A Minkowski problem for electrostatic capacity

by

DAVID JERISON

Massachusetts Institute of Technology
Cambridge, MA, U.S.A.

Dedicated to my father, Meyer Jerison

Introduction

The Minkowski problem is to find a convex polyhedron from data consisting of normals to the faces and their surface areas. The corresponding problem for convex bodies with smooth boundaries is to find the convex body given the Gauss curvature of its boundary as a function of the unit normal. There is a natural notion of variation of a convex domain introduced by Minkowski, which is essentially variation of the boundary of the domain in the direction normal to the domain. Under this definition, the first variation of the volume of a convex body is the surface area measure on its boundary. The purpose of this paper is to develop a theory analogous to the one for the Minkowski problem in which volume is replaced by electrostatic capacity and surface area is replaced by the first variation of capacity. This theory was proposed in the paper [J].

Let $N \geqslant 3$ and let Ω be a bounded, convex, open subset of $\mathbf{R}^N$. Let Ω' be the complement of the closure of Ω. The equilibrium potential of Ω is the continuous function U defined in $\bar{\Omega}'$ satisfying

$$\Delta U = 0 \ \text{ in } \Omega' \quad \text{and} \quad U = 1 \ \text{ on } \partial\Omega'$$

and such that U tends to zero at infinity. Let $n = N - 1$ and define the dimensional constant

$$a_N = \frac{1}{(N-2)\operatorname{vol}(S^n)},$$

where $\operatorname{vol}(S^n)$ is the volume of the unit sphere S^n in $\mathbf{R}^N$. Then $a_N|x|^{2-N}$ is the fundamental solution to Laplace's equation, i.e., $\Delta a_N|x|^{2-N} = -\delta_0$. It is well known that U

has the asymptotic expansion

$$U(x) = \gamma a_N |x|^{2-N} + O(|x|^{1-N}) \quad \text{as } x \to \infty \tag{0.1}$$

for some positive constant γ. The constant γ is known as the electrostatic capacity of Ω,

$$\gamma = \operatorname{cap} \Omega.$$

Let P be a support plane of Ω, that is, a hyperplane tangent to $\partial\Omega$. We will always use the convention that the unit normal ξ to P points away from Ω and into Ω'. The Minkowski support function of Ω is the function $u(\xi)$ given by

$$u(\xi) = x \cdot \xi$$

for any point $x \in P \cap \partial\Omega$. In other words, $u(\xi)$ is the (signed) distance from P to the origin. Let $g: \partial\Omega \to S^n$ be the Gauss map, that is, the mapping from $x \in \partial\Omega$ to the outer unit normal. The mapping g is defined almost everywhere with respect to surface measure $d\sigma$ on $\partial\Omega$. We will frequently identify the boundary of Ω with the unit sphere by the Gauss map. In particular, we will abuse notation by considering the support function as a function on $\partial\Omega$: $u(x) = u(g(x))$ is defined almost everywhere on $\partial\Omega$.

The starting point for this article is a formula due to Poincaré [P],

$$\operatorname{cap} \Omega = \frac{1}{N-2} \int_{\partial\Omega} u |\nabla U|^2 \, d\sigma. \tag{0.2}$$

In the case of a smooth domain, the formula can be proved by integration by parts. We will show in the first section that it is valid for arbitrary convex domains. This formula bears a strong resemblance to the formula for volume of Ω,

$$\operatorname{vol} \Omega = \frac{1}{N} \int_{\partial\Omega} u \, d\sigma. \tag{0.3}$$

The analogy extends to the first variation. Let Ω_1 be another bounded, convex domain with Minkowski support function u_1. The algebraic sum

$$\Omega + t\Omega_1 = \{x + ty : x \in \Omega \text{ and } y \in \Omega_1\}$$

gives a region whose boundary is the variation of $\partial\Omega$ by the distance $tu_1(g(x))$ in the outer normal direction $g(x)$ at x. We then have [BF]

$$\frac{d}{dt} \operatorname{vol}(\Omega + t\Omega_1)\Big|_{t=0} = \int_{\partial\Omega} u_1 \, d\sigma \tag{0.4}$$

and Hadamard's variational formula [GS]

$$\frac{d}{dt} \operatorname{cap}(\Omega + t\Omega_1)\Big|_{t=0} = \int_{\partial\Omega} u_1 |\nabla U|^2 \, d\sigma. \tag{0.5}$$

Consider the measure $d\mu = g_*(d\sigma)$ defined on the unit sphere by

$$\mu(E) = \int_{g^{-1}(E)} d\sigma \quad \text{for any Borel set } E \subset S^n. \tag{0.6}$$

This is well-defined because g is defined almost everywhere with respect to $d\sigma$. Note, however, that the measure μ need not be continuous with respect to the uniform measure on S^n. For example, when Ω is a polyhedron the measure is a sum of point masses at the normals to each of the faces and the coefficient at a normal is the surface area of that face. Minkowski posed the problem of discovering which measures arise in this way. Comprehensive existence, uniqueness and regularity results for this problem can be stated as follows.

THEOREM 0.7. *Let $N \geqslant 2$. Let μ be a positive measure on the sphere S^n. There exists a bounded, convex domain Ω such that $g_*(d\sigma) = d\mu$ if and only if μ satisfies*

$$\int_{S^n} \xi_k \, d\mu(\xi) = 0, \quad k = 1, ..., N,$$

and μ is not supported on any equator (the intersection of the sphere with any hyperplane through the origin). Moreover, the domain Ω is unique up to translation. Suppose that k is a nonnegative integer and $0 < \alpha < 1$. If, in addition, $d\mu = (1/K)\, d\xi$ for some strictly positive function $K \in C^{k,\alpha}(S^n)$, then Ω is $C^{k+2,\alpha}$.

The case of polyhedra was solved by Minkowski and existence was proved, in general, by Alexandrov. C^∞ regularity was proved by Pogorelov, Nirenberg, Cheng and Yau [CY]. The precise gain of two derivatives and the treatment of small values of k is due to Caffarelli [C1], [C2], [C3], [C4]. The function K is the Gauss curvature of the boundary $\partial\Omega$, which explains why, in the smooth case, the Minkowski problem can also be phrased in terms of Gauss curvature.

The main result of this paper is the analogous theorem to Theorem 0.7 with the density $|\nabla U|^2 \, d\sigma$ in place of $d\sigma$. The theorem of Dahlberg [D] implies that $|\nabla U|^2 \, d\sigma$ is mutually absolutely continuous with $d\sigma$. It follows that $g_*(|\nabla U|^2 \, d\sigma)$ is a well-defined measure on the unit sphere. The main result is stated as follows.

THEOREM 0.8. *Let $N \geqslant 4$. Let μ be a positive measure on the sphere S^n. There exists a bounded convex body Ω such that $g_*(|\nabla U|^2 \, d\sigma) = d\mu$ if and only if μ satisfies*

$$\int_{S^n} \xi_k \, d\mu(\xi) = 0, \quad k = 1, ..., N,$$

and μ is not supported on any equator (the intersection of the sphere with any hyperplane through the origin). Moreover, the body Ω is unique up to translation. Suppose that k is a nonnegative integer and $0<\alpha<1$. If, in addition, $d\mu=R\,d\xi$ for some strictly positive density $R\in C^{k,\alpha}(S^n)$, then Ω is $C^{k+2,\alpha}$.

If $N=3$, then the theorem has a slightly different statement because the problem is not only translation invariant, but also dilation invariant. For each μ satisfying the necessary conditions of Theorem 0.8, there is a unique $\lambda>0$ such that $g_*(|\nabla U|^2\,d\sigma)=\lambda\mu$ for some convex body Ω. The body Ω is unique up to translation and dilation. We have not treated the case of logarithmic capacity ($N=2$).

The theorems described here fall into three parts: existence, uniqueness and regularity. The basic outline of the proof is closely linked to Minkowski's original ideas, the work of Cheng and Yau [CY], Pogorelov, and the direct approach of Caffarelli. However, most of the details are different. We will describe the main steps in the proof of Theorem 0.8 and this description will serve as an outline of the contents of the paper.

In §1 we review the properties of the equilibrium potential U, including its boundary behavior in the nonsmooth case. We deduce formula (0.2) in the nonsmooth case. In §2, we prove formulas for the first and second variation of capacity in the smooth case. In §3, we prove a delicate continuity property for the first variation that is used later in the existence part of Theorem 0.8 and also to extend the validity of the first variation formula to nonsmooth convex domains. In §4 we prove *a priori* estimates on the inradius and diameter of Ω needed for the proof of existence. In §5, we prove existence for polyhedra by formulating a variational problem. Existence for general convex bodies is then deduced from the polyhedral case by a weak limiting procedure. We emphasize that on the one hand, the *a priori* estimates of §4 depend on the fact that the power of $|\nabla U|$ is at least 2, and on the other hand the limiting procedure does not work and the definition of the density does not even make sense for a power of $|\nabla U|$ greater than 2.

In §6 we begin the proof of regularity by proving new estimates for the density $|\nabla U|^2$ on the boundary of a convex domain. As in the treatment of the interior problem in [J] it is necessary to estimate this density on "slices" of the region by any hyperplane. However, in contrast to the case of interior estimates, the doubling condition fails. Instead there is a weaker estimate (Theorem 6.5). We then develop a stronger version of the regularity theory of the Monge–Ampère equation (Lemma 7.3) that can make use of this weaker estimate on the right-hand side. The conclusion of Theorem 6.5 and the hypothesis of Lemma 7.3 are in the nature of best possible, and it is lucky that they are the same. §8 illustrates this lucky coincidence by giving examples to show that the estimates of the paper are sharp and several reasons why the exponent 2 of $|\nabla U|^2$ is the only one possible.

The *a priori* inequalities used in the proof of existence depend on an isoperimetric inequality due to Christer Borell. (Note, however, that we do not need the exact constant in the isoperimetric inequality.) This isoperimetric inequality is a consequence of Borell's Brunn–Minkowski inequality for capacity:

THEOREM 0.9. *Let* $\Omega_t = t\Omega_1 + (1-t)\Omega_0$ *be a convex combination of two convex regions in* $\mathbf{R}^N$. *Then*

$$(\operatorname{cap} \Omega_t)^{1/(N-2)} \geqslant (1-t) \operatorname{cap} \Omega_0^{1/(N-2)} + t \operatorname{cap} \Omega_1^{1/(N-2)}$$

for $0 \leqslant t \leqslant 1$.

(Theorem 0.9 was conjectured in the paper [J, p. 398]. It was pointed out to us by G. Philippin and L. E. Payne that the theorem had been proved long before in [B].) We will also use the Borell inequality to assist in the calculation of the first variation of capacity in §3. This is more a matter of convenience than essence. However, the Borell inequality is essential for uniqueness.

As we will show in [CJL]:

THEOREM 0.10. *There is equality in the inequality of Theorem 0.9 if and only if* Ω_1 *is a translate and dilate of* Ω_0.

Theorems 0.9 and 0.10 imply the uniqueness part of Theorem 0.8 in essentially the same way that the case of equality in the ordinary Brunn–Minkowski inequality implies uniqueness in the classical Minkowski problem. The details will not be carried out here: we refer the reader to [CJL].

1. Capacity and the equilibrium distribution

In this section we review properties of surface measure, harmonic measure and the equilibrium potential U. Our main goal is to confirm formula (0.2) for capacity. We will first prove it for smooth domains. We will deduce it for general convex domains by taking a limit. We will use the abbreviation

$$|\nabla U(x)| = h \quad \text{on } \partial\Omega.$$

PROPOSITION 1.1. *Let* Ω *be a smooth, bounded convex domain in* $\mathbf{R}^N$. *Let* V *be a harmonic function in* $\Omega' = \mathbf{R}^N \setminus \bar{\Omega}$ *that is continuous in* $\bar{\Omega}'$ *and tends to zero at infinity. Then there is a number* V_∞ *such that*

$$V(x) = V_\infty a_N |x|^{2-N} + O(|x|^{1-N}) \quad \text{as } x \to \infty$$

and

$$V_\infty = \int_{\partial\Omega} Vh\, d\sigma.$$

Proof. A standard removable singularities theorem implies that the Kelvin transform of V, $|x|^{2-N}V(x/|x|^2)$, is infinitely differentiable at the origin. It follows that

$$V(x) = V_\infty a_N |x|^{2-N} + O(|x|^{1-N}) \quad \text{as } x \to \infty,$$

$$x \cdot \nabla V(x) = V_\infty a_N (2-N)|x|^{2-N} + O(|x|^{1-N}) \quad \text{as } x \to \infty.$$

In particular, these asymptotics are valid for U. Let B_R be the ball around the origin of radius R. Choose R sufficiently large that $\Omega \subset B_R$. Green's formula implies

$$0 = \int_{B_R \setminus \Omega} (V(x)\Delta(1-U(x)) - \Delta V(x)(1-U(x)))\, dx$$

$$= \int_{\partial B_R} \left(V(x)\frac{x}{R}\nabla(1-U(x)) - \frac{x}{R}\nabla V(x)(1-U(x)) \right) d\sigma(x)$$

$$- \int_{\partial\Omega} V(x)g(x)\cdot\nabla(1-U(x))\, d\sigma(x).$$

Taking the limit as R tends to infinity, we find the formula in Proposition 1.1.

The value V_∞ is the value at the origin of the Kelvin transform of the harmonic function V. In that sense, $h\, d\sigma$ can be viewed as harmonic measure at infinity. However, as the special case $V = U$ shows,

$$\operatorname{cap}\Omega = \int_{\partial\Omega} h\, d\sigma. \tag{1.2}$$

In other words, the total mass of the measure $h\, d\sigma$ is normalized to be $\operatorname{cap}\Omega$ rather than 1.

The second important special case of Proposition 1.1 is the case $V = x \cdot \nabla U$. In that case, $V_\infty = -(N-2)\operatorname{cap}\Omega$ and $V(x) = -u(\xi)h(x)$ for $x \in \partial\Omega$, where $\xi = -\nabla U(x)/h = g(x)$. Thus we have

$$\operatorname{cap}\Omega = \frac{1}{N-2}\int_{\partial\Omega} uh^2\, d\sigma. \tag{1.3}$$

Capacity is translation invariant, but the function u is not. Note that translation $\Omega \to \Omega + x^0$ changes u to $u(\xi) + x^0\cdot\xi$. It follows that

$$\int_{\partial\Omega} x^0\cdot g(x)h(x)^2\, d\sigma(x) = 0$$

Since the equation holds for all x^0, we have the vector equation

$$\int_{\partial\Omega} g(x)h(x)^2\, d\sigma(x) = 0. \tag{1.4}$$

This gives N linear constraints on the measure $d\mu = g_*(h^2\, d\sigma)$

$$\int_{S^n} \xi\, d\mu(\xi) = 0. \tag{1.4'}$$

PROPOSITION 1.5. *Proposition 1.1 and (1.2), (1.3) and (1.4) are valid for arbitrary open, bounded convex domains.*

Proof. The natural distance between convex domains is the Minkowski metric

$$\text{dist}(\Omega_0, \Omega_1) = \max_{\xi \in S^n} |u_0(\xi) - u_1(x)|$$

where u_0 and u_1 are the support functions for Ω_0 and Ω_1, respectively. Thus, the distance we have defined is invariant if both domains are translated simultaneously. It is easy to see that this metric is the same as the Hausdorff metric, which is defined on any pair of compact sets of $\mathbf{R}^N$ by

$$\text{dist}_H(\bar{\Omega}_0, \bar{\Omega}_1) = \max\Big\{ \max_{x \in \bar{\Omega}_0} \min_{y \in \bar{\Omega}_1} |x - y|, \ \max_{x \in \bar{\Omega}_1} \min_{y \in \bar{\Omega}_0} |x - y| \Big\}.$$

Let Ω be a convex domain that is not necessarily smooth. Suppose that the origin is contained in Ω. Then $x \cdot g(x) \geqslant c > 0$, with a constant depending only on the diameter of Ω and the distance from the origin to $\partial \Omega$. The surface area can be written in polar coordinates $\theta = x/|x|$ as

$$d\sigma(x) = \frac{|x|^N}{x \cdot g(x)} \, d\theta$$

where $d\theta$ is the uniform measure on the sphere.

