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Finite loop spaces are manifolds

by

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1. Introduction and statement of results

One of the motivating questions for surgery theory was whether every finite H-space is homotopy equivalent to a Lie group. This question was answered in the negative by Hilton and Roitberg's discovery of some counterexamples [18]. However, the problem remained whether every finite H-space is homotopy equivalent to a closed, smooth manifold.

This question is still open, but in case the H-space admits a classifying space we have the following theorem.

THEOREM. *Let B be a* CW*-complex and denote by X the loops on B, ΩB. If $H_*(X)=\bigoplus_i H_i(X)$ is a finitely generated abelian group, then X is homotopy equivalent to a compact, smooth, parallelizable manifold.*

This condition on $H_*(X)$ is often called *quasifiniteness.* We will briefly discuss the history of smoothing H-spaces in this introduction.

Suppose given a quasifinite space $X=\Omega B$, B a CW-complex. It follows from [22] that X is finitely dominated, since it is a simple space with finitely generated homology. Finitely dominated means that up to homotopy it is a retract of a finite complex.

Recall [38] that an oriented, n-dimensional Poincaré duality space Y is a finitely dominated space Y, together with a class $[Y]\in H_n(Y,\mathbf{Z})$ such that if $[\widetilde{Y}]$ is the transfer of $[Y]$ to $H_n^{\mathrm{l.f.}}(\widetilde{Y},\mathbf{Z})$ then

$$[\widetilde{Y}]\cap -: H^*_{\mathrm{c.s.}}(\widetilde{Y},\mathbf{Z}) \longrightarrow H_{n-*}(\widetilde{Y},\mathbf{Z})$$

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is an isomorphism from cohomology with compact supports to homology of the universal cover. Obviously all oriented manifolds satisfy this kind of Poincaré duality, so a first step to prove that a loop space is a manifold is to prove that it is a Poincaré duality space.

Since X is finitely dominated, $H^1(X;\mathbf{Z})$ is a free abelian group. Choose a classifying map $X\to T^k$ representing a basis. This map has a section, using the H-space structure, given by the composite $T^k\to X^k\to X$, defining a basis of $\pi_1(X)$ modulo torsion. Denoting the homotopy fiber by X' it is easy to see that $X'\times T^k\to X\times X\to X$ induces isomorphisms on homotopy groups, and hence is a homotopy equivalence. Since X' inherits an H-space structure, it thus suffices to consider X with finite fundamental group for the question of Poincaré duality.

Assuming $\pi_1(X)$ finite, we consider $H^*(\widetilde{X};F)$, where $F=\mathbf{F}_p$ or the rational numbers. The induced product map $\widetilde{X}\times\widetilde{X}\to\widetilde{X}$ induces a Hopf algebra structure on $H^*(\widetilde{X};F)$. It follows from the classification of finitely generated, connected, graded Hopf algebras over a field F [4, Theorem 6.1], [19] that $H^*(\widetilde{X};F)$ is a tensor product of exterior algebras and truncated polynomial algebras. The top dimension is generated by a product of the algebra generators to their maximal nonzero power. Denoting the top dimension by n_p for $F=\mathbf{F}_p$ and by n_0 for $F=\mathbf{Q}$, it follows that cap product with the homology dual of this top-dimensional class induces an isomorphism

$$H^*(\widetilde{X};F)\longrightarrow H_{n_p-*}(\widetilde{X};F).$$

Clearly $n_0\leqslant n_p$. Since $H_1(\widetilde{X};F)$ is 0 so is $H^{n_p-1}(\widetilde{X};F)$. Hence the top class is the reduction mod p of a $\mathbf{Z}$-summand in integral cohomology, which means $n_p\leqslant n_0$, so all n_p's are the same. Denoting this common dimension by n, there must be an integral class $[\widetilde{X}]\in H_n(\widetilde{X};\mathbf{Z})$ so that cap product with the induced element in $H_n(\widetilde{X};F)$ induces an isomorphism

$$H^*(\widetilde{X};F)\longrightarrow H_{n-*}(\widetilde{X};F)$$

for all F, and hence an isomorphism

$$H^*(\widetilde{X};\mathbf{Z})\longrightarrow H_{n-*}(\widetilde{X};\mathbf{Z}).$$

Precisely the same arguments may be applied to $H^*(X;F)$ to obtain dimensions n'_p for X. The spectral sequence of the fibration $\widetilde{X}\to X\to B\pi_1(X)$ is a spectral sequence of Hopf algebras, and analyzing this [5], it is shown that $n'_p=n_p$. Hence all n'_p's are the same, but that is only possible if $H_n(X;\mathbf{Z})=\mathbf{Z}$.

Denoting the generator by $[X]$, we now only need to see that the transfer of $[X]$ is $[\widetilde{X}]$. For this purpose we factor $\widetilde{X}\to X$ as a sequence of covering spaces

$$\widetilde{X}=X_n\longrightarrow X_{n-1}\longrightarrow\ldots\longrightarrow X_0=X,$$

where each $X_i \to X_{i-1}$ is a p-fold covering for some prime p. Each of the X_i's are H-spaces, and the maps are maps of H-spaces, so in mod p cohomology we get an induced map of Hopf algebras. Since in all cases there will be some class going to 0, and the top class is a product of all classes to maximal degree, it follows that the induced map is 0 on the top-dimensional class. In integral homology, transfer followed by the induced map is multiplication by p, but the induced map is 0 in mod p cohomology, so the transfer must send a generator to a generator in the top dimension.

What we have described here is a slight modification of Browder's argument [6], [5] that X is a Poincaré duality space.

The S-dual of a Poincaré duality space is the Thom space of the Spivak normal fibration. Being a Poincaré duality space, X can be written as an $(n-1)$-dimensional complex with one n-cell attached. Browder and Spanier [9] used the map $X \times X \to X \to S^n$ to show that X is self-dual in the sense of S-duality, so stably the Thom space of the Spivak normal