbf{R})$ or $q\in C^1(\mathbf{R})$, respectively, i.e., they do not at all depend on periodicity of q. While (3.38) is a standard result (see, e.g., [32]) the differential equation (3.43) appears to be new.

Equations (3.38) and (3.43) are the main ingredients for the following

THEOREM 3.2. *Let q be a differentiable nonconstant periodic function of period $\Omega>0$ on $\mathbf{R}$. Then for every $E\in\mathbf{C}$,*

$$p_i(E)=r_i(E), \tag{3.44}$$

$$d(E)-p_i(E)-r_i(E)\geqslant 0. \tag{3.45}$$

Proof. By Proposition 3.1, $p_i(E)$ and $r_i(E)$ are either both zero or else both positive. Hence we only have to consider the case when they are both positive in which case E is also a periodic or an antiperiodic eigenvalue.

The asymptotic behavior of μ_n, ν_n, and E_n in (3.25)–(3.27) shows that the products in (3.28)–(3.30) are independent of the order of the factors. Hence we may write

$$s(E,x_0+\Omega,x_0)=F_D(E,x_0)D(E), \tag{3.46}$$

$$c'(E,x_0+\Omega,x_0)=F_N(E,x_0)N(E), \tag{3.47}$$

where

$$D(E)=\prod_{\lambda\in\mathbf{C}}\left(1-\frac{E}{\lambda}\right)^{p_i(\lambda)}, \tag{3.48}$$

$$N(E)=\prod_{\lambda\in\mathbf{C}}\left(1-\frac{E}{\lambda}\right)^{r_i(\lambda)}, \tag{3.49}$$

$$F_D(E,x_0)=c_1(x_0)\prod_{\lambda\in\mathbf{C}}\left(1-\frac{E}{\lambda}\right)^{p_m(\lambda,x_0)}, \tag{3.50}$$

$$F_N(E,x_0)=c_2(x_0)\prod_{\lambda\in\mathbf{C}}\left(1-\frac{E}{\lambda}\right)^{r_m(\lambda,x_0)}. \tag{3.51}$$

Hence

$$G(E,x_0,x_0)=\frac{-F_D(E,x_0)D(E)}{2[\Delta(E)^2-1]^{1/2}}, \tag{3.52}$$

$$H(E,x_0,x_0)=\frac{F_N(E,x_0)N(E)}{2[\Delta(E)^2-1]^{1/2}} \tag{3.53}$$

and from (3.38) and (3.52),

$$U(E,x_0)D(E)^2=4(\Delta(E)^2-1), \tag{3.54}$$

$$V(E,x_0)N(E)^2=4(E-q(x_0))^3(\Delta(E)^2-1), \tag{3.55}$$

where

$$U(E,x_0)=4(E-q(x_0))F_D(E,x_0)^2-2F_D(E,x_0)F_D''(E,x_0)+F_D'(E,x_0)^2 \tag{3.56}$$

and

$$\begin{aligned} V(E,x_0) = 4(E-q(x_0))^2 F_N(E,x_0)^2 - 2q'(x_0)F_N(E,x_0)F_N'(E,x_0) \\ -2(E-q(x_0))F_N(E,x_0)F_N''(E,x_0) + (E-q(x_0))F_N'(E,x_0)^2. \end{aligned} \tag{3.57}$$

Equation (3.54) shows that the multiplicity $2p_i(E)$ of a zero E of $D(\cdot)^2$ cannot be larger than the multiplicity $d(E)$ of the zero E of $\Delta(\cdot)^2-1$. Similarly, (3.55) shows that $2r_i(E) \leqslant d(E)$ assuming that x_0 is chosen such that $E-q(x_0)\neq 0$. This yields (3.45).

In order to prove (3.44) we first note that equations (3.5), (3.7), and (3.8) imply

$$\Delta(E)^2 - 1 - \frac{1}{4}\left(\frac{d}{dx_0}s(E,x_0+\Omega,x_0)\right)^2 = c'(E,x_0+\Omega,x_0)s(E,x_0+\Omega,x_0), \tag{3.58}$$

$$(q(x_0)-E)^2(\Delta(E)^2-1) - \frac{1}{4}\left(\frac{d}{dx_0}c'(E,x_0+\Omega,x_0)\right)^2 \tag{3.59}$$

$$= (q(x_0)-E)^2 c'(E,x_0+\Omega,x_0)s(E,x_0+\Omega,x_0). \tag{3.60}$$

From (3.46) and (3.47) we get for all $E\in\mathbf{C}$ and all $x\in\mathbf{R}$,

$$\frac{\Delta(E)^2-1}{D(E)^2} - \frac{1}{4}F_D'(E,x_0)^2 = \frac{N(E)}{D(E)}F_D(E,x_0)F_N(E,x_0), \tag{3.61}$$

$$(q(x_0)-E)^2\frac{\Delta(E)^2-1}{N(E)^2} - \frac{1}{4}F_N'(E,x_0)^2 = (q(x_0)-E)^2\frac{D(E)}{N(E)}F_D(E,x_0)F_N(E,x_0). \tag{3.62}$$

Since, according to the first part of the proof, the left-hand sides of (3.61) and (3.62) are entire functions with respect to E for every $x_0\in\mathbf{R}$, so must be the right-hand sides. Suppose λ is a zero of $D(E)$ or $N(E)$ and assume there is an x_0 such that

$$(q(x_0)-\lambda)^2 F_D(\lambda,x_0)F_N(\lambda,x_0) \neq 0. \tag{3.63}$$

Then (3.61) shows that $p_i(\lambda)\leqslant r_i(\lambda)$ while (3.62) shows the converse inequality and hence (3.44). To show that there is an $x_0\in\mathbf{R}$ satisfying (3.63) observe that equation (3.38) implies that zeros of $F_D(E,\cdot)$ are isolated since $F_D'(E,x_0)\neq 0$ if $F_D(E,x_0)=0$ and similarly, (3.43) implies that a zero x_0 of $F_N(E,\cdot)$ is isolated provided $q(x_0)\neq E$. □

4. Floquet theory and finite-gap potentials

In this section we prove that q is a finite-gap potential if the equation $\psi''+q\psi=E\psi$ has two linearly independent Floquet solutions for all but finitely many values of the spectral parameter E. The proof reveals a number of other properties, notably about the Green's function of H.