Remark 1.6. Define $\varrho \colon S^n \to \partial \Omega$ as the radial projection mapping $\varrho(\theta) = r(\theta)\theta$, where $r(\theta)$ is the unique positive number such that $r(\theta)\theta \in \partial \Omega$. If we identify the surface measure $d\sigma$ with the measure on S^n induced from $\partial \Omega$ by ϱ, then

$$d\sigma = \frac{r^N}{u(g(\varrho(\theta)))} \, d\theta.$$

Moreover, suppose that Ω_j converges to Ω in the Minkowski metric, then with the analogous notations ϱ_j, r_j, and $d\sigma_j$ and u_j associated to Ω_j, we have

$$\frac{d\sigma_j}{d\sigma}(\theta) = \frac{r_j(\theta)^N u(g(\varrho(\theta)))}{r(\theta)^N u_j(g_j(\varrho_j(\theta)))}.$$

This density is bounded above and below by positive constants and tends to 1 almost everywhere with respect to $d\theta$ as j tends to infinity.

The remark is proved by noting that almost every point $g(\varrho(\theta))$ of $\partial \Omega$ has a unique tangent plane. At those points, $g_j(\varrho_j(\theta)))$ tends to $g(\varrho(\theta))$ as j tends to infinity.

Green's function for the domain Ω', with pole at infinity, is $G(x) = 1 - U(x)$. A convex domain is a Lipschitz domain and the Lipschitz constant depends only on the ratio of the diameter to the inradius of the domain. Because a convex domain is a Lipschitz domain, $\nabla U = -\nabla G$ has nontangential boundary values and maximal function estimates. Indeed, for $x \in \partial \Omega$, let $\Gamma(x)$ denote the nontangential cone

$$\Gamma(x) = \{y \in \Omega' : |x - y| < C \, \text{dist}(y, \partial \Omega')\}.$$

THEOREM 1.7 ([D], [JK1]). *Let Ω be a convex domain and let U be the equilibrium potential defined in Ω' as above. Then the nontangential limit*

$$\lim_{y \to x} \nabla U(y), \quad y \in \Gamma(x),$$

exists for almost every $x \in \partial\Omega$ with respect to surface measure $d\sigma$. (This limit is denoted $\nabla U(x)$.) Furthermore, the nontangential maximal function $|\nabla U|^*$ belongs to $L^2(\partial\Omega, d\sigma)$:

$$|\nabla U|^*(y) \equiv \sup_{x \in \Gamma(y)} |\nabla U(x)| \in L^2(\partial\Omega, d\sigma).$$

Let Ω be a convex domain that is not necessarily smooth. Let

$$\Omega'(t) = \{x \in \Omega' : U(x) < t\},$$
$$\Omega(t) = \{x : x \in \bar{\Omega} \text{ or } x \in \Omega' \text{ and } U(x) > t\}.$$

The theorem of Gabriel [Ga], [CS] says that the regions $\Omega(t)$ are convex. Without loss of generality we may assume that the origin belongs to Ω. (This assumption is for notational convenience only, as in Remark 1.6.) It follows from the maximum principle that $x \cdot \nabla U$ is strictly negative in Ω'. Thus, by the implicit function theorem, the domains $\Omega(t)$ are C^∞ for all $t < 1$. Proposition 1.1 implies that for $t < 1$,

$$V_\infty = \int_{\partial\Omega(t)} V(x)|\nabla U(x)| \, d\sigma_t(x).$$

Since the left- and right-hand sides are translation invariant, we can choose the origin in Ω. Continuity of U at $\partial\Omega$ implies that $\Omega(t)$ tends to Ω in the Minkowski metric as t tends to 1. Therefore, it follows from Remark 1.6, Theorem 1.7 and the dominated convergence theorem that

$$\lim_{t \to 1} \int_{\partial\Omega(t)} V(x)|\nabla U(x)| \, d\sigma_t(x) = \int_{\partial\Omega} V(x)|\nabla U(x)| \, d\sigma(x).$$

Thus we have proved Proposition 1.1 without the smoothness hypothesis and (1.2) follows. However, the extension of (1.3) to the nonsmooth case requires an extra argument because the function $V = x \cdot \nabla U$ is not, in general, continuous up to the boundary.

The function $U(x)/t$ is the equilibrium potential for $\Omega(t)$ and its asymptotics show that $\text{cap}\,\Omega(t) = (1/t)\,\text{cap}\,\Omega$. Thus, (1.3) implies that for $t < 1$,

$$\frac{\text{cap}\,\Omega}{t} = \text{cap}\,\Omega(t) = \int_{\partial\Omega(t)} (x \cdot \nabla U(x))|\nabla U(x)| \, d\sigma_t(x).$$

The desired formula follows from the dominated convergence theorem. (Note that in this case we need the full strength of Theorem 1.7, namely, that the square of the gradient $|\nabla U|^2$ has an integrable nontangential maximal function, rather than just the first power.) Now we have confirmed (1.3) in general, and (1.4) and (1.4') are immediate consequences.

The physical interpretation of the equilibrium potential is that it is the potential energy (voltage) of the electrical field induced when the body $\bar{\Omega}$ is a conductor, normalized so that the voltage difference between $\bar{\Omega}$ and infinity is 1 (see Kellogg [K]). The equilibrium distribution is the distribution of charge on $\bar{\Omega}$ in equilibrium.

The mathematical formulation of this physical model is as follows. (None of the facts from this discussion are needed later in the paper, so they are stated only.) Consider a positive measure ν supported on $\bar{\Omega}$ interpreted physically as a distribution of charge. The induced electrical potential is

$$V_\nu(x) = \int a_N |x-y|^{2-N} \, d\nu(y).$$

It is well known that among all positive measures ν with $V_\nu \leqslant 1$, there is a unique ν of largest total mass. Moreover, for this measure ν, V_ν is continuous in all of $\mathbf{R}^N$ and $V_\nu(x)=1$ for all $x\in\bar{\Omega}$ (see [C]). The extremal measure ν is called the equilibrium distribution. Since V_ν is harmonic in Ω', we see that $V_\nu=U$ in Ω'. From this it is not hard to prove that

$$d\nu = h \, d\sigma$$

and that the total mass of ν is the capacity of Ω. Thus the capacity can also be defined as the largest charge that can be carried by a body $\bar{\Omega}$ if the voltage drops by at most 1.

2. The first and second variation of capacity

In this section we calculate the first and second variation of capacity for smooth domains. Following [J], [CY], let $e_1, ..., e_n$ be an orthonormal frame for S^n, and let covariant derivatives with respect to this frame be denoted ∇_i and ∇_{ij}. Denote $\mathcal{U}=\{u\in C^\infty(S^n): \nabla_{ij}u+u\delta_{ij}\gg 0\}$. For each $u\in\mathcal{U}$, define the domain $\Omega=\{x\in\mathbf{R}^N: x\cdot\xi<u(\xi) \text{ for all } \xi\in S^n\}$. Then u is the support function of Ω. This correspondence gives a one-to-one correspondence between C^∞ convex domains with strictly positive Gauss curvature and functions of $\mathcal{U}$. Let $b\in\mathbf{R}^N$. Translation of the domain Ω to $\Omega+b$ corresponds to the change in u to $u+b\cdot\xi$. Denote the N-dimensional space $\mathcal{P}_1=\text{span}\{\xi_1, ..., \xi_N\}$. The Gauss mapping g is a diffeomorphism and we denote the inverse mapping by $F: S^n\to\partial\Omega$. It is given by the formula $F=\nabla\bar{u}$, where $\bar{u}$ is the extension of u from S^n to $\mathbf{R}^N$ as homogeneous function of degree 1: $\bar{u}(r\xi)=ru(\xi)$ for all $\xi\in S^n$. The Gauss curvature K can be defined as a

function of the unit normal by $g_*(d\sigma)=(1/K(\xi))\,d\xi$, where $d\xi$ is the uniform measure on the Gauss sphere. The density $1/K$ can be computed in terms of u and written

$$\frac{1}{K(\xi)} = \det(\nabla_{ij}u(\xi)+u(\xi)\delta_{ij}). \tag{2.1}$$

K is unchanged by translation of Ω. In fact, each individual entry of the matrix whose determinant is $1/K$ is unchanged by translation: if $v\in\mathcal{P}_1$, then

$$(\nabla_{ij}v(\xi)+v(\xi)\delta_{ij}) = 0 \quad \text{for all } i,j. \tag{2.2}$$

Define the coefficients c_{ij} of the cofactor matrix of $\nabla_{ij}u+u\delta_{ij}$ by

$$c_{ij}(\nabla_{jl}u+u\delta_{jl}) = \delta_{il}\det(\nabla_{pq}u+u\delta_{pq}) = \frac{\delta_{il}}{K}. \tag{2.3}$$

Here and in subsequent formulas we follow the convention that repeated indices are summed. Define the density $S\in C^\infty(S^n)$ by $g_*(|\nabla U|^2\,d\sigma)=S\,d\xi$, define the mapping $\mathcal{F}:\mathcal{U}\to C^\infty(S^n)$ by $\mathcal{F}(u)=S$. We have the formula

$$S(\xi) = \frac{h(F(\xi))^2}{K(\xi)}, \tag{2.4}$$

where $h(x)=|\nabla U(x)|$ for $x\in\partial\Omega$. The Hessian of U is given by

LEMMA 2.5. (a) $\nabla^2 U(F(\xi))(e_i,e_j)=-Khc_{ij}$,
(b) $\nabla^2 U(F(\xi))(\xi,\xi)=Kh\operatorname{Tr}(c_{ij})$,
(c) $\nabla^2 U(F(\xi))(e_i,\xi)=-Kc_{ij}\nabla_j h$.
($\nabla^2 U(x)$ is the Hessian in $\mathbf{R}^N$ as a bilinear form on $\mathbf{R}^N$ and the vectors e_i are vectors in $\mathbf{R}^N$ orthogonal to ξ, i.e., tangent vectors to $\partial\Omega$ at $x=F(\xi)$.)

Proof. Let $G=1-U$. Then G is a harmonic function that vanishes on $\partial\Omega$ and $\xi\cdot\nabla G=h$. Thus the formulas of Lemma A of [J] for the Hessian of G are valid. The formulas given here have the opposite sign because $\nabla^2 U=-\nabla^2 G$.

Let $f\in C^\infty(S^n)$ and let w be the harmonic function in Ω' that vanishes at infinity and has boundary values $f(\xi)$ at $x=F(\xi)$ on $\partial\Omega'$. Define the operator Λ acting on $C^\infty(S^n)$ by $\Lambda(f)=\xi\cdot\nabla w(F(\xi))$, the normal derivative of the harmonic extension. Let $v\in C^\infty(S^n)$. For t sufficiently small, $u+tv\in\mathcal{U}$. Furthermore, if v is the support function of a domain Ω_1, then $u+tv$ is the support function of $\Omega+t\Omega_1$.

PROPOSITION 2.6. *The directional derivative of $\mathcal{F}$ is given by*

$$\frac{d}{dt}\mathcal{F}(u+tv)\Big|_{t=0} = Lv,$$

where $L=L_u$ is defined as

$$Lv = \nabla_i(h^2 c_{ij}\nabla_j v) - \frac{2}{K}h\Lambda(hv) - h^2\operatorname{Tr}(c_{ij})v.$$

Proof. We follow the proof of Proposition 2 of [J]. There are a few sign changes and h^2 replaces h. For each t sufficiently small, we define the functions U, and F correspond to the support function $u+tv$. Let "dot", as in $\dot{U}$, $\dot{F}$, etc., denote the derivative with respect to t at $t=0$. Then formulas (3.10) of [J] becomes

$$h = -\xi\cdot\nabla U, \tag{2.7}$$

and (3.11) and (3.12) become

$$\dot{U} + \nabla U\cdot\dot{F} = 0, \quad \dot{F} = (\nabla_i)e_i + v\xi.$$

We deduce the variants of (3.13) and (3.14)

$$\dot{U} = hv \quad \text{and} \quad \xi\cdot\nabla\dot{U} = \Lambda(hv).$$

Next, taking the t derivative of (2.7) as in (3.15) of [J], we obtain

$$\dot{h} = -\xi\cdot\nabla\dot{U} - \nabla^2 U(\xi,\dot{F}) = -\Lambda(hv) + Kc_{ij}(\nabla_i h)(\nabla_j v) - Kh\operatorname{Tr}(c_{ij})v.$$

Recall also that

$$(1/K)\dot{} = c_{ij}(\nabla_{ij}v + v\delta_{ij}) = \nabla_i(c_{ij}\nabla_j v) + \operatorname{Tr}(c_{ij})v.$$

The latter equation follows from the fact that $\nabla_i(c_{ij})=0$ for all j; see [CY], [J, p. 386]. Combining the last two formulas we have

$$\begin{aligned}\frac{d}{dt}\mathcal{F}(u+tv)\Big|_{t=0} &= \frac{2h\dot{h}}{K} + h^2\nabla_i(c_{ij}\nabla_j v) + h^2\operatorname{Tr}(c_{ij})v \\ &= -\frac{2h\Lambda(hv)}{K} + 2hc_{ij}(\nabla_i h)(\nabla_j v) - h^2\operatorname{Tr}(c_{ij})v + h^2\nabla_i(c_{ij}\nabla_j v) = Lv.\end{aligned}$$

Green's formula implies that Λ is selfadjoint on $L^2(\partial\Omega, d\sigma)$. It follows that:

Remark 2.8. L is selfadjoint on $L^2(S^n, d\xi)$.

Dilation gives $\mathcal{F}((1+t)u) = (1+t)^{N-3}\mathcal{F}(u)$, so that

$$Lu = (N-3)\mathcal{F}(u). \tag{2.9}$$

We can now deduce

PROPOSITION 2.10. *Let Ω and Ω_1 be C^∞ strictly convex domains with support functions u and v, respectively. Then at $t=0$,*

(a) $\displaystyle \frac{d}{dt} \operatorname{cap}(\Omega + t\Omega_1) = \int_{\partial\Omega} v(g(x))|\nabla U(x)|^2 \, d\sigma(x),$

(b) $\displaystyle \frac{d^2}{dt^2} \operatorname{cap}(\Omega + t\Omega_1) = \int_{S^n} vLv \, d\xi.$

Proof. Let $f(t) = \operatorname{cap}(\Omega + t\Omega_1)$. Proposition 1.9 says

$$f(t) = \frac{1}{N-2} \int_{S^n} (u+tv)\mathcal{F}(u+tv) \, d\xi.$$

Hence, by Remark 2.8 and (2.9),

$$f'(0) = \frac{1}{N-2} \int_{S^n} v\mathcal{F}(u) \, d\xi + \frac{1}{N-2} \int_{S^n} uLv \, d\xi = \int_{S^n} v\mathcal{F}(u) \, d\xi.$$

The same calculation gives

$$f'(t) = \int_{S^n} v\mathcal{F}(u+tv) \, d\xi.$$

Consequently, Proposition 2.6 implies

$$f''(0) = \int_{S^n} vLv \, d\xi.$$

3. Continuity of the first variation

In this section we will prove the continuity of the first variation $h^2 \, d\sigma$ in the Minkowski metric. We will then deduce the first variation formula (0.4) for nonsmooth domains and some further consequences. We have already proved (0.4) in the smooth case in part (a) of Proposition 2.10. Surface measure varies continuously even when the domain collapses to a closed $(N-1)$-dimensional set, provided both flat "sides" of the limiting domain are counted. However, as we shall see in §4, the total mass of $h^2 \, d\sigma$ tends to infinity when the domain collapses. Thus we will assume here that the limiting domain is bounded and has nonempty interior. Then there are uniform upper and lower bounds on the Lipschitz constant of the approximating domains.

THEOREM 3.1. *Let Ω_j be convex domains tending to an open, bounded convex domain Ω in the Minkowski metric. Let g_j denote the Gauss map on $\partial\Omega_j$. Let U_j be the equilibrium potential on Ω_j'. Let $d\sigma_j$ denote surface measure on $\partial\Omega_j$, and let $h_j = |\nabla U_j|$ be the density of harmonic measure on $\partial\Omega_j$. Let g, U, $d\sigma$ and h be the corresponding*

objects associated with Ω. *Then* $(g_j)_*(h_j^2 \, d\sigma_j)$ *tends weakly to* $g_*(h^2 \, d\sigma)$ *as measures on the sphere* S^n.

