THE INVERSE STURM—LIOUVILLE PROBLEM WITH SYMMETRIC POTENTIALS

BY

OLE H. HALD

University of California, Berkeley, U.S.A.

Introduction

The purpose of this paper is to present an algorithm for solving the inverse Sturm–Liouville problem on a finite interval. The main idea is to reduce the problem to a system of finitely many ordinary nonlinear differential equations.

The inverse Sturm–Liouville problem is primarily a model problem. Typically, in an inverse eigenvalue problem, one measures the frequencies of a vibrating system, and tries to infer some physical properties of the system. Because of the difficulties in obtaining the higher eigenvalues in practice, only a finite amount of data will in general be available. On the other hand, one might have an a priori guess for the solution. The question is therefore whether the model, i.e., the initial guess, is compatible with the data, and, if this is not the case, how it should be modified. The results, which we will derive, are well suited to answer this kind of question.

There are at least four different versions of the inverse Sturm–Liouville problem. The best known is the one studied by Gel'fand and Levitan [6], in which the potential and the boundary conditions are uniquely determined by the spectral function. This case has also been investigated by Marčenko [17], Krein [13] and Žikov [22]. In the second version, the potential and the boundary conditions are uniquely determined by two spectra. This case can be reduced to the previous one as shown by Marčenko [17], Levitan [15], Gasymov and Levitan [5] and Žikov [22]. In the third version, the potential is uniquely determined by the boundary conditions and two-possible reduced-spectra. This case has been studied by Borg [3], Levinson [14] and Hochstadt [8]. The fact, that the boundary conditions are known implies that the lowest eigenvalue in one of the spectra is superfluous. Finally, Borg [3], Levinson [14], and Hochstadt [8] have shown that if the boundary conditions and one-possible reduced-spectrum are given, then the potential is uniquely determined, provided it is an even function around the middle of the interval.

In this paper we will present a constructive method for the last case. However, some of the results can be extended to the other versions as well. The basic result is an extension of a formula due to Hochstadt [8] for the difference of two potentials and our proof rests on the technique developed by Hochstadt [8]. This formula leads directly to several uniqueness theorems due to Borg [3], Levinson [14], Hochstadt [8] and Hald [7], as well as a new well-posedness result. Hochstadt [9] has pointed out that his formula leads to an algorithm for solving the inverse Sturm–Liouville problem. The trick is to reduce the problem to solving a system of ordinary differential equations. However, the original suggestion contains an oversight and in this paper we will prove that a modified version of Hochstadt's algorithm will always provide a solution of the inverse Sturm–Liouville problem. The algorithm can therefore be used instead of the Gel'fand–Levitan technique for this particular kind of problem.

Finally, we consider the dualism between the lowest eigenvalue and the boundary conditions. This investigation shows very clearly why the lowest eigenvalue cannot be prescribed if the boundary conditions are given and of mixed type. In addition it leads to a natural generalization of Borg's original formulation of the inverse Sturm–Liouville problem and provides a link between the four versions mentioned above.

1. The difference of two potentials

In this section we will consider two Sturm–Liouville problems with different potentials and different boundary conditions. We will assume that the potentials are even functions around the middle of the interval. The main result is that if the sum of the absolute value of the differences of the eigenvalues of the two Sturm–Liouville problems is finite, then the potentials differ by a continuous function.

THEOREM 1. *Consider the eigenvalue problems*

$$-u'' + q(x)u = \lambda u \tag{1.1}$$
$$hu(0) - u'(0) = 0, \quad hu(\pi) + u'(\pi) = 0$$

$$-u'' + \tilde{q}(x)u = \tilde{\lambda} u \tag{1.2}$$
$$\tilde{h}u(0) - u'(0) = 0, \quad \tilde{h}u(\pi) + u'(\pi) = 0,$$

where q and $\tilde{q}$ are integrable on $[0, \pi]$ and satisfy the symmetry conditions $q(x) = q(\pi - x)$

and $\tilde{q}(x)=\tilde{q}(\pi-x)$ *almost everywhere in the interval* $0\leqslant x\leqslant\pi$. *Let* λ_j *and* $\tilde{\lambda}_j$ *be the eigenvalues of* (1.1) *and* (1.2). *Let* $\tilde{u}_j$ *and* $\tilde{v}_j$ *be the solutions of*

$$u''+(\lambda-\tilde{q})u=0 \tag{1.3}$$

$$u(0)=1,\quad u'(0)=\tilde{h} \tag{1.4}$$

$$v(\pi)=1,\quad v'(\pi)=-\tilde{h} \tag{1.5}$$

with $\lambda=\lambda_j$. *Define the functions* $\tilde{y}$ *by*

$$\tilde{y}_j=2\cdot\frac{\tilde{v}_j-k_j\tilde{u}_j}{\omega'(\lambda_j)}. \tag{1.6}$$

Here $k_j/\omega'(\lambda_j)=1/\int_0^\pi u_j^2\,dx$ *where* $k_j=(-1)^j$ *and* $u_j(x)$ *are the eigenfunctions of* (1.1) *normalized such that* $u_j(0)=1$. *If* $\sum_j|\lambda_j-\tilde{\lambda}_j|<\infty$ *then*

$$h-\tilde{h}=\tfrac{1}{2}\sum_j\tilde{y}_j(0) \tag{1.7}$$

$$q-\tilde{q}=\sum_j(\tilde{y}_ju_j)'\quad\text{a.e.} \tag{1.8}$$

Remark. This result is a generalization of a theorem due to Hochstadt [8], who assumes that $h=\tilde{h}$ and that only a finite number of the eigenvalues λ_j and $\tilde{\lambda}_j$ are different. In this case eq. (1.7) is trivially satisfied, and the summation in eq. (1.8) is only over those j for which $\lambda_j\neq\tilde{\lambda}_j$. Our extension seems quite innocent, but is crucial in order to prove that the algorithm presented in Section 4 has a solution and that this solution is unique. The case $h=\tilde{h}=\infty$, which should be interpreted as Dirichlet boundary conditions, has been discussed by Hochstadt [8]. The extension to infinite many eigenvalues is straightforward.

The proof below is based on three ingredients. The first is the Cauchy integral technique for deriving the Sturm–Liouville expansion of an integrable function. The basic idea goes back to Poincaré, but the implementation is due to Kneser, Birkhoff and Tamarkin among others. For an elementary presentation see Titchmarsh [21]. The second ingredient is a very clever device by Levinson [14], who modifies the Cauchy integral technique by replacing one entire function by another which has the same asymptotic expansion. Levinson's proof is based on Titchmarsh's presentation, and so is ours. The final ingredient is closely related to Hochstadt's approach [8]. In his proof, Hochstadt introduces two Hilbert spaces each spanned by those eigenfunctions of (1.1) and (1.2) for which $\lambda_j=\tilde{\lambda}_j$. It is then natural to consider the mapping T which takes the eigenfunctions of one Sturm–Liouville problem onto the eigenfunctions of the other. Hochstadt finds an explicit representation of this mapping and his result follows directly from this representation.

Proof. Let $u(x, \lambda)$ and $v(x, \lambda)$ be the solutions of eq. (1.3) with initial conditions (1.4) and (1.5), where $\tilde{h}$ and $\tilde{q}$ are replaced by h and q. Then u satisfies the Volterra integral equation

$$u(x) = \cos \sqrt{\lambda} x + \frac{h}{\sqrt{\lambda}} \sin \sqrt{\lambda} x + \frac{1}{\sqrt{\lambda}} \int_0^x \sin \sqrt{\lambda}(x-t)\, q(t)\, u(t)\, dt. \tag{1.9}$$

Let $\lambda = s^2$ where $s = \sigma + i\tau$. From eq. (1.9) follows that for each x, $u(x, \lambda)$ is an entire function of λ of order $\frac{1}{2}$ and asymptotically we have

$$u(x, \lambda) = \cos sx + O\left(\frac{e^{|\tau| x}}{|s|}\right) \tag{1.10}$$

$$u'(x, \lambda) = -s \sin sx + O(e^{|\tau| x}) \tag{1.11}$$

see Titchmarsh [21, p. 10]. Since $q(x) = q(\pi - x)$ a.e. we find that $v(x) = u(\pi - x)$ and thus eq. (1.10) and eq. (1.11) provide the asymptotic expansions for v and v' as well. We introduce now the Wronskian

$$\omega(\lambda) = -hu(\pi, \lambda) - u'(\pi, \lambda) \tag{1.12}$$

and note that λ is an eigenvalue of (1.1) iff $\omega(\lambda) = 0$. From (1.10) and (1.11) follows that the asymptotic expansion for $\omega(\lambda)$ is

$$\omega(\lambda) = s \sin s\pi + O(e^{|\tau|\pi}).$$

Let f be an absolutely continuous function and assume that f' is square integrable. We consider now the meromorphic function

$$\Phi(x, \lambda) = \frac{\tilde{v} \int_0^x uf\, dy + \tilde{u} \int_x^\pi vf\, dy}{\omega(\lambda)}.$$

Here $\tilde{u}$ and $\tilde{v}$ are the solutions of (1.3) with initial conditions (1.4) and (1.5), and have the same asymptotic expansions as u and v. We will integrate Φ along a large contour in the λ plane.