THEOREM 4.1. *Assume that $q(x)$ is a continuous nonconstant periodic function of period $\Omega>0$ on $\mathbf{R}$ and that $Ly=y''+qy=Ey$ has two linearly independent Floquet solutions for all but finitely many values of $E\in\mathbf{C}$. Then the following statements hold:*

(i) *Suppose $\{\widehat{E}_j\}_{j=1}^{\widehat{M}}$ for some $\widehat{M}\in\mathbf{N}$ is the set where two linearly independent Floquet solutions do not exist. Then the function $\Delta(E)^2-1$ has a zero at each point $\widehat{E}_j$, i.e., $d(\widehat{E}_j)>0$ for $1\leqslant j\leqslant\widehat{M}$. Moreover, none of the $\widehat{E}_j$, $j=1,\dots,\widehat{M}$, is an immovable Dirichlet (or Neumann) eigenvalue.*

(ii) *The inequality*

$$d(E)-p_i(E)-r_i(E)\geqslant 0 \tag{4.1}$$

is strict only on a finite set $\{\widehat{E}_j\}_{j=1}^{M}$, $M\geqslant\widehat{M}$, which includes the numbers $\widehat{E}_1,\dots,\widehat{E}_{\widehat{M}}$.

(iii) *The number of movable Dirichlet eigenvalues and the number of movable Neumann eigenvalues are finite. More precisely, there exists an integer $g\in\mathbf{N}_0$ such that for any given $x_0\in\mathbf{R}$,*

$$\sum_{E\in\mathbf{C}} p_m(E,x_0)=g, \tag{4.2}$$

$$\sum_{E\in\mathbf{C}} r_m(E,x_0)=g+1. \tag{4.3}$$

This number g satisfies

$$2g+1=\sum_{j=1}^{M}(d(\widehat{E}_j)-p_i(\widehat{E}_j)-r_i(\widehat{E}_j))=\sum_{j=1}^{M}\hat{q}_j, \tag{4.4}$$

where

$$\hat{q}_j=d(\widehat{E}_j)-p_i(\widehat{E}_j)-r_i(\widehat{E}_j)\quad \text{for } j=1,\dots,M. \tag{4.5}$$

(iv) *$q\in C^\infty(\mathbf{R})$.*

(v) *q is an algebro-geometric finite-gap potential associated with the compact (possibly singular) hyperelliptic curve K_g of (arithmetic) genus g obtained upon one-point compactification of the curve*

$$F^2=\widehat{R}_{2g+1}(E)=\prod_{j=1}^{M}(E-\widehat{E}_j)^{\hat{q}_j}. \tag{4.6}$$

Equivalently, there exists a monic ordinary differential expression $\widehat{P}_{2g+1}$ of order $2g+1$, i.e.,

$$\widehat{P}_{2g+1}=\sum_{l=0}^{2g+1}p_l(x)\frac{d^l}{dx^l},\quad p_{2g+1}(x)=1, \tag{4.7}$$

which commutes with L, i.e.,

$$[\widehat{P}_{2g+1}, L] = 0 \tag{4.8}$$

and satisfies the Burchnall–Chaundy relation

$$\widehat{P}_{2g+1}^2 = \widehat{R}_{2g+1}(L) = \prod_{j=1}^{M} (L - \widehat{E}_j)^{\hat{q}_j}. \tag{4.9}$$

(vi) *The diagonal Green's function $G(E,x,x)$ of H is defined and continuous for all $E \in \mathbf{C} \setminus \{\widehat{E}_j\}_{j=1}^M$ and is of the type*

$$G(E,x,x) = \frac{-\frac{1}{2}\widehat{F}_g(E,x)}{\widehat{R}_{2g+1}(E)^{1/2}}, \tag{4.10}$$

where

$$\widehat{F}_g(E,x) = \prod_{j \in J_g} [E - \mu_j(x)] \tag{4.11}$$

and where J_g (of cardinality g) is the set of indices $j \in \mathbf{N}$ such that $p_m(\mu_j(x), x) > 0$ (we set $\widehat{F}_g(E,x) = 1$ for $g = 0$).

(vii) *Let $\lambda \in \mathbf{C}$, $B(\lambda; \varepsilon) = \{E : |E - \lambda| < \varepsilon\}$ and let $f_\pm(E,x)$ be two Floquet solutions of $L\psi = E\psi$ which are linearly independent for each $E \in B(\lambda;\varepsilon) \setminus \{\lambda\}$ and which, together with their x-derivatives, are continuous as functions of E in $B(\lambda;\varepsilon)$. Then the Wronskian $W(f_-, f_+)$ vanishes at λ if and only if $\lambda \in \{\widehat{E}_1, ..., \widehat{E}_M\}$.*

(viii) *The spectrum of H consists of finitely many bounded spectral arcs $\tilde{\sigma}_n$, $1 \leqslant n \leqslant \tilde{g}$, for some $\tilde{g} \leqslant g$ and one unbounded (semi-infinite) arc $\tilde{\sigma}_\infty$ which tends to $-\infty + \langle q \rangle$, with $\langle q \rangle = \Omega^{-1} \int_{x_0}^{x_0+\Omega} q(x)\, dx$, i.e.,*

$$\sigma(H) = \Big(\bigcup_{n=1}^{\tilde{g}} \tilde{\sigma}_n \Big) \cup \tilde{\sigma}_\infty, \tag{4.12}$$

where each $\tilde{\sigma}_n$ and $\tilde{\sigma}_\infty$ is a union of some of the spectral arcs σ_n in (3.34).