Proof. This continuity would be relatively easy if it concerned the first power $h \, d\sigma$ rather than $h^2 \, d\sigma$. (See [J, §6] for a such a case.) The extra difficulty of the second power requires a convergence theorem due to Jerison and Kenig [JK2]. The theorem says that $\log h$ is continuous in the BMO norm under C^1 perturbations of the domain. We will show first that convergence of convex domains in the Minkowski metric implies C^1 convergence except on a set of small measure. We will then use Dahlberg's reverse Hölder inequality to complete the proof.

Dahlberg's theorem says:

PROPOSITION 3.2 [D]. *There exist constants* $p_0 > 2$ *and* C *depending only on the Lipschitz constant of* Ω *such that for any* $r > 0$ *and any* $x \in \partial\Omega$

$$\left(r^{-n} \int_{B(x,r) \cap \partial\Omega} h^{p_0} \, d\sigma \right)^{1/p_0} \leqslant C r^{-n} \int_{B(x,r) \cap \partial\Omega} h \, d\sigma.$$

In particular, there is a constant C *depending only on the diameter and inradius of* Ω *such that*

$$\int_{\partial\Omega} h^{p_0} \, d\sigma \leqslant C.$$

LEMMA 3.3. *For any* $\varepsilon > 0$, *there exists* $\delta > 0$ *and a finite disjoint collection of balls* $B(z_i, \alpha_i)$ *such that* $z_i \in \partial\Omega$ *and for every convex* Ω_1 *for which* $\mathrm{dist}(\Omega_1, \Omega) < \delta$:

(a) $\sigma\big(\partial\Omega \backslash \bigcup_i B(z_i, \alpha_i)\big) < \varepsilon$;

(b) *After a suitable translation and rotation, depending on* i, *we have that* $\partial\Omega$ *and* $\partial\Omega_1$ *are given in* $B(z_i, \alpha_i/\varepsilon)$ *by the graphs of functions* ϕ *and* ϕ_1, *respectively.*

$$|\nabla\phi(x)| + |\nabla\phi_1(x)| \leqslant \varepsilon$$

for all $x \in \mathbf{R}^n$ *such that* $|x| < \alpha_i/\varepsilon$.

This lemma says that on a set of large measure, $\partial\Omega$ and $\partial\Omega_1$ are flat. To prove it, we need several observations. It suffices to consider a portion of $\partial\Omega$ represented by the graph of a single convex function. Let $B \subset \mathbf{R}^n$ be ball and suppose that

$$\widehat{B} \cap \partial\Omega = \{(x, \phi(x)) : x \in B\}$$

where ϕ is a convex function and $\widehat{B}$ is an appropriate cylinder in $\mathbf{R}^N$ whose projection is B. It is well known that convex functions (and even Lipschitz functions) are differentiable almost everywhere [S]. Thus, for almost every $x \in B$ and every $\varepsilon_1 > 0$ there exists $\delta_1 > 0$ such that

$$|\phi(y) - \phi(x) - \nabla\phi(x) \cdot (y - x)| \leqslant \varepsilon_1 |y - x|$$

whenever $|y-x|<\delta_1$ and we can assume that $\nabla\phi(x)$ is uniformly bounded on B. By Vitali's covering theorem, there exists a finite collection of disjoint balls $B(x_i,\alpha_i)$ such that

$$|B\backslash\bigcup B(x_i,\alpha_i)|<\varepsilon$$

and

$$|\phi(x)-\phi(x_i)-\nabla\phi(x_i)\cdot(x-x_i)|<\varepsilon\alpha_i$$

for all $x\in B(x_i,\alpha_i/\varepsilon)$.

Remark 3.4. Suppose that ψ is convex and $0\leqslant\psi\leqslant\eta$ in $|x|<r$. Then

$$|\nabla\psi(x)|\leqslant\frac{2\eta}{r}\quad\text{for almost all }|x|<\tfrac{1}{2}r.$$

Proof. Let x be a point where ψ is differentiable and $|x|<\tfrac{1}{2}r$. We may assume that $\nabla\psi(x)\neq0$, so that the unit vector $v=\nabla\psi(x)/|\nabla\psi(x)|$ can be defined. Since $|x+\tfrac{1}{2}rv|<r$,

$$\eta\geqslant\psi\big(x+\tfrac{1}{2}rv\big)\geqslant\psi(x)+\nabla\psi(x)\cdot\big(\tfrac{1}{2}rv\big)\geqslant\tfrac{1}{2}r|\nabla\psi(x)|,$$

which proves the remark.

Remark 3.4 applied to the convex function $\psi(x)=\phi(x)-\phi(x_i)\nabla\phi(x_i)\cdot(x-x_i)$ gives

$$|\nabla\phi(x)-\nabla\phi(x_i)|\leqslant2\varepsilon^2$$

for all $x\in B(x_i,\alpha_i/2\varepsilon)$. The fundamental theorem of calculus implies

$$|\phi(x)-\phi(x_i)-\nabla\phi(x_i)\cdot(x-x_i)|\leqslant2|x-x_i|\varepsilon^2\leqslant2\varepsilon\alpha_i \tag{3.5}$$

for all $x\in B(x_i,\alpha_i/\varepsilon)$. Now choose $\delta=\min_i\varepsilon\alpha_i$. It follows that for every $x\in B(x_i,\alpha_i/2\varepsilon)$,

$$|\phi_1(x)-\phi(x_i)-\nabla\phi(x_i)\cdot(x-x_i)|\leqslant C\varepsilon\alpha_i$$

for some constant C depending only on the Lipschitz constant of Ω. An application of Remark 3.4 to the function $\psi(x)=\phi_1(x)-\phi(x_i)\nabla\phi(x_i)\cdot(x-x_i)$ shows that for all $x\in B(x_i,\alpha_i/4\varepsilon)$,

$$|\nabla\phi_1(x)-\nabla\phi(x_i)|\leqslant C\varepsilon^2.$$

Finally, rotate the coordinate system so that $\nabla\phi(x_i)=0$. In the new coordinate system we need to take the graph over a ball $B(x_i,C\alpha_i)$ to be sure to cover the same set as before the rotation. On the other hand, in the new coordinate system,

$$|\nabla\phi_1(x)|+|\nabla\phi(x)|\leqslant C\varepsilon^2$$

for all $x\in B(x_i,\alpha_i/4\varepsilon)$. After a suitable renaming of ε and α_i, we have proved Lemma 3.3.

Next, we recall (see [JK2, Theorem 2.1]):

LEMMA 3.6. *For any $\varepsilon_2 > 0$ there exists $\varepsilon > 0$, such that for any $r < r_0$, if $\partial\Omega$ is given locally in a neighborhood of the origin by*

$$\partial\Omega = \{(x, \phi(x)) : |x| < r\}$$

and ϕ satisfies $\phi(0) = 0$, and

$$|\nabla\phi(x)| \leqslant \varepsilon$$

for all $|x| < r$, then for every $s < \varepsilon r$ and every $|x| < \varepsilon r$, there is a constant $a(x, s)$ such that

$$s^{-n} \int_{|y-x|<s} |\log h(y, \phi(y)) - a(x, s)| \, dy < \varepsilon_2.$$

Assume that the origin is contained in Ω. Recall the radial projection $\varrho_j : S^n \to \partial\Omega_j$ was defined in Remark 1.4. Define $\tilde{h}_j$ on S^n by

$$\tilde{h}_j(\theta) = h_j(\varrho_j(\theta)) \sqrt{\left(\frac{d\sigma_j}{d\theta}\right)(\theta)}.$$

Then $(\varrho_j)_* (\tilde{h}_j^2 \, d\theta) = h_j^2 \, d\sigma_j$. Similarly, define $\tilde{h}$ so that $\varrho_* (\tilde{h}^2 \, d\theta) = h^2 \, d\sigma$.

LEMMA 3.7. *For any $\varepsilon > 0$ and any $p < \infty$, there exist numbers $s_0 > 0$ and $\delta > 0$ and family of balls $\mathcal{B}$ on S^n such that:*

(a) *Every ball of family $\mathcal{B}$ has radius s_0.*

(b) *There is a constant C depending only on the eccentricity of Ω such that every point of S^n belongs to at most C balls of $\mathcal{B}$.*

(c) *$\int_{S^n \setminus F} d\theta < \varepsilon$ where $F = \bigcup_{B \in \mathcal{B}} B$.*

(d) *If $\mathrm{dist}(\Omega_j, \Omega) < \delta$, then for any $B \in \mathcal{B}$,*

$$s_0^{-n} \int_B \left| \frac{\tilde{h}_j(\theta)}{\tilde{h}(\theta)} - 1 \right|^p d\theta + s_0^{-n} \int_B \left| \frac{\tilde{h}(\theta)}{\tilde{h}_j(\theta)} - 1 \right|^p d\theta < \varepsilon.$$

Proof. Note that the mappings ϱ and ϱ_j preserve distance up to a factor. Choose $s_0 < (\min \alpha_i)$, and sufficiently small that the Jacobians $d\sigma_j / d\theta$ and $d\sigma / d\theta$ of the change of variables ϱ_j and ϱ vary by at most ε when θ varies by the distance s and $\varrho(\theta)$ is contained in one of the balls $B(z_i, \alpha_i / \varepsilon)$ of Lemma 3.3. Then we can choose $\mathcal{B}$ satisfying properties (a), (b) and (c), and it follows from Lemma 3.6 that for every $s < s_0$ and every ball B of radius s in the concentric $1/\varepsilon$ multiple of any ball of $\mathcal{B}$, there exists a constant a_B such that

$$s^{-n} \int_B |\log \tilde{h}(\theta) - a_B| \, d\theta < \varepsilon. \tag{3.8}$$

In other words, $\log \bar{h}$ has small BMO norm in a $(1/\varepsilon)$-neighborhood of every ball of $\mathcal{B}$. Moreover, as Lemma 3.3 shows, we can choose δ sufficiently small that the same estimate holds for $\log \tilde{h}_j$.

Next, (3.8) and the John–Nirenberg inequality [JN] imply that for any $p < \infty$ and any $\varepsilon_3 > 0$ one can choose $\delta > 0$ and s_0 sufficiently small that for every $B \in \mathcal{B}$, there exists a constant A_B such that

$$s_0^{-n} \int_B \left| A_B \frac{\tilde{h}_j(\theta)}{\tilde{h}(\theta)} - 1 \right|^p d\theta < \varepsilon_3. \tag{3.9}$$

Because of Remark 1.4, we can choose δ sufficiently small and s_0 sufficiently small that $d\sigma_j/d\theta$ and $d\sigma/d\theta$ are arbitrarily close to the same constant on every ball $B \in \mathcal{B}$. In other words, for every $B \in \mathcal{B}$,

$$\frac{\int_B \tilde{h}_j(\theta)\, d\theta}{\int_B \tilde{h}(\theta)\, d\theta} = (1 + O(\varepsilon_3)) \frac{\int_{\varrho_j(B)} h_j(x)\, d\sigma_j(x)}{\int_{\varrho(B)} h(x)\, d\sigma(x)}. \tag{3.10}$$

In order to get rid of the factor A_B in (3.9), we must now fix s_0 and permit δ to depend on s_0. The ratio on the right-hand side of (3.10) is the ratio of the harmonic measures of the sets $\varrho(B)$ and $\varrho_j(B)$. Using the maximum principle to compare harmonic functions in the complement of Ω_j to harmonic functions in the complement of dilations of Ω, or using [JK2, Lemma 2.6], we can choose δ smaller still, depending on s_0 so that

$$\left| \frac{\int_B \tilde{h}_j(\theta)\, d\theta}{\int_B \tilde{h}(\theta)\, d\theta} - 1 \right| = O(\varepsilon_3) \tag{3.11}$$

for every $B \in \mathcal{B}$. Recall, that since $p_0 > 2$ in Proposition 3.2, we have the "reverse Schwarz" inequality

$$\left(\frac{1}{\sigma(\varrho(B))} \int_{\varrho(B)} h^2\, d\sigma \right)^{1/2} \leqslant \frac{C}{\sigma(\varrho(B))} \int_{\varrho(B)} h\, d\sigma, \tag{3.12}$$

valid for all balls B centered on $\partial\Omega$. There is a similar inequality for $\partial\Omega_1$. It follows that for every $B \in \mathcal{B}$,

$$\left(s_0^{-n} \int_B \tilde{h}^2\, d\theta \right)^{1/2} \leqslant C s_0^{-n} \int_B \tilde{h}\, d\theta \tag{3.13}$$

and similarly for $\tilde{h}_j$. We can now show that $A_B - 1 < \varepsilon$ for all $B \in \mathcal{B}$. Using the Schwarz inequality, the "reverse Schwarz" inequality (3.13), and (3.9) we obtain

$$\frac{\int_B A_B \tilde{h}_j\, d\theta}{\int_B \tilde{h}\, d\theta} - 1 = \int_B \frac{(A_B \tilde{h}_j/\tilde{h} - 1)\tilde{h}\, d\theta}{\int_B \tilde{h}\, d\theta} \leqslant \frac{\left(\int_B (A_B \tilde{h}_j/\tilde{h} - 1)^2\, d\theta \right)^{1/2} \left(\int_B \tilde{h}^2\, d\theta \right)^{1/2}}{\int_B \tilde{h}\, d\theta}$$

$$\leqslant C \left(s_0^{-n} \int_B (A_B \tilde{h}_j/\tilde{h} - 1)^2\, d\theta \right)^{1/2} < C\varepsilon_3.$$

Similarly, if we use the analogous statements to (3.9) with $\tilde{h}/A_B\tilde{h}_j$ in place of $A_B\tilde{h}_j/\tilde{h}$ we obtain

$$\frac{\int_B \tilde{h}\,d\theta}{\int_B A_B\tilde{h}_j\,d\theta} - 1 < C\varepsilon_3.$$

Finally, using (3.11) we see that $|A_B - 1| < C\varepsilon_3$. It follows that we can replace A_B by 1 in the inequality (3.9). This bounds the first term in part (d) of Lemma 3.7, and the second term is similar.

PROPOSITION 3.14. *Let Ω_j tend to Ω in the Minkowski metric. Suppose (for convenience) that Ω contains a ball around the origin. With the notations above,*

$$\int_{S^n} |\tilde{h}_j^2 - \tilde{h}^2|\,d\theta \to 0 \quad \text{as } j \to \infty.$$

Proof. Since $p_0 > 2$, we can choose q, $1 < q < \infty$, so that $2q' = p_0$, where $1/q + 1/q' = 1$. Let $\varepsilon > 0$ and $p = 2q$, and choose δ according to Lemma 3.7. Then using Schwarz's inequality, Proposition 3.2 and Hölder's inequality, we find

$$\int_F |\tilde{h}_j^2 - \tilde{h}^2|\,d\theta \leqslant \left(\int_F |\tilde{h}_j - \tilde{h}|^2\,d\theta\right)^{1/2} \left(\int_F |\tilde{h}_j + \tilde{h}|^2\,d\theta\right)^{1/2} \leqslant C\left(\int_F |\tilde{h}_j - \tilde{h}|^2\,d\theta\right)^{1/2}$$

$$\leqslant C\left(\int_F \left|\frac{\tilde{h}_j}{\tilde{h}} - 1\right|^{2q}\,d\theta\right)^{1/2q} \left(\int_F \tilde{h}^{2q'}\right)^{1/2q'}.$$

The first factor is bounded by Lemma 3.7 and the second by Proposition 3.2, so

$$\int_F |\tilde{h}_j^2 - \tilde{h}^2|\,d\theta \leqslant C\varepsilon.$$

On the other hand,

$$\int_{S^n \setminus F} |\tilde{h}_j^2 - \tilde{h}^2|\,d\theta \leqslant \int_{S^n \setminus F} (\tilde{h}_j^2 + \tilde{h}^2)\,d\theta.$$

Hölder's inequality, Proposition 3.2 and Lemma 3.7 give

$$\int_{S^n \setminus F} \tilde{h}^2\,d\theta \leqslant \left(\int_{S^n \setminus F} \tilde{h}^{p_0}\,d\theta\right)^{1/q'} \left(\int_{S^n \setminus F} d\theta\right)^{1/q} \leqslant C\varepsilon^{1/q},$$

and there is a similar estimate with $\tilde{h}_j$ in place of $\tilde{h}$. This proves Proposition 3.14.