In the s plane we let R be the rectangle with vertices at $\pm d + i0$ and $\pm d + id$ where $d = n + 1/2$ and we let Γ be the contour in the λ plane which corresponds to the points of R for which $\tau > 0$. By using the asymptotic estimates for u, $\tilde{v}$ and ω we find that

$$\frac{\tilde{v}(x, \lambda)\, u(y, \lambda)}{\omega(\lambda)} = \frac{\cos s(\pi - x) \cos sy}{s \sin s\pi} + O\left(\frac{e^{\tau(y-x)}}{|s|^2}\right).$$

This is precisely what is needed to make Titchmarsh's arguments apply, see [21, p. 13], and we conclude that

$$\frac{1}{2\pi i}\int_{\Gamma}\Phi(x,\lambda)\,d\lambda-\frac{1}{2\pi i}\int_{\Gamma}\int_{0}^{x}\frac{\cos s(\pi-x)\cos sy}{s\sin s\pi}f(y)\,dy\,d\lambda$$
$$-\frac{1}{2\pi i}\int_{\Gamma}\int_{x}^{\pi}\frac{\cos sx\cos s(\pi-y)}{s\sin s\pi}f(y)\,dy\,d\lambda$$

converges uniformly to zero on the interval $0\leqslant x\leqslant\pi$ as $n\to\infty$. It follows from the residue theorem that the sum of the last two terms is the first $n+1$ terms of the Fourier cosine expansion of the function f. It is therefore natural to extend f as an even, 2π periodic function and since f' is square integrable and $f(-\pi)=f(\pi)$ we know that the Fourier series for f converges uniformly, see Zygmund [23, p. 242]. By using the residue theorem to evaluate the first term in the above expression and letting $n\to\infty$ we obtain

$$f(x)=\sum_{j=0}^{\infty}\frac{\tilde{v}_j\int_0^x u_j f\,dy+\tilde{u}_j\int_x^{\pi}v_j f\,dy}{\omega'(\lambda_j)}. \tag{1.13}$$

We note that u_j and v_j represent the same eigenfunction, whereas $\tilde{u}_j$ and $\tilde{v}_j$ are just solutions of eq. (1.3) with $\lambda=\lambda_j$. Since $q(x)=q(\pi-x)$ we see that $v_j=k_ju_j$ where $k_j=(-1)^j$. If $q=\tilde{q}$ and $h=\tilde{h}$ then (1.13) reduces to the Sturm–Liouville expansion and consequently $k_j/\omega'(\lambda_j)=1/\int_0^\pi u_j^2\,dx$. Let now f be equal to the first eigenfunction u_0 of (1.1). From (1.13) and definition (1.6) follows that

$$u_0=\tilde{u}_0+\tfrac{1}{2}\sum_j\tilde{y}_j\int_0^x u_ju_0\,dt. \tag{1.14}$$

We can now obtain the results stated in the theorem by differentiating eq. (1.14) formally. To realize that let $f_j=\tilde{y}_j\int_0^x u_ju_0\,dt$. Thus $f_j(0)=0$ and $f_j'(0)=\tilde{y}_j(0)$. Since u_j and u_0 are eigenfunctions of (1.1) and $\tilde{y}_j$ is a solution of (1.3) with $\lambda=\lambda_j$ we find by differentiating f_j twice and using integration by parts that

$$f_j''+(\lambda_0-\tilde{q})f_j=2(\tilde{y}_ju_j)'u_0. \tag{1.15}$$

We can now derive (1.7) and (1.8). By differentiating eq. (1.14) and using eq. (1.15) we obtain

$$u_0'-\tilde{u}_0'=\tfrac{1}{2}\sum f_j'$$
$$u_0''-\tilde{u}_0''=(\tilde{q}-\lambda_0)(u_0-\tilde{u}_0)+\sum(\tilde{y}_ju_j)'u_0.$$

Thus eq. (1.7) follows by setting $x=0$ in the first equation. To derive (1.8) from the second equation we use that $u_0''=(q-\lambda_0)u_0$ and $\tilde{u}_0''=(\tilde{q}-\lambda_0)\tilde{u}_0$ and note that the eigenfunction u_0 is positive in the whole interval.

To establish the validity of the above formal argument we must show that the series under consideration actually converges. This requires a number of fairly detailed estimates. An important byproduct of these estimates is a well-posedness result for the inverse Sturm–Liouville problem, see Section 3.

We will show that $\sqrt{\lambda_j}\tilde{y}_j$ and $\tilde{y}_j'$ are $O(\lambda_j-\tilde{\lambda}_j)$. Let z_j be the eigenfunction of (1.2) corresponding to the eigenvalue $\tilde{\lambda}_j$. We will compare $\tilde{u}_j$ with z_j and let $w_j=(\tilde{u}_j-z_j)/(\lambda_j-\tilde{\lambda}_j)$. Thus $w=w_j$ satisfies the differential equation

$$w''+(\lambda-\tilde{q})w=-z_j$$
$$w(0)=w'(0)=0$$

with $\lambda=\lambda_j$. Let φ_1 and φ_2 be solutions of the homogeneous equation, i.e. (1.3), with initial conditions $\varphi_{i+1}^{(j)}=\delta_{ij}$ at $x=0$. The solution of the inhomogeneous equation is then given by

$$w(x)=\int_0^x[\varphi_1(x)\varphi_2(y)-\varphi_2(x)\varphi_1(y)]z_j(y)\,dy. \tag{1.16}$$

To estimate w we must estimate φ_1, φ_2 and z_j. We first observe that φ_1 is the solution of the Volterra integral equation (1.9) with $h=0$ and q replaced by $\tilde{q}$ and $\lambda=\lambda_j$. Let $\|\tilde{q}\|_1=\int_0^\pi|\tilde{q}|\,dx$ and $\|\varphi\|_\infty=\text{ess sup}\,|\varphi|$. If $\sqrt{\lambda}$ is larger than $6\|\tilde{q}\|_1$ then we conclude from eq. (1.9) that $\|\varphi_1\|_\infty\leq 6/5$ and that $\|\varphi_1'\|_\infty\leq(6/5)\sqrt{\lambda}$, see Titchmarsh [21, p. 10]. In a similar manner we find that $\sqrt{\lambda}\|\varphi_2\|_\infty$ and $\|\varphi_2'\|_\infty$ are less than 6/5. To estimate z_j we replace h, q and λ in eq. (1.9) with $\tilde{h}$, $\tilde{q}$ and $\tilde{\lambda}_j$. Thus if $\sqrt{\tilde{\lambda}_j}$ is larger than $6\|\tilde{q}\|_1$ and $6|\tilde{h}|$ then we have $\|z_j\|_\infty\leq 7/5$ and $\|z_j'\|_\infty\leq(7/5)\sqrt{\tilde{\lambda}_j}$. From these estimates and the solution formula (1.16) for w we conclude that $\sqrt{\lambda_j}\|w_j\|_\infty$ and $\|w_j'\|_\infty$ are less than $(504/125)\pi$.

Since the potential $\tilde{q}$ is symmetric we find that $z_j(\pi-z)=k_jz_j(x)$ and $\tilde{v}_j(x)=\tilde{u}_j(\pi-x)$. It therefore follows from the definition of w_j and formula (1.6) that

$$\tilde{y}_j(x)=2\frac{\lambda_j-\tilde{\lambda}_j}{\omega'(\lambda_j)}[w_j(\pi-x)-k_jw_j(x)]. \tag{1.17}$$

To complete our study of $\tilde{y}_j$ we must find a lower bound for $|\omega'(\lambda_j)|=\|u_j\|_2^2$. Here u_j is the eigenfunction of (1.1) normalized such that $u_j(0)=1$. From eq. (1.9) we see that $u_j(x)=\cos\sqrt{\lambda_j}x+\theta(x)$ where $|\theta|\leq 2/5$ provided $\sqrt{\lambda_j}$ is larger than $6|h|$ and $6\|q\|_1$. If we assume in addition that $\sqrt{\lambda_j} > 1/\pi$ then we find by using the triangle inequality that $|\omega'(\lambda_j)| \geq \pi/100$. By using the bounds for w_j and w_j' we conclude from eq. (1.17) that

$$\|\tilde{y}_j\|_\infty \leq 1613 \frac{|\lambda_j - \tilde{\lambda}_j|}{\sqrt{\lambda_j}} \tag{1.18}$$

$$\|\tilde{y}_j'\|_\infty \leq 1613 |\lambda_j - \tilde{\lambda}_j|. \tag{1.19}$$

We can now show that $\sum (\tilde{y}_j u_j)'$ actually converges. To begin with we choose the integer N so that $\sqrt{\lambda_N}$ and $\sqrt{\tilde{\lambda}_N}$ are larger than $1/\pi$ and 6 times the maximum of $|h|$, $|\tilde{h}|$, $\|q\|_1$ and $\|\tilde{q}\|_1$. In this case all the above estimates hold simultaneously and moreover, we have, as in our bound for the eigenfunction z_j, that $\|u_j\|_\infty \leq 7/5$ and $\|u_j'\|_\infty \leq (7/5)\sqrt{\lambda_j}$. Thus by using (1.18) and (1.19) we see that

$$\left|\frac{1}{2} \sum_{j=N}^{\infty} \tilde{y}_j(0)\right| \leq 404\pi \sum_{j=N}^{\infty} |\lambda_j - \tilde{\lambda}_j| \tag{1.20}$$

$$\left\|\sum_{j=N}^{\infty} (\tilde{y}_j u_j)'\right\|_\infty \leq 4520 \sum_{j=N}^{\infty} |\lambda_j - \tilde{\lambda}_j|. \tag{1.21}$$

We can now prove the validity of the formal arguments leading to (1.7) and (1.8). From the solution of equation (1.15) follows that

$$\tfrac{1}{2} \sum f_j = \tfrac{1}{2} \sum \tilde{y}_j(0)\, \varphi_2(x) - \int_0^x G(x,y) \sum (\tilde{y}_j u_j)'\, u_0\, dy. \tag{1.22}$$

Here $G(x, y) = \varphi_1(x)\varphi_2(y) - \varphi_2(x)\varphi_1(y)$, where φ_1 and φ_2 are solutions of the homogeneous eq. (1.3) with $\lambda = \lambda_0$ and satisfy the initial conditions $\varphi_{i+1}^{(j)} = \delta_{ij}$ at $x = 0$. To interchange the order of integration and summation we have used that $\sum (\tilde{y}_j u_j)'$ converges in L^∞, see (1.21). On the other hand, $u = u_0 - \tilde{u}_0$ satisfies the differential equation $u'' + (\lambda_0 - \tilde{q})u = (q - \tilde{q})u_0$ with the initial conditions $u(0) = 0$ and $u'(0) = h - \tilde{h}$ and consequently

$$u_0 - \tilde{u}_0 = (h - \tilde{h})\, \varphi_2(x) - \int_0^x G(x,y)\,(q - \tilde{q})\, u_0\, dy.$$

The theorem now follows from (1.14) by comparing the last two equations and using the uniqueness theorem for ordinary differential equations with summable coefficients, see Neumark [19, § 15, Satz 2]. This completes the proof.

2. Uniqueness and regularity results

The explicit formula (1.8) for the difference between two potentials is well suited for deriving some well-known uniqueness results. We will show that the potential and the boundary conditions are uniquely determined by the full spectrum. Moreover, we will prove that if the boundary conditions are known then the potential is uniquely determined by the reduced spectrum. Here the reduced spectrum is the full spectrum with the lowest eigenvalue omitted. Finally, we will show that the assumption $\sum |\lambda_j - \tilde{\lambda}_j|$ in Theorem 1 is quite restrictive. In particular, it implies that if the comparison spectrum $\{\tilde{\lambda}_j\}$ corresponds to a Sturm–Liouville problem with constant coefficients, then the Fourier series for the potential $q(x)$ will be absolutely convergent.

COROLLARY 1. *Consider the eigenvalue problem* (1.1) *where q is integrable in* $[0, \pi]$. *If $q(x) = q(\pi - x)$ almost everywhere in $0 < x < \pi$ then $q(x)$ and h are uniquely determined by the spectrum.*

Proof. Assume that we have two Sturm–Liouville problems with the same eigenvalues $\lambda_j = \tilde{\lambda}_j$. From equations (1.3) and (1.4) follows that $\tilde{u}_j$ is an eigenfunction, and since the potential $\tilde{q}$ is symmetric we conclude that $\tilde{v}_j = k_j \tilde{u}_j$. This shows that all $\tilde{y}_j$ vanish identically and the right hand sides of equations (1.7) and (1.8) are zero. This completes the proof.