Proof. We introduce the following sets

$$D_m(x_0) = \{E : p_m(E, x_0) > 0\}, \tag{4.13}$$

$$N_m(x_0) = \{E : r_m(E, x_0) > 0\}, \tag{4.14}$$

$$B_1 = \{E : 0 = p_i(E) = r_i(E); 0 < d(E)\}, \tag{4.15}$$

$$B_2 = \{E : 0 < p_i(E), 0 < r_i(E); p_i(E) + r_i(E) < d(E)\}. \tag{4.16}$$

First we prove

$$B_1 = \{\widehat{E}_1, ..., \widehat{E}_{\widehat{M}}\}. \tag{4.17}$$

To see this let $E \in B_1$. In this case at least one of the entries in the off-diagonal of the monodromy matrix $M(E,x_0)$ is different from zero by part (ii) of Proposition 3.1. Also, $M(E,x_0)$ has a double eigenvalue since $\Delta(E)^2-1=0$. This implies that $M(E,x_0)$ has only one linearly independent eigenvector and hence only one Floquet solution (up to constant multiples) exists. Thus $E \in \{\widehat{E}_j\}_{j=1}^{\widehat{M}}$. Conversely, if $E \in \{\widehat{E}_j\}_{j=1}^{\widehat{M}}$ then $M(E,x_0)$ is not diagonalizable. Therefore not both of its off-diagonal entries can be zero. This finishes the proof of part (i) of the theorem and shows that the inequality (4.1) is strict for $E \in B_1$.

By (3.25)–(3.27) there exists a $\Lambda > 0$ such that for $|E| \geqslant \Lambda$ the inequalities $p(E,x_0) \leqslant 1$, $r(E,x_0) \leqslant 1$, and $d(E) \leqslant 2$ hold. Assume also that $\max\{|\widehat{E}_1|, ..., |\widehat{E}_{\widehat{M}}|\} < \Lambda$. Then we have $d(E)=2$ for all (anti)periodic eigenvalues E for which $|E| \geqslant \Lambda$, since otherwise there would be only one linearly independent Floquet solution for $L\psi = E\psi$.

Assume now that $E \in B_2$ and that $|E| \geqslant \Lambda$. Then $p_i(E) \geqslant 1$, $r_i(E) \geqslant 1$ and $p(E,x_0) \leqslant 1$, $r(E,x_0) \leqslant 1$ and hence $p_i(E) = r_i(E) = 1$. Also $d(E) = 2 = p_i(E) + r_i(E)$ which contradicts the definition of B_2. Therefore no E whose absolute value is larger than Λ can be in B_2, i.e., B_2 is a finite set.

Next we show that $D_m(x_0)$ and $N_m(x_0)$ are also finite sets. Consider the periodic or antiperiodic eigenvalue E_{2n} where n is such that $|E_{2n}| \geqslant \Lambda$. Then $M(E_{2n},x_0)$ has a double eigenvalue ± 1 with geometric multiplicity two. This forces the diagonal elements of $M(E_{2n},x_0)$ to be zero, i.e., E_{2n} is both a Dirichlet and a Neumann eigenvalue for every $x_0 \in \mathbf{R}$. The asymptotic behavior of these eigenvalues shows that $E_{2n} = \mu_n(x_0) = \nu_n(x_0)$ for every $x_0 \in \mathbf{R}$. The same argument shows that $\mu_k(x_0)$ and $\nu_k(x_0)$ are immovable for all $k > n$. Thus, if $\mu_k(x_0)$ or $\nu_k(x_0)$ actually depend on x_0, then $k < n$ and its absolute value must be smaller than $|E_{2n}|$. Note that this conclusion only holds because B_1 is finite, since there are only finitely many values of E where $L\psi = E\psi$ has only one linearly independent Floquet solution.

Choosing n_0 to be any integer such that $|E_{2n_0}| \geqslant \Lambda$, the above considerations also show that the numbers

$$\sum_{E \in \mathbf{C}} p_m(E,x_0) = n_0 - \sum_{|E| \leqslant |E_{n_0}|} p_i(E), \tag{4.18}$$

$$\sum_{E \in \mathbf{C}} r_m(E,x_0) = n_0 + 1 - \sum_{|E| \leqslant |E_{n_0}|} r_i(E) \tag{4.19}$$

are independent of x_0. We call the former number g and the latter $\tilde{g}$. Thus the function $F_D(E,x_0)$ introduced in the proof of Theorem 3.2 is a polynomial in E of degree g. Moreover, the same considerations as in that proof show that $U(E,x_0)$ in (3.56) is a polynomial in E of degree $2g+1$ and satisfies

$$U(E,x_0)=\frac{4(\Delta(E)^2-1)}{D(E)^2}. \tag{4.20}$$

In particular, $U(E,x_0)$ is independent of x_0.

Let $\gamma(x_0)$ denote the leading coefficient of $F_D(E,x_0)$. From (3.56) we conclude that the leading coefficient of $U(E,x_0)$ equals $4\gamma(x_0)^2$ and hence $\gamma(x_0)$ does not depend on x_0. Therefore the function

$$\widehat{F}_g(E,x_0)=\frac{1}{\gamma(x_0)}F_D(E,x_0)=\prod_{\lambda\in D_m(x_0)}(E-\lambda)^{p_m(\lambda,x_0)} \tag{4.21}$$

satisfies

$$4(E-q(x))\widehat{F}_g(E,x)^2-2\widehat{F}_g(E,x)\widehat{F}_g''(E,x)+\widehat{F}_g'(E,x)^2=4\widehat{R}_{2g+1}(E), \tag{4.22}$$

where

$$\widehat{R}_{2g+1}(E)=\frac{\Delta(E)^2-1}{D(E)^2\gamma(x_0)^2}. \tag{4.23}$$

Hence the assumptions of Proposition 2.1 are satisfied. This proves that $q\in C^\infty(\mathbf{R})$. Now we may also apply Theorem 3.2 and obtain that $p_i(E)=r_i(E)\leqslant\frac{1}{2}d(E)$ for all $E\in\mathbf{C}$. Thus, except for $E\in B_1$, inequality (4.1) is strict only when $E\in B_2$. This proves parts (ii) and (iv) of the theorem. We denote the points in B_2 henceforth by $\widehat{E}_{\widehat{M}+1},\dots,\widehat{E}_M$ (B_2 may be empty). This implies that $\widehat{R}_{2g+1}$ in (4.23) agrees with the one given in (4.6).

We now turn to part (iii). Equation (4.2) is satisfied by definition of g since its left-hand side is independent of x_0. Equation (4.3), i.e., $\tilde{g}=g+1$ follows from $p_i(E)=r_i(E)$ and from equations (4.18) and (4.19). Equation (4.4), finally, follows from the fact that $U(E,x_0)$ is of degree $2g+1$ in E, from equation (4.20), from the fact that $p_i(E)=r_i(E)$, and from part (ii). Thus part (iii) is proved.

