PERTURBATIONS OF NONLINEAR SYSTEMS

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1. Introduction

In section 3 of this paper we extend the author's results [4] concerning perturbations of real linear systems of the form

$$\dot{x} = [A(t) + B(t)]\,x, \tag{1.1}$$

where $A(t)$ and $B(t)$ are continuous and uniformly bounded matrices. In particular we obtain conditions on the matrix $A(t)$ which will assure us that the characteristic exponents of system (1.1) are continuous at $B(t) \equiv 0$ as functions of $B(t)$ (Corollary 3.1). In section 4 we obtain results concerning the existence and variation of bounded solutions of nonlinear differential equations (Theorem 4.1). In section 5 we apply these results to almost periodic nonlinear systems and extend the author's previous results (Theorem 5.1).

2. Elementary Transformations and Definitions

A counter example due to Perron [7] shows that even in the case where the matrix $A(t)$ in (1.1) is a diagonal matrix the characteristic exponents of (1.1) need not be continuous at $B(t) \equiv 0$. One notes, however, that one of the diagonal terms in Perron's example fails to possess a mean value. We shall find that if one restricts oneself to matrices $A(t)$ which are kinematically similar [5] to upper triangular matrices whose diagonal elements have the following property I, then the characteristic exponents of system (1.1) are continuous at $B(t) \equiv 0$.

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(1) This research was sponsored by the Office of Ordnance Research, U.S. Army Contract No. DA-23-072-ORD-1289.

DEFINITION 2.1. *A continuous function* $a(t)$ *is said to possess property* I *with constant* λ *(real) if for every* $\mu>0$ *there exists a constant* $M(\mu)$ *such that*

$$\left|\int_s^t [a(t)-\lambda]\,dt\right| < M(\mu)+\mu|t-s| \text{ for all } -\infty<t,\ s<\infty.$$

Let M_n be the set of $n\times n$ matrices whose entries are real valued functions of a real variable t, continuous and bounded on the line, $-\infty<t<\infty$ (the results of the first three sections hold if we restrict ourselves to $0\leqslant t<\infty$).

DEFINITION 2.2. *For* $A\in M_n$ *we say* A *possesses property* II *with the real constants* $\lambda_1, \ldots, \lambda_n$ *if there exists a diagonal matrix* $D=\{d_{ij}(t)\}\in M_n$, *where the* $d_{ii}(t)$ *possess property* I *with constants* λ_i, *and if for any* $\varepsilon>0$ *there exists a matrix* $E(t)\in M_n$ *such that* $\|E\|\leqslant\varepsilon$ *and* $A\sim D+E$.

We recall the definition of kinematic similarity.

DEFINITION 2.3. *For* $A, B\in M_n$ *we say* $A\sim B$ *(read: A is kinematically similar to B) in case there exists a matrix* P *with* P, P^{-1}, *and* $\dot P=dP/dt\in M_n$ *and* $-P^{-1}[\dot P-AP]=B$ *for* $-\infty<t<\infty$.

We note without proof that property I for a continuous function $a(t)$ with constant 0 is equivalent to $a(t)$ satisfying the following two conditions.

1) $\lim\limits_{\Delta\to\infty}\left|\int_s^{s+\Delta} a(t)\,dt\right|/\Delta=0$ uniformly for $-\infty<s<\infty$.

2) $\sup\limits_{\substack{-\infty<s,\,s+\Delta<\infty\\ |\Delta|<1}}\left|\int_s^{s+\Delta} a(t)\,dt\right|<\infty$.

We next note that the results of Perron [6] and Diliberto [2] assure us that the problem of studying the continuity of the characteristic exponents of the n-dimensional linear system (1.1) for $\|B\|$ small is equivalent to the problem for a system of the form

$$\dot z=[C(t)+D(t)]z, \tag{2.1}$$

where $C(t)$, $D(t)\in M_n$, $\|D(t)\|$ is small, and $C(t)$ is an upper triangular matrix. Here we use for the norm of a matrix $A(t)$: $\|A(t)\|=\sum\limits_{i,j=1}^{n}|a_{ij}(t)|$, $\|A\|=\operatorname*{lub}\limits_{-\infty<t<\infty}\|A(t)\|$. We use the analogous norm for vectors. The study of system (1.1) can be simplified further. If $H(t)=\operatorname{diag}\,(c_{11}(t), \ldots, c_{nn}(t))$, then by the change of coordinates $\xi=Tz$, where $T=\operatorname{diag}\,(1, \gamma, \ldots, \gamma^{n-1})$ one has

$$\dot\xi=[H(t)+T(C(t)-H(t)+D(t))T^{-1}]\xi=[H(t)+E(t)]\xi. \tag{2.2}$$

If $\|C-H\|=M$, then for any small $\delta>0$, if we set $\gamma=2M/\delta$, we have for any $D(t)$, $\|D\|<\delta^{n+1}/(2^{n+1}M^n)$, that the matrix $E(t)$ is such that $\|E\|<\delta$. Thus we see that the question of continuity of characteristic exponents for systems of the form (1.1) is equivalent to the same problem for systems of the form (1.1), where $A(t)$ is a diagonal matrix.

3. Basic Perturbation Theorem

In this section we consider the two systems

$$\dot{x}=A(t)x \tag{3.1}$$

$$\dot{z}=[A(t)+B(t)]z \tag{3.2}$$

where $A, B \in M_n$. In what follows the jth row (ith column) of a matrix $C(t)$ will be denoted by $[C(t)]_j$ ($[C(t)]^i$).

DEFINITION 3.1. *System* (3.1) *is said to possess property* III *with constants* (*real*) $\lambda_1, \ldots, \lambda_n$ *if for any* $\mu>0$ *there exists a set of* n *independent solution vectors* $x^j(t)$ *and a fundamental solution* $\Phi(t)$ *of* (3.1) *such that*

1) $\|x^i(t)\|\leqslant h(\mu)\exp[\lambda_i t+\mu|t|]$ $\quad(i=1, \ldots, n;\ 0\leqslant t<\infty)$,