We are now ready to prove Theorem 3.1. We follow the procedure of [CY], [J, p. 392]. Let E be a closed subset of S^n. We redefine the Gauss map as a set-valued function:

$$g(x) = \{\xi \in S^n : (x' - x) \cdot \xi < 0 \text{ for every } x' \in \Omega\}.$$

In other words, $g(x)$, is the set of all outer normals ξ of the support planes of Ω at x. The mapping g is single-valued except on a set of surface measure zero. Define the inverse image

$$g^{-1}(E) = \{x \in \partial\Omega : g(x) \cap E \neq \varnothing\}.$$

Then

$$\mu(E) = \int_{g^{-1}(E)} h^2 \, d\sigma$$

and similarly for μ_j. Our goal is to prove that μ_j tends weakly to μ. Note that the densities $\tilde{h}^2$ and $\tilde{h}_j^2$ are the densities of these same measures but with respect to the polar coordinate θ and the radial projections ϱ_j and ϱ, not the Gauss maps. We will try to avoid confusion by using the variable ξ to denote the variable on the Gauss sphere. Proposition 3.14 implies, in particular, that

$$\mu_j(S^n) - \mu(S^n) = \int_{S^n} \tilde{h}_j^2(\theta) \, d\theta - \int_{S^n} \tilde{h}^2(\theta) \, d\theta \to 0 \quad \text{as } j \to \infty. \qquad (3.15)$$

Recall from [CY], [J] that $g^{-1}(E)$ is a closed subset of $\partial\Omega$. Moreover, if $\xi_j \in g_j(x_j)$, ξ_j tends to ξ, and x_j tends to x, then $x \in \partial\Omega$ and $\xi \in g(x)$. Let U be an open neighborhood in $\partial\Omega$ of the closed set $g^{-1}(E)$. It follows that $\varrho_j^{-1}(g_j^{-1}(E)) \subset \varrho^{-1}(U)$ for sufficiently large j. Therefore,

$$\limsup_{j\to\infty} \mu_j(E) \leqslant \lim_{j\to\infty} \int_{\varrho^{-1}(U)} \tilde{h}_j(\theta)^2 \, d\theta \leqslant \int_{\varrho^{-1}(U)} \tilde{h}(\theta)^2 \, d\theta.$$

Taking the infimum over all $U \supset g^{-1}(E)$ we find that

$$\limsup_{j\to\infty} \mu_j(E) \leqslant \mu(E).$$

It then follows from (3.15) that for every open set $V \subset S^n$,

$$\liminf_{j\to\infty} \mu_j(V) \geqslant \mu(V).$$

Let $\tilde{\mu}$ be any weak limit of a subsequence (or subnet) of μ_j. Then $\tilde{\mu}(E) \leqslant \mu(E)$ for every closed set E and $\tilde{\mu}(V) \geqslant \mu(V)$ for every open set V, and since

$$\mu(E) = \inf_{V \supset E} \mu(V)$$

and similarly for $\tilde{\mu}$, we have $\tilde{\mu} = \mu$.

COROLLARY 3.16. *Let Ω_0 and Ω_1 be open, bounded convex domains. Let g_0 be the Gauss mapping for Ω_0, $h_0 = |\nabla U_0|$ where U_0 is the equilibrium potential for Ω_0 and let $d\sigma_0$ be surface measure on $\partial\Omega_0$. Finally let u_1 be the support function of Ω_1. Then*

$$\lim_{t\to 0^+} \frac{\mathrm{cap}(\Omega_0 + t\Omega_1) - \mathrm{cap}\,\Omega_0}{t} = \int_{\partial\Omega_0} u_1(g_0(x))h_0(x)^2\,d\sigma_0.$$

Proof. For any $v \in C(S^n)$ and any bounded open convex domain Ω, define

$$M(v,\Omega) = \int_{\partial\Omega} v(g(X))|\nabla U(X)|^2\,d\sigma(X)$$

where U is the equilibrium potential for Ω, g is the Gauss map and $d\sigma$ is surface measure. As a consequence of Theorem 3.1, M is jointly continuous when v varies in $C(S^n)$ and Ω varies in the Minkowski metric, provided the diameter and inradius of the convex bodies are uniformly bounded above and below. Consider a sequence of smooth strongly convex domains $\Omega_0(k)$ and $\Omega_1(k)$ tending to Ω_0 and Ω_1, respectively, in the Minkowski metric. Let

$$\Omega_t = (1-t)\Omega_0 + t\Omega_1, \quad \Omega_t(k) = (1-t)\Omega_0(k) + t\Omega_1(k),$$

$$f(t) = \mathrm{cap}(\Omega_0 + t\Omega_1), \quad m(t) = \mathrm{cap}(\Omega_t)^{1/(N-2)}$$

$$f_k(t) = \mathrm{cap}(\Omega_0(k) + t\Omega_1(k)), \quad m_k(t) = \mathrm{cap}(\Omega_t(k))^{1/(N-2)}.$$

Note that

$$m_k(t) = (1-t)f_k(t/(1-t))^{1/(N-2)}.$$

Let u_t^k be the support function of $\Omega_t(k)$. Proposition 2.10 (a), the chain rule and formula (1.3) imply

$$m_k'(t) = \frac{1}{N-2}\,\mathrm{cap}\,\Omega_t(k)^{-1+1/(N-2)}M(u_1^k - u_0^k, \Omega_t(k)).$$

Because, by Theorem 0.9, m_k is a concave function of t, we have

$$m_k'(t) \leqslant \frac{m_k(t) - m_k(0)}{t} \leqslant m_k'(0)$$

for $0 \leqslant t \leqslant 1$. Take the limit as k tends to infinity with t fixed. Because M is continuous and because capacity is continuous in the Minkowski metric,

$$\frac{\mathrm{cap}\,\Omega_t^{-1+1/(N-2)}M(u_1 - u_0, \Omega_t)}{N-2} \leqslant \frac{m(t) - m(0)}{t} \leqslant \frac{\mathrm{cap}\,\Omega_0^{-1+1/(N-2)}M(u_1 - u_0, \Omega_0)}{N-2}.$$

Now take the limit as t tends to zero to find

$$\lim_{t\to 0^+} \frac{m(t) - m(0)}{t} = \frac{1}{N-2}\,\mathrm{cap}\,\Omega_0^{-1+1/(N-2)}M(u_1 - u_0, \Omega_0). \tag{3.17}$$

This is equivalent to the assertion in Corollary 3.16.

For future reference, we write a dilated version of the first variation formula that is equivalent to the formula in Corollary 3.16.

$$\lim_{t \to 0^+} \frac{\operatorname{cap}\Omega_t - \operatorname{cap}\Omega_0}{t} = \int_{\partial\Omega_0} (u_1 - u_0)(g_0(X))h_0(X)^2 \, d\sigma_0(X). \tag{3.18}$$

We will now deduce an isoperimetric inequality from Borell's theorem.

COROLLARY 3.19. *There is a dimensional constant C_N such that for any convex domain Ω in $\mathbf{R}^N$, $N \geqslant 3$,*

$$(\operatorname{cap}\Omega)^{(N-3)/(N-2)} \leqslant C_N \int_{\partial\Omega} h^2 \, d\sigma$$

and the constant C_N can be chosen so that there is equality when Ω is a ball.

Proof. Note that if a_N is the constant of (0.1), and B is a ball of radius r, then $\operatorname{cap}B = r^{N-2}/a_N$. Choose r so that $\operatorname{cap}\Omega = \operatorname{cap}B$. Then Borell's theorem implies

$$\operatorname{cap}((1-t)\Omega + tB) \geqslant \operatorname{cap}B$$

for $0 \leqslant t \leqslant 1$. In particular, the derivative at $t=0$ is nonnegative. By (3.18) and Proposition 3.16 this can be written as

$$\int_{\partial\Omega} (r-u)h^2 \, d\sigma \geqslant 0$$

with equality if Ω is a ball. But this inequality can be seen to be the same as the one in Corollary 3.19 by writing

$$r \int_{\partial\Omega} h^2 \, d\sigma \geqslant \int_{\partial\Omega} uh^2 \, d\sigma = (N-2)\operatorname{cap}\Omega = \frac{(N-2)r^{N-2}}{a_N}$$

and dividing by r.

Note that the analogy with perimeter is valid if one keeps in mind that the degree of homogeneity of capacity is $N-2$ and the degree of homogeneity of $h^2 \, d\sigma$ is $N-3$. In particular, in the special case $N=3$,

$$\int_{\partial\Omega} h^2 \, d\sigma \geqslant 4\pi = \operatorname{vol}S^2 \tag{3.20}$$

for any convex domain Ω in $\mathbf{R}^3$ with equality if Ω is a ball.

4. Estimates on eccentricity

The bounds on eccentricity of convex bodies that we are going to derive will depend on comparisons with ellipsoids, and, in addition, on comparisons to a semi-infinite strip.

LEMMA 4.1. *Let Ω be a bounded, open convex domain. There is a translation and rotation of Ω and an ellipsoid*

$$\mathcal{E} = \left\{ x \in \mathbf{R}^N : \sum_{j=1}^{N} \frac{x_j^2}{b_i^2} < 1 \right\}$$

such that

$$\mathcal{E} \subset \Omega \subset N\mathcal{E}$$

where $N\mathcal{E} = \{Nx : x \in \mathcal{E}\}$.

This lemma is due to F. John; see also the elementary proof of Córdoba and Gallegos [Gu, pp. 133–134]. We will always use the convention that $b_1 \leqslant b_2 \leqslant \ldots \leqslant b_N$. Lemma 2.1 implies in particular that the diameter of Ω is comparable to b_N and the inradius is comparable to b_1. Thus we can define the eccentricity of Ω as the ratio b_N/b_1.

The equilibrium potential for an ellipsoid has a simple explicit form, which we now derive using a separation of variables. (See [K] for the case $N=3$.) Let $\Omega' = \mathbf{R}^N \backslash \mathcal{E}$. For $x \in \Omega'$, define $\lambda(x) \geqslant 0$ by

$$\sum_{j=1}^{N} \frac{x_j^2}{b_j^2 + \lambda(x)} = 1. \tag{4.2}$$

Note that $\lambda(x) = 0$ for $x \in \partial\mathcal{E}$. Let

$$\phi(s) = \prod_{j=1}^{N} (b_j^2 + s).$$

Then one can calculate that

$$\frac{\Delta\lambda}{|\nabla\lambda|^2} = \frac{1}{2} \cdot \frac{\phi'(\lambda)}{\phi(\lambda)}.$$

It follows that $\Delta F(\lambda(x)) = 0$ if and only if

$$\phi(\lambda)F''(\lambda) + \tfrac{1}{2}\phi'(\lambda)F'(\lambda) = 0.$$

The solution

$$F(\lambda) = \int_{\lambda}^{\infty} \frac{ds}{\sqrt{\phi(s)}} \tag{4.3}$$

satisfies $F(\lambda)\to 0$ as $\lambda\to\infty$. Therefore, $U(x)=F(\lambda(x))/F(0)$ is the equilibrium potential for $\mathcal{E}$. Next, observe that

$$\lim_{x\to\infty}|x|^{N-2}F(\lambda(x))=\tfrac{1}{2}(N-2).$$

Thus there is a dimensional constant c_N such that

$$\operatorname{cap}\mathcal{E}=\frac{c_N}{F(0)}. \tag{4.4}$$

It is easy to calculate that if $b_N\leqslant C$, then $F(0)\geqslant c\log(1/b_2)$. In other words,

Remark 4.5. If $b_1\leqslant b_2\leqslant\ldots\leqslant b_N\leqslant C$ and $b_2<1$, then there is a constant C' depending on C for which

$$\operatorname{cap}\mathcal{E}\leqslant\frac{C'}{\log(1/b_2)}.$$

Consider the conformal mapping

$$f(z)=\int_0^z (w+1)^{1/2}(w-1)^{1/2}\,dw$$

in which the integral is defined in the upper half-plane with the branches of the square roots specified by $0\leqslant\arg(w+1)\leqslant\pi$ and $0\leqslant\arg(w-1)\leqslant\pi$. The upper half-plane is mapped to the complement of a semi-infinite strip

$$S=\{s+it:s>0,\,|t|<c\},\quad\text{where }c=-if(1)=\int_0^1 (w+1)^{1/2}(1-w)^{1/2}\,dw.$$

If $f(x+iy)=s+it$, then we define a positive harmonic function in the region $\mathbf{C}\backslash S$ that vanishes on the boundary by $F(s+it)=y$. The lemma that follows gives the size of this variant of Green's function at a unit distance from the boundary.

LEMMA 4.6. *There is an absolute constant A such that for every $s>1$,*

$$A^{-1}s^{-1/2}\leqslant|F'(s+ic)|\leqslant As^{-1/2}\quad\text{and}\quad A^{-1}s^{1/2}\leqslant F(-s)\leqslant As^{1/2}.$$

Proof. Note that

$$\lim_{y\to\infty}\frac{f(iy)}{y^2}=-\frac{1}{2}.$$

Thus, by Harnack's inequality, if $s=-\tfrac{1}{2}y^2$, $F(-s)$ is comparable to y. This proves the second estimate. For the first estimate, let $x>1$ and define

$$s(x)=f(x)-f(1)=f(x)-ic=\int_1^x (w+1)^{1/2}(w-1)^{1/2}\,dw.$$

Then $f(x)=s(x)+ic$, and one can compute that $s(x)/x^2\to\tfrac{1}{2}$ as $x\to\infty$. In addition,

$$|F'(s(x)+ic)|=\frac{1}{|f'(x)|}=(x+1)^{-1/2}(x-1)^{-1/2}=x^{-1}+o(x^{-1})\quad\text{as }x\to\infty.$$

This implies the first estimate.

Next, let us recall several theorems of Dahlberg, which will be used to estimate the density h in terms of the equilibrium potential U.

THEOREM 4.7. *Let $\phi: \mathbf{R}^n \to \mathbf{R}$ satisfy $|\nabla\phi(x)| \leqslant L$ in $|x| < 2r$. Let V be a positive harmonic function in the region $\{(x,y) : |x| < 2r, \phi(x) < y < 10r\}$ satisfying $u(x, \phi(x)) = 0$ for $|x| < 2r$. Let $X = (0, \phi(0) + r)$. Then the following are comparable with constants depending only on the Lipschitz constant L and on the dimension n:*

$$\frac{V(X)}{r} \approx r^{-n} \int_{|x| < r} |\nabla u(x, \phi(x))| \, dx \approx \left(r^{-n} \int_{|x| < r} |\nabla V(x, \phi(x))|^2 \, dx \right)^{1/2}.$$

Moreover, if W is a function satisfying the same hypotheses as V, then

$$\frac{V(Y)}{W(Y)} \approx \frac{V(X)}{W(X)}$$

for all $Y = (x, \phi(x) + t)$ such that $|x| < r$ and $0 < t < r$.

For proofs see [D], [JK1].

Denote $f_+(\xi) = \max(f(\xi), 0)$. Denote

$$A_1 = \int_{S^n} d\mu, \quad A_2 = \inf_{e \in S^n} \int_{S^n} (e \cdot \xi)_+ \, d\mu.$$

PROPOSITION 4.8. *Let Ω be a convex domain in $\mathbf{R}^N$ such that $g_*(h^2 \, d\sigma) = d\mu$. Assume that $A_2 > 0$. If $N \geqslant 4$, then the diameter and inradius of Ω are bounded above and below by constants depending only on A_1 and A_2. If $N = 3$, then $g_*(h^2 \, d\sigma)$ is unchanged after dilation of Ω, and Ω can be replaced by a dilate for which the diameter and inradius are bounded above and below by constants depending only on A_1 and A_2.*

Proof. Translate Ω so that the origin is the midpoint of a diameter. Let $e \in S^n$ be a unit vector in the direction of that diameter. Then

$$(\xi \cdot e)_+ \operatorname{diam} \Omega \leqslant 2u(\xi).$$

Therefore

$$2 \operatorname{cap} \Omega = 2 \int_{S^n} u \, d\mu \geqslant \operatorname{diam} \Omega \int_{S^n} (\xi \cdot e)_+ \, d\mu \geqslant A_2 \operatorname{diam} \Omega.$$

This can be rewritten as

$$\operatorname{diam} \Omega \leqslant \frac{2 \operatorname{cap} \Omega}{A_2}. \tag{4.9}$$

When $N = 3$ dilate Ω so that $\operatorname{cap} \Omega = 1$ to obtain an upper bound for the diameter. When $N \geqslant 4$, apply the isoperimetric inequality (Corollary 3.19),

$$\operatorname{diam} \Omega \leqslant 2C_N \frac{A_1^{(N-2)/(N-3)}}{A_2}. \tag{4.10}$$

Thus we have an upper bound on diameter in all dimensions.