A straightforward application of Proposition 2.1 shows that q is an algebro-geometric finite-gap potential, i.e., that (4.8) is satisfied for a suitable differential expression $\widehat{P}_{2g+1}$. The validity of (4.9) follows from the formalism in (2.10)–(2.15) proving part (v).

Using (4.21) and (4.23) we may express the diagonal Green's function as

$$G(E,x_0,x_0)=\frac{-F_D(E,x_0)D(E)}{2[\Delta(E)^2-1]^{1/2}}=\frac{-\widehat{F}_g(E,x_0)}{2\widehat{R}_{2g+1}(E)^{1/2}}. \tag{4.24}$$

Note that this yields the known asymptotic behavior of the diagonal Green's function

$$G(E,x,x)=-\tfrac{1}{2}E^{-1/2}(1+O(1))\quad\text{as } E\to\infty. \tag{4.25}$$

(This asymptotic behavior is valid for any bounded $q\in C^0(\mathbf{R})$. In the special case of a periodic $q\in C^0(\mathbf{R})$ it also follows directly from iterating Volterra integral equations as in the proof of (3.22), (3.24).) This completes the proof of part (vi).

The proof of part (vii) relies on the relationship (3.36) between the Green's function of H and the Wronskian of Floquet solutions. It implies that

$$W(\psi_-(E,x,x_0),\psi_+(E,x,x_0))=\frac{-2\widehat{R}_{2g+1}(E)^{1/2}}{\widehat{F}_g(E,x_0)} \tag{4.26}$$

on the resolvent set of the operator H and, by continuity, for all $E\in\mathbf{C}\backslash D_m(x_0)$. Now given any $E\in\mathbf{C}$ choose x_0 such that $E\notin D_m(x_0)$. Then the Wronskian in (4.26) is equal to zero if and only if E is a zero of $\widehat{R}_{2g+1}$, i.e., if and only if $E\in\{\widehat{E}_1,...,\widehat{E}_M\}$. Thus $W(\psi_-,\psi_+)$ is not only zero for $E\in B_1$, where only one Floquet solution exists, but also for $E\in B_2$, where two Floquet solutions exist but where linearly independent Floquet solutions in the vicinity of E converge to (multiples of) just one of them.

To prove the final claim (viii) observe that according to (3.34) and (3.5) the spectral arcs are given by

$$\varrho(E)=e^{i\theta}=\Delta(E)+\sqrt{\Delta(E)^2-1}\,,\quad \theta\in\mathbf{R}. \tag{4.27}$$

In some vicinity of any point $\lambda\in\sigma(H)$ we have

$$\Delta(E)+\sqrt{\Delta(E)^2-1}=\varrho(\lambda)+(E-\lambda)^{k/2}A(E), \tag{4.28}$$

where k is a positive integer and A is analytic and nonzero. If k is even then $\frac{1}{2}k$ spectral arcs intersect in λ ($k=2$ being the generic case). If k is odd then k semi-arcs meet at λ. In particular, when $k=1$ then the arc ends in λ. Of course k may be odd only when $\Delta(\lambda)^2-1=0$. In this case, however, $k=d(\lambda)$. Since $d(\lambda)=p_i(\lambda)+r_i(\lambda)=2p_i(\lambda)$, except when $\lambda\in B_1\cup B_2$, we infer that k may be odd only for $\lambda\in B_1\cup B_2=\{\widehat{E}_1,...,\widehat{E}_M\}$. This proves that only finitely many spectral arcs can have an end point, i.e., a point where the arc cannot be analytically continued. □

To the best of our knowledge, Theorem 4.1 appears to be new in the present generality. Especially, our method of proof, relying on (3.38) and the new equation (3.43), appears to be without precedent. Moreover, for M to be strictly larger than $\widehat{M}$, i.e., for B_2 to be nonempty, it is necessary that $d(\lambda)\geqslant 3$ for some (anti)periodic eigenvalue λ. While it seems difficult to construct an explicit example where $B_2\neq\varnothing$, the very existence of this phenomenon has not been considered in previous work on the subject. In particular, references [30], [31], [40], [60], [61] treat potentials with $d(E)\leqslant 2$ and references [12], [13] require that algebraic and geometric multiplicities of all (anti)periodic eigenvalues coincide and hence also that $d(E)\leqslant 2$. Generically one has $\hat{q}_j=1$, $1\leqslant\widehat{M}=M$ (cf. [72]).

Remark 4.2 (pole structure of the Green's function). As Theorem 4.1 shows, it is precisely the multiplicity $\hat{q}_j$ of the branch and singular points in the Burchnall–Chaundy polynomial (4.9) which determines the singularity structure of the diagonal Green's function $G(E,x,x)$ of H. Moreover, since (see, e.g., [56])

$$G(E,x,x') = [G(E,x,x)G(E,x',x')]^{1/2} \exp\left[-\frac{1}{2}\int_{\min(x,x')}^{\max(x,x')} G(E,s,s)^{-1}\,ds\right], \quad (4.29)$$

this observation extends to the off-diagonal Green's function $G(E,x,x')$ of H as well.

While Theorem 4.1 concentrates on the finite-gap situation, the main objective in this paper, it is clear that an appropriate extension to $g\to\infty$ exists and produces a corresponding two-sheeted noncompact (open) Riemann surface of the type

$$F^2 = R_\infty(E) = (E-\widehat{E}_0)^{\hat{q}_0} \prod_{j\in\mathbf{N}} [(E-\widehat{E}_j)\Omega^2 j^{-2}\pi^{-2}]^{\hat{q}_j}. \quad (4.30)$$

The corresponding diagonal Green's function $G(E,x,x)$ of H then takes on the form

$$G(E,x,x) = C\,\frac{\prod_{m\in\mathbf{N}}\{[E-\mu_m(x)]\Omega^2 m^{-2}\pi^{-2}\}}{R_\infty(E)^{1/2}}. \quad (4.31)$$

We omit further details at this point.