2) $\|[\Phi(t)\Phi^{-1}(s)]_j\|\leqslant h(\mu)\exp[\lambda_j(t-s)+\mu|t-s|]$ $\quad(j=1, \ldots, n;\ 0\leqslant s,\ t<\infty)$,

where, for a fixed value of μ, $h(\mu)$ *is a positive constant.*

THEOREM 3.1. *If system* (3.1) *possesses property* III *with constants* $\lambda_1, \ldots, \lambda_n$, *then given* $\varepsilon>0$ *there exists* $\delta>0$ *such that for* $\|B\|<\delta$ *system* (3.2) *possesses* n *independent solutions* $z^j(t)$ *satisfying the inequalities*

$$\|z^j(t)\|\leqslant k(\varepsilon)\exp[(\lambda_j+\varepsilon)t] \quad (j=1, \ldots, n;\ 0\leqslant t<\infty),$$

where $k(\varepsilon)$ *for a fixed value of* ε *is a positive constant.*

Proof. Set $\lambda=\min\limits_{\lambda_l\neq\lambda_j}|\lambda_l-\lambda_j|/4$ and assume that $\varepsilon<\lambda/4$. We now consider a fixed $x^j(t)$ and let $S_1=\{l\,|\,\lambda_l>\lambda_j\}$, $S_2=\{l\,|\,\lambda_l\leqslant\lambda_j\}$. We also define the matrices $H(t-s)$ and $\gamma(t-s)$ as follows:

$$[H(t-s)]_j=\begin{cases}[\Phi(t)\Phi^{-1}(s)]_j & (j\in S_1),\\ \text{zero vector} & (j\in S_2);\end{cases}$$

$$\gamma(t-s)=\Phi(t)\Phi^{-1}(s)-H(t-s).$$

Next we define for system (3.2) the formal solution matrix $\psi_j(t)=\sum_{k=0}^{\infty}\psi_j^k(t)$, where

1) $[\psi_j^0(t)]^k=\delta_{jk}x^j(t)$ (δ_{jk} denotes the Kronecker delta)

2) $\psi_j^{l+1}(t)=\int_0^t \gamma(t-s)B(s)\psi_j^l(s)\,ds-\int_t^\infty H(t-s)B(s)\psi_j^l(s)\,ds.$

We now show that $\psi_j(t)$ is an actual solution matrix for $0\leqslant t<\infty$. Taking $\mu=\varepsilon/4$ we have, since $\|x_j(t)\|\leqslant h(\mu)\exp[(\lambda_j+\mu)t]$,

$$\|\psi_j^0(t)\|\leqslant h(\mu)\exp[(\lambda_j+\mu)t]\leqslant h(\mu)\exp[(\lambda_j+\varepsilon)t].$$

Assuming that $\|\psi_j^l(t)\|\leqslant h(\mu)k^l\exp[(\lambda_j+\varepsilon)t]$, $k=h(\mu)n\|B\|\cdot\mu^{-1}$, we establish that

$$\|\psi_j^{l+1}(t)\|\leqslant h(\mu)k^{l+1}\exp[(\lambda_j+\varepsilon)t].$$

From its definition we have

$$\begin{aligned}\|\psi_j^{l+1}(t)\|&\leqslant\|B\|\{\int_0^t\|\gamma(t-s)\|\,\|\psi_j^l(s)\|\,ds+\int_t^\infty\|H(t-s)\|\,\|\psi_j^l(s)\|\,ds\}\\&\leqslant h(\mu)k^l\|B\|n\,h(\mu)\,\{\int^t\exp[(\lambda_j+\mu)(t-s)+(\lambda_j+\varepsilon)s]\,ds\\&\quad+\int_t^\infty\exp[\lambda_k(t-s)+(\lambda_j+\varepsilon)s+\mu(s-t)]\,ds\},\end{aligned}$$

where $\lambda_k=\min_{l\in S_1}\{\lambda_l\}$.

Thus we have:

$$\|\psi_j^{l+1}(t)\|\leqslant\{h(\mu)k^l\|B\|n\,h(\mu)\exp[(\lambda_j+\varepsilon)t]\}/\mu\leqslant h(\mu)k^{l+1}\exp[(\lambda_j+\varepsilon)t].$$

This completes our induction.

However, once $\varepsilon>0$ has been fixed the quantities μ and $h(\mu)$ are fixed constants, and so, if we take $\|B\|=\mu/[n^2h(\mu)]$, we have that the formal series converges uniformly for $0\leqslant t\leqslant T$ (T arbitrary) and so by the usual arguments is an actual solution matrix for (3.2), where $0\leqslant t<\infty$. It is clear that the jth column vector $z^j(t)$ of $\psi_j(t)$ satisfies the inequality of Theorem 3.1. We note that the coordinates of $z^j(0)$ tend to those of $x^j(0)$ as $\|B\|\to 0$. Thus for sufficiently small values of $\|B\|$ the linear independence of the $z^j(t)$ follows from that of the $x^j(t)$. This completes the proof of Theorem 3.1.

DEFINITION 3.2. *System* (3.1) *will be said to possess property* IV *with constants* (*real*) $\lambda_1, \ldots, \lambda_n$ *if for any* $\mu>0$ *its adjoint system*

$$\dot{x} = -A^*(t)\,x \tag{3.3}$$

possesses n independent solutions $x^{i*}(t)$ and a fundamental solution $\Phi^*(t)$ such that

1) $\|x^{i*}(t)\| \leq n(\mu) \exp[-\lambda_i t + \mu|t|]$ $(i=1, \ldots, n;\ 0 \leq t < \infty)$

2) $\|[\Phi^*(t)\,\Phi^{*-1}(s)]_j\| \leq n(\mu) \exp[-\lambda_j(t-s) + \mu|t-s|]$ $(j=1, \ldots, n;\ 0 \leq s, t < \infty)$,

where, for a fixed value of μ, $n(\mu)$ is a positive constant.

By the method used to prove Theorem 3.1 we now have:

THEOREM 3.2. *If system* (3.1) *possesses property* IV *with constants $\lambda_1, \ldots, \lambda_n$, then given any $\varepsilon > 0$ there exists $\delta > 0$ such that for $\|B\| < \delta$ the adjoint system*

$$\dot{z} = -[A(t) + B(t)]^* z \tag{3.4}$$

for system (3.2) *possesses n independent solutions $z^{j*}(t)$ satisfying the inequalities*

$$\|z^{j*}(t)\| \leq k(\varepsilon) \exp[(-\lambda_j + \varepsilon)t] \quad (j=1, \ldots, n;\ 0 \leq t < \infty),$$

where, for a fixed ε, $k(\varepsilon)$ is a positive constant.