To obtain the lower bound on the inradius, we first derive a lower bound on $\operatorname{cap}\Omega$. When $N=3$, the normalization $\operatorname{cap}\Omega=1$ gives the lower bound. When $N\geqslant 4$, note that $\Omega\subset B$ for a ball of radius $\operatorname{diam}\Omega$ and hence

$$\operatorname{cap}\Omega \leqslant \operatorname{cap} B \leqslant c_N(\operatorname{diam}\Omega)^{N-2}. \tag{4.11}$$

But (4.9) implies $A_2\operatorname{diam}\Omega\leqslant 2\operatorname{cap}\Omega$, so

$$A_2 \leqslant 2c_N(\operatorname{diam}\Omega)^{N-3}.$$

Since $N\geqslant 4$, this gives a lower bound $\operatorname{diam}\Omega$. Combining this bound with (4.9), there is a dimensional constant $c>0$ such that

$$cA_2^{(N-2)/(N-3)} \leqslant \operatorname{cap}\Omega. \tag{4.12}$$

Thus in all dimensions we have a lower bound on $\operatorname{cap}\Omega$.

Let $\mathcal{E}$ denote an ellipsoid with semi-axes $b_1\leqslant b_2\leqslant...\leqslant b_N$ such that $\mathcal{E}\subset\Omega\subset N\mathcal{E}$. Since b_N is comparable to $\operatorname{diam}\Omega$ the upper bound on $\operatorname{diam}\Omega$, the lower bound on $\operatorname{cap}\Omega$ and (4.11) imply that $b_N\approx 1$. Remark 4.5 implies that if b_2 (and hence b_1) tends to zero and b_N is bounded above, then $\operatorname{cap} N\mathcal{E}$ tends to zero. But we have just shown that the capacity of Ω is bounded below, so it must be that $b_2\geqslant c>0$. Finally, if $0<C^{-1}<b_2\leqslant b_N\leqslant C$ and b_1 tends to zero, then we will show that A_1 tends to infinity.

LEMMA 4.13. *Suppose that* $b_1\leqslant b_2\leqslant...\leqslant b_N$ *are the axes of the ellipsoid associated to the convex body* Ω *by John's lemma. Let* C *be a constant such that* $C^{-1}<b_2\leqslant b_N\leqslant C$. *There is a constant* $c>0$ *depending only on* C *and* N *such that*

$$A_1 = \int_{\partial\Omega} h^2\,d\sigma > c\log(1/b_1).$$

Proof. Rotate and translate Ω so that $\mathcal{E}\subset\Omega\subset N\mathcal{E}$. Denote $G(x)=1-U(x)$. Notice that for points at a unit distance from Ω, $G(x)$ is comparable to 1. This follows from Harnack's inequality, the fact that $G_{N\mathcal{E}}\leqslant G\leqslant G_{\mathcal{E}}$ and easy, explicit estimates on the size of the equilibrium potential for ellipsoids.

Let $\omega=\{x\in\mathbf{R}^n:(s,x)\in\Omega$ for some $s\in\mathbf{R}\}$. In other words, ω is the projection of Ω in the direction perpendicular to the x_1-axis. It follows that ω is convex and the ellipsoid comparable to ω has semi-axes comparable to $b_2,b_3,...,b_N$. We assert that the "top" and "bottom" of the domain Ω as a graph over ω are given by Lipschitz functions with a uniform Lipschitz bound away from $\partial\omega$. More precisely,

Remark 4.14. $\Omega = \{(s, x) \in \mathbf{R} \times \omega : f_1(x) < s < f_2(x)\}$ for a convex function f_1 and a concave function f_2, and for all $x \in \omega$ such that $\text{dist}(x, \partial \omega) > b_1$,

$$|\nabla f_i(x)| \leqslant 2N.$$

Proof. A tangent plane $P = \{(f_2(x) + a \cdot (z-x), z) : z \in \mathbf{R}^n\}$ to the graph of f_2 through $(f_2(x), x)$ does not meet the cone of rays from $(f_2(x), x)$ through $(f_1(z), z)$ for all z such that $|z-x| < b_1$. But because $\Omega \subset N\mathcal{E}$, $|f_2(x) - f_1(z)| < 2Nb_1$. It follows that the slope with respect to the vertical axis x_1 of any line of P through $(f_2(x), x)$ is at most $2N$.

Let $b = Nb_1$. Consider x as in Remark 4.14 and let z be a point of $\partial \omega$ with $r = |x-z| = \text{dist}(x, \partial \omega) > 2b$. We claim that there is a constant $c' > 0$ such that

$$G((f_2(x) + b_1, x)) > c' b r^{-1/2}. \tag{4.15}$$

To prove (4.15), translate and rotate ω so that the projection of z onto $\partial \omega$ is the origin and the tangent plane to $\partial \omega$ at the origin is the plane $x_2 = 0$. (The rotation scrambles $b_2, ..., b_N$, but we are assuming that these numbers are all comparable—the region ω has bounded eccentricity.) Let $b = Nb_1$. Ω is contained in the region

$$\mathcal{S} = \{x \in \mathbf{R}^N : |x_1| < b, \, x_2 > 0\}.$$

Let F and c be the function and constant of Lemma 4.6. Then

$$u(x) = b^{1/2} F(cx_2/b + icx_1/b)$$

is positive and harmonic in the complement of $\mathcal{S}$ and vanishes on $\partial \mathcal{S}$. The first bound of Lemma 4.6 may be restated to say that there is an absolute constant c_1 such that for all $x \in \partial \mathcal{S}$ with $x_2 > 1$,

$$|\nabla u(x)| > c_1 x_2^{-1/2}. \tag{4.16}$$

The second bound implies that for any constant C there is a constant C_1 such that if $x \in \mathbf{R}^N \backslash \mathcal{S}$ and $|x| < C$, then

$$u(x) < C_1. \tag{4.17}$$

Let $G = 1 - U$, then the maximum principle implies that there is a constant C_2 such that

$$u < C_2 G \tag{4.18}$$

for all $|x| < C$ in the complement of $\mathcal{S}$. Let $r > 2b$ and consider points

$$x^0 = (b, r, x_3, ..., x_N), \quad x^1 = x^0 + (b, 0, ..., 0).$$

Let $B = \{x \in \mathbf{R}^N : x_1 = b, |x - x^0| < b\}$, the surface ball on $\partial \mathcal{S}$ of radius b around x^0. Recall that the surface area of B is comparable to b^n with $n = N - 1$. A change of variable of Proposition 4.7 from unit scale to scale b yields

$$\int_B |\nabla u| \, d\sigma \approx u(x^1) b^{n-1}$$

so that (4.16) implies there is a constant $c_2 > 0$ such that

$$c_2 b r^{-1/2} \geqslant u(x^1).$$

Hence, by (4.18), there is a constant c_3 such that

$$G(x^1) > c_3 r^{-1/2} b$$

which, using Harnack's inequality, is the same as assertion (4.15). Consider the surface ball

$$B(X) = \{Z \in \partial\Omega : |Z - X| < b_1\} \quad \text{where } X = (f_2(x), x).$$

For x at a distance r from $\partial\omega$, $r > 2b$, Remark 4.14 says that f_2 is a Lipschitz function, so the rescaled version of Proposition 4.7 applies to the function G and implies

$$\int_{B(X)} |\nabla G| \, d\sigma \geqslant b^{n-1} G(X + (b, 0, ..., 0)) \geqslant r^{-1/2} b^n.$$

It follows from Schwarz's inequality that

$$\int_{B(X)} |\nabla G|^2 \, d\sigma \geqslant r^{-1} \sigma(B(X)).$$

Let $A_k = \{(f_2(x), x) : 2^k b \leqslant \text{dist}((x_2, ..., x_N), \partial\omega) \leqslant 2^{k+1} b\}$. For $1 < 2^k < 1/b$, the regions A_k have area comparable to $r = 2^k b$ and

$$\int_{A_k} |\nabla G|^2 \, d\sigma \geqslant c_4 > 0$$

independent of k. Summing over k we find the estimate of Lemma 4.13. This also concludes the proof of Proposition 4.8.

5. Existence of solutions

We begin with the case of polyhedra. Consider a collection $\{\xi(1), ..., \xi(m)\}$ of unit vectors in $\mathbf{R}^N$ with the properties:

$$\text{for any unit vector } v, \text{there exists } j \text{ such that } v \cdot \xi(j) > 0, \tag{5.1}$$

$$|\xi(j) + \xi(k)| > 0. \tag{5.2}$$

Consider any $p \in \mathbf{R}^m$ with all nonnegative entries, and define

$$\Omega(p) = \{x \in \mathbf{R}^N : x \cdot \xi(k) \leqslant p_k, \, k = 1, ..., m\}.$$

Then $\Omega(p)$ is a closed, convex subset of $\mathbf{R}^N$ and (5.1) implies that $\Omega(p)$ is bounded. Denote

$$\mathcal{M} = \{p \in \mathbf{R}^m : p_k \geqslant 0 \text{ for all } k, \, \mathrm{cap}(\Omega(p)) \geqslant 1\}.$$

Let $c_1, ..., c_m$ be a sequence of positive real numbers satisfying the compatibility condition

$$\sum_{k=1}^{m} c_k \xi(k) = 0. \tag{5.3}$$

(This is a vector equation in $\mathbf{R}^N$, so it imposes N conditions on the numbers c_k.) Denote

$$\Phi(p) = \sum_{k=1}^{m} c_k p_k.$$

THEOREM 5.4. *Let $\xi(k)$ and p_k satisfy the conditions (5.1), (5.2) and (5.3). There exists $p^* \in \mathcal{M}$ such that*

$$\min_{\mathcal{M}} \Phi = \Phi(p^*) > 0$$

and the polyhedron $\Omega(p^*)$ has faces F_k with outer normal ξ_k satisfying

$$c_k = \frac{\Phi(p^*)}{N-2} \int_{F_k} |\nabla U|^2 \, d\sigma$$

and these are the only faces.

Before we begin the proof let us remark that while conditions (5.1) and (5.3) are necessary, condition (5.2) is not. We have imposed it for our convenience in proving that the minimizing convex body has nonempty interior. Even without (5.2), this can be proved using the inradius estimate (Proposition 4.8). There is no need to carry out that argument because we can easily approximate any sum of point masses by a sum at points satisfying (5.2). Perhaps a more interesting remark is that if properly formulated, this variational procedure gives the solution to the problem in the general (nonpolyhedral) case. The proof of this relies on the uniqueness in the Borell inequality (Theorem 0.10) and will be carried out in [CJL]. However, this technique does not give an independent proof of existence because the proof uses the fact that the solution has been constructed already, by the method given here.

Proof of Theorem 5.4. It is obvious that a minimizing sequence is bounded. So one can take a convergent subsequence. The limit will be a minimizer because capacity is continuous in the Minkowski metric. Also, we have

$$\operatorname{cap}\Omega(p^*)=1. \tag{5.5}$$

Furthermore,

$$\min_{\mathcal{M}}\Phi>0.$$

Indeed, assume that $\Phi(p^*)=0$. Then $p^*=0$ and condition (5.1) implies that $\Omega(p^*)$ is a single point (the origin). This contradicts the fact that its capacity is 1. Moreover, we claim that $\Omega(p^*)$ has nonempty interior. If $\Omega(p^*)$ had empty interior, then (5.2) would imply that it is contained in a plane of dimension $N-2$. But such a set has zero capacity.

Note that the vector p^* is not unique. Let $x^0 \in \mathbf{R}^N$. If $p \in \mathcal{M}$, then the vector q defined by $\tilde{p}_k = p_k + x^0 \cdot \xi_k$ satisfies

$$\Omega(q)=\Omega(p)+x^0 \quad \text{and} \quad \Phi(q)=\Phi(p).$$

(This is where we use the assumption (5.3).) Because $\Omega(p^*)$ has nonempty interior, one can translate the origin to the interior of $\Omega(p^*)$ so that

$$p_k^*>0 \quad \text{for all } k. \tag{5.6}$$

Consider $p \in P$ for which $p_k \geqslant 0$, then $\Phi(tp+(1-t)p^*)=\lambda$ and the fact that λ is the minimum of Φ over $\mathcal{M}$ implies that $\operatorname{cap}\Omega(tp+(1-t)p^*)\leqslant 1$ for $0\leqslant t\leqslant 1$. Moreover,

$$t\Omega(p)+(1-t)\Omega(p^*)\subset\Omega(tp+(1-t)p^*).$$

Since $\operatorname{cap}\Omega(p^*)=1$, we have

$$\lim_{t\to 0^+}\frac{\operatorname{cap}(t\Omega(p)+(1-t)\Omega(p^*))-\operatorname{cap}\Omega(p^*)}{t}\leqslant 0.$$

For p near p^*, the support function v of $\Omega(p)$ satisfies

$$v(\xi(k))=p_k$$

for all k such that $w_k>0$. Therefore, Proposition 3.16 and (3.17) imply

$$\lim_{t\to 0^+}\frac{\operatorname{cap}(t\Omega(p)+(1-t)\Omega(p^*))-\operatorname{cap}\Omega(p^*)}{t}=\sum_k(p_k-p_k^*)w_k.$$

Therefore,

$$\sum_k(p_k-p_k^*)w_k\leqslant 0$$

for all $p \in P$ near p^*. Property (5.6) says that the values of p for which $p_k \geqslant 0$ form a full open neighborhood of p^* in P. Therefore,

$$\sum_k(p_k-p_k^*)w_k=0$$

for all $p \in P$. In other words, $P=P'$ and $w_k=(N-2)c_k/\lambda$.

LEMMA 5.7. *Let μ be a positive measure on the unit sphere S^n which is a finite sum of point masses. Suppose that μ satisfies*

(a) *$\mu(H) > 0$ for every hemisphere $H = \{\xi \in S^n : \xi \cdot v > 0\}$,*

(b) *if μ assigns a positive mass to a point, then it assigns no mass to the antipodal point,*

(c) *$\int_{S^n} \xi \, \delta\mu(x) = 0$.*

If $N = n+1 \geqslant 4$, then there exists a bounded convex polyhedron Ω in $\mathbf{R}^N$ such that $g_*(h^2 \, d\sigma) = d\mu$. If $N = 3$, then there exists $t > 0$ and Ω such that $g_*(h^2 \, d\sigma) = t \, d\mu$.

Proof. The measure μ can be written

$$\mu = \sum_{k=1}^{m} c_k \delta_{\xi_k}$$

for some positive constants c_k. Conditions (a), (b) and (c) are equivalent to (5.1), (5.2) and (5.3). Therefore, by Theorem 5.4, there is a number $t > 0$ and a convex body Ω such that

$$g_*(h^2 \, d\sigma) = t \, d\mu.$$

If $N \geqslant 4$, then one can dilate Ω to achieve any multiple of μ. (If $N = 3$, then $g_*(h^2 \, d\sigma)$ is dilation invariant.)