From the symmetry of the potential follows that the eigenfunctions are either odd or even functions around $\pi/2$. We can therefore decompose the eigenvalue problem on $[0, \pi]$ into two problems on $[0, \pi/2]$ with either Dirichlet or Neumann boundary conditions at $x = \pi/2$. It has been shown by Marčenko [17, 18], Krein [12] and Levitan [15], that the potential and the boundary conditions are uniquely determined by two spectra, and can be reconstructed from this data. The result is sometimes credited to Borg, see Levitan [15], and Gasymov and Levitan [5]. However, Borg proved a different, but equally precise result, namely that if the boundary conditions are given then two-possible reduced-spectra determine the potential uniquely. It is not obvious that the two problems are equivalent, although our results in Section 7 indicate that this is indeed the case. For symmetric potentials we have

COROLLARY 2. *Consider the eigenvalue problem* (1.1) *where q is integrable on* $[0, \pi]$ *and satisfies $q(x) = q(\pi - x)$ a.e. If $q(x)$ is replaced by another symmetric potential $\tilde{q}(x)$ and the two problems have the same reduced spectrum* (*i.e. the full spectrum with the lowest eigenvalue omitted*) *then $q = \tilde{q}$ a.e.*

Remark. The case $h = 0$ was discussed by Borg [3, p. 69]. For $h \neq 0$ the result follows from Borg's theorem concerning two spectra, but was first stated explicitly by Hochstadt [8]. The proof below is due to Hochstadt [8]. A less precise version of the theorem has been given by Levinson [14].

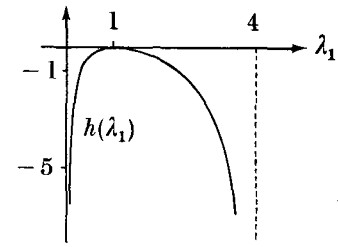

Figure 1. Dependence on the constant h in the boundary conditions on the eigenvalue λ_1

Proof. Let λ_j and $\tilde{\lambda}_j$ be the eigenvalues corresponding to q and $\tilde{q}$. If $h=\tilde{h}$ and $\lambda_j=\tilde{\lambda}_j$ for $j=1, 2, \ldots$, then we see from Theorem 1 that $0=\tilde{y}_0(0)$ and $q-\tilde{q}=(\tilde{y}_0 u_0)'$. We will show that $\tilde{y}_0$ vanishes identically. From (1.6) follows that $\tilde{y}_0(\pi-x)=-\tilde{y}_0(x)$ and thus $\tilde{y}_0$ vanishes at $x=0$, $\pi/2$ and π. Since $\lambda_0<\lambda_1=\tilde{\lambda}_1$ we conclude from Sturm comparison theorem that if $\tilde{y}_0 \not\equiv 0$ then the eigenfunction $\tilde{u}(x, \tilde{\lambda}_1)$ will have at least two zeros in $(0, \pi)$. But this contradicts Sturm's oscillation theorem [4, p. 210], and the result follows by contraposition.

It should be pointed out that the problems discussed in Corollary 1 and 2 may both arise in applications. Thus the inverse eigenvalue problem for a cylinder can be reduced to solving two inverse Sturm–Liouville problems, see [7]. In this case the boundary conditions in the second eigenvalue problem can be determined from the spectrum of the first eigenvalue problem by using Corollary 1.

We have seen in Corollary 2 that the lowest eigenvalue plays a special role. Thus the question arises whether the potential can be uniquely determined by the boundary conditions and say, $\lambda_0, \lambda_2, \lambda_3, \ldots$. In general the answer is no. To see this let $\lambda_j=j^2$ for all $j\neq 1$. For each $h<0$ there exist two potentials having the requested boundary conditions and eigenvalues, see Figs. 1 and 2. The natural comparison potential is $\tilde{q}=0$, and from Theorem 1 follows that the graph in Fig. 1 is

$$h(\lambda)=\frac{\lambda-1}{\sqrt{\lambda}}\cot\left(\sqrt{\lambda}\frac{\pi}{2}\right).$$

In Fig. 2 we give the two potentials which correspond to $h=-1$, i.e. $\lambda_1=0.316$ and $\lambda_1=2.365$. For $h=0$ there is only one potential, namely $q\equiv 0$. This is a direct consequence of a theorem by Borg [3, p. 70 and p. 88], and closely related to a result of Ambarzumian [1]. The same phenomenon occurs for other eigenvalues as well.

The proof of Theorem 1 reveals more than stated in the theorem. To realize this we observe that both $\tilde{y}_j'$ and u_j' are equal to absolutely continuous functions almost everywhere. Thus after modifying $\tilde{y}_j'$ and u_j' on a set of measure zero we see that $\sum (\tilde{y}_j u_j)'$ converges uniformly to a continuous function. This implies that if $\sum |\lambda_j-\tilde{\lambda}_j|$ converges then

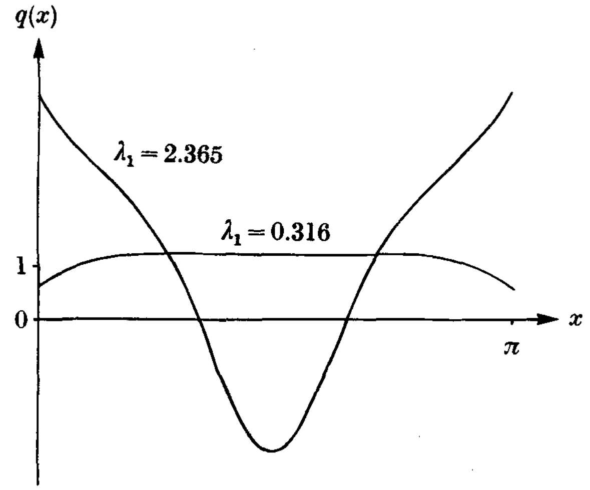

Figure 2. Two potentials having h and λ_0, λ_2, λ_3, ... in common

the difference between the potentials is a continuous function a.e., even though the potentials are only summable. In particular we have

COROLLARY 3. *Consider the eigenvalue problem* (1.1) *where q is integrable on $[0,\pi]$ and satisfies the symmetry condition $q(x)=q(\pi-x)$ almost everywhere. Let $Q=1/\pi\int_0^\pi q(x)dx$. The Fourier cosine series for $q(x)$ converges absolutely if and only if*

$$\sum_j \left|\lambda_j-\left(j^2+Q+\frac{4h}{\pi}\right)\right|<\infty. \tag{2.1}$$

Remark. Let $\tilde{\lambda}_j$ be the eigenvalues of (1.1) with the potential $q(x)$ replaced by the constant Q. Borg has shown, see [3, p. 26], that if $1<p\leqslant 2$ and $\sum|\lambda_j-\tilde{\lambda}_j|^p$ converges then the potential is in $L^{p'}$ where $1/p+1/p'=1$. Borg's proof cannot be extended to $p=1$, as it is based on the Hausdorff–Young theorem and the asymptotic expansion

$$\lambda_j-\tilde{\lambda}_j=\tfrac{1}{2}a_{2j}+O\left(\frac{1}{j}\right) \tag{2.2}$$

where a_{2j} is the $2j$th coefficient in the Fourier cosine expansion of $q(x)$. Our corollary is therefore an extension of Borg's result.

Proof. Borg has shown, see [3, p. 21], that if $q\in L^2$ then $\sum|\lambda_j-\tilde{\lambda}_j|$ converges iff $\sum|a_{2j}|$ converges. Since

$$\tilde{\lambda}_j=j^2+Q+\frac{4h}{\pi}+O\left(\frac{1}{j^2}\right) \tag{2.3}$$

see Borg [3, p. 15], we conclude that $\sum |\lambda_j - \tilde{\lambda}_j|$ converges iff (2.1) is satisfied. We have seen that if $\sum |\lambda_j - \tilde{\lambda}_j| < \infty$ then $q(x)$ is continuous a.e., and hence in L^2. The same conclusion holds if $\sum |a_{2j}| < \infty$. The proof is therefore completed by using Borg's result.

3. Well-posedness

In this section we will show that the difference between two symmetric potentials can be bounded in terms of the difference between the corresponding spectra. We have already obtained one result in this direction namely the inequality (1.21). In general we have

THEOREM 2. *Consider the eigenvalue problems* (1.1) *and* (1.2) *where* q *and* $\tilde{q}$ *are integrable on* $[0, \pi]$ *and satisfy the symmetry conditions* $q(x) = q(\pi - x)$ *and* $\tilde{q}(x) = \tilde{q}(\pi - x)$ *almost everywhere in the interval* $0 \leqslant x \leqslant \pi$. *Let* λ_j *and* $\tilde{\lambda}_j$ *be the eigenvalues of* (1.1) *and* (1.2). *Let* $M = \max(|h|, |\tilde{h}|, \|q\|_1, \|\tilde{q}\|_1)$. *Then*

$$|h - \tilde{h}| \leqslant 15 \cdot 10^{6+38M+11M^2} \sum_{j=0}^{\infty} |\lambda_j - \tilde{\lambda}_j|$$

$$\|q - \tilde{q}\|_\infty \leqslant 15 \cdot 10^{6+38M+11M^2} \sum_{j=0}^{\infty} |\lambda_j - \tilde{\lambda}_j|.$$

Remark. There is a deplorable lack of estimates of this kind in the literature on the inverse Sturm–Liouville problem. For the inverse Sturm–Liouville problem with Dirichlet boundary conditions, Barcilon [2] has given a quite explicit well-posedness result, under the assumption that the potentials are symmetric and have a small L^2 norm. A more general result, proved for two spectra, is hidden in Borg's paper, see [3, p. 78, formula (29)]. Roughly speaking Borg proves that if the potentials are symmetric then

$$\|q - \tilde{q}\|_2 \leqslant 2\sqrt{2\pi} \cdot K \sqrt{\sum_j |\lambda_j - \tilde{\lambda}_j|^2}$$

provided the right-hand side is sufficiently small. Here the constant K depends only on q and can be characterized as a lower bound for the quotient of two infinite quadratic forms, but no specific estimate is available. Finally, Hochstadt [10] has obtained a well-posedness result in L^∞ assuming that $h = \tilde{h}$ and that only a finite number of the eigenvalues differs. Hochstadt's proof is based on eq. (1.8) and so is ours.