COROLLARY 3.1. *If system* (3.1) *possesses properties* III *and* IV *with constants $\lambda_1, \ldots, \lambda_n$ which are also the characteristic exponents of* (3.1), *then the characteristic exponents of system* (3.1) *are continuous functions of $B(t)$ at $B(t) \equiv 0$.*

Proof. The upper semi-continuity of the characteristic exponents is an immediate consequence of Theorem 3.1. We now establish the lower semi-continuity of the characteristic exponents. Let $\psi(t)$ be any fundamental matrix of (3.2). Then for small $\|B\|$ we may assume, without loss of generality, that the $[\psi(t)]^j$ satisfy the inequalities

$$\|[\psi(t)]^j\| \leq p(\varepsilon) \exp[(\lambda_j + \varepsilon)t] \quad (j=1, \ldots, n;\ 0 \leq t < \infty),$$
$$\lambda_{j+1} \leq \lambda_j \quad (j=1, \ldots, n-1),$$

where $p(\varepsilon)$ is a positive constant for fixed ε. We next establish that $\|[\psi(t)]^j\| \geq k_2 \exp[(\lambda_j - \varepsilon)t]$. Since $\psi^{*-1}(t)$ satisfies the adjoint equation we have $\psi^{*-1}(t) = R(t)\,C^{-1}$, where C^{-1} is a constant nonsingular matrix and $[R(t)]^j = z^{j*}(t)$ defined in Theorem 3.2. We again assume that the λ_i have been linearly ordered as described above. Thus $\psi^*(t)\,R(t) = C = (c_{ij})$, and if $c_{jj} \neq 0$ we have

$$\|[\psi(t)]^j\| \geq |c_{jj}| / \|[R(t)]^j\| \geq k_2 \exp[(\lambda_j - \varepsilon)t].$$

The case $c_{jj} = 0$ can be reduced to the above case by a dimension argument (Bellman [1, p. 50]), which we shall not repeat here except to note that this was the reason for the ordering of the λ_i. This completes the proof of Corollary 3.1.

COROLLARY 3.2. *If in system* (3.1) $A(t)$ *possesses property* II, *then the characteristic exponents of* (3.2) *are continuous at* $B\equiv 0$.

Proof. By the definition of property II and the results of section 2, the continuity problem for (3.2) is equivalent to the same problem in which we assume that $A(t)=\operatorname{diag}\,(d_{11}(t), \ldots, d_{nn}(t))$. But it is easily seen that in this case system (3.1) has properties III and IV; and the result follows from the previous corollary.

In preparation for Lemma 4.1 in the next section, we introduce the following definitions and results.

DEFINITION 3.3. *System* (3.1) *is said to possess property* III^{+} *if it possesses property* III *for* $-\infty<s,\ t<\infty$.

DEFINITION 3.4. *System* (3.1) *is said to possess property* IV^{+} *if it possesses property* IV *for* $-\infty<s,\ t<\infty$.

THEOREM 3.3. *If system* (3.1) *possesses properties* III^{+} *and* IV^{+} *with constants* $\lambda_1, \ldots, \lambda_n$ *then for any* $\varepsilon>0$ *there exists a* $\delta>0$ *such that for* $\|B\|<\delta$ *to every solution* $x^j(t)$ *described in Theorem* 3.1 *there corresponds a solution* $z^j(t)$ *of system* (3.2) *which satisfies the inequalities:*

$$k_1 \exp\,[\lambda_j t-\varepsilon|t|]\leqslant\|z^j(t)\|\leqslant k_2 \exp\,[\lambda_j t+\varepsilon|t|] \quad \text{for } -\infty<t<\infty.$$

Proof. Notation is as in Theorem 3.1, except now with respect to λ_j we define the sets $S_1=\{k\,|\,\lambda_k>\lambda_j\}$, $S_2=\{k\,|\,\lambda_k<\lambda_j\}$, $S_3=\{k\,|\,\lambda_k=\lambda_j\}$ and the matrices $H^1(t-s)$, $H^2(t-s)$, $\gamma(t-s)$ as follows:

$$[H^i(t-s)]_m=\begin{cases}[\Phi(t)\,\Phi^{-1}(s)]_m & (m\in S_i), \quad (m=1,2,\ldots,n,\ i=1,2)\\ \text{zero vector} & (m\notin S_i)\end{cases}$$

$$\gamma(t-s)=\Phi(t)\,\Phi^{-1}(s)-H^1(t-s)-H^2(t-s).$$

Again we define the formal solution matrix $\psi_j(t)=\sum_{k=0}^{\infty}\psi_j^k(t)$, where

1. $[\psi_j^0(t)]^k=\delta_{jk}\,x^j(t)$, ($\delta_{jk}$ denotes the Kronecker delta)
2. $$\psi_j^{k+1}(t)=\int_0^t\gamma(t-s)\,B(s)\,\psi_j^k(s)\,ds-\int_t^{\infty}H^1(t-s)\,B(s)\,\psi_j^k(s)\,ds + \int_{-\infty}^{t}H^2(t-s)\,B(s)\,\psi_j^k(s)\,ds.$$

Now one proceeds as in the proof of Theorem 3.1 to obtain the inequality $\|z^j(t)\| \leqslant k_2 \exp(\lambda_j t + \varepsilon|t|)$ for $-\infty < t < \infty$. The remaining inequality is obtained as in the proof of Corollary 3.1 after the obvious changes in Theorem 3.2.

COROLLARY 3.3. *If system* (3.1) *possesses properties* III$^+$ *and* IV$^+$ *with constants* $\lambda_i \neq 0$ $(i=1, \ldots, n)$, *then there exists* $\delta_1 > 0$ *such that every nontrivial solution of* (3.2) *is unbounded for* $\|B\| < \delta_1$.

4. Perturbations of Bounded Solutions

In this section we shall consider an n-dimensional system of the form

$$\dot{x} = F(x, \mu, t). \tag{4.1}$$

DEFINITION 4.1. *System* (3.1) *is said to possess property* V *if there exists a constant matrix* $D = \operatorname{diag}(d_{11}, \ldots, d_{nn})$, $|d_{ii}| > \delta > 0$ $(i=1, \ldots, n)$ *and if for any* $\varepsilon > 0$ *there exists a matrix* $H(t) \in M_n$ *such that* $A(t) \sim D + H(t)$, *where* $\|H\| < \varepsilon$.

LEMMA 4.1. *If system* (3.1) *possesses property* V, *then there exists a* $\delta_2 > 0$ *such that every nontrivial solution of* (3.2) *is unbounded for* $\|B\| < \delta_2$.

Proof. We observe that the system $\dot{\xi} = D\xi$ clearly possesses properties III$^+$ and IV$^+$. Thus all the nontrivial solutions of the system $\dot{\xi} = [D + H(t) + P^{-1}(t) B(t) P(t)]\xi$ are unbounded for $\|H\| < \frac{1}{2}\delta_1$ and $\|B\| < \delta_1 / (2\|P\| \, \|P^{-1}\|)$, where δ_1 is the constant described in Corollary 3.3. But now the conclusion of Lemma 4.1 is an immediate consequence of the fact that there exists a $P(t)$ such that $D + H(t) + P^{-1} BP = -P^{-1}[\dot{P} - (A+B)P]$, where $P, P^{-1}, \dot{P} \in M_n$. This completes the proof of Lemma 4.1.