5. A characterization of elliptic finite-gap potentials

In this section we prove our principal result, an explicit characterization of all elliptic finite-gap potentials, a problem posed, e.g., in [59, p. 152]. One of the two key ingredients in our main Theorem 5.7 (the other being Theorem 4.1) is a systematic use of a powerful theorem of Picard (see Theorem 5.1 below) concerning the existence of solutions which are elliptic functions of the second kind of ordinary differential equations with elliptic coefficients. As will be pointed out in Remark 5.8 at the end of this section, Theorem 5.7 sheds new light on Picard's theorem and can be viewed as a complement to this classical result.

We start with Picard's theorem. Due to our focus on finite-gap solutions of the KdV hierarchy we only state the result for second-order differential equations.

THEOREM 5.1 (see, e.g., [3, pp. 182–187], [44, pp. 375–376]). *Let Q be an elliptic function with fundamental periods $2\omega_1$ and $2\omega_3$. Consider the differential equation*

$$\psi''(z)+Q(z)\psi(z)=0, \quad z\in\mathbf{C}, \quad (5.1)$$

and assume that (5.1) *has a meromorphic fundamental system of solutions. Then there exists at least one solution* ψ_1 *which is elliptic of the second kind, i.e.,* ψ_1 *is meromorphic and*

$$\psi_1(z+2\omega_j)=\varrho_j\psi_1(z), \quad j=1,3, \tag{5.2}$$

for some constants $\varrho_1, \varrho_3 \in \mathbf{C}$. *If in addition, the characteristic equation corresponding to the substitution* $z\to z+2\omega_1$ *or* $z\to z+2\omega_3$ *(see* [44, pp. 358, 376]) *has distinct roots then there exists a fundamental system of solutions of* (5.1) *which are elliptic functions of the second kind.*

The characteristic equation associated with the substitution $z\to z+2\omega_j$ alluded to in Theorem 5.1 is given by

$$\det[A-\varrho I]=0, \tag{5.3}$$

where

$$\phi_l(z+2\omega_j)=\sum_{k=1}^{2} a_{l,k}\phi_k(z), \quad A=(a_{l,k})_{1\leqslant l,k\leqslant 2}, \tag{5.4}$$

and ϕ_1, ϕ_2 is any fundamental system of solutions of (5.1).

What we call Picard's theorem following the usual convention in [3, pp. 182–185], [16, pp. 338–343], [41, pp. 536–539], [48, pp. 181–189], appears, however, to have a longer history. In fact, Picard's investigations [63], [64], [65] were inspired by earlier work of Hermite in the special case of Lamé's equation [42, pp. 118–122, 266–418, 475–478] (see also [8, §3.6.4] and [80, pp. 570–576]). Further contributions were made by Mittag-Leffler [55], and Floquet [27], [28], [29]. Detailed accounts on Picard's differential equation can be found in [41, pp. 532–574], [48, pp. 198–288].

In this context it seems appropriate to recall the well-known fact (see, e.g., [3, pp. 185–186]) that ψ_1 is elliptic of the second kind if and only if it is of the form

$$\psi_1(z)=Ce^{\lambda z}\prod_{j=1}^{m}\frac{\sigma(z-a_j)}{\sigma(z-b_j)} \tag{5.5}$$

for suitable $m\in\mathbf{N}$ and $C,\lambda,a_j,b_j\in\mathbf{C}$, $1\leqslant j\leqslant m$. Here $\sigma(z):=\sigma(z;\omega_1,\omega_3)$ is the Weierstrass sigma function associated with the period lattice Λ spanned by $2\omega_1$ and $2\omega_3$ (see [1, Chapter 18]).

Picard's Theorem 5.1 motivates the following definition.

Definition 5.2. Let q be an elliptic function. Then q is called a *Picard potential* if and only if the differential equation

$$\psi''(z)+q(z)\psi(z)=E\psi(z) \tag{5.6}$$

has a meromorphic fundamental system of solutions (with respect to z) for each value of the spectral parameter $E\in\mathbf{C}$.

For completeness we recall the following result proven in [38].

THEOREM 5.3. (i) *Any nonconstant Picard potential q has a representation of the form*

$$q(z)=C-\sum_{j=1}^{m}s_j(s_j+1)\mathcal{P}(z-b_j) \tag{5.7}$$

for suitable $m, s_j\in\mathbf{N}$ and $C, b_j\in\mathbf{C}$, $1\leqslant j\leqslant m$, where the b_j are pairwise distinct $\operatorname{mod}(\Lambda)$ *and $\mathcal{P}(z):=\mathcal{P}(z;\omega_1,\omega_3)$ denotes the Weierstrass $\mathcal{P}$-function associated with the period lattice* Λ ([1, Chapter 18]).

(ii) *Let $q(z)$ be given as in* (5.7). *If $\psi''+q\psi=E\psi$ has a meromorphic fundamental system of solutions for a number of distinct values of E which exceeds* $\max\{s_1,\dots,s_m\}$, *then q is a Picard potential.*

We emphasize that while any Picard potential is necessarily of the form (5.7), a potential q of the type (5.7) is a Picard potential only if the constants b_j satisfy a series of additional intricate constraints, see, e.g., §3.2 in [38].

The following result indicates the connection between Picard potentials and elliptic finite-gap potentials.

THEOREM 5.4 ([46], [49], [50], [67]). *Every elliptic finite-gap potential q is a Picard potential.*

Proof. For nonsingular curves $K_g\colon F^2=\prod_{j=0}^{2g}(E-\widehat{E}_j)$, where $\widehat{E}_l\neq\widehat{E}_{l'}$ for $l\neq l'$, associated with q (see (2.21)) this fact is obvious from the standard representation of the Baker–Akhiezer function in terms of the Riemann theta function of K_g ([21], [46], [49], [50]). For singular curves K_g the result follows from the τ-function representation of the Floquet solutions $\psi_\pm(E,x)$ associated with q,

$$\psi_\pm(E,x)=e^{\pm k(E)x}\frac{\tau_\pm(E,x)}{\tau(x)}, \tag{5.8}$$

where

$$q(x)=C+2\{\ln[\tau(x)]\}'' \tag{5.9}$$

and from the fact that $\tau(x)$ and $\tau_\pm(E,x)$ are entire with respect to x (cf. [67]). □

Naturally, one is tempted to conjecture that the converse of Theorem 5.4 is true as well. The rest of this section will be devoted to a proof of this conjecture.