Proof. The proof of Theorem 1 is based on the asymptotic behavior of the solutions of eq. (1.3). For example, the estimates (1.18) and (1.19) of $\tilde{y}_j$ and $\tilde{y}_j'$ are only valid for the solutions associated with the higher eigenvalues and will here be replaced by quite crude bounds for the solutions corresponding to the lower end of the spectrum. Let z be the solution of the differential equation (1.3) with initial conditions (1.4). Then

$$|z(x,\lambda)| \leq \max(1,|\tilde{h}|)\exp(R\tilde{r}(x)) \tag{3.1}$$

$$|z'(x,\lambda)| \leq \max(1,|\tilde{h}|)R\exp(R\tilde{r}(x)). \tag{3.2}$$

The estimates (3.1) and (3.2) are obtained when the existence of solutions of eq. (1.3) is proved by using the method of successive approximations. See K. Jürgens [11, § 4, p. 3]. Specifically, one may choose

$$\tilde{r}(x) = \int_0^x \max(1,|\tilde{q}|)\,dt$$

$$R \geq \sqrt{1+|\tilde{\lambda}|}. \tag{3.3}$$

Let N be chosen such that the estimates (1.18) and (1.19) are valid for $j \geq N$. We will derive a bound for $j<N$. Assume that R is so large that (3.3) holds for λ equal to λ_j and $\tilde{\lambda}_j$ for $j=0, 1, \ldots, N-1$. To estimate the functions w_j in eq. (1.16) we note that the solutions φ_1 and φ_2 of the homogeneous eq. (1.3) satisfy

$$|\varphi(x)| \leq \exp(R\tilde{r}(x)), \quad |\varphi'(x)| \leq R\exp(R\tilde{r}(x)).$$

From these estimates and inequality (3.1) we conclude by using that $1 \leq \tilde{r}'(x)$ and integrating with respect to y that

$$|w_j(x)| \leq \max(1,|\tilde{h}|)R^{-1}\exp(3R\tilde{r}(x)) \tag{3.4}$$

$$|w_j'(x)| \leq \max(1,|\tilde{h}|)\exp(3R\tilde{r}(x)). \tag{3.5}$$

Our next step is to bound the denominator $\omega'(\lambda_j) = \pm\|u_j\|^2$ in eq. (1.17) from below. This is accomplished by showing that the function u_j cannot oscillate arbitrarily quickly. Let $r(x) = \int_0^x \max(1, |q|)\,dt$. Since u_j are the solutions of (1.3)–(1.4) with $\tilde{h}$ and $\tilde{q}$ replaced by h and q we see that $u_j(x)$ also satisfy the inequalities (3.1) and (3.2) with $\tilde{h}$ and $\tilde{r}$ replaced by h and r. Let

$$C = \max(1,|h|)R\exp(Rr(\pi/2)).$$

Thus $|u'| \leq C$ for all x in $[0, \pi/2]$ and since $C > \pi/2$ we conclude that $u(x) \geq 1 - Cx$ for all $x < 1/C$. By squaring and integrating with respect to x we find that $\|u_j\|^2 \geq 2/(3C)$. Thus

$$\frac{1}{|\omega'(\lambda_j)|} \leq \tfrac{3}{2}\max(1,|h|)R\exp(Rr(\pi/2)). \tag{3.6}$$

We can now estimate $\tilde{y}_j$ and $\tilde{y}'_j$ for $j<N$. By combining eq. (1.17) with the inequalities (3.4), (3.5) and (3.6) we see that

$$\|\tilde{y}_j\|_\infty \leqslant 6|\lambda_j-\tilde{\lambda}_j| \max(1,|h|)\max(1,|\tilde{h}|)\exp(Rr(\pi/2)+3R\tilde{r}(\pi))$$
$$\|\tilde{y}'_j\|_\infty \leqslant \text{as above}\cdot R.$$

Since $\sum(\tilde{y}_ju_j)'$ is an even function around $\pi/2$ it follows that it is sufficient to estimate the sum in the interval $0\leqslant x\leqslant\pi/2$. We have already seen that u_j and u'_j are bounded by the right-hand side of (3.1) and (3.2) with $\tilde{h}$ and $\tilde{r}$ replaced by h and r. By using the estimates for $\tilde{y}_j$ and $\tilde{y}'_j$ we find that

$$\left\|\sum_0^{N-1}(\tilde{y}_ju_j)'\right\|_\infty \leqslant 12\sum_0^{N-1}|\lambda_j-\tilde{\lambda}_j|\max(1,|h|)^2\max(1,|\tilde{h}|)R\exp(2Rr(\pi/2)+3R\tilde{r}(\pi)). \tag{3.7}$$

The corresponding estimate for $\frac{1}{2}\sum\tilde{y}_j(0)$ follows from our bound for $\tilde{y}_j$. Since $r(x)\geqslant x$ and $R\geqslant 1$ we see that if h and $\tilde{h}$ are zero and q and $\tilde{q}$ vanish identically then the constant in (3.7) is at least $12\exp(4\pi)\doteq 3.4\cdot 10^6$. This is much larger than the constant 4520 in (1.21), and we have therefore proved our theorem in principle.

To complete the proof we must give a specific choice of N and estimate R in terms of h, $\tilde{h}$, q and $\tilde{q}$. Since the bound for $h-\tilde{h}$ is smaller than the bound for $q-\tilde{q}$, we will restrict our attention to the latter. We will first give a lower bound for the eigenvalues λ_0 and $\tilde{\lambda}_0$. Let u be the eigenfunction corresponding to λ_0. By multiplying eq. (1.1) with u and integrating with respect to x we find

$$\lambda_0=\frac{\int_0^\pi(u'^2+qu^2)\,dx+h[u^2(\pi)+u^2(0)]}{\int_0^\pi u^2\,dx}. \tag{3.8}$$

Since $\pi u^2(\pi)=\int_0^\pi(xu^2)'\,dx$ we see by using Schwarz inequality that

$$u^2(\pi)\leqslant\frac{1}{\pi}\|u\|^2+2\|u\|\,\|u'\|. \tag{3.9}$$

The same inequality holds with $u^2(\pi)$ replaced by $u^2(0)$. By writing q as $(\int_0^x q)'$ a.e. and integrating $\int qu^2$ by parts we obtain

$$\left|\int_0^\pi qu^2\,dx\right|\leqslant\|q\|_1(u^2(\pi)+2\|u\|\cdot\|u'\|) \tag{3.10}$$

where $\|q\|_1 = \int_0^\pi |q|\,dx$. We can now combine (3.9) and (3.10) and find by using eq. (3.8) that

$$\lambda_0 \|u\|^2 \geq \|u'\|^2 - \frac{1}{\pi}(2|h| + \|q\|_1)\|u\|^2 - (2|h| + 2\|q\|_1)\, 2\|u\|\,\|u'\|.$$

This shows that λ_0 is larger than $-S/(2\pi) - 4S^2/9$. Here $S = 6M$ where M is the maximum of $|h|$, $|\tilde{h}|$, $\|q\|_1$ and $\|\tilde{q}\|_1$. The lower bound for λ_0 is equally valid for $\tilde{\lambda}_0$. Consequently, we have for all the negative eigenvalues λ_j and $\tilde{\lambda}_j$ in the two spectra that

$$\sqrt{1 + |\lambda|} \leq 1 + S. \tag{3.11}$$

We will now investigate the positive eigenvalues. In particular we are interested in finding the integer N which separates the lower eigenvalues from the upper eigenvalues. Let N be chosen such that $S + 0.22 < N \leq S + 1.22$. We will investigate the Wronskian $\omega(\lambda)$ in the interval $N - 0.22 \leq \sqrt{\lambda} \leq N + 0.22$. Since $S \geq 0$ we see that $\sqrt{\lambda}$ is larger than $6|h|$, $6\|q\|_1$ and $1/\pi$. We can therefore use our estimate $\|u\|_\infty \leq 7/5$ from the proof of Theorem 1 and conclude from the definition (1.12) and eq. (1.9) that

$$\omega(\lambda)/\sqrt{\lambda} = \sin \sqrt{\lambda}\pi + \theta(\sqrt{\lambda})$$

where $|\theta| \leq 19/30$ for all $\sqrt{\lambda} \geq S$. Thus for each integer $j \geq N$ there exists a $\sqrt{\lambda_j}$ in the interval $(j - 0.22,\ j + 0.22)$ such that $\omega(\lambda_j) = 0$. There can only be one root in each interval because the eigenvalues are simple and depend continuously on the potential and the boundary conditions. If λ_{N-1} is positive then $\sqrt{\lambda_{N-1}}$ is certainly less than $S + 1.44$, but a closer look at the graph of $\sin \sqrt{\lambda}\pi$ reveals that $\sqrt{\lambda_{N-1}}$ is actually less than $S + 0.44$. Thus

$$\sqrt{1 + |\lambda_{N-1}|} \leq 1.1 + S. \tag{3.12}$$

By combining (3.11) and (3.12) we are lead to a choice of the constant R in (3.3) namely

$$R = 1.1 + 6 \max(|h|;\ |\tilde{h}|,\ \|q\|_1,\ \|\tilde{q}\|_1).$$

It is now straightforward to derive the estimates given in the theorem. We need only observe that $\max(1, x) \leq \exp(x/e)$, that $R \leq \exp(0.1 + 6M)$ and that $\tilde{r}(\pi) \leq \pi + \|\tilde{q}\|_1$ and use the inequality (3.7). This completes the proof.

We remark that Corollary 1 can be obtained from Theorem 2 by inspection, but we cannot derive Corollary 2 in this manner. It should be emphasized that Theorem 2 does not show that if h and q are known then we can bound $h - \tilde{h}$ and $q - \tilde{q}$ in terms of $\sum |\lambda_j - \tilde{\lambda}_j|$. This is only true if $\sum |\lambda_j - \tilde{\lambda}_j|$ is sufficiently small and will be proved in Section 6.

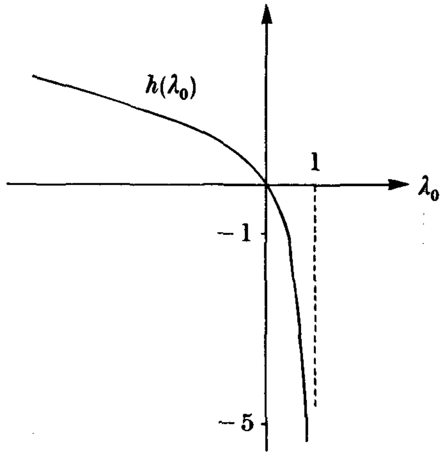

Figure 3. Dependence of the constant h in the boundary conditions on the eigenvalue λ_0

The estimates (3.1) and (3.2) can be sharpened considerably provided the potential is continuous, see Titchmarsh [21, p. 6]. Such an improvement will render the term M^2 in the exponent superfluous. One might therefore question whether there exists a constant independent of M as in (1.21) such that Theorem 2 is still valid. We will now show that this cannot be the case.