DEFINITION 4.2. *System* 4.1 *is said to possess property* VI *with respect to a bounded curve* $p(0, t)$ *if there exists a* $\Delta > 0$ *such that for* $0 \leqslant |\mu| \leqslant \Delta$ *and* $0 \leqslant \|\eta\| \leqslant \Delta$ *it may be written in the form*

$$\dot{\eta} = F(p(0, t), \mu, t) - F(p(0, t), 0, t) + [A(\mu, t) + \mathfrak{F}(\eta, \mu, t)]\eta, \tag{4.2}$$

where

1) $\eta = x - p(0, t)$,
2) $\mathfrak{F}_{qi} = \sum_{j=1}^{n} [F^q_{x_i x_j}(p(0, t), \mu, t) + F^q_{x_i x_j}(p(0, t) + \theta(t, \eta)\eta, \mu, t)]\eta_j$, *where* F^q *is the* q-*th component of the vector function* $F(x, t, \mu)$ *and* $|\theta_i(t, \eta)| < 1$,

3) *$A(\mu,t)$ and $F^q_{x_i x_j}(i,j,q=1,\ldots,n)$ are bounded, continuous in x,μ, t and jointly continuous in μ, x uniformly for* $-\infty<t<\infty$,

4) $\dot\eta=A(0,t)\eta$ *possesses property* V, *and* $\dot p(0,t)=F(p(0,t),0,t)$.

We next introduce the system

$$\dot z=f(\mu,t)+[D+\sigma(\mu,t)]z+c(z,\mu,t)z, \tag{4.3}$$

where

1) $\|\sigma(\mu,t)\|<\sigma(\mu)+\varepsilon$ for $-\infty<t<\infty$, where $\lim\limits_{\mu\to 0}\sigma(\mu)=0$,

2) $\|f(\mu,t)\|<m(\mu)$ for $-\infty<t<\infty$, where $\lim\limits_{\mu\to 0} m(\mu)=0$,

3) there exists a $\Delta_1>0$ and M such that $\|c(z,\mu,t)\|<M\|z\|$ for $0\leqslant\|z\|\leqslant\Delta_1$ and $0\leqslant|\mu|<\Delta_1$,

4) $D=\operatorname{diag}(d_{11},\ldots,d_{nn})$, where $|d_{ii}|>\delta$ for $i=1,\ldots,n$.

LEMMA 4.2. *If system* (4.1) *possesses property* VI *with respect to a bounded curve* $p(0,t)$, *then for any* $\varepsilon>0$ *there exists a transformation* $z=P^{-1}(t)\eta$ *which for* $0\leqslant\|\eta\|\leqslant\Delta$ *and* $0\leqslant|\mu|\leqslant\Delta$ *reduces system* (4.1) *to a system of the form* (4.3).

Proof. The result is an immediate consequence of definitions 4.1 and 4.2.

THEOREM 4.1. *There exists* $\Delta>0$ *such that to every* μ *in the interval* $(-\Delta,\Delta)$ *there corresponds a bounded solution* $z(\mu,t)$ *of* (4.3) *such that* $\|z\|$ *tends to* 0 *as* μ *tends to* 0.

Proof. Select $\varepsilon>0$ and then Δ_2 and Δ_3 such that for $|\mu|\leqslant\Delta_2$ and $\|z\|\leqslant\Delta_3$, we have $|[2(\sigma(\mu)+\varepsilon)n]/\delta|<\frac{1}{2}$ and $[m(\mu)+M\Delta_3^2]8n/\delta<\Delta_3$.
We next introduce system $(\overline{4.3})$ which for a fixed μ_1, $|\mu_1|\leqslant\Delta_2$, is defined in the following way. Let $z(\mu_1,t)$ denote any solution of (4.3) such that $\|z(\mu_1,0)\|\leqslant\Delta_3$. Then for any value of t we define a corresponding solution $\hat z(\mu_1,t)$ of system $(\overline{4.3})$ as follows:

$$\hat z(\mu_1,t)=\begin{cases} z(\mu_1,t) & \text{if } \|z(\mu_1,t)\|\leqslant\Delta_3,\\ z(\mu_1,t)\,\Delta_3/\|z(\mu_1,t)\| & \text{if } \|z(\mu_1,t)\|\geqslant\Delta_3.\end{cases} \tag{$\overline{4.3}$}$$

Thus all the solution curves of system $(\overline{4.3})$ are contained in a cylinder of radius Δ_3 about the t axis. We further observe that any solution $\hat z(\mu_1,t)$ such that $\|\hat z(\mu_1,t)\|<\Delta_3$ for all t is an actual solution of system (4.3). Now corresponding to each $\hat z(\mu_1,t)$ we define

$$x(\mu_1, t) = \sum_{i=0}^{\infty} x^i(\mu_1, t), \tag{4.4}$$

where

1) $x^0(\mu_1, t) = \int_{\infty}^{t} \Phi_1(t-s)\,[f(\mu_1, s) + c(\hat{z}(\mu_1, s), \mu_1, s)\,\hat{z}(\mu_1, s)]\,ds$
$$+ \int_{-\infty}^{t} \Phi_2(t-s)\,[f(\mu_1, s) + c(\hat{z}(\mu_1, s), \mu_1, s)\,\hat{z}(\mu_1, s)]\,ds,$$

2) $x^{i+1}(\mu_1, t) = \int_{\infty}^{t} \Phi_1(t-s)\,\sigma(\mu_1, s)\,x^i(\mu_1, s)\,ds + \int_{-\infty}^{t} \Phi_2(t-s)\,\sigma(\mu_1, s)\,x^i(\mu_1, s)\,ds.$

Here
$$\Phi_1(t-s) = \operatorname{diag}\,(a_{11}\exp[d_{11}(t-s)], \ldots, a_{nn}\exp[d_{nn}(t-s)]$$
$$\Phi_2(t-s) = \exp\,(D(t-s)) - \Phi_1(t-s),$$

where a_{ii} is 0 or 1 according as $d_{ii} < 0$ or $d_{ii} > 0$. Since $\|\Phi_1(t-s)\| \leqslant n \exp[\delta(t-s)]$ for $t \leqslant s$ and $\|\Phi_2(t-s)\| \leqslant n \exp[-\delta(t-s)]$ for $s \leqslant t$, we have the following estimates on $\|x^i(\mu_1, t)\|$:

$$\|x^0(\mu_1, t)\| \leqslant [m(\mu_1) + M\Delta_3^2]\,n/\delta + [m(\mu_1) + M\Delta_3^2]\,n/\delta;$$
$$\|x^i(\mu_1, t)\| \leqslant (2n/\delta)\,[m(\mu_1) + M\Delta_3^2]\,[2n(\sigma(\mu_1) + \varepsilon)/\delta]^i \quad (i = 1, \ldots).$$