We start with a bit of notation. Let $q(z)$ be an elliptic function with fundamental periods $2\omega_1, 2\omega_3$ and assume, without loss of generality, that $\operatorname{Re}(\omega_1)>0$, $\operatorname{Re}(\omega_3)\geqslant 0$,

$\mathrm{Im}(\omega_3/\omega_1)>0$. The fundamental period parallelogram Δ consists then of the points $z=2\omega_1 s+2\omega_3 t$ where $0\leqslant s,t<1$.

We introduce

$$e^{i\theta}=\frac{\omega_3}{\omega_1}\left|\frac{\omega_1}{\omega_3}\right|,\quad \theta\in(0,\pi), \tag{5.10}$$

and

$$t_j=\frac{\omega_j}{|\omega_j|},\quad j=1,3, \tag{5.11}$$

and define

$$q_j(x):=t_j^2 q(t_j x+z_0),\quad j=1,3, \tag{5.12}$$

for a $z_0\in\mathbf{C}$ which we choose in such a way that no pole of q_j, $j=1,3$, lies on the real axis. (This is equivalent to the requirement that no pole of q lies on the line through the points z_0 and $z_0+2\omega_1$ or on the line through z_0 and $z_0+2\omega_3$. Since q has only finitely many poles in the fundamental period parallelogram Δ this can always be achieved.) For such a choice of z_0 we infer that $q_j(x)$ are real-analytic and periodic of period $\Omega_j=2|\omega_j|$, $j=1,3$. Comparing the differential equations

$$\psi''(z)+q(z)\psi(z)=E\psi(z) \tag{5.13}$$

and

$$w''(x)+q_j(x)w(x)=\lambda w(x),\quad j=1,3, \tag{5.14}$$

connected by the variable transformation

$$z=t_j x+z_0,\quad \psi(z)=w(x), \tag{5.15}$$

one concludes that w is a solution of (5.14) if and only if ψ is a solution of (5.13) with

$$\lambda=t_j^2 E,\quad j=1,3. \tag{5.16}$$

Next, consider $\tilde{q}\in C^0(\mathbf{R})$ of period $\tilde{\Omega}>0$ and let $\tilde{c}(\lambda,x),\tilde{s}(\lambda,x)$ be the corresponding fundamental system of solutions of $\tilde{w}''+\tilde{q}\tilde{w}=\lambda\tilde{w}$ defined by

$$\tilde{c}(\lambda,0)=\tilde{s}'(\lambda,0)=1,\quad \tilde{c}'(\lambda,0)=\tilde{s}(\lambda,0)=0. \tag{5.17}$$

The corresponding Floquet discriminant is then given by

$$\tilde{\Delta}(\lambda)=\tfrac{1}{2}[\tilde{c}(\lambda,\tilde{\Omega})+\tilde{s}(\lambda,\Omega)] \tag{5.18}$$

and the same techniques that lead to the asymptotic expansion (3.24) also yield

$$\tilde{\Delta}(\lambda)=\cos[i\tilde{\Omega}\lambda^{1/2}(1+O(\lambda^{-1}))] \tag{5.19}$$

as $|\lambda|$ tends to infinity.

PROPOSITION 5.5. *Let $\tilde{\lambda}_n$ be a periodic or antiperiodic eigenvalue of $\tilde{q}$. Then there exists an $m\in\mathbf{Z}$ such that*

$$|\tilde{\lambda}_n+m^2\pi^2\tilde{\Omega}^{-2}|\leqslant\tilde{C} \tag{5.20}$$

for some $\tilde{C}>0$ independent of $n\in\mathbf{N}_0$. In particular, all periodic and antiperiodic eigenvalues $\tilde{\lambda}_n$, $n\in\mathbf{N}_0$, of $\tilde{q}$ are contained in a half-strip $\tilde{S}$ given by

$$\tilde{S}=\{\lambda\in\mathbf{C}:|\mathrm{Im}(\lambda)|\leqslant\tilde{C},\,\mathrm{Re}(\lambda)\leqslant\tilde{M}\} \tag{5.21}$$

for some $\tilde{M}\in\mathbf{R}$.

Proof. The periodic and antiperiodic eigenvalues are precisely the points λ_0 where $\tilde{\Delta}(\lambda_0)=\pm1$. Let

$$\tilde{\Omega}\tilde{\lambda}_n^{1/2}=a_n+ib_n,\quad a_n,b_n\in\mathbf{R}. \tag{5.22}$$

Then (5.19) implies

$$m\pi+b_n-ia_n=O((a_n+ib_n)^{-1}) \tag{5.23}$$

for some $m\in\mathbf{Z}$. Hence

$$|m\pi+b_n|\leqslant c_1,\quad |a_n|\leqslant c_1 \tag{5.24}$$

and (multiplying (5.23) by a_n+ib_n and taking real and imaginary parts)

$$|a_n(m\pi+2b_n)|\leqslant c_2,\quad |b_nm\pi+b_n^2-a_n^2|\leqslant c_2 \tag{5.25}$$

for some constants $c_1,c_2>0$. For $|\tilde{\lambda}_n|=(a_n^2+b_n^2)\tilde{\Omega}^{-2}$ sufficiently large we conclude that $|b_n|$ and hence $|m\pi|$ and $|m\pi+2b_n|$ are also large since $|a_n|$ stays bounded. By (5.24) one obtains that $|m\pi|\leqslant|b_n|+c_1$ and hence (5.25) and $|c_1|\leqslant\frac{1}{2}|b_n|$ imply

$$|a_n|\leqslant\frac{c_2}{|m\pi+2b_n|}\leqslant\frac{c_2}{2|b_n|-|m\pi|}\leqslant\frac{c_2}{|b_n|-c_1}\leqslant\frac{2c_2}{|b_n|} \tag{5.26}$$

and

$$|b_n(m\pi+b_n)|\leqslant a_n^2+c_2\leqslant c_1^2+c_2, \tag{5.27}$$

i.e.,

$$|m\pi+b_n|\leqslant\frac{c_1^2+c_2}{|b_n|}. \tag{5.28}$$

Consequently, we infer from (5.26), (5.28) and from the fact that $|a_n|$ is bounded that