Let $h=0$ and $\tilde{q}\equiv 0$. We will only perturb the lowest eigenvalue λ_0 and assume that $\lambda_j=j^2$ for $j\geqslant 1$. By using the algorithm presented in Section 4 we find the graphs in Fig. 3 and 4. It follows from eq. (1.7) that

$$h(\lambda_0)=\begin{cases}\sqrt{|\lambda_0|}\tanh\left(\sqrt{|\lambda_0|}\,\frac{\pi}{2}\right), & \lambda_0\leqslant 0\\ -\sqrt{\lambda_0}\tan\left(\sqrt{\lambda_0}\,\frac{\pi}{2}\right), & 0<\lambda_0<1.\end{cases}$$

This shows that $h\to-\infty$ as $\lambda_0\to 1$ and the bound in Theorem 2 must therefore depend on h. This conclusion can also be derived from Fig. 1. Fig. 4 shows that the potential q becomes large near 0 and π as $\lambda_0\to 1$. Since $\lambda_j=j^2$ for all $j\geqslant 1$ we conclude from Corollary 3 that $4h=-\int_0^\pi q\,dx$ and since $h\to-\infty$ we find that $\|q\|_1\to\infty$. The bound in Theorem 2 must therefore depend on $\|q\|_1$ as well.

4. Hochstadt's algorithm

In this section we will derive an algorithm which is well-suited for solving the inverse Sturm–Liouville problem numerically. It is based on a very clever idea due to Hochstadt

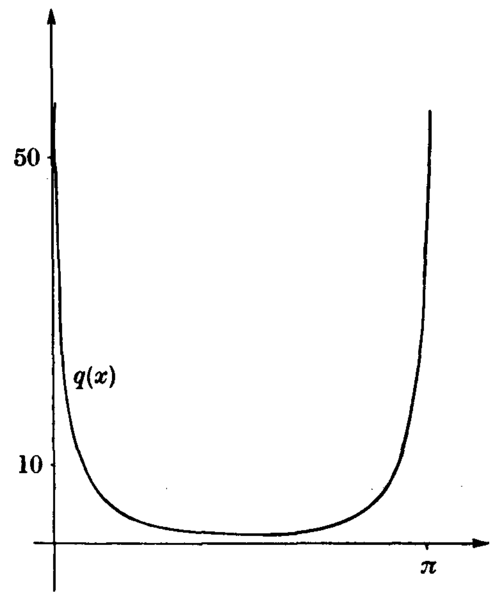

Figure 4. Potential corresponding to $\lambda_0 = 0.8$

[9]. The fundamental observation is that by using Theorem 1, the constructive problem can be reduced to solving a system of ordinary nonlinear differential equations. Hochstadt formulated his result for two spectra, but the specialization to symmetric potentials and one spectrum is immediate.

Assume that $\tilde{h}$ and $\tilde{q}(x)$ are given and let $\tilde{\lambda}_j$ be the eigenvalues of eq. (1.2). We want to find a potential $q(x)$ and a constant h in the boundary conditions such that the eigenvalues of (1.1) are $\lambda_0, \ldots, \lambda_n$ and $\lambda_j = \tilde{\lambda}_j$ for $j > n$. Thus we perturb only a finite number of eigenvalues and let Λ_0 be the corresponding index set. From eq. (1.8) we see that q can be expressed in terms of $\tilde{q}$, $\tilde{y}_j$ and u_j. Because all the eigenvalues λ_j are given we can in principle compute the denominator $\omega'(\lambda_j)$ in eq. (1.6). A more elegant method is given below. To find $\tilde{y}_j$ we solve eq. (1.3) with initial conditions (1.4) and (1.5). The constant h in the boundary conditions can now be obtained by using eq. (1.7), i.e.

$$h = \tilde{h} + \tfrac{1}{2} \sum_{\Lambda_0} \tilde{y}_j(0).$$

To compute the potential $q(x)$ we need in addition the eigenfunctions u_j. These are naturally unknown, but can be obtained by solving the following system of nonlinear differential equations

$$u_i'' + [\lambda_i - \tilde{q}(x) - \sum_{\Lambda_0} (\tilde{y}_j u_j)'] u_i = 0 \tag{4.1}$$

$$u_i(0) = 1, \quad u_i'(0) = h \tag{4.2}$$

for all i in Λ_0. Here we have used eq. (1.8). It is always possible to use this technique, the question is whether it will give the solution of the inverse Sturm–Liouville problem. For example, what guarantees that the asserted eigenfunctions u_i satisfy the right-hand boundary condition in eq. (1.2)?

These questions are not just academic. To realize this we note that Hochstadt bases his algorithm on a representation theorem in which $h=\tilde{h}$, see [9]. Thus Hochstadt uses $\tilde{h}$ in eq. (4.2) instead of h. Numerical experiments by the author show that for this choice the computed potentials are not in general symmetric and the functions u_i do not satisfy the right-hand boundary condition. Moreover, if the perturbation of the eigenvalues is sufficiently strong then the solution of (4.1) may fail to exist in the whole interval. Hochstadt's version is therefore a recovering procedure and our modification is crucial for the success of the algorithm.

The above outline is somewhat inconvenient from a numerical point of view. For example, we need not find $\tilde{v}_j$ since $\tilde{v}_j(x)=\tilde{u}_j(\pi-x)$ for all x. To evaluate $\omega'(\lambda_j)$ we note that

$$\omega(\lambda)=a\prod_{\Lambda_0}(\lambda-\lambda_j)\prod_{\Lambda}\left(1-\frac{\lambda}{\bar{\lambda}_j}\right). \tag{4.3}$$

This follows from Hadamard factorization theorem. By replacing a and λ_j by $\tilde{a}$ and $\tilde{\lambda}_j$ we obtain the corresponding factorization of $\tilde{\omega}$, where $\tilde{\omega}$ is the Wronskian for eq. (1.2). Here we have assumed that $\tilde{\lambda}_j$ are different from zero for all j in Λ. Otherwise the factor $1-\lambda/0$ must be replaced by λ. Consequently

$$\frac{\omega(\lambda)}{\tilde{\omega}(\lambda)}=\frac{a}{\tilde{a}}\prod_{\Lambda_0}\frac{\lambda-\lambda_j}{\lambda-\tilde{\lambda}_j}. \tag{4.4}$$

To determine the constant a we use that $\lambda_j=\tilde{\lambda}_j$ for all j sufficiently large and that $(-1)^j\omega'(\lambda_j)\to\pi/2$ as $j\to\infty$ and similarly for $\tilde{\omega}$. Thus $a=\tilde{a}$. By differentiating eq. (4.4) we find that if λ_j is not an eigenvalue of eq. (1.2) then

$$\omega'(\lambda_j)=\prod_{i\neq j}\frac{\lambda_j-\lambda_i}{\lambda_j-\tilde{\lambda}_i}\,\frac{\tilde{\omega}(\lambda_j)-\tilde{\omega}(\tilde{\lambda}_j)}{\lambda_j-\tilde{\lambda}_j}. \tag{4.5}$$

If $\lambda_j\to\tilde{\lambda}_j$ then the last term must be replaced by $\tilde{\omega}'(\tilde{\lambda}_j)$. Let z_j be the eigenfunction of (1.2) corresponding to $\tilde{\lambda}_j$. We will consider the function $w_j=(\tilde{u}_j-z_j)/(\lambda_j-\tilde{\lambda}_j)$. Since $\tilde{\omega}(\lambda)=-\tilde{h}\tilde{u}(\pi)-\tilde{u}'(\pi)$ we conclude that the last quotient in (4.5) is equal to $-\tilde{h}w_j(\pi)-w_j'(\pi)$ where w_j satisfies the differential equation

$$w_j''+(\tilde{\lambda}_j-\tilde{q})w_j=-\tilde{u}_j$$

$$w_j(0)=w_j'(0)=0$$

see also Section 1. If $\tilde{\lambda}_j \to \lambda_k$ with $k \neq j$ then we replace $\tilde{\lambda}_j$ and z_j in the above arguments by $\tilde{\lambda}_k$ and z_k.

We can now give the complete recipe for solving the inverse Sturm–Liouville problem with symmetric potentials.

Step 1°: For each j in Λ_0 determine a k in Λ_0 such that

$$|\lambda_j - \tilde{\lambda}_k| = \min_{i \in \Lambda_0} |\lambda_j - \tilde{\lambda}_i|$$

Step 2°: For each j in Λ_0 solve

$$\begin{bmatrix} \tilde{u}_j \\ \tilde{u}_j' \\ w_j \\ w_j' \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \tilde{q} - \lambda_j & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & \tilde{q} - \tilde{\lambda}_k & 0 \end{bmatrix} \begin{bmatrix} \tilde{u}_j \\ \tilde{u}_j' \\ w_j \\ w_j' \end{bmatrix}$$

$$\begin{bmatrix} \tilde{u}_j \\ \tilde{u}_j' \\ w_j \\ w_j' \end{bmatrix}_{x=0} = \begin{bmatrix} 1 \\ \tilde{h} \\ 0 \\ 0 \end{bmatrix}$$

Step 3°: For each j in Λ_0 compute

$$\omega'(\lambda_j) = \frac{\prod_{i \neq j} (\lambda_j - \lambda_i)}{\prod_{i \neq k} (\lambda_j - \tilde{\lambda}_i)} [-\tilde{h} w_j(\pi) - w_j'(\pi)]$$

Step 4°: Set

$$h = \tilde{h} + \sum_{\Lambda_0} (\tilde{u}_j(\pi) - (-1)^j)/\omega'(\lambda_j)$$

Step 5°: Solve the system of differential equations

$$\begin{bmatrix} \tilde{y}_j \\ \tilde{y}_j' \\ u_j \\ u_j' \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \tilde{q} - \lambda_j & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \tilde{q} + \sum_{\Lambda_0} (\tilde{y}_i' u_i + \tilde{y}_i u_i') - \lambda_j & 0 \end{bmatrix} \begin{bmatrix} \tilde{y}_j \\ \tilde{y}_j' \\ u_j \\ u_j' \end{bmatrix}$$

$$\begin{bmatrix} \tilde{y}_j \\ \tilde{y}_j' \\ u_j \\ u_j' \end{bmatrix}_{x=0} = \begin{bmatrix} 2(\tilde{u}_j(\pi) - (-1)^j)/\omega'(\lambda_j) \\ -2(\tilde{u}_j'(\pi) + (-1)^j \tilde{h})/\omega'(\lambda_j) \\ 1 \\ h \end{bmatrix}$$

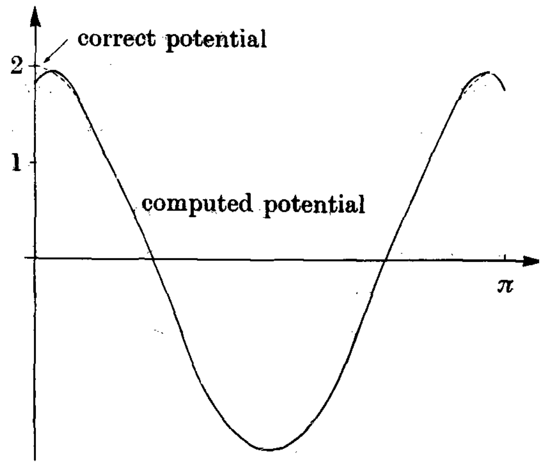

Figure 5. Reconstruction of the Mathieu equation from five eigenvalues

Step 6°: Set

$$q = \tilde{q} + \sum_{\Lambda_0} (\tilde{y}_j' u_j + \tilde{y}_j u_j').$$

It should be pointed out that it is not really necessary to compute $\tilde{y}_j$ in Step 5° since it can be expressed in terms of $\tilde{u}_j$ already found in Step 2°. However, $\tilde{y}_j$ and $\tilde{y}_j'$ are needed in eq. (4.1) anyway and it is easier to recompute them than storing $\tilde{u}_j$ and $\tilde{u}_j'$ for all j in Λ_0.