Thus

$$\begin{aligned}\|x(\mu_1, t)\| &\leqslant [m(\mu_1) + M\Delta_3^2]\,(2n/\delta)\,(1 - [\sigma(\mu_1) + \varepsilon]\,2n/\delta)^{-1} \\ &< (4n/\delta)\,[m(\mu_1) + M\Delta_3^2] < \tfrac{1}{2}\Delta_3\end{aligned} \tag{4.5}$$

We shall next show in Lemma 4.3 that there exists a $\hat{z}^*(\mu_1, t)$ such that $\hat{z}^*(\mu_1, 0) = x^*(\mu_1, 0)$, where $x^*(\mu_1, t)$ is the function obtained by using $\hat{z}^*(\mu_1, t)$ in the definition of $x^{0*}(\mu_1, t)$. However, from the definition of $x^*(\mu_1, t)$, as long as $\|x^*(\mu_1, t)\|$ remains less then Δ_3, it is clear that $x^*(\mu_1, t) \equiv \hat{z}^*(\mu_1, t)$. But by (4.5) $\|x^*(\mu_1, t)\| < \frac{1}{2}\Delta_3$ for $-\infty < t < \infty$, and so $x^*(\mu_1, t) = \hat{z}^*(\mu_1, t) = z(\mu_1, t)$, where $z(\mu_1, t)$ is an actual solution of system (4.3). Thus we will have established for every μ_1, $|\mu_1| < \Delta_2$, that there exists a bounded solution $z(\mu_1, t)$ of system (4.3). From the inequality (4.5) it follows that this solution satitfies

$$\|z(\mu_1, t)\| \leqslant (4n/\delta)\,[m(\mu_1) + M\Delta_3^2] < \tfrac{1}{2}\Delta_3.$$

Since $\lim_{\mu \to 0} m(\mu) = 0$, we may conclude that Δ_3 also tends to zero. Thus the proof of Theorem 4.1 will be complete once Lemma 4.3 has been established.

LEMMA 4.3. *The mapping $\psi(\hat{z}(\mu_1, 0)) = x(\mu_1, 0)$ of the n cell $\|\hat{z}(\mu_1, 0)\| \leqslant \frac{1}{2}\Delta_3$ into itself is continuous and so possesses a fixed point.*

Proof. From the definition of the set of functions $x(\mu_1, t)$ it is clear that there exists an $N(\varepsilon_1)$, independent of $\hat{z}(\mu_1, t)$, such that $\sum_{j=N}^{\infty} \| x^j(\mu_1, t) \| < \varepsilon_1 / 16$ for all t. We set $M_1 = [2m(\mu_1) + 2M\Delta_3^2]$ and choose $T > 0$ such that $n M_1 \exp(-\delta T)/\delta \leqslant \delta_1$. Since

$$I = \operatorname*{lub}_{|t| \leqslant (N+2)T} \| c(\hat{z}_1(\mu_1, t), \mu_1, t) \hat{z}_1(\mu_1, t) - c(\hat{z}_2(\mu_1, t), \mu_1, t) \hat{z}_2(\mu_1, t) \|$$

is a continuous function of $\hat{z}_i(\mu_1, 0)$, $i = 1,2$ and is zero for $\hat{z}_1(\mu_1, t) = \hat{z}_2(\mu_2, t)$, for any $\delta_2 > 0$ there exists a $\delta_3 > 0$ such that $\mathrm{I} < \delta_2$ if $\| \hat{z}_1(\mu_1, 0) - \hat{z}_2(\mu_1, 0) \| \leqslant \delta_3$. Thus for any t, $|t| \leqslant (N+1)T$, we have

$$\begin{aligned} \| x_1^0(\mu_1, t) - x_2^0(\mu_1, t) \| \leqslant & - \int_{(N+2)T}^{t} \| \Phi_1(t-s) \| I\, ds + \int_{-(N+2)T}^{t} I \| \Phi_2(t-s) \| ds \\ & - \int_{\infty}^{(N+2)T} M_1 \| \Phi_1(t-s) \| ds + \int_{-\infty}^{-(N+2)T} \| \Phi_2(t-s) \| M_1\, ds \leqslant 2(\delta_1 + n\delta_2/\delta). \end{aligned}$$

For any t, $|t| \leqslant NT$, we have

$$\begin{aligned} \| x_1^1(\mu_1, t) - x_2^1(\mu_1, t) \| \leqslant & - \int_{(N+1)T}^{t} \| \Phi_1(t-s) \| [\sigma(\mu_1) + \varepsilon]\, 2(\delta_1 + n\delta_2/\delta)\, ds \\ & + \int_{-(N+1)T}^{t} \| \Phi_2(t-s) \| [\sigma(\mu_1) + \varepsilon]\, 2(\delta_1 + n\delta_2/\delta)\, ds + 2\delta_1. \end{aligned}$$

In general for t, $|t| \leqslant (N - i + 1)T$, we obtain

$$\| x_1^i(\mu_1, t) - x_2^i(\mu_1, t) \| \leqslant 2\delta_1 \sum_{j=0}^{i} \{2[\sigma(\mu_1) + \varepsilon] n/\delta\}^j + (2n\delta_2/\delta)\, \{2[\sigma(\mu_1) + \varepsilon] n/\delta\}^i.$$

Thus we obtain for t, $|t| \leqslant 2T$,

$$\begin{aligned} \sum_{i=1}^{N-1} \| x_1^i(\mu_1, t) - x_2^i(\mu_1, t) \| \leqslant & (2n\delta_2/\delta) \sum_{i=1}^{N-1} \{2[\sigma(\mu_1) + \varepsilon] n/\delta\}^i \\ & + 2\delta_1 \sum_{i=0}^{N-1} (N-i)\, \{2[\sigma(\mu_1) + \varepsilon] n/\delta\}^i. \end{aligned}$$

Now $k_1 = 2\sum_{i=0}^{N-1} (N-i)\, \{2[\sigma(\mu_1) + \varepsilon] n/\delta\}^i$ and $k_2 = (2n/\delta) \sum_{i=1}^{N-1} 2[\sigma(\mu_1) + \varepsilon] M/\delta\}^i$ are fixed numbers for a given μ_1 and $N(\varepsilon_1)$. Hence we may choose T in such a way that $\delta_1 k_1 < \varepsilon_1/8$. Then having determined T we may choose δ_3 in such a way that $k_2\delta_2 < \varepsilon_1/8$. Thus for any $\varepsilon_1 > 0$ and $T > 0$ there exists a δ_3 such that $\| x_1(\mu_1, t) -$

$-x_2(\mu_1, t)\| < \varepsilon_1$ for $\|\hat{z}_1(\mu_1, 0) - \hat{z}_2(\mu_1, 0)\| \leqslant \delta_3$ and $|t| < 2T$. It then follows that for any finite value of t, $x(\mu_1, t)$ is a continuous function of $\hat{z}(\mu_1, 0)$. If we set $t = 0$ it follows that ψ is a continuous map of an n-cell into itself and so possesses a fixed point. This completes the proof of Lemma 4.3 and so of Theorem 4.1.