$$|\tilde{\Omega}\tilde{\lambda}_n^{1/2}+i\pi m|=|a_n+ib_n+i\pi m|\leqslant|a_n|+|b_n+\pi m|\leqslant c|b_n|^{-1}\leqslant c'|\tilde{\lambda}_n|^{-1/2} \tag{5.29}$$

for some constants c and c'. Multiplying (5.29) by $|\tilde{\Omega}\tilde{\lambda}_n^{1/2}-i\pi m|$ finally results in

$$|\tilde{\Omega}^2\tilde{\lambda}_n+\pi^2m^2|\leqslant c'|\tilde{\lambda}_n|^{-1/2}|\tilde{\Omega}\tilde{\lambda}_n^{1/2}-i\pi m| \tag{5.30}$$

$$\leqslant c'|\tilde{\lambda}_n|^{-1/2}[2\tilde{\Omega}|\tilde{\lambda}_n|^{1/2}+|-\tilde{\Omega}\tilde{\lambda}_n^{1/2}-i\pi m|] \tag{5.31}$$

$$\leqslant c'|\tilde{\lambda}_n|^{-1/2}[2\tilde{\Omega}|\tilde{\lambda}_n|^{1/2}+c'|\tilde{\lambda}_n|^{-1/2}]\leqslant \tilde{C}. \tag{5.32}$$

Hence $\tilde{\lambda}_n$ is in a disk around $-m^2\pi^2\tilde{\Omega}^{-2}$ whose radius is independent of n. □

In order to apply Proposition 5.5 to q_1 and q_3 we note that according to (5.19),

$$\Delta_j(\lambda)=\cos[i\Omega_j\lambda^{1/2}(1+O(\lambda^{-1}))], \quad j=1,3, \tag{5.33}$$

as $|\lambda|$ tends to infinity, where, in obvious notation, $\Delta_j(\lambda)$ denotes the discriminant of $q_j(x)$, $j=1,3$. Next, denote by $\lambda_{j,n}$ an Ω_j-(anti)periodic eigenvalue of $w''+q_jw=\lambda w$. Then $E_{j,n}=t_j^{-2}\lambda_{j,n}$ is a $2\omega_j$-(anti)periodic eigenvalue of $\psi''+q\psi=E\psi$ and vice versa. Hence Proposition 5.5 immediately yields the following result.

PROPOSITION 5.6. *Let $j=1$ or 3. Then all the $2\omega_j$-(anti)periodic eigenvalues $E_{j,n}$, $n\in\mathbf{N}_0$, associated with q lie in the half-strip S_j given by*

$$S_j=\{E\in\mathbf{C}:|\mathrm{Im}(t_j^2E)|\leqslant C_j,\ \mathrm{Re}(t_j^2E)\leqslant M_j\} \tag{5.34}$$

for suitable constants $C_j>0$, $M_j\in\mathbf{R}$. The angle between the axes of the strips S_1 and S_3 equals $2\theta\in(0,2\pi)$.

Propositions 5.5 and 5.6 apply to any elliptic potential whether or not they are finite-gap. In our final step we shall now invoke Picard's Theorem 5.1 to obtain our characterization of elliptic finite-gap potentials.

THEOREM 5.7. *q is an elliptic finite-gap potential if and only if q is a Picard potential (i.e., if and only if for each $E\in\mathbf{C}$ every solution of $\psi''(z)+q(z)\psi(z)=E\psi(z)$ is meromorphic with respect to z).*

Proof. By Theorem 5.4 it remains to prove that a Picard potential is finite-gap. Hence we assume in the following that q is a Picard potential. Since all $2\omega_j$-(anti)periodic eigenvalues $E_{j,n}$ of q yield zeros $\lambda_{j,n}=t_j^2E_{j,n}$ of the entire functions $\Delta_j(\lambda)^2-1$, the $E_{j,n}$ have no finite limit point. Next we choose $R>0$ large enough such that the exterior of the closed disk $\overline{D(0,R)}$ centered at the origin of radius $R>0$ contains no intersection of S_1 and S_3 (defined in (5.34), i.e.,

$$(\mathbf{C}\backslash\overline{D(0,R)})\cap(S_1\cap S_3)=\varnothing. \tag{5.35}$$

Let $\varrho_{j,\pm}(\lambda)$ be the Floquet multipliers of $q_j(x)$, i.e., the solutions of

$$\varrho_j^2 - 2\Delta_j \varrho_j + 1 = 0, \quad j = 1, 3. \tag{5.36}$$

Then (5.35) implies that for $E \in \mathbf{C} \setminus \overline{D(0,R)}$, at most one of the numbers $\varrho_1(t_1 E)$ and $\varrho_3(t_3 E)$ can be in $\{-1, 1\}$. In particular, at least one of the characteristic equations corresponding to the substitution $z \to z + 2\omega_1$ or $z \to 2\omega_3$ (cf. (5.3) and (5.4)) has two distinct roots. Since by hypothesis q is a Picard potential, Picard's Theorem 5.1 applies and guarantees for all $E \in \mathbf{C} \setminus \overline{D(0,R)}$ the existence of two linearly independent solutions $\psi_1(E,z)$ and $\psi_2(E,z)$ of $\psi'' + q\psi = E\psi$ which are elliptic of the second kind. Then $w_{j,k}(x) = \psi_k(t_j x + z_0)$, $k = 1, 2$, are linearly independent Floquet solutions associated with q_j. Therefore the points λ for which $w'' + q_j w = \lambda w$ has only one Floquet solution are necessarily contained in $\overline{D(0,R)}$ and hence finite in number. This is true for both $j = 1$ and $j = 3$. Applying Theorem 4.1 then proves that both q_1 and q_3 are finite-gap potentials.