To illustrate the power of the method we have tried to reconstruct the Mathieu equation from its first five eigenvalues. The result is given in Figure 5. The correct potential is $2 \cos(2x)$ and the comparison potential is identically zero.

We will now compare Hochstadt's algorithm with the Gel'fand–Levitan technique [6]. The numerical solution of the Gel'fand–Levitan equation requires the solution of a sequence of linear systems of equations followed by a numerical differentiation. Assume that the potential is wanted at N points in the interval $[0, \pi]$. In the most obvious implementation of the Gel'fand–Levitan technique all matrices are full and the number of operations grows like $N^4/12$, but a less obvious approach can reduce the work to $N^3/3$. The cost of solving the integral equations becomes critical long before the loss of accuracy in the numerical differentiation becomes a problem. On the other hand, the cost of Hochstadt's algorithm is $10Nn$. Here n is the number of perturbed eigenvalues and we have assumed that the differential equations are solved by a method which requires two function-evaluations per step. Thus if only a few eigenvalues are perturbed and the potential is wanted at many points, then Hochstadt's algorithm is the most economical of the two.

5. Existence of solutions

It is easier to prove that the solution of the inverse Sturm–Liouville problem is unique than giving necessary and sufficient conditions for its existence. The first existence result is due to Borg [3, p. 71]. He proved that if the boundary conditions are fixed and the reduced spectrum is slightly perturbed (in l^2), then there exists a symmetric potential which gives rise to the perturbed eigenvalues. Later existence results have been based on the technique due to Gel'fand and Levitan [6]. They gave necessary and sufficient conditions (with a slight gap) for a given stepfunction to be the spectral function of a regular Sturm–Liouville operator. The gap was closed by Krein [13]. The conditions are formulated as differentiability properties of a certain function Φ, see eq. (5.2) below. The theory has been further extended to non-self-adjoint operators by Marčenko [18]. By using Marčenko's technique Žikov [22] succeeded in formulating the necessary and sufficient conditions in terms of the eigenvalues and the normalizing constants separately. However, Žikov admits potentials which fall outside the class studied in Theorem 1, and for this reason we cannot use his otherwise very convenient results. Finally, it should be mentioned that Žikov obtains similar necessary and sufficient conditions for two spectra to give rise to a potential, thus extending the sufficient conditions due to Levitan [15]. In this section we will show that the inverse Sturm–Liouville problem under consideration does have a solution. The modified version of Hochstadt's algorithm will therefore always be successful.

LEMMA 1. *Consider the eigenvalue problem* (1.2) *where the function* $\tilde{q}$ *is integrable and satisfies* $\tilde{q}(x)=\tilde{q}(\pi-x)$ *almost everywhere in* $[0,\pi]$. *Let* $\tilde{\lambda}_0, \tilde{\lambda}_1, \ldots$ *be the eigenvalues of* (1.2). *Let* $\lambda_0<\lambda_1<\ldots$ *be given and assume that* $\lambda_j=\tilde{\lambda}_j$ *for all* $j\geq N$. *Then there exist a constant* h *and an integrable function* $q(x)$, *which satisfies* $q(x)=q(\pi-x)$ *almost everywhere, such that* λ_j *are the eigenvalues of* (1.1).

Proof. Let h and H be two real constants and let $q(x)$ be an integrable, but not necessarily symmetric function. We will consider the eigenvalue problem

$$-u''+q(x)u=\lambda u \tag{5.1}$$
$$hu(0)-u'(0)=0, \quad Hu(\pi)+u'(\pi)=0.$$

Let λ_n and u_n be the eigenvalues and eigenfunctions of (5.1) with $u_n(0)=1$ and let $\varrho_n=\int_0^\pi u_n^2\,dx$ be the normalizing constants. Assume now that two sequences $\{\lambda_n\}$ and $\{\varrho_n\}$ are given. From a theorem by Krein [13] follows that λ_n and ϱ_n are the eigenvalues and the normalizing constants of a boundary value problem of form (5.1) if and only if

$$\Phi(x) = \sum\left(\frac{\cos\sqrt{\lambda_n}\,x}{\varrho_n\lambda_n} - \frac{2}{\pi}\,\frac{\cos nx}{n^2}\right) \tag{5.2}$$

has two absolutely continuous derivatives, see also Gel'fand and Levitan [6, § 11], Levitan [16, p. 103] and Žikov [22]. Here the sum is taken over those n for which $\lambda_n > 0$. To prove Lemma 1 we will first show that $\Phi(x)$ is twice differentiable. Secondly, we will prove that the constants h and H in eq. (5.1) are equal and that the potential is symmetric.

Let λ_j be the perturbed eigenvalues of eq. (1.2). Since $\lambda_j = \tilde{\lambda}_j$ for $j \geq N$ we conclude from the proof of Weierstrass's factorization theorem, see [20, p. 246], that the function $\omega(\lambda)$ defined by eq. (4.3) is an entire function. Here we let $a = \tilde{a}$ where $\tilde{a}$ is the constant in the corresponding factorization of the Wronskian $\tilde{\omega}(\lambda)$ for eq. (1.2). Thus eq. (4.4) holds and

$$\frac{\omega'(\lambda_j)}{\tilde{\omega}'(\lambda_j)} = 1 + \frac{\sum_{\Lambda_*}(\tilde{\lambda}_i - \lambda_i)}{\lambda_j} + O\left(\frac{1}{\lambda_j^2}\right). \tag{5.3}$$

We have seen in the proof of Theorem 1 that if $\omega(\lambda)$ is the Wronskian of eq. (5.1) with a symmetric potential and $h = H$ then $(-1)^j\omega'(\lambda_j) = \int_0^\pi u_j^2\,dx$. It is therefore natural to choose $\varrho_j = (-1)^j\omega'(\lambda_j)$. From the asymptotic expansion (1.10) follows that $(-1)^j\tilde{\omega}'(\tilde{\lambda}_j) = \pi/2 + O(1/j)$. Since the λ_j's are real and distinct we conclude by using eq. (5.3) that $\varrho_j > 0$ for all j and that

$$\varrho_j = \frac{\pi}{2} + O\left(\frac{1}{j}\right). \tag{5.4}$$

We can now show that the function Φ in eq. (5.2) is twice differentiable. Let $\tilde{\lambda}_n$ and $\tilde{\varrho}_n$ be the eigenvalues and normalizing constants of (1.2). We define the function $\tilde{\Phi}(x)$ by replacing λ_n and ϱ_n in eq. (5.2) with $\tilde{\lambda}_n$ and $\tilde{\varrho}_n$. Since $\tilde{q}$ is integrable it follows from Krein's theorem that $\tilde{\Phi}$ has two absolutely continuous derivatives. It is therefore sufficient to consider the difference $\Phi - \tilde{\Phi}$ and to show that

$$F(x) = \sum_{n=N}^{\infty}\left(\frac{\cos\sqrt{\lambda_n}\,x}{\varrho_n\lambda_n} - \frac{\cos\sqrt{\lambda_n}\,x}{\tilde{\varrho}_n\lambda_n}\right)$$

has two absolutely continuous derivatives. Here we have used that $\lambda_n = \tilde{\lambda}_n$ for all $n \geq N$. From equations (5.3) and (5.4) follows that

$$F(x) = A\sum\frac{\cos\sqrt{\lambda_n}\,x}{\lambda_n^2} + \sum O\left(\frac{1}{n}\right)\frac{\cos\sqrt{\lambda_n}\,x}{\lambda_n^2} \tag{5.5}$$

where $A=(2/\pi)\sum_{\Lambda_0}(\lambda_j-\tilde{\lambda}_j)$. The last sum in (5.5) is three times differentiable and the third derivative continuous. The second derivative of the first sum in (5.5) is absolutely convergent and can be written in the form

$$\sum\int_0^x \frac{\sin\sqrt{\lambda_n}\,t}{\sqrt{\lambda_n}}\,dt-\sum\frac{1}{\lambda_n}.$$

It follows from eq. (2.2) and (2.3) that $\sqrt{\lambda_n}=n+O(1/n)$. Since the partial sums of $n^{-1}\sin nx$ are bounded we can use Lebesgue theorem on dominated convergence to interchange the order of summation and integration. This shows that F'' is absolutely continuous. We can therefore conclude from Krein's theorem that there exist two constants h and H and an integrable function $q(x)$ such that λ_n and ϱ_n are the eigenvalues and the normalizing constants of eq. (5.1).

To complete the proof we must show that $h=H$ and $q(x)=q(\pi-x)$ a.e. Let ν be the Wronskian of equation (5.1), i.e. $\nu(\lambda)=-Hu(\pi)-u'(\pi)$. Here $u(x)$ satisfies the initial conditions $u=1$ and $u'=h$ at $x=0$. Since ν is an entire function of order $\frac{1}{2}$ it is completely determined except for a multiplicative constant by its roots. We will factor ν as in (4.3) with a replaced by α. We will show that $\alpha=a$ and thus $\nu(\lambda)=\omega(\lambda)$. Let $v=v(x,\lambda)$ be the solution of the differential equation in (5.1) and satisfy the initial condition $v=1$ and $v'=-H$ at $x=\pi$. If $\lambda=\lambda_j$ then $v_j(x)=k_ju_j(x)$ in the whole interval. By using the asymptotic expansion (1.10) with $x=\pi$ we conclude that $k_j=(-1)^j+O(1/j)$. We note now that the derivation of equation (1.13) is independent of the symmetry of the potentials and that the dependence on $h=H$ is not essential. Thus if $q=\tilde{q}$ and $f=u_j$ then eq. (1.13) implies that

$$\frac{k_j}{\nu'(\lambda_j)}\int_0^\pi u_j^2\,dx=1. \tag{5.6}$$

From the asymptotic expansion (1.10) follows that $\int_0^\pi u_j^2\,dx\to\pi/2$ and we can therefore conclude that $(-1)^j\nu'(\lambda_j)\to\pi/2$ as $j\to\infty$. But by construction this holds for $\omega(\lambda)$ as well and consequently $\alpha=a=\tilde{a}$ and $\nu(\lambda)=\omega(\lambda)$. Since the normalizing constants ϱ_j are equal to $(-1)^j\omega'(\lambda_j)$ we infer from eq. (5.6) that $k_j=(-1)^j$.