As an immediate consequence of Theorem 4.1 and Lemma 4.2 we have the following corollary.

COROLLARY 4.1. *If the system* (4.1) *possesses property* VI *with respect to a bounded curve* $p(0, t)$, *then there exists a* $\Delta > 0$ *such that for each* μ, $|\mu| < \Delta$, *there corresponds a bounded solution* $p(\mu, t)$ *of* (4.1). *The solution* $p(\mu, t)$ *tends to* $p(0, t)$ *uniformly as* μ *tends to zero.*

LEMMA 4.4. *If system* (4.1) *possesses property* VI *with respect to* $p(0, t)$, *then there exist* $\Delta_3 > 0$ *and* $r_3 > 0$ *such that for* $|\mu| < \Delta_3$ *there is at most one bounded solution* $p(\mu, t)$ *contained in a cylinder of radius* r_3 *about* $p(0, t)$.

Proof. Since system (4.1) possesses property VI, it is clear that there exist $\Delta_1 > 0$ and $r_1 > 0$ such that for any bounded solution $p(\mu, t)$, where $|\mu| < \Delta_1$, $0 \leqslant \|\eta\| \leqslant r_1$, $0 \leqslant \|p(0, t) - p(\mu, t)\| \leqslant r_1$, system (4.1) may be rewritten in the form

$$\dot{\eta} = [A(0, t) + c(t, \mu) + G(\eta, \mu, t)]\eta, \tag{4.6}$$

where 1) $\eta = x - p(\mu, t)$,

2) $G_{qi} = \sum_{j=1}^{n} [F^q_{x_i x_j}(p(\mu, t), \mu, t) + F^q_{x_i x_j}(p(\mu, t) + \theta(t, \eta)\eta, \mu, t)]\eta_j$,

where $|\theta_i(t, \eta)| < 1$ for $-\infty < t < \infty$; $i, j, q = 1, \ldots, n$,

3) $A(0, t) + c(\mu, t) = \dfrac{\partial(F^1, \ldots, F^n)}{\partial(x_1, \ldots, x_n)} \,|\, p(\mu, t)$ and $c(0, t) = 0$.

If we set $B(\mu, t) = [c(\mu, t) + G(\eta, \mu, t)]$, then by Lemma 4.1 there exists $\sigma > 0$ such that for $\|B\| < \sigma$ the system $\dot{x} = [A(0, t) + B(\mu, t)]x$ possesses no nontrival bounded solutions. By the joint continuity in x and μ imposed by condition VI it follows that there exist $\Delta_2 (\Delta_1 > \Delta_2 > 0)$, and $r_2 (r_1 > r_2 > 0)$, such that $\|c(\mu, t)\| < \frac{1}{2}\sigma$ and $\|G(\eta, \mu, t)\| < \frac{1}{2}\sigma$ for $0 < \|\eta\| < r_2, 0 < \|p(\mu, t) - p(0, t)\| < r_2, 0 \leqslant |\mu| < \Delta_2$, and $-\infty < t < \infty$. Thus, if we set $r_3 = \frac{1}{2} r_2$ and $\Delta_3 = \Delta_2$, we have our desired result.

THEOREM 4.2. *If system* 4.1 *possesses property* VI *with respect to a bounded curve* $p(0, t)$, *then there exist a* Δ *and* r *such that to every value of* μ, $|\mu| < \Delta$, *there* *corresponds a unique bounded solution* $p(\mu, t)$ *contained in the cylinder of radius* r *about* $p(0, t)$.

Proof. In Corollary 4.1 we select Δ_1 so that for $|\mu| < \Delta_1$ the $p(\mu, t)$ there described are contained in a cylinder of radius r_3, where r_3 is as described in Lemma 4.4. Then, if we set $\Delta = \min(\Delta_1, \Delta_3)$ and $r = r_3$, the desired result follows.

5. Almost Periodic Systems

We consider the perturbation problem for the continuing of real almost periodic solutions of the real nonlinear differential system

$$\dot{x} = F(x, \mu, t). \tag{5.1}$$

It is assumed that for $\mu = 0$ system (5.1) possesses an almost periodic solution $p(0, t)$. It is further assumed that there exists $\gamma > 0$ such that for μ sufficiently small $F(x, \mu, t)$ is almost periodic in t uniformly with respect to x in $R(\gamma)$, where $R(\gamma) = \{x \mid \operatorname{lub}_{-\infty < t < \infty} \|x - p(0, t)\| \leqslant \gamma\}$. Here the function $F(x, \mu, t)$ for a fixed μ is said to be almost periodic in t uniformly with respect to x in a set R if for any $\varepsilon > 0$ there exists a relatively dense set $S(\varepsilon)$ such that if $x \in R$, $\tau \in S$, then τ is an ε translation number of $F(x, \mu, t)$. In the future we shall say that $F(x, \mu, t)$ is almost periodic for $x \in R$, or if the range of x is clear, we shall simply say $F(x, \mu, t)$ is almost periodic. Our main result is that if system (5.1) also possesses property VI* with respect to $p(0, t)$, then for μ in a sufficiently small interval about $\mu = 0$ there exists a one parameter family of almost periodic solutions $p(\mu, t)$ which tend uniformly to $p(0, t)$ as μ tends to 0. In preparation for this result we first establish a number of elementary lemmas.

Let the function $F(x, \mu, t, \tau)$ be defined by the equality $F(x, \mu, t, \tau) = F(x, \mu, t + \tau)$. We now assume that there exists $\Delta > 0$ such that for any μ, $|\mu| < \Delta$, $F(x, \mu, t)$ is almost periodic for x in a compact set R, and $F(x, \mu, t)$ is jointly continuous in x and μ uniformly for $-\infty < t < \infty$, $x \in R$, $|\mu| < \Delta$. Then for any fixed μ the set of all translates of the almost periodic function $F(x, \mu, t)$ is clearly the set $P = \{F(x, \mu, t, \tau) \mid -\infty < \tau < \infty\}$. We now consider the closure $\bar{P}$ of P under the uniform norm ($\operatorname{lub}_{-\infty < t < \infty} \|x(t)\| =$ uniform norm of $x(t)$). $\bar{P}$ is called the closed hull of $F(x, \mu, t)$. It isclear that if $F^*(x, \mu, t) \in \bar{P}$, then $F^*(x, \mu, t)$ is almost periodic; and there exists a sequence $\{t_i\}$ of real numbers such that $\lim_{i \to \infty} [\operatorname{lub}_{-\infty < t < \infty} \|F(x, \mu, t, t_i) - F^*(x, \mu, t)\|] = 0$ exists uniformly for $x \in R$.