THEOREM 5.8. *Let μ be any positive Borel measure on the unit sphere S^n, satisfying (a) and (c) of Lemma 5.7. If $N = n+1 \geqslant 4$, then there exists a bounded convex domain Ω in $\mathbf{R}^N$ such that $g_*(h^2 \, d\sigma) = d\mu$. If $N = 3$, then there exists $\lambda > 0$ and Ω such that $g_*(h^2 \, d\sigma) = \lambda \, d\mu$.*

Proof. It follows from (a) that

$$\inf_{e \in S^n} \int_{S^n} (e \cdot \xi)_+ \, d\mu(\xi) > 0.$$

Let μ_j be a sequence of positive measures that are finite sums of point masses such that μ_j tends weakly to μ, each μ_j satisfies Lemma 5.7 (a), (b) and (c), and

$$\inf_{e \in S^n} \int_{S^n} (e \cdot \xi)_+ \, d\mu_j(\xi) \geqslant c > 0 \quad \text{for all } j.$$

Let $N \geqslant 4$. Lemma 5.7 implies that there is a convex body Ω_j associated to each μ_j and Proposition 4.8 implies that there is a uniform lower bound on the inradius and upper bound on the diameter of the domains Ω_j. Therefore, by the Blaschke selection theorem there is a subsequence that converges in the Minkowski metric to a domain Ω. Finally, Theorem 3.1 implies that the measure associated to Ω is μ. The case $N = 3$ is

similar. Dilate all the domains Ω_j to have diameter 1 and replace Ω_j by a subsequence that converges in the Minkowski metric. The domains Ω_j are associated to measures $t_j \mu_j$, and Theorem 3.1 implies that the measures $t_j \mu_j$ converge weakly to a measure $t\mu$ associated to Ω.

6. Estimates for harmonic measure

Consider a convex domain Ω of bounded eccentricity. After translation we may assume that

$$B_{r_0} \subset \Omega \subset B_{R_0}. \tag{6.1}$$

The ratio R_0/r_0 is known as the eccentricity of Ω. The boundary of a convex domain is represented locally as the graph of a Lipschitz function with Lipschitz constant depending only on the eccentricity of the convex domain.

LEMMA 6.2. *Let V, X and ϕ be as in Theorem 4.7. Suppose further that the function ϕ is concave. Then the following are comparable with a constant depending only on dimension and the Lipschitz constant of ϕ:*

$$\frac{V(X)}{r} \approx \min_{|x|<r} |\nabla V(x, \phi(x))|.$$

Proof. Fix any point $Z=(x, \phi(x))$ with $|x|<r$, and choose a new coordinate system so that Z is the origin and $y_1 < 0$ for every $y \in \Omega$. The half-ball $B_r^+ = \{y \in \mathbf{R}^N : |y| < r, y_1 > 0\}$ is such that its flat boundary is tangent to the graph at Z. Let $Y = \{y \in \partial B_r^+ : |y| = r, y_1 > \frac{1}{2} r\}$. By Harnack's inequality, $V(y) \geqslant cV(X)$ for $y \in Y$. It follows from explicit calculation of the Poisson kernel of the half-ball that if the function v satisfies $\Delta v = 0$ in B_r^+, $v(y) = 1$ for $y \in Y$, and $v = 0$ on $\partial B_r^+ \setminus \overline{Y}$, then $(\partial/\partial y_1)v(0) > c/r$. On the other hand, the maximum principle implies $V(y) \geqslant cV(X)v(y)$. Since $v(0) = 0$, we have $(\partial/\partial y_1)V(0) > cV(X)/r$, as desired.

Note that when Lemma 6.2 and Theorem 4.7 are applied to the function $V = G = 1 - U$ defined on Ω' we find the following comparability depending only on eccentricity of Ω:

$$\left(\frac{1}{\sigma(S)} \int_S h^2 \, d\sigma\right)^{1/2} \approx \frac{1}{\sigma(S)} \int_S h \, d\sigma \approx \frac{G((1+r)x)}{r} \approx \min_S h \tag{6.3}$$

where, as usual $h = |\nabla U| = |\nabla G|$ and $S = B(x, r) \cap \partial\Omega$ for any $r < \frac{1}{10} r_0$ and $x \in \partial\Omega$.

Make a dilation so that $r_0 = 1$ and R_0 is the eccentricity of Ω. We will retain this normalization through much of this section. Cover $\partial\Omega$ by finitely many balls of radius

comparable to $r_0 = 1$. It then follows from (6.3) that there is a constant C depending only on the eccentricity of Ω such that

$$\int_{\partial\Omega} h^2 \, d\sigma \leqslant C \quad \text{and} \quad \min_{\partial\Omega} h^2 \geqslant C^{-1}. \tag{6.4}$$

The main point of this section is to derive estimates like (6.3) and (6.4) for certain "slices" or cross sections of $\partial\Omega$ that need not be comparable to balls. These slices are needed for Caffarelli's regularity theory for the Monge–Ampère equation.

Consider a bounded, open, convex subset E of $\mathbf{R}^n$. The John lemma can be restated to say that there is an affine linear transformation T (namely the one that takes the ellipsoid $\mathcal{E}$ to the unit ball) for which

$$B_1 \subset TE \subset B_n$$

where B_r denotes the ball of radius r about the origin in $\mathbf{R}^n$. We will call TE the normalization of E. Let $S(x, E)$ denote the family of all pairs of points (x^0, x^1) in ∂E for which x is on the segment joining x^0 to x^1. Define

$$\delta(x, E) = \min_{(x^0, x^1) \in S(x, E)} \frac{|x - x^0|}{|x - x^1|}.$$

It is invariant under all linear transformations (not just dilations), because x, x^0 and x^1 are collinear. On the other hand, it is easy to see that the distance $\delta(Tx, TE)$ is comparable to the distance from Tx to $\partial(TE)$ and also that it is comparable to the radial distance $|Tx - cTx|$ where c is chosen so that $cTx \in \partial(TE)$. Thus we refer to the distance $\delta(x, E)$ as the normalized distance of x to ∂E. The distance $\delta(x, E)$ is defined in exactly the same way if E is a convex subset of an n-plane in $\mathbf{R}^N$.

Suppose that H is a half-space in $\mathbf{R}^N$, and $H \cap B_{r_0} = \varnothing$. Denote $\Pi = \partial H$ and $F = H \cap \partial\Omega$. Define P as the radial projection onto Π, that is, if $y \in \Pi$ and $P(x) = y$, then there is a scalar $\alpha(x)$ such that $x = \alpha(x)y$. Let $E = P(F)$. Then E is a convex subset of (a copy of) $\mathbf{R}^n$. Define a normalized distance to the boundary on F by

$$\delta(x, F) = \delta(P(x), E)$$

with $\delta(x, E)$ as above. It is easy to see that this distance is changed at most by a bounded factor for different choices of P depending on the location of the origin, provided the distance from the origin to $\partial\Omega$ is bounded below by a fixed constant times the inradius.

THEOREM 6.5. *Let Ω be a convex domain in $\mathbf{R}^N$, $N \geqslant 3$. There are constants C and $\varepsilon > 0$, depending only on dimension and the eccentricity of Ω such that*

$$\int_F \delta(x, F)^{1-\varepsilon} h(x)^2 \, d\sigma(x) \leqslant C\sigma(F) \min_F h^2.$$

for every set F of the form $F=\partial\Omega\cap H$ for some half-space H. (Recall the notation $h=|\nabla U|$ on $\partial\Omega$.)

The estimate of Theorem 6.5 is dilation invariant, so without loss of generality, we will make a dilation so that $r_0=1$ and R_0 is the eccentricity of Ω. Note also that the assumption $H\cap B_{r_0}=H\cap B_1=\varnothing$ is not a significant one: As (6.4) shows, a better estimate is true for slices F of unit size. From (6.3) we see that the only difficulty in proving Theorem 6.5 will be that the shape of the set F can be very different from a ball.

Let θ denote the unit normal to Π pointing into H. Because $H\cap B_1=\varnothing$, we have

$$1\leqslant x\cdot\theta\leqslant R_0\quad\text{for all }x\in E.\tag{6.6}$$

LEMMA 6.7. *Let a denote the inradius of E. There is a constant C_1 depending only on eccentricity such that for every $x\in F$,*

$$|x-Px|\leqslant C_1 a.$$

Proof. Suppose that $|x-Px|>C_1 a$. The convex hull of x with $B_{1/2}$ is contained in Ω. The cross section at Px in the direction perpendicular to x is an n-disk centered at Px of radius at least $C_1 a/R_0$. It follows that E contains a ball of similar radius and for C_1 sufficiently large this contradicts the assumption that a is the inradius of E.

LEMMA 6.8. *If x^0 and x^1 belong to F, $\delta(x^1,F)\approx 1$, and a is the inradius of E as in Lemma 6.7, then*

$$\min_{B_a(x^1)\cap\partial\Omega} h\leqslant C\min_{B_a(x^0)\cap\partial\Omega} h.$$

Proof. Choose $x^2\in\partial E\cap\partial F$ such that Px^1 is on the segment joining Px^0 and x^2. Then since the normalized distance of Px^1 to the boundary is comparable to 1,

$$|Px^0-Px^1|\leqslant C_2|Px^1-x^2|.\tag{6.9}$$

Define the dilation

$$\Phi(x)=s(x-x^2)+x^2$$

with s chosen so that

$$\Phi(Px^0)=Px^1.\tag{6.10}$$

It follows from (6.9) that $1\geqslant s>1/(1+C_2)$. The function $G(\Phi^{-1}(x))$ is a positive harmonic function in the complement of $\Phi(\Omega)$ that vanishes on the boundary and tends to 1 at infinity. Since $\Phi(\Omega)\subset\Omega$, the maximum principle implies

$$G(x)\leqslant G(\Phi^{-1}(x))$$

for every $x \in \Omega'$. We will choose a number L sufficiently large depending on R_0 such that the points $z = (1 + La)Px^0 \in \Omega'$ and $x = \Phi(z) \in \Omega'$ satisfy

$$\text{dist}(z, \partial\Omega) \approx a \quad \text{and} \quad |z - x^0| \leqslant Ca, \tag{6.11}$$

and

$$\text{dist}(x, \partial\Omega) \approx a \quad \text{and} \quad |x - x^1| \leqslant Ca. \tag{6.12}$$

The second assertion of (6.11) is obvious with $C = L$. Lemma 6.7 implies that the distance to E of every point of F is less than a multiple of a. On the other hand, the distance from z to Π is $\theta \cdot (z - Px^0) = La\theta \cdot Px^0$ and (6.6) implies $\theta \cdot Px^0 \geqslant 1$. Hence the distance from z to $\partial\Omega$ is comparable to a. The proof of (6.12) is somewhat similar. First of all,

$$x = \Phi(z) = Px^1 + sLaPx^0.$$

Thus $|x - x^1| \leqslant |Px^1 - x^1| + sLa|Px^0| \leqslant C_1 a + sLa$. The distance from x to Π is $sLa\theta \cdot Px^0 \geqslant sLa$. It follows by similar reasoning to the proof of (6.11) that for L sufficiently large the distance from x to $\partial\Omega$ is comparable to a.

Finally, (6.11), (6.12) and Lemma 6.2 imply that $G(z)$ is comparable to $\min_{B_a(x^0)} h$ and $G(x)$ is comparable to $\min_{B_a(x^1)} h$. Since $G(x) \leqslant G(z)$, Lemma 6.8 follows.

Our main lemma is

LEMMA 6.13. *There are constants C and $\varepsilon > 0$ depending only on dimension and R_0 such that for every $x^1 \in F$,*

$$\frac{1}{\sigma(B(x^1, a))} \int_{B(x^1, a)} h \, d\sigma \leqslant C\delta(x^1, F)^{-1+\varepsilon} \min_F h.$$

Proof. Let x^0 be a central point, $\delta(x^0, F) \approx 1$. Choose $x^2 \in \partial F \cap \partial E$ such that Px^1 is on the segment with endpoint Px^0 and x^2. We will use the same dilation as in Lemma 6.8, but note that this time the factor s may be arbitrarily small. It is x^0 that is near the "middle" of F, not x^1. Let $r = |x^0 - x^2|$. Then

$$\delta(x^1, F) \approx \frac{|Px^1 - x^2|}{r}.$$

If $\delta(x^1, F) \approx 1$, then Lemma 6.8 implies

$$\min_{B_a(x^1) \cap \partial\Omega} h \leqslant C \min_{B_a(x^0) \cap \partial\Omega} h \leqslant C \min_F h,$$

so we are done. Let us assume from now on that $\delta(x^1, F)$ is much smaller than 1. In particular, we can suppose that $|x^1 - x^2| < \frac{1}{10}r$. Let $U = \Omega \cap B(x^0, \frac{1}{2}r)$. Let Γ denote the cone $\{x^2 + t(x - x^2) : x \in U, t > 0\}$. Denote by G_Γ a positive harmonic function on the complement of Γ that vanishes on the boundary. The function G_Γ is unique up to a multiple and homogeneous of the form

$$G_\Gamma(x) = |x - x^2|^\beta f((x - x^2)/|x - x^2|)$$

with $\beta > 0$ bounded below by a constant depending only on the eccentricity of Ω. Choose a point x^3 at distance r from Γ and at most $2r$ from x^2. Normalize G_Γ so that $G_\Gamma(x^3) = G(x^3)$. We claim that there is a constant C depending only on eccentricity for which

$$G(x) \leqslant CG_\Gamma(x) \tag{6.14}$$

for all $x \in B(x^2, \frac{1}{4}r) \backslash \Omega$. To prove this, let Γ_1 be the convex hull of U with x^2 and let G_1 be the positive harmonic function in the complement of Γ_1 that vanishes on $\partial \Gamma_1$ normalized by $G_1(x^3) = G(x^3)$. Note that $\Gamma_1 \cap B(x^2, \frac{1}{2}r) = \Gamma \cap B(x^2, \frac{1}{2}r)$. Because Ω is convex, it follows that $B(x^2, \frac{1}{2}r) \backslash \Gamma$ is a Lipschitz domain. It follows from Theorem 4.7 that G_1 and G_Γ are comparable at all points of $B(x^2, \frac{1}{4}r) \backslash \Gamma$. On the other hand, Harnack's inequality implies that G_1 is comparable to $G(x^3)$ on all of $\partial B(x^2, 3r)$. Since $\Gamma_1 \subset \Omega$, we see that $G_1 \geqslant 0$ on $\partial \Omega$. Therefore, the maximum principle applied to the region $B(x^2, 3r) \backslash \Omega$ implies that $G(x) \leqslant CG_1(x)$ for all $x \in B(x^2, 3r) \backslash \Omega$. This proves (6.14).

Next, the same argument as in the last line of the preceding paragraph shows that $G_\Gamma(x) \leqslant CG_1(x)$ for all $x \in B(x^2, 3r) \backslash \Gamma$. On the other hand, $B(x^0, \frac{1}{2}r) \cap \Omega = B(x^0, \frac{1}{2}r) \cap \Gamma_1$, so an argument similar to the one above using Theorem 4.7 implies that G_1 and G_Γ are comparable at all points of $B(x^0, \frac{1}{4}r) \backslash \Gamma$. Putting these two estimates together, we have

$$G_\Gamma(x) \leqslant CG(x) \tag{6.15}$$

for all $x \in B(x^0, \frac{1}{4}r) \backslash \Gamma$.

For sufficiently large C depending only on eccentricity, the segment $S = \{(1 + Ca)x : x = tPx^0 + (1 - t)x^2, 0 \leqslant t \leqslant 1\}$ has the property that every point $x \in S$ is at a distance comparable to a from Γ and from Ω. This can be seen from the construction of Γ and Lemma 6.7. It follows that $(1 + Ca)x^1 \in B(x^2, \frac{1}{4}r) \backslash \Omega$ and by (6.14),

$$G((1 + Ca)x^1) \leqslant CG_\Gamma((1 + Ca)x^1).$$

Similarly, (6.15) implies

$$G_\Gamma((1 + Ca)x^0) \leqslant CG((1 + Ca)x^0).$$

Let z be δr distance along the ray from x^2 to $(1+Ca)x^0$. In other words, z is at a height δa above Γ and very near Px^1. Hemisphere plus Carleson lemma comparison implies

$$\frac{G_\Gamma(z)}{\delta a} \geqslant \frac{G_\Gamma((1+Ca)x^1)}{a},$$

and homogeneity of degree ε implies

$$G_\Gamma(z) = \delta^\varepsilon G_\Gamma((1+Ca)x^0).$$

Combining these inequalities, we have

$$G((1+Ca)x^1) \leqslant C\delta^{-1+\varepsilon} G((1+Ca)x^0),$$

which, by (6.3), is the same as the conclusion to Lemma 6.13.

Lemma 6.13 is not quite sufficient to prove Theorem 6.5 because a covering of F by balls of radius a can fatten F significantly. We take care of this excess with the following geometric lemma.