We observe now that the derivation of eq. (1.14) only depends on the symmetry of the potential via the requirement $k_j=(-1)^j$. We may therefore also use it in this case and by replacing u_0 by u_m where m is not in Λ_0 we get

$$u_m=\tilde{u}_m+\tfrac{1}{2}\sum_{\Lambda_0}\tilde{y}_j\int_0^x u_ju_m\,dt \tag{5.7}$$

$$u_m'=\tilde{u}_m'+\tfrac{1}{2}\sum_{\Lambda_0}\tilde{y}_ju_ju_m+\tilde{y}_j'\int_0^x u_ju_m\,dt. \tag{5.8}$$

By setting $x=0$ in eq. (5.8) we arrive at eq. (1.7). Since $\tilde{q}(x)=\tilde{q}(\pi-x)$ we find from definition (1.6) that $\tilde{y}_j(x)=-k_j\tilde{y}_j(\pi-x)$. Thus by setting $x=\pi$ in eq. (5.8) and using the orthogonality of the eigenfunctions we obtain

$$H-\tilde{h}=\tfrac{1}{2}\sum_{\Lambda_0}\tilde{y}_j(0).$$

By comparing this result with eq. (1.7) we conclude finally that $h=H$. We will now show that $q(x)=q(\pi-x)$ a.e. We note first that the difference between the potentials q and $\tilde{q}$ is given by eq. (1.8). This follows by differentiating eq. (5.8) and using equations (1.2), (5.1), and (5.7) as in Section 1. We can therefore find the eigenfunctions u_i by solving the nonlinear system of equations (4.1) with initial conditions (4.2). Let $\zeta_j(x)=v_j(\pi-x)$. Since $\tilde{q}(x)=\tilde{q}(\pi-x)$ and $\tilde{y}_j(x)=-k_j\tilde{y}_j(\pi-x)$ we see by using $v_j=k_ju_j$ and manipulating eq. (4.1) that

$$\zeta_i''+[\lambda_i-\tilde{q}(x)-\sum_{\Lambda_0}(\tilde{y}_j\zeta_j)']\,\zeta_i=0$$

$$\zeta_i(0)=1,\quad \zeta_i'(0)=h.$$

Thus u_i and ζ_i satisfy the same differential equations and have the same initial conditions. We can therefore conclude that $u_i(x)=\zeta_i(x)=v_i(\pi-x)$ for all i in Λ_0. This completes the proof because

$$q(x)-\lambda_i=\frac{u_i''(x)}{u_i(x)}=\frac{v_i''(\pi-x)}{v_i(\pi-x)}=q(\pi-x)-\lambda_i.$$

It should be noted that the perturbation of a finite number of eigenvalues implies that infinitely many of the normalizing constants are perturbed. But this is not important since we do not use the Gel'fand–Levitan technique to solve the inverse eigenvalue problem. We also remark that the comparison spectrum and the normalizing constants need not satisfy the asymptotic requirements used by Levitan [15]. On the other hand, the perturbation is of a very special kind.

It is well known that the smoothness of the solution is closely connected to the number of terms in the asymptotic expansion of the eigenvalues and the normalizing constants. In our problem we find that as long as only a finite number of eigenvalues are perturbed then q and $\tilde{q}$ have the same regularity properties. The reason is that the difference $q-\tilde{q}$ has always one derivative more than $\tilde{q}$.

6. Well-posedness revisited

In the previous section we proved a global existence theorem, in the sense that the perturbation of a finite number of eigenvalues may be arbitrarily large. In this section we will show that if the spectrum corresponding to a symmetric potential is perturbed slightly, then there exists a potential which gives rise to the perturbed spectrum. This is a local result and is similar to Borg's existence theorem, see [3, p. 71]. The main difference lies in the choice of norms and that we obtain a very explicit well-posedness result.

THEOREM 3. *Consider the eigenvalue problem* (1.2) *where* $\tilde{q}$ *is an integrable function and satisfies* $\tilde{q}(x)=\tilde{q}(\pi-x)$ *almost everywhere. Let* $\tilde{\lambda}_j$ *be the eigenvalues and set* $M=\max(|\tilde{h}|, \|\tilde{q}\|_1)$. *Assume that*

$$\sum_{j=0}^{\infty}|\lambda_j-\tilde{\lambda}_j|\leqslant 2\cdot 10^{-10-39M-11M^2} \tag{6.1}$$

Then there exist a constant h *and an integrable function* q *with* $q(x)=q(\pi-x)$ *for almost all* x, *such that* λ_j *are the eigenvalues of* (1.1). *In addition*

$$|h-\tilde{h}|\leqslant 2\cdot 10^{8+38M+11M^2}\sum_{j=0}^{\infty}|\lambda_j-\tilde{\lambda}_j| \tag{6.2}$$

$$\|q-\tilde{q}\|_\infty\leqslant 2\cdot 10^{8+38M+11M^2}\sum_{j=0}^{\infty}|\lambda_j-\tilde{\lambda}_j| \tag{6.3}$$

Remark. It follows from the example in Section 3 that we cannot replace the bound in (6.1) by 1 and still have estimates like (6.2) and (6.3). The bounds are very pessimistic, and in practice one should investigate the sensitivity of the solution by using an a posteriori perturbation analysis. The proof below is based on the existence result for the perturbation of finitely many eigenvalues and the well-posedness result in Theorem 2.

Proof. We will first consider the case in which $\lambda_j=\tilde{\lambda}_j$ for all j sufficiently large. According to Lemma 1 there exist a constant h and an integrable function $q(x)$ such that λ_j are the eigenvalues of (1.1). Let $\delta=\max(|h-\tilde{h}|), \pi\|q-\tilde{q}\|_\infty)$. From Theorem 2 follows that

$$\begin{aligned}\delta &\leqslant 15\pi\, 10^{6+38(M+\delta)+11(M+\delta)^2}\sum|\lambda_j-\tilde{\lambda}_j| \\ &= e^{a+b\delta+c\delta^2}\varepsilon.\end{aligned}$$

The question is now: what is the largest value of $\varepsilon=\sum|\lambda_j-\tilde{\lambda}_j|$ for which we can estimate δ in terms of ε? Since $\varepsilon\exp(a+b\delta+c\delta^2)$ is a convex function of δ we see that it is enough to find the value of δ for which $\varepsilon\exp(a+b\delta+c\delta^2)$ is equal to δ and its derivative is equal to 1. These conditions lead to

$$\delta=\frac{b}{4c}\left[\sqrt{1+\frac{8c}{b^2}}-1\right] \tag{6.4}$$

$$\varepsilon=\delta/e^{a+b\delta+c\delta^2}. \tag{6.5}$$

Let $z = 8c/b^2$. From the explicit values of b and c follows that $0 < z < 0.03$. By using the mean value theorem in eq. (6.4) we see that $\delta = b^{-1}(1+\theta z)^{-1/2}$ where $0 < \theta < 1$ and conclude that $\delta > 0.98/b$. We also note that $b\delta + c\delta^2$ is equal to $\frac{1}{2} + (\sqrt{1+z} - 1)/z$ and thus less than 1. By combining these results we find from eq. (6.5) that if

$$\sum |\lambda_j - \tilde{\lambda}_j| < \frac{0.98}{b \cdot e^{a+1}}$$

then we can estimate δ by $e^{a+1} \sum |\lambda_j - \tilde{\lambda}_j|$. This argument is only valid if the solution of the inverse problem depends continuously on the perturbed eigenvalues, but this is a direct consequence of the algorithm in Section 4. Observe here that we cannot use Theorem 2 unless we have an a priori bound for $|h|$ and $\|q\|_1$. To obtain the constants in (6.1), (6.2) and (6.3) we go back to base 10 and use elementary estimates.

Assume now that $\{\lambda_j\}$ are given and satisfy the inequality (6.1). Let $q_n(x)$ be the symmetric potential which corresponds to the eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_n, \tilde{\lambda}_{n+1}, \ldots$ and let h_n be the constant in the boundary conditions. By using (6.3) and the triangle inequality we see that

$$\|q_n - q_m\|_\infty \leqslant K \sum_{j=n+1}^{m} |\lambda_j - \tilde{\lambda}_j|$$

where K only depends on $|\tilde{h}|$ and $\|\tilde{q}\|_1$. The same estimate holds for $|h_n - h_m|$. This shows that $\{h_n\}$ and $\{q_n\}$ are Cauchy sequences. There exist therefore a constant h and an integrable function $q(x)$ such that $h_n \to h$ and $q_n \to q$. It follows from the symmetry of q_n that $q(x) = q(\pi - x)$ a.e. Because the eigenvalues depend continuously on the boundary conditions and the potential, see K. Jürgens [11, § 5. Th. 3], we conclude that λ_j are indeed the eigenvalues of eq. (1.1). Finally we note that since all h_n and q_n satisfy the inequalities (6.2) and (6.3) the same inequalities are valid for their limits. This completes the proof.

7. Prescribed boundary conditions

In this section we will show that if the constant h in the boundary conditions is given then the lowest eigenvalue is superfluous. More precisely we will prove that to each perturbation of a finite number of the eigenvalues in the reduced spectrum there corresponds a potential. This is in some respects more general, in others less general than Borg's result, see [3, p. 71]. For example, in our case there is no restriction on the size of the perturbation of the eigenvalues and the boundary conditions can be perturbed as well. On the other hand, Borg permits a small perturbation of all the eigenvalues. Another difference lies in the class of potentials considered. Borg admits all square-integrable functions as comparison potentials, while we restrict our attention to potentials which are essentially bounded. This restriction is imposed by our method of proof, and the result can presumably be extended to all integrable comparison potentials.

LEMMA 2. *Consider the eigenvalue problem* (1.2) *where* $\tilde{q}$ *is essentially bounded and satisfy* $\tilde{q}(x)=\tilde{q}(\pi-x)$ *for almost all* x. *Let* $\tilde{\lambda}_0, \tilde{\lambda}_1, \ldots$ *be the eigenvalues of* (1.2). *Let the constant* h *and* $\lambda_1<\lambda_2<\ldots$ *be given and assume that* $\lambda_j=\tilde{\lambda}_j$ *for all* $j\geqslant N$. *Then there exists an essentially bounded function* $q(x)$, *which satisfies* $q(x)=q(\pi-x)$ *almost everywhere, such that* λ_j *are the eigenvalues of* (1.1).