By (2.8) (in slight abuse of notation)

$$\sum_{k=0}^{g} c_{g-k} \frac{df_{k+1}(q_1(x))}{dx} = 0, \tag{5.37}$$

where $g \in \mathbf{N}_0$, f_{k+1}, $k = 0, \ldots, g$, are differential polynomials in q_1 homogeneous of degree $2k+2$ (cf. (2.9)), and c_k, $k = 0, \ldots, g$, are complex constants. Since

$$q_1^{(l)}(x) = t_1^{l+2} q^{(l)}(z) \tag{5.38}$$

(where $z = t_1 x + z_0$), we obtain

$$\sum_{k=0}^{g} c_{g-k} t_1^{2k+3} \frac{df_{k+1}(q(z))}{dz} = 0, \tag{5.39}$$

i.e., q is a finite-gap potential as well. A similar argument would have worked using the relationship between q_3 and q. In particular, the order of the operators commuting with $d^2/dz^2 + q(z)$, $d^2/dx^2 + q_1(x)$, and $d^2/dx^2 + q_3(x)$, respectively, is the same in all cases, namely $2g+1$. □

We add a series of remarks further illustrating the significance of Theorem 5.7.

Remark 5.8 (complementing Picard's theorem). Theorem 5.7 extends and complements Picard's Theorem 5.1 in the sense that it determines the elliptic functions $q(z)$ which satisfy the hypothesis of the theorem precisely as (elliptic) finite-gap solutions of the stationary KdV hierarchy.

Remark 5.9 (characterization of elliptic finite-gap potentials). While an explicit proof of the finite-gap property of q in general is highly nontrivial (see, e.g., the references cited in connection with special cases such as the Lamé–Ince and Treibich–Verdier potentials in Remark 5.11 below), the fact of whether or not $\psi''(z)+q(z)\psi(z)=E\psi(z)$ has a fundamental system of solutions meromorphic in z for a finite (but sufficiently large) number of the spectral parameter $E\in\mathbf{C}$ can be decided by means of an elementary Frobenius-type analysis (see, e.g., [36] and [37]). To date Theorem 5.7 appears to be the only effective tool to identify general elliptic finite-gap solutions of the KdV hierarchy. Thus Theorem 5.7 provides an explicit characterization of all elliptic finite-gap solutions of the stationary KdV hierarchy, a problem posed, e.g., by Novikov et al. in [59, p. 152].

Remark 5.10 (reduction of Abelian integrals). Theorem 5.7 is also relevant in the context of the Weierstrass theory of reduction of Abelian to elliptic integrals, a subject that attracted considerable interest, see, e.g., [6], [7], [8, Chapter 7], [9], [10], [23], [24], [25], [45], [47], [51], [68], [70]. In particular, the theta functions corresponding to the hyperelliptic curves derived from the Burchnall–Chaundy polynomials (2.15), associated with Picard potentials, reduce to one-dimensional theta functions.

Remark 5.11 (computation of genus and branch points). While Theorem 5.7 characterizes all elliptic finite-gap potentials as Picard potentials, it does not yield an effective way to compute the underlying hyperelliptic curve K_g; in particular, its proof provides no means to compute the branch and singular points nor the (arithmetic) genus g of K_g. To the best of our knowledge K_g has been computed only for Lamé–Ince potentials and certain Treibich–Verdier potentials (see, e.g., [5], [10], [47], [59], [68], [70], [77], [79], [80]). Even the far simpler task of computing g has only been achieved in the case of Lamé–Ince potentials (see [43] and [74] for the real and complex-valued case, respectively). In [36], [37], and [38] we have treated these problems for Lamé–Ince, Treibich–Verdier, and elliptic finite-gap potentials even with respect to some $z_0\in\mathbf{C}$, respectively. In particular, in [37] we computed g for all Treibich–Verdier potentials and in [38] we reduced the computation of the branch and singular points of K_g for any even elliptic finite-gap potential to the solution of linear algebraic eigenvalue problems. We refrain from reproducing a detailed discussion of this matter here, instead we just recall an example taken from [37] which indicates some of the subtleties involved: Consider the potentials

$$q_4(z)=-20\mathcal{P}(z-\omega_j)-12\mathcal{P}(z-\omega_k), \tag{5.40}$$

$$\hat{q}_4(z)=-20\mathcal{P}(z-\omega_j)-6\mathcal{P}(z-\omega_k)-6\mathcal{P}(z-\omega_l), \tag{5.41}$$

$$q_5(z)=-30\mathcal{P}(z-\omega_j)-2\mathcal{P}(z-\omega_k), \tag{5.42}$$

$$\hat{q}_5(z)=-12\mathcal{P}(z-\omega_j)-12\mathcal{P}(a-\omega_k)-6\mathcal{P}(z-\omega_l)-2\mathcal{P}(z-\omega_m), \tag{5.43}$$

where $j,k,l,m\in\{1,2,3,4\}$ ($\omega_2=\omega_1+\omega_3$, $\omega_4=0$) are mutually distinct. Then q_4 and $\hat{q}_4$ correspond to (arithmetic) genus $g=4$ while q_5 and $\hat{q}_5$ correspond to $g=5$. However, we emphasize that all four potentials contain precisely 16 summands of the type $-2\mathcal{P}(x-b_n)$ (cf. the discussion following (1.10)). q_5 and $\hat{q}_5$ are isospectral (i.e., correspond to the same curve K_5) while q_4 and $\hat{q}_4$ are not.

Remark 5.12 (generalizations). Finally, we remark that Theorems 5.1 and 5.4 and Propositions 5.5 and 5.6 extend to nth order operators L_n. We have decided to restrict this paper to the second-order case $L_2=d^2/dx^2+q(x)$ since the corresponding generalization of Theorem 4.1 to algebro-geometric finite-gap solutions of the stationary Gelfand–Dickey (GD) hierarchy is beyond the scope of this paper. (Even though a recursion relation formalism for the GD hierarchy analogous to the KdV case in (2.1)–(2.21) exists in principle, the explicit construction of a monic differential expression P_r of order r (r and n relatively prime) commuting with L_n, along the lines of our proof of Theorem 4.1, is a formidable task which obscures the remarkable simplicity of our argument displayed in the proof of Proposition 5.5.) Here we just mention the fact that if L_n is a Picard differential expression (in the sense that $L_n\psi(z)=E\psi(z)$ has a fundamental system of solutions meromorphic in z for each $E\in\mathbf{C}$) then the number of E-values where there exist less than n Floquet solutions for $L_n\psi=E\psi$ is finite in number.

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