If Q is a cube in $\mathbf{R}^n$ of sidelength s, denote by Q^* the concentric cube of sidelength $b_n s$. (The multiple b_n will be chosen later.) Let $|S|$ denote the Lebesgue measure of a set S.

LEMMA 6.16. *Let E be a convex subset of $\mathbf{R}^n$ with inradius a. Choose coordinate axes parallel to the axes of an optimal inscribed ellipsoid in E. Let $\mathcal{Q}$ be a tiling of E by cubes with sides of length s parallel to the coordinate axes. Assume that $s<a$. For each cube $Q\in\mathcal{Q}$, denote*

$$\delta^*(Q) = \max_{x\in Q^*\cap E} \delta(x,E).$$

Then,

$$\sum_{\{Q\,:\,\delta^*(Q)<\delta\}} |Q| \leqslant C_n \delta |E|.$$

Proof. The proof is by induction on dimension. The case $n=1$ is easy. Let $P_k: \mathbf{R}^n \to \mathbf{R}^{n-1}$ be the orthogonal projection onto the plane perpendicular to the x_k-axis, that is, $P_k(x_1, x_2, ..., x_n) = (x_1, x_2, ..., x_{k-1}, x_{k+1}, ..., x_n)$. To prove the induction step we first prove

LEMMA 6.17. *There are dimensional constants $c_n>0$ and C_n such that if $E\subset\mathbf{R}^n$ is a convex set and Q is a cube satisfying*

(a) $Q\cap E\neq\varnothing$,

(b) *for any $x\in Q^*\cap E$, $\delta(x,E)<\delta$,*

(c) $|Q^*\cap E|<c_n|Q|$,

then there exists k such that

$$\delta(x', P_k(E)) < C_n \delta \quad \text{for any } x' \in P_k(Q)^* \cap P_k(E).$$

Proof. The John lemma and (c) imply that $Q^* \cap E$ is contained in an ellipsoid of volume at most $c_n n^n s^n$. Translate all the sets so that the ellipsoid is centered at the origin. If c_n is chosen sufficiently small, then at least one axis of the ellipsoid has length less than s. In other words, there is a unit vector v such that $Q^* \cap E \subset \{x \in \mathbf{R}^n : |v \cdot x| < s\}$. There is a component v_k of v of length greater than $1/\sqrt{n}$. We will show that the conclusion of Lemma 6.17 is valid for this index k. For notational simplicity, assume that $k = n$.

Let $x' \in P_n(Q)^* \cap P_n(E)$, and let I denote the set of all $x \in E$ such that $P_n(x) = x'$. We claim that $x \in Q^*$. Suppose not. We will derive a contradiction. Since $P_n(x) \in P_n(Q)^*$, it must be that $|x_n| > b_n s$. Without loss of generality we suppose that $x_n > b_n s$. Hypothesis (a) says that there is a point $y \in E$ such that $|y_i| < s$ for all i. Let $z = tx + (1-t)y$ with $t = bs/x_n$. Then $|z_n - bs| \leqslant s$. Since $P_n(x) \in P_n(Q)^*$, we have $|x_i| < b_{n-1}$ for all $i < n$. It follows that

$$|z_i| < \left(1 + \frac{bb_{n-1}}{x_n}\right)s < \left(1 + \frac{bb_{n-1}}{b_n}\right)s$$

for all $i < n$. If we choose b so that $b + 1 < b_n$ and $1 + bb_{n-1}/b_n < b_n$, then $z \in Q^*$. By convexity, we also have $z \in E$. Therefore, $v \cdot z < s$. But

$$v \cdot z \geqslant v_n z_n - \sum_{i=1}^{n-1} |v_i z_i| \geqslant \frac{bs}{\sqrt{n}} - n\left(1 + \frac{bb_{n-1}}{b_n}\right)s.$$

If we choose $b_n = 10(n!)^2$ and $b = 10n^2$, then for $n \geqslant 2$,

$$\frac{bs}{\sqrt{n}} - n\left(1 + \frac{bb_{n-1}}{b_n}\right)s > \frac{bs}{\sqrt{n}} - n\left(1 + \frac{b}{n^2}\right)s > s.$$

This is a contradiction. Thus we have shown that $I \subset Q^*$.

Hypothesis (b) implies that $\delta(x, E) < \delta$ for every $x \in I$. We will now prove that this implies $\delta(x', P_n(E)) < C\delta$. Without loss of generality we can assume that E is normalized. If $I \cap B_{1/2} \neq \varnothing$, then $\delta > \delta'(x, E) > \frac{1}{2}$ for some $x \in I$, so the assertion is trivial. Suppose that $I \cap B_{1/2} = \varnothing$. Then x' does not belong to $P_n(B_{1/2})$. For each $x \in I$, denote by $\alpha(x)$ the scalar such that $(1 + \alpha(x))x \in \partial E$. Since $\delta(x, E) < \delta$ and $\alpha(x)$ is comparable to the radial distance to the boundary, we have $\alpha(x) < C\delta$ for all $x \in I$. Let x^0 and x^1 be the endpoints of I. Because the segment I is contained in E, the normal $g(x^0)$ to any support plane at x^0 must satisfy $g(x^0) \cdot e_n \geqslant 0$. Similarly, $g(x^1) \cdot e_n \leqslant 0$. There is a (not necessarily

unique) continuous choice of support plane along the curve $g((1+\alpha(x))x)$ as x varies in I. Moreover, $\alpha(x^0)=\alpha(x^1)=0$ so the e_n component is positive at one endpoint and negative at the other. In particular, there exists $x \in I$ such that $g((1+\alpha(x))x) \cdot e_n=0$. The associated support plane projects to a support plane of $P_n(E)$, so that $P_n((1+\alpha(x))x)$ is a boundary point of $P_n(E)$. Since the distance $P_n(x)$ to $P_n((1+\alpha(x))x)$ is less than $C\delta$ we are done.

We can now prove the induction step of Lemma 6.16. Define

$$\mathcal{Q}' = \{Q \in \mathcal{Q} : \delta^*(Q) < \delta \text{ and } |Q^* \cap E| \geqslant c_n|Q|\},$$
$$\mathcal{Q}'' = \{Q \in \mathcal{Q} : \delta^*(Q) < \delta \text{ and } |Q^* \cap E| < c_n|Q|\}.$$

Then

$$\sum_{Q \in \mathcal{Q}'} |Q| \leqslant \sum_{Q \in \mathcal{Q}'} c_n^{-1}|Q^* \cap E| = c_n^{-1} \int_E \sum_{Q \in \mathcal{Q}'} \chi_{Q^*}(x)\, dx \leqslant C|\{x \in E : \delta(x, E) < \delta\}|$$

because a point x can belong to at most finitely many sets Q^*. Moreover, $|\{x \in E : \delta(x, E) < \delta\}| \leqslant C\delta|E|$. This can be checked by reducing by a linear transformation to the case of a normalized convex domain.

Next, define

$$\mathcal{Q}_k'' = \Big\{Q \in \mathcal{Q}'' : \max_{x' \in P_k(Q)^*} \delta(x', P_k(E)) < C_n\delta\Big\}.$$

Lemma 6.17 says that every cube of $\mathcal{Q}''$ belongs to some $\mathcal{Q}_k''$. Hence,

$$\sum_{Q \in \mathcal{Q}''} |Q| \leqslant \sum_{k=1}^n \sum_{Q \in \mathcal{Q}_k''} s|P_k(Q)|.$$

Furthermore, the induction hypothesis implies

$$\sum_{Q \in \mathcal{Q}_k''} s|P_k(Q)| \leqslant Cs\delta|P_k(E)|.$$

Finally, $|P_k(E)|$ is comparable to the product of the lengths of all but the kth axis of the ellipsoid associated to E. In particular, $|P_k(E)| \leqslant C|E|/a$ where a is the inradius of E. But $s < a$, so, combining the inequalities above, we have

$$\sum_{Q \in \mathcal{Q}''} |Q| \leqslant C\delta|E|$$

as desired.

By summing separately for cubes Q for which $\delta^*(Q) \approx 2^{-k}$, one can deduce from Lemma 6.16 that for any $\varepsilon > 0$,

$$\sum_{Q \in \mathcal{Q}} \delta^*(Q)^{-1+\varepsilon} |Q| \leqslant C \sum_{k=1}^{\infty} (2^k)^{1-\varepsilon} 2^{-k} |E| \leqslant C_\varepsilon |E|. \qquad (6.18)$$

If we write $\widehat{Q} = P^{-1}(Q)$, and note that $\sigma(\widehat{Q}) \approx |Q|$ and $\sigma(F) \approx |E|$, then we deduce

$$\int_F h^2 \delta(x, F)^{1-\varepsilon} \, d\sigma \leqslant \sum_{Q \in \mathcal{Q}} \int_{\widehat{Q}} h^2 \delta(x, F)^{1-\varepsilon} \, d\sigma \leqslant \sum_{Q \in \mathcal{Q}} \delta^*(Q)^{1-\varepsilon} \int_{\widehat{Q}} h^2 \, d\sigma$$

$$\leqslant C \sum_{Q \in \mathcal{Q}} \delta^*(Q)^{1-\varepsilon} \sigma(\widehat{Q}) \left(\frac{1}{\sigma(\widehat{Q})} \int_{\widehat{Q}} h \, d\sigma \right)^2$$

$$\leqslant C \sum_{Q \in \mathcal{Q}} \delta^*(Q)^{1-\varepsilon} \sigma(\widehat{Q}) \delta^*(Q)^{-2+2\varepsilon} \min_F h^2$$

$$= C \sum_{Q \in \mathcal{Q}} \delta^*(Q)^{-1+\varepsilon} \sigma(\widehat{Q}) \min_F h^2 \leqslant C \min_F h^2.$$

The first and second inequalities are trivial, the third follows from (6.3), the fourth from Lemma 6.13, and the fifth from (6.18). This proves Theorem 6.5.

7. Regularity of solutions

Let ϕ be a convex function defined on an open set $O \subset \mathbf{R}^n$. Define the set $\nabla f(x)$ as the set of all $y \in \mathbf{R}^n$ such that the plane $\{(z, z_N) \in \mathbf{R}^n \times \mathbf{R} : z_N = \phi(x) + y \cdot (z - x)\}$ is tangent to the graph of ϕ at $(x, \phi(x))$. For a subset $F \subset \mathbf{R}^n$ we define $\nabla \phi(F) = \bigcup \{\nabla \phi(x) : x \in F\}$. The function ϕ is said to satisfy the Monge–Ampère equation

$$\det \nabla^2(\phi) = d\nu \qquad (7.1)$$

in the sense of Alexandrov if

$$\operatorname{vol} \nabla \phi(F) = \nu(F) \qquad (7.2)$$

for every Borel set $F \subset O$. The set-valued mapping $\nabla \phi$ is directly related to the set-valued Gauss mapping by

$$g((x, \phi(x))) = \{\xi = (y, -1)/\sqrt{1+|y|^2} \in S^n : y \in \nabla \phi(x)\}.$$

The coordinate $\xi_N = -1/\sqrt{1+|y|^2}$ gives the Jacobian of the change of variable: $dy = |\xi_N|^{-N} d\xi$. Thus (7.2) can be rewritten

$$\nu(F) = \int_{g(\widetilde{F})} |\xi_N|^{-N} \, d\xi \qquad (7.3)$$

where $\widetilde{F} = \{(x, \phi(x)) : x \in F\}$.

THEOREM 7.1. *Let E be a compact, convex domain in $\mathbf{R}^n$. Let $\varepsilon > 0$ and suppose that ϕ is a convex function in E that vanishes on ∂E and satisfies*

$$\det D_{ij}\phi = d\nu$$

on E in the sense of Alexandrov. Consider $a \in \mathbf{R}^n$ and $b \in \mathbf{R}$ such that $a \cdot x + b \leqslant 0$ on E. Suppose that for every set $F = \{x : \phi(x) < a \cdot x + b\}$,

$$\int_F \delta(x, F)^{1-\varepsilon}\, d\nu(x) \leqslant C\nu\left(\tfrac{1}{2}F\right).$$

Suppose further that there exist a and b such that $V(a,b) = \{x : \phi(x) = a \cdot x + b\}$ contains more than one interior point of E. Then $V(a,b) \cap \partial E$ is nonempty. If ϕ is strictly convex, then $\phi \in C^{1,t}$ for some $t > 0$.

The theorem of Caffarelli is the case $\varepsilon = 1$. We need this stronger result for our application because in Theorem 6.5, ε can be arbitrarily small. The proof follows the same outline as [C4], so we only need to explain the details that are different. First, recall

LEMMA 7.2. *Let E be an open convex set and suppose that u is a convex function satisfying*

$$\det D_{ij}u = d\mu$$

in the Alexandrov sense on E and $u = 0$ on ∂E. There is a dimension constant C such that

$$|u(x)|^n \leqslant C\delta(x, E)|E|\mu(E).$$

Proof. An outline of the proof is given for completeness (see [C4]). After changing variables by an affine linear transformation of E, it suffices to prove the lemma in the case $B_1 \subset E \subset B_n$. After multiplication of u by a suitable constant, we may assume that $u(x) = -1$. It is not hard to check that the image of ∇u is a convex set with diameter greater than a multiple of $1/\delta(x, E)$ and inradius greater than 1. It follows that

$$\mu(E) = |\nabla u(E)| \geqslant \frac{C}{\delta(x, E)} = \frac{C|u(x)|^n}{\delta(x, E)}.$$

This proves the lemma.

The next lemma is the new element in the proof.

LEMMA 7.3. *Suppose that in addition to the hypotheses of Lemma 7.2, $B_1 \subset E \subset B_R$. Let $1 \geqslant \varepsilon > 0$. Then there is a constant C depending on ε and R such that*

$$|u(x^0)|^n \leqslant C \delta(x^0, E)^\varepsilon \int_E \delta(x, E)^{1-\varepsilon} \, d\mu(x).$$

(Note that Lemma 7.2 is the case $\varepsilon = 1$.)

Proof. Without loss of generality, we may multiply u by a constant so that $u(x^0) = -1$. Define

$$s_k = s 2^{-k\beta}$$

with $s > 0$ and $\beta > 0$ sufficiently small that

$$\beta n \leqslant \varepsilon \quad \text{and} \quad \sum_{k=1}^\infty s_k \leqslant \tfrac{1}{2}. \tag{7.4}$$

Denote

$$A = \delta(x^0, E)^\varepsilon \int_E \delta(x, E)^{1-\varepsilon} \, d\mu(x).$$

Our goal is to show that A is larger than a constant depending on s. Define, for $k = 0, 1, \ldots,$

$$E_k = \{x \in \Omega : u(x) \leqslant \lambda_k = -1 + s_1 + \ldots + s_k\} \quad \text{and} \quad \delta_k = \text{dist}(\partial E_k, \partial E).$$

We will prove that if A is sufficiently small, then δ_k tends to zero as k tends to infinity. This contradicts the continuity of u at the boundary of E because it follows that u tends to a limit less than or equal $-1 + \sum s_k \leqslant -\tfrac{1}{2}$.

If the sequence δ_k does not tend to zero, then there is a smallest value of k for which

$$\delta_{k+1} > \tfrac{1}{2} \delta_k.$$

Let $x^k \in \partial E_k$ be a point closest to ∂E. The segment from x^k to ∂E of length δ_k meets ∂E_{k+1}, so

$$\text{dist}(x_k, \partial E_{k+1}) < \tfrac{1}{2} \delta_k < \delta_{k+1}.$$

By Lemma 7.2, applied to the function $u(x) - \lambda_{k+1}$ on E_{k+1},

$$s_{k+1}^n = |u(x^k) - \lambda_{k+1}|^n \leqslant C \delta(x_k, E_{k+1}) |E_{k+1}| \mu(E_{k+1}).$$

Let L be a shortest segment from x^k to ∂E_{k+1}, and let z be the endpoint on ∂E_{k+1}. Since E_{k+1} is convex, the hyperplane Π perpendicular to L at z is a support plane for E_{k+1}. Let ϱ be the length of L. Let Π' be the support plane parallel to Π on the opposite

side of E_{k+1} and suppose that the distance between them is r. Thus there is a constant C depending on R such that

$$|E_{k+1}| \leqslant Cr.$$

If T is an affine transformation that normalizes E_{k+1}, then the distance between the parallel planes $T(\Pi)$ and $T(\Pi')$ is comparable to 1 and by a similar triangle argument, the distance from $T(x^k)$ to $T(\Pi)$ is comparable to ϱ/r. On the other hand,

$$\delta(x^k, E_{k+1}) \leqslant C \operatorname{dist}(T(x^k), T(\Pi)).$$

It follows that

$$r \leqslant \frac{C\varrho}{\delta(x^k, E_{k+1})}.$$

Combining this with the previous inequalities, we see that

$$s_{k+1}^n \leqslant C\delta_{k+1}\mu(E_{k+1}).$$

But because k was chosen smallest,

$$\delta_{k+1} \leqslant \delta_k \leqslant 2^{-k}\delta_0 \leqslant C2^{-k}\delta(x^0, E).$$

Therefore,

$$\begin{aligned}
\delta_{k+1}\mu(E_{k+1}) &\leqslant C\delta_{k+1}^\varepsilon \int_{E_{k+1}} \delta(x, E)^{1-\varepsilon}\, d\mu(x) \\
&\leqslant C2^{-k\varepsilon}\delta(x^0, E)^\varepsilon \int_{E_{k+1}} \delta(x, E)^{1-\varepsilon}\, d\mu(x) = C2^{-k\varepsilon}A.
\end{aligned}$$

Hence,

$$s^n 2^{-(k+1)\beta n} \leqslant C2^{-k\varepsilon}A$$

and since $\beta n \leqslant \varepsilon$, we deduce that $s^n \leqslant CA$. For A sufficiently small, depending on s, this is a contradiction. This proves Lemma 7.3.

PROPOSITION 7.5. *Let μ be a positive measure on S^n satisfying*

$$\int_{g^{-1}(E)} |\nabla U|^2\, d\sigma = \int_E d\mu$$

for every Borel set $E \subset S^n$. Suppose that $d\mu = S(\xi)\, d\xi$ for some integrable function S and $S(\xi) \geqslant c > 0$. Let ϕ denote the convex, Lipschitz function defined on an open subset O

of $\mathbf{R}^n$, *whose graph* $\{(x, \phi(x)): x \in O\}$ *is a portion of* $\partial\Omega$. *Then* ϕ *satisfies the Monge–Ampère equation*

$$\det(\phi_{ij}(x)) = (1 + |\nabla\phi(x)|^2)^{(N+1)/2} \frac{|\nabla U(x, \phi(x))|^2}{S(\xi)};$$

$$\xi = \frac{(-1, \nabla\phi(x))}{\sqrt{1 + |\nabla\phi(x)|^2}}$$

in the sense of Alexandrov.

Proof. First of all, g^{-1} is single-valued except on Borel set $E_1 \subset S^n$ of $d\xi$-measure 0. Let $F_1 = g^{-1}(E_1)$. Because $d\mu \ll d\xi$ and $d\sigma \ll |\nabla U|^2 d\sigma$, we have $\sigma(F_1) = 0$. Next, g is single-valued except on a Borel set $F_2 \subset \partial\Omega$ of σ-measure 0. Let $F_3 = F_2 \backslash F_1$, and let $E_3 = g(F_3)$. Since g^{-1} is single-valued on E_3, $g^{-1}(E_3) = F_3$, so that

$$0 = \int_{F_3} |\nabla U|^2 \, d\sigma = \int_{E_3} d\mu.$$

Now let $V = \partial\Omega \backslash (F_1 \cup F_3)$ and $W = S^n \backslash (E_1 \cup E_3)$. Then $\sigma(V) = \sigma(\partial\Omega)$, $\mu(W) = \mu(S^n)$, g is a single-valued bijective mapping from V to W, and

$$\int_F |\nabla U|^2 \, d\sigma = \int_{g(F)} S(\xi) \, d\xi$$

for every Borel subset $F \subset V$. So far we have only used the absolute continuity of $d\mu$. Now, we invoke the assumption that S is uniformly bounded from below and $|\nabla\phi(x)|$, defined almost everywhere, is bounded from above. It follows from the Radon–Nikodym theorem that the measure

$$(1 + |\nabla\phi(x)|^2)^{N/2} |\nabla U((x, \phi(x))|^2 S(g(x, \phi(x)))^{-1} d\sigma(x, \phi(x))$$

corresponds by g to the measure $|\xi_N|^{-N} d\xi$. Recall that $d\sigma = (1 + |\nabla\phi(x)|^2) \, dx$. It follows that

$$\int_F (1 + |\nabla\phi(x)|^2)^{(N+1)/2} |\nabla U((x, \phi(x))|^2 S(\xi)^{-1} \, dx = \int_{g(\widetilde{F})} |\xi_N|^{-N} d\xi$$

with $\xi = (\nabla\phi(x), -1)/\sqrt{1 + |\nabla\phi(x)|^2}$ for every Borel subset F of O. This is the statement that the Monge–Ampère equation is satisfied in the sense of Alexandrov.

Now we recall that we have proved the estimate for slices for the density $|\nabla U|^2 \, d\sigma$ in Theorem 6.5, and in Lemma 7.3 we have shown that the method used by Caffarelli in [C4] to obtain $C^{1,\varepsilon}$ for Alexandrov solutions applies. Therefore, we conclude that under the assumption that S is bounded above and below by positive constants, $\partial\Omega$ is $C^{1,\varepsilon}$.

Now let us suppose that $S \in C^\alpha(S^n)$ and $S > 0$. We have already shown that Ω is a $C^{1,\varepsilon}$ domain for some $\varepsilon > 0$. Therefore, $|\nabla U|$ is bounded above and below by positive constants and belongs to C^ε. Thus the right-hand side of our Monge–Ampère equation belongs to C^ε, so that by [C2], [C3], $\phi \in C^{2,\varepsilon}$. Finally, this implies that ∇U belongs to $C^{1,\varepsilon}$ and the regularity of the right-hand side of the Monge–Ampère equation is now C^α. The remainder of the regularity estimates of Theorem 0.8 now follow from [C1], [C2], [C3].

8. Final remarks

We show, by example, that Theorem 6.5 is sharp. Let $0 < t_1 < s$. Consider a domain $\bar\Omega$ in $\mathbf{R}^3$ defined as the convex hull of the square

$$Q = \{(x, y, z) : |x| \leqslant s, \, |y| \leqslant s, \, z = t_1\}$$

and the segment

$$S = \{(x, y, z) : |x| \leqslant 1, \, y = z = 0\}.$$

(A corresponding example in $\mathbf{R}^{3+k}$ can be obtained by taking the product $\Omega \times I$ where I is a unit cube in $\mathbf{R}^k$.) Let

$$F = \{(x, y, z) \in \partial\Omega : z < t\}.$$

Remark 8.1. For every $\varepsilon > 0$, there exist s and t_1 such that

$$\lim_{t \to 0} \frac{\int_F \delta(X, F)^{1-\varepsilon} h^2 \, d\sigma}{\sigma(F) \min_F h^2} = \infty.$$

Proof. Define the rectangle

$$R = \{(x, y) : |x| < (1 - t/t_1) + ts/t_1, \, |y| < ts/t_1\}.$$

Then (x, y, t) belongs to Ω if and only if $(x, y) \in R$. In other words, R corresponds to the cross section $z = t$ of Ω. Define $F_1 = \{(x, y, z) \in F : (x, y) \in R\}$. Let Γ denote smallest infinite cone with vertex $(1, 0, 0)$ containing Ω. As s tends to zero this cone tends to a ray, so for sufficiently small s, the positive harmonic function W with zero boundary values defined on the complement of Γ behaves like $W(X) \approx |X - (1, 0, 0)|^\eta$ for some $\eta < \frac{1}{4}\varepsilon$. It follows from the maximum principle that $1 - U(X) \geqslant C|X - (1, 0, 0)|^\eta$ as X tends to $(1, 0, 0)$ and hence

$$|\nabla U(X)| \geqslant C|X - (1, 0, 0)|^{-1+\eta}. \tag{8.2}$$

(Note that the constant C may depend on t_1, but η can be made arbitrarily small depending only on how small s is, provided $t_1 < s$.) From now on s will be fixed.

Choose t_1 sufficiently small that $t_1/s < \frac{1}{100}\varepsilon$. Then the angle between the two faces of Ω that meet at the edge S is within $2t_1/s$ of ψ. Thus the vanishing rate of Green's function at the origin is bounded above by $Cr^{1-\varepsilon/4}$. Let $B_t(x)$ be the ball around $(x,0,0)$ of radius t. Using Lemma 6.2, we have

$$\min_{B_t(0)\cap\partial\Omega} h \leqslant Ct^{-\varepsilon/4}$$

and therefore,

$$\sigma(F) \min_F h^2 \leqslant Ct^{1-\varepsilon/2}. \tag{8.3}$$

On the other hand, we have a lower bound for h on each from (8.2),

$$\min_{B_t(x)\cap\partial\Omega} h \geqslant c(1-x)^{-1+\eta}$$

for $0 \leqslant x \leqslant 1-2t/t_1$. Furthermore, if $0 \leqslant x \leqslant 1-2t/t_1$, then

$$\min_{X\in B_t(x)} \delta(X,F) \geqslant \tfrac{1}{2}(1-x).$$

Therefore,

$$2\int_F h^2 \delta(X,F)^{1-\varepsilon}\,d\sigma \geqslant \int_{F_1} h^2(1-x)^{1-\varepsilon}\,d\sigma$$
$$\geqslant ct \int_0^{1-2t/t_1} (1-x)^{-1-\varepsilon+2\eta}\,dx \geqslant c't^{1-\varepsilon+2\eta}$$

where $c' > 0$ depends on s and t_1. It follows from (8.3) that the ratio whose limit we are evaluating is greater than a positive multiple of $t^{-\varepsilon/2+2\eta}$ which tends to infinity as t tends to 0.

We see from Remark 8.1 that the best that we could hope for in Theorem 6.5 was a power $\delta(X,F)^{1-\varepsilon}$ for some positive ε. Moreover, this power is at the borderline for the regularity theory of the Monge–Ampère equation. The main step there was the observation that there was a scale-invariant control, given by Lemma 7.3 on the rate at which a solution u to the Monge–Ampère equation vanishes at the boundary. The next example shows that there can be no control on the vanishing rate in the case $\varepsilon = 0$.

Let $Q = \{X = (x_1, ..., x_n) \in \mathbf{R}^n : |x_i| \leqslant 1\}$. Let e_k denote the standard basis of unit vectors. Let $F_{\pm k}$ be the face of ∂Q containing $\pm e_k$. Let $P = (1-s, 0, ..., 0)$. Let S_k be the convex hull of F_k and P. Define the function u as the convex function satisfying

$$u(X) = 0 \quad \text{for } X \in \partial Q, \quad u(P) = -1$$

and u is linear on each sector S_k.

$$\det \nabla^2 u = d\mu_P$$

in the sense of Alexandrov, where the measure $d\mu_P$ is a multiple of the delta function at P. To calculate the total mass of $d\mu_P$, observe that $u(X) = -1 + Y^k \cdot (X - P)$, for all $X \in S_k$, where

$$Y^{\pm k} = \pm e_k \quad \text{for } k = 2, 3, ..., n$$

and $Y^1 = e_1/s$, $Y^{-1} = -e_1/(1+s)$. The set-valued mapping ∇u defined before (7.1) takes the point P to the convex hull K of the points Y^k. Therefore,

$$\mu_P(\{P\}) = \operatorname{vol} K \leqslant Cs^{-1}. \tag{8.4}$$

Let $\delta(\cdot, Q)$ be the normalized distance to the boundary of Q as in Lemma 7.3. Since $\delta(P, Q) = s$, we have

$$\int_Q \delta(X, Q) \, d\mu_P \leqslant C. \tag{8.5}$$

On the other hand, since $u(P) = -1$, and the distance s from P to ∂Q can be arbitrarily small, we see that there can be no estimate on the rate at which u vanishes at the boundary in terms the expression (8.4).

Taken together, our remarks on the limitations of estimates for h^2 and for the Monge–Ampère equation show that h^2 is the largest power for which the regularity theory can work. On the other hand, the existence theory depends on special variational formulas, so the power 2 on h^2 is essential. Furthermore, the fundamental inradius estimate of Proposition 4.8 and Lemma 4.13 that controls the limiting process for existence depends on the fact that the power of h is *at least* 2. If $p < 2$, then the convex body can collapse to a convex set with empty interior but the integral

$$\int_{\partial\Omega} h^p \, d\sigma$$

remains bounded. The logarithmic divergence in Lemma 4.13 does not arise. This is because it is based on the limiting case of a domain slit in $\mathbf{R}^2$, for which $h \approx x^{-1/2}$ and $h^p \approx x^{-p/2}$ is integrable in x.

References

[B]   BORELL, C., Capacitary inequalites of the Brunn–Minkowski type. *Math. Ann.*, 263 (1983), 179–184.

[BF]   BONNESEN, T. & FENCHEL, W., *Theory of Convex Bodies*. BCS Associates, Moscow, ID, 1987 (translation from German).

[C]   CARLESON, L., *Selected Problems on Exceptional Sets*. Van Nostrand, Princeton, NJ, 1967.

[C1]   CAFFARELLI, L. A., Interior a priori estimates for solutons of fully non-linear equations. *Ann. of Math.*, 131 (1989), 189–213.

[C2]   — A localization property of viscosity solutions to the Monge–Ampère equation and their strict convexity. *Ann. of Math.*, 131 (1990), 129–134.

[C3]   — Interior $W^{2,p}$ estimates for solutions of the Monge–Ampère equation. *Ann. of Math.*, 131 (1990), 135–150.

[C4]   — Some regularity properties of solutions to the Monge–Ampère equation. *Comm. Pure Appl. Math.*, 44 (1991), 965–969.

[CJL]   CAFFARELLI, L. A., JERISON, D. & LIEB, E. H., On the case of equality in the Brunn–Minkowski inequality for capacity. To appear in *Adv. in Math.*, 1996.

[CS]   CAFFARELLI, L. A. & SPRUCK, J., Convexity properties of solutions to some classical variational problems. *Comm. Partial Differential Equations*, 7 (1982), 1337–1379.

[CY]   CHENG, S.-Y. & YAU, S.-T., On the regularity of the solution of the n-dimensional Minkowski problem. *Comm. Pure Appl. Math.*, 29 (1976), 495–516.

[D]   DAHLBERG, B. E. J., Estimates for harmonic measure. *Arch. Rational Mech. Anal.*, 65 (1977), 275–283.

[Ga]   GABRIEL, R., A result concerning convex level surfaces of the Green's function for a 3-dimensional convex domain. *J. London Math. Soc.*, 32 (1957), 286–294 and 303–306.

[GS]   GARABEDIAN, P. R. & SCHIFFER, M., On estimation of electrostatic capacity. *Proc. Amer. Math. Soc.*, 5 (1954), 206–211.

[Gu]   GUZMÁN, M. DE, *Differentiation of Integrals in $\mathbf{R}^n$*. Lecture Notes in Math., 481. Springer-Verlag, Berlin–New York, 1975.

[J]   JERISON, D., Prescribing harmonic measure on convex domains. *Invent. Math.*, 105 (1991), 375–400.

[JK1]   JERISON, D. & KENIG, C. E., Boundary behavior of harmonic functions in nontangentially accessible domains. *Adv. in Math.*, 46 (1982), 80–147.

[JK2]   — The logarithm of the Poisson kernel of a C^1 domain has vanishing mean oscillation. *Trans. Amer. Math. Soc.*, 273 (1982), 781–794.

[JN]   JOHN, F. & NIRENBERG, L., On functions of bounded mean oscillation. *Comm. Pure Appl. Math.*, 14 (1961), 415–426.

[K]   KELLOGG, O. D., *Foundations of Potential Theory*. Dover, New York, 1929.

[P]   POINCARÉ, H., *Figures d'équilibre d'une masse fluide*. Paris, 1902.

[S]   STEIN, E. M., *Singular Integrals and Differentiability Properties of Functions*. Princeton Univ. Press, Princeton, NJ, 1970.

DAVID JERISON
Department of Mathematics
Massachusetts Institute of Technology
Cambridge, MA 02139-4307
U.S.A.
jerison@math.mit.edu