Proof. Let $\lambda_0<\lambda_1$. From Lemma 1 follows that there exist a constant h and a function q such that λ_j are the eigenvalues of (1.1). Since $\tilde{q}$ is essentially bounded we conclude from Theorem 2 that $\|q\|_\infty$ is finite. To prove Lemma 2 it is sufficient to show that if h^* is given and $\lambda_j=\tilde{\lambda}_j$ for all $j\geqslant 1$ then there exists a $\lambda_0<\tilde{\lambda}_1$ such that

$$h^*-\tilde{h}=(\lambda_0-\tilde{\lambda}_0)\frac{\tilde{u}(\pi,\lambda_0)-1}{\tilde{\omega}(\lambda_0)}. \tag{7.1}$$

Assume namely that there exists a λ_0 in $(-\infty,\tilde{\lambda}_1)$ such that (7.1) is satisfied. According to Lemma 1 there exist a constant h and a function q such that $\lambda_0, \lambda_1, \ldots$ are the eigenvalues of (1.1). We can then use eq. (4.5) and conclude by comparing eq. (1.7) and (7.1) that $h=h^*$. That λ_0 is uniquely determined from h and $\lambda_1, \lambda_2, \ldots$ follows from Corollary 2.

Let $h=h^*$ be given. To show that eq. (7.1) does have a solution we will show that the graph in Fig. 3 is qualitatively correct. Since $\tilde{q}$ is symmetric we see that $u(\pi)\to-1$ and $\tilde{\omega}\to 0$ as $\lambda_0\to\tilde{\lambda}_1$. Thus the right-hand side of (7.1) converges to $-\infty$ when $\lambda_0\to\tilde{\lambda}_1$. We will now show that the right-hand side of (7.1) tends toward $+\infty$ as $\lambda_0\to-\infty$. Since $\tilde{\omega}=-\tilde{h}\tilde{u}-\tilde{u}'$ we can rewrite eq. (7.1) as

$$h-\tilde{h}=(\lambda_0-\tilde{\lambda}_0)\frac{\tilde{u}/\tilde{u}'-1/\tilde{u}'}{-\tilde{h}\tilde{u}/\tilde{u}'-1}. \tag{7.2}$$

Consider the auxiliary equations $\varphi''+(|\lambda_0|\pm\|\tilde{q}\|_\infty)\varphi=0$ with initial conditions $\varphi=1$ and $\varphi'=\tilde{h}$ at $x=0$. If λ_0 is less than $-(3|\tilde{h}|+0.1)^2-\|\tilde{q}\|_\infty$ then it follows from Sturm's comparison theorem that

$$\frac{1}{2\sqrt{|\lambda_0|+\|\tilde{q}\|_\infty}}\leqslant\frac{\tilde{u}}{\tilde{u}'}\leqslant\frac{2}{\sqrt{|\lambda_0|-\|\tilde{q}\|_\infty}}. \tag{7.3}$$

We will now show that $1/\tilde{u}'$ is less than $(4/\pi)/(|\lambda_0|-\|\tilde{q}\|_\infty)$ when $-\lambda_0$ is sufficiently large. By combining this result with (7.2) and (7.3) we see that the right-hand side of (7.1) tends to $+\infty$ as $\lambda_0 \to -\infty$, and the proof will be complete. Let q be a bounded function and let φ be the solution of $\varphi'' = (q-\lambda)\varphi$ with initial conditions $\varphi = 1$ and $\varphi' = h$ at $x = 0$. We will assume that

$$\lambda_0 < -\max\left(\frac{3}{2}h^2, \frac{4}{\pi}|h|\right) - \|q\|_\infty. \tag{7.4}$$

Let $\psi = \varphi'$. To estimate φ and ψ we use the method of successive approximations and define

$$\varphi_{n+1} = 1 + \int_0^x \psi_n \, dt$$

$$\psi_{n+1} = h + \int_0^x (|\lambda| + q)\varphi_n \, dt.$$

If φ_0 and ψ_0 are continuous functions then φ_n and ψ_n converge uniformly to φ and ψ. Specifically we choose

$$\varphi_0 = 1 + hx + \frac{\alpha}{4}x^2$$

$$\psi_0 = h + \frac{\alpha}{2}x$$

where $\alpha = |\lambda| - \|q\|_\infty > 0$. Thus $\varphi_1 = \varphi_0$. It follows from (7.4) that $\varphi_0 > 0$. Since $|\lambda| + q > \alpha$ we can estimate ψ_1 by

$$\psi_1 \geq h + \frac{\alpha x}{2} + \frac{x}{12}(\alpha x + 3h)^2 + \frac{x}{2}\left(\alpha - \frac{3h^2}{2}\right).$$

Thus $\psi_1 \geq \psi_0$ and we find by induction that $\varphi_n \geq \varphi_0$ and $\psi_n \geq \psi_0$ and consequently $\varphi \geq \varphi_0$ and $\psi \geq \psi_0$. By evaluating ψ_0 at $x = \pi$ and using (7.4) we see finally that $1/\psi(\pi)$ is less than $4/(\pi\alpha)$. This completes the proof.

The proof of Lemma 2 provides a numerical method for solving the inverse Sturm–Liouville problem with prescribed boundary conditions. A more practical technique is to find the root λ_0 of the nonlinear equation in Step 4°, see Section 4. Here we note that the calculations in Step 2° need only be performed once for each $j > 0$, and that the modifications in Step 3° are trivial.

The results presented in this section are incomplete. We have not proved a global well-posedness result like Theorem 2 nor a local existence result like Theorem 3. In the case $h = \tilde{h}$ Borg has obtained a local existence and well-posedness result, see [3, p. 71]. The graph in Fig. 3 indicates that if $h - \tilde{h}$ is very large then the lowest eigenvalue λ_0 is quite sensitive to small changes of h.

8. Further results

The theory for the inverse Sturm–Liouville problem with fixed boundary conditions is in many ways simpler than the theory for mixed boundary conditions. The reason is that the potential is uniquely determined by all the eigenvalues and can be constructed from this data, see Borg [3, p. 71]. We will therefore not present any details, but mention that Hochstadt's representation of $q-\tilde{q}$ for the inverse Sturm–Liouville problem with Dirichlet boundary conditions can be extended by using the same method as in the proof of Theorem 1, thus permitting all eigenvalues to differ, see [8]. In this connection it would be of interest to discover the weakest possible assumptions on the eigenvalues under which Theorem 1 is valid. The representation theorem leads naturally to Hochstadt's algorithm, see [9]. Since the boundary conditions are fixed there is no need to modify the original algorithm. Finally we note that Borg has given a local existence and well-posedness result in the case q is a square integrable function and $\sum |\lambda_j-\tilde{\lambda}_j|^2$ is finite and small. If q is only integrable then one can derive a well-posedness result like Theorem 2 but we have not been able to cast the result in an equally explicit form.

Acknowledgements

The author wants to thank Professor Harry Hochstadt for clarifying discussions and for making his results available before publication. The research has been supported by the National Science Foundation under Grant No. MCS75-10363 A01.

References

[1]. Ambarzumian, V., Über eine Frage der Eigenwerttheorie. *Z. Physik.*, 53 (1929), 690–695.

[2]. Barcilon, V., Iterative solution of the inverse Sturm–Liouville problem. *J. Mathematical Phys.*, 15 (1974), 287–298.

[3]. Borg, G., Eine Umkehrung der Sturm–Liouvilleschen Eigenwertaufgabe. *Acta Math.*, 78 (1946), 1–96.

[4]. Coddington, E. & Levinson, N., *Theory of ordinary differential equations.* McGraw-Hill, New York, 1955.

[5]. Gasymov, G. M. & Levitan, B. M., On Sturm–Liouville differential operators with discrete spectra. *Amer. Math. Soc. Transl. Series 2*, 68 (1968), 21–33.

[6]. Gel'fand, I. M. & Levitan, B. M., On the determination of a differential equation from its spectral function. *Amer. Math. Soc. Transl. Series 2*, 1 (1955), 253–304.

[7]. Hald, O. H., Inverse eigenvalue problems for layered media. *Comm. Pure Appl. Math.*, 30 (1977), 69–94.

[8]. Hochstadt. H., The inverse Sturm–Liouville problem. *Comm. Pure Appl. Math.*, 26 (1973), 715–729.

[9]. —— Well-posed inverse spectral problems. *Proc. Nat. Acad. Sci.*, 72 (1975), 2496–2497.

[10]. —— On the well-posedness of the inverse Sturm–Liouville problem. *J. Differential Equations*, 23 (1977), 402–413.
[11]. Jörgens, K., Spectral theory of second-order ordinary differential operators, Lectures delivered at Aarhus Universitet, 1962/63, Aarhus, 1964.
[12]. Krein, M. G., Solution of the inverse Sturm–Liouville problem. *Dokl. Akad. Nauk SSSR (N.S.)*, 76 (1951), 21–24.
[13]. —— On the transfer function of a one-dimensional boundary problem of second order. *Dokl. Akad. Nauk SSSR (N.S.)*, 88 (1953), 405–408.
[14]. Levinson, N., The inverse Sturm–Liouville problem. *Mat. Tidsskr. B.*, (1949), 25–30.
[15]. Levitan, B. M., On the determination of a Sturm–Liouville equation by two spectra. *Amer. Math. Soc. Transl. Series 2*, 68 (1968), 1–20.
[16]. —— Generalized translation operators, Israel Program for Scientific Translations, Jerusalem, 1964.
[17]. Marčenko, V. A., Concerning the theory of a differential operator of the second order. *Dokl. Akad. Nauk SSSR (N.S.)*, 72 (1950), 457–460.
[18]. —— Expansion into eigenfunctions of non-selfadjoint singular differential operators of second order. *Amer. Math. Soc. Transl. Series 2*, 25 (1963), 77–130.
[19]. Neumark, M. A., *Lineare Differential Operatoren*. Akademie-Verlag, Berlin, 1963.
[20]. Titchmarsh, E. C., *The theory of functions*. Oxford University Press, London, 1939.
[21]. —— *Eigenfunction expansions associated with second order differential equations*. Oxford University Press, London, 1946.
[22]. Žikov, V. V., On inverse Sturm–Liouville problems on a finite segment. *Math. USSR—Izv.* 1 (1967), 923–934.
[23]. Zygmund, A., *Trigonometric series, vol. I, 2nd ed.* Cambridge University Press, London, 1959.

Received December 20, 1976

Received in revised form May 5, 1978