LEMMA 5.1. *Suppose that $F(x, \mu, t)$ in system (5.1) for a fixed μ is almost periodic for $x \in R$, $R = \{x \mid \|x\| \leqslant \sigma\}$. Assume also that for any $F^*(x, \mu, t)$ contained in the hull of $F(x, \mu, t)$ the system*

$$\dot{x} = F^*(x, \mu, t) \tag{5.1*}$$

possesses a unique bounded sulution $p^*(\mu, t)$ such that $\|p^*(\mu, t)\| \leqslant \sigma$. Then this solution is almost periodic.

Proof. We establish that the family G of functions formed by $p(\mu, t)$ and its translates is a normal family. If we set $p(\mu, t, \tau) = p(\mu, t+\tau)$, then $G = \{p(\mu, t, \tau) \mid -\infty < \tau < \infty\}$. Clearly G is an equicontinuous family, and it will suffice to show that for any sequence $\{\tau_i\}$ of real numbers there exists a subsequence $\{\tau_i^*\}$ such that the sequence $\{p(\mu, t, \tau_i^*)\}$ converges uniformly on the real line. To do this one first picks a subsequence $\{\tau_j^*\}$ of $\{\tau_i\}$ in such a way that the sequence $\{F(x, \mu, t, \tau_j^*)\}$ converges uniformly to a function $F^*(x, \mu, t)$ which is again almost periodic for $\|x\| \leqslant \sigma$. If the sequence $\{p(\mu, t, \tau_j^*)\}$ does not converge uniformly, then system (5.1*) can be shown to possess at least two distinct bounded solutions which are contained in the cylinder of radius σ about $x = 0$. The proof of this fact follows by means of a well known argument due to Favard [3] and will not be repeated here. However, the existence of two bounded solutions in the cylinder of radius σ contradicts the hypothesis and the result follows. But if $p(\mu, t)$ is almost periodic, then the limit function $p^*(\mu, t)$ is also almost periodic and the proof of Lemma 5.1 is complete.

We next obtain two lemmas which will allow us to recast our problem for systems of the form (5.1) in a more manageable form.

LEMMA 5.2. *If $F(x, \mu, t)$ and $p(0, t)$ are almost periodic for $x \in R(\Delta)$ and $F(x, \mu, t)$ is continuous in x uniformly with respect to $x \in R(\Delta)$ and $-\infty < t < \infty$, then $F(\eta + \|p(0, t), \mu, t)$ is almost periodic for $\eta \in R, R = \{\eta \mid \|\eta\| < \Delta\}$.*

Proof. Let $E(\gamma)$ denote the set of common γ-translation numbers of $F(x, \mu, t)$ and $p(0, t)$. By the uniform continuity of $F(x, \mu, t)$ for any $\varepsilon > 0$ there exists $\sigma(\varepsilon) > 0$ such that $\|F(x_1, \mu, t) - F(x_2, \mu, t)\| < \varepsilon/2$ if $\|x_1 - x_2\| < \sigma$.

Setting $\gamma = \min(\varepsilon/2, \sigma(\varepsilon))$, we have

$$\begin{aligned}\|F(\eta + p(0,t), \mu, t) - F(\eta + p(0, t+\tau), \mu, t+\tau)\| &\leqslant \|F(\eta + p(0,t), \mu, t) - F(\eta + p(0,t), \mu, t+\tau)\| \\ &+ \|F(\eta + p(0,t), \mu, t+\tau) - F(\eta + p(0, t+\tau), \mu, t+\tau)\| \leqslant \varepsilon/2 + \varepsilon/2 = \varepsilon\end{aligned}$$

for any $\tau \in E(\gamma)$. Thus the set of ε-translation numbers of $F(\eta + p(0, t), \mu, t)$ contains $E(\gamma)$ and so is relatively dense. This completes the proof of Lemma 5.2.

LEMMA 5.3. *If for a fixed μ there exists $\sigma > 0$ such that $F(x, \mu, t)$ is almost periodic for $\|x - x_0\| < \sigma$ and $F^q_{x_i}(x_0, \mu, t)$ is continuous in x uniformly for $\|x - x_0\| < \sigma$ and $-\infty < t < \infty$, then $F^q_{x_i}(x_0, \mu, t)$ is almost periodic.*

Proof. For any $\Delta < \sigma$ we have

$$[F^q(x_0 + \hat{\Delta}, \mu, t) - F^q(x_0, \mu, t)] / \Delta = F^q_{xi}(x_0 + \theta(t)\hat{\Delta}, \mu, t), \theta(t) < 1,$$

where $\hat{\Delta} = \Delta(\delta_{1i}, \ldots, \delta_{ni})^T$. Since $F^q(x_0 + \hat{\Delta}, \mu, t)$ and $F^q(x_0, \mu, t)$ are almost periodic, $F^q_{x_i}(x_0 + \theta(t)\hat{\Delta}, \mu, t)$ is almost periodic. Thus $\lim_{\Delta \to 0} F^q_{x_i}(x_0 + \theta(t)\hat{\Delta}, \mu, t) = F^q_{x_i}(x_0, \mu, t)$ is the uniform limit of a sequence of almost periodic functions and so is almost periodic.

In preparation for the proof of Theorem 5.1 which follows we have the following definitions.

DEFINITION 5.1. *For $A, B \in M_n$ we say $A \approx B$ in case there exists a matrix P with P, P^{-1}, and $\dot{P} \in M_n$, $\dot{P}$ uniformly continuous and $-P^{-1}[\dot{P} - AP] = B$ for $-\infty < t < \infty$.*

DEFINITION 5.2. *System 5.1 is said to satisfy condition VI* if it satisfies condition VI with $\sim$ replaced by $\approx$.*

In light of the preceeding lemmas we note that if for $x \in R(\gamma)$ the function $F(x, \mu, t)$ is almost periodic and possesses second order partial derivatives with respect to x which are jointly continuous in x and μ, uniformly for $-\infty < t < \infty$, then the problem described at the beginning of this section for system (5.1) may be reduced to an equivalent problem for systems of the form (5.2) below. This is accomplished first by the change of variable $\eta = x - p(0, t)$ and then by developing $F(x, \mu, t)$ in a Taylor series about $\eta = 0$ to obtain the system

$$\dot{\eta} = F(p(0, t), \mu, t) - F(p(0, t), 0, t) + [A(\mu, t) + \mathfrak{F}(\eta, \mu, t)]\eta, \tag{5.2}$$

where $F(p(0, t), \mu, t)$, $A(\mu, t)$ are almost periodic and $\mathfrak{F}(\eta, \mu, t)$ is almost periodic for $\eta < \gamma$. Here $A(\mu, t)$ and $\mathfrak{F}(\eta, \mu, t)$ are as defined in definition 4.2. If we further assume that system (5.1) possesses property VI* with respect to $p(0, t)$, then there exists a bounded transformation $T(t)$ which reduces system (5.2) to

$$\dot{z} = f(t, \mu) + [D + \sigma(\mu, t)]z + c(z, \mu, t)z \tag{5.3}$$

which is of the same form as (4.3) and satisfies the same restrictions. Thus by Theorem 4.2 there exist $\Delta > 0$, $r > 0$ such that for every μ in the interval $(-\Delta, \Delta)$ there corresponds a unique bounded solution $\eta(\mu, t)$ of (5.2) which is contained in the cylinder of radius r about $\eta = 0$. We next observe that for any system

$$\dot{\eta} = F(p(0, t+t_i), \mu, t+t_i) - F(p(0, t+t_i), 0, t+t_i) + [A(\mu, t+t_i), + \mathfrak{F}(\eta, \mu, t+t_i)]\,\eta \tag{5.2 i}$$

the transformation $T(t+t_i)$ reduces system (5.2 i) to one of the form

$$\dot{z} = f(t+t_i, \mu) + [D + \sigma(\mu, t+t_i)]\,z + c(z, \mu, t+t_i)\,z \tag{5.3 i}$$

which again satisfies exactly the same restrictions as (5.3). Thus we have that for any $\mu \in (-\Delta, \Delta)$ there corresponds a unique bounded solution $p(\mu, t+t_i)$ of (5.2 i) which is contained in the cylinder of radius r about $z = 0$. Thus in order that the hypothesis of Lemma 5.1 be satisfied it remains only to establish that any system $\{(5.2^* \mathrm{i})\}$ which is the uniform limit of a sequence of systems $\{(5.2\,\mathrm{i})\}$ associated with a given sequence $\{t_i\}$ possesses for $\mu \in (-\Delta, \Delta)$ a unique bounded solution $p^*(\mu, t)$ which is contained in the cylinder of radius r about $\eta \equiv 0$. Since $T(t)$, $T^{-1}(t)$, and $\dot{T} \in M_n$ and $\dot{T}$ is uniformly continuous for $-\infty < t < \infty$, it is clear that for any sequence $\{T(t+t_i)\}$ we can choose a subsequence $\{T(t+t_j)\}$ which converges uniformly on all finite intervals to $T^*(t)$ where $T^*(t)$, $T^{*-1}(t)$ and $\dot{T}^*(t) \in M_n$. It is further noted that $T^*(t)$ reduces system (5.2^*) to a system of the form (5.3^*), where the functions $f^*(\mu, t)$, $\sigma^*(\mu, t)$, and $c^*(z, \mu, t)$ satisfy exactly the same conditions as the functions $f(\mu, t)$, $\sigma(\mu, t)$ and $c(z, \mu, t)$ in system (5.3). Thus system (5.2^*) possesses for μ in $(-\Delta, \Delta)$ a unique bounded solution $p^*(\mu, t)$ which is contained in the cylinder of radius r about $\eta = 0$. It then follows by Lemma 5.1 that the bounded solution $p(\mu, t)$ of (5.2) is almost periodic. Using the equivalence of systems (5.1) and (5.2) we have established our desired result. We now collect the conditions imposed on system (5.1) and the results obtained in Definition 5.3 and Theorem 5.1 below.

DEFINITION 5.3. *System* (5.1) *is said to possess property* VII *with respect to* $p(0, t)$ *if*

1) *for* $\mu = 0, p(0, t)$ *is an almost periodic solution of system* (5.1),
2) *there exist* $\Delta > 0$, $\gamma > 0$, *such that for any* μ, $|\mu| < \Delta$, $F(x, \mu, t)$ *is almost periodic in t uniformly with respect to x in* $R(\gamma)$,
3) *system* (5.1) *possesses property* VI^* *with respect to* $p(0, t)$.

THEOREM 5.1. *If for $\mu=0$ system (5.1) possesses property VII with respect to the almost periodic solution $p(0,t)$, then there exist $\sigma>0$, $r>0$ such that to every μ, $|\mu|<\sigma$, there corresponds a unique almost periodic solution $p(\mu,t)$ of system 5.1 which is contained in a cylinder of radius r about $p(0,t)$. As $\mu\to 0, p(\mu,t)$ tends uniformly to $p(0,t)$.*

It is in general not known whether there exists a transformation $T(t)$ such that $T(t)$, $T^{-1}(t)$ and $\dot{T}(t)\in M_n$ and $\dot{T}(t)$ is uniformly continuous which reduces the linear part of (5.2) to a system which is "close" to a linear system with constant coefficients. There are cases, however, in which such a transformation is known to exist. We consider two of these cases in the following corollaries to Theorem 5.1.

COROLLARY 5.1. *For $\mu=0$ let $F(x,\mu,t)$ of system (5.1) be periodic in t and let system (5.1) possess a periodic solution $p(0,t)$ whose variational equation has no characteristic roots with zero real parts. Further assume that system (5.1) satisfies the almost periodic, differentiability, and continuity restrictions of property VII. Under these assumptions there exist $\sigma>0$ and $r>0$ such that for every μ, $|\mu|<\sigma$, system (5.1) possesses a unique almost periodic solution $p(\mu,t)$ which is contained in a cylinder of radius r about $p(0,t)$. Furthermore, the $p(\mu,t)$ tend to $p(0,t)$ uniformly as μ tends to zero.*

COROLLARY 5.2. *For $\mu=0$ let system (5.1) possess an almost periodic solution $p(0,t)$ whose variational equation is a linear system with constant coefficients, none of whose characteristic roots have zero real parts. Further assume that system (5.1) satisfies the almost periodic, differentiability, and continuity restrictions of property VII. Under these assumptions there exists $\sigma>0$ and $r>0$ such that for every μ, $|\mu|<\sigma$, system (5.1) possesses a unique almost periodic solution $p(\mu,t)$ which is contained in a cylinder of radius r about $p(0,t)$. Furthermore, the $p(\mu,t)$ tend to $p(0,t)$ uniformly as μ tends to zero.*

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Received April 3, 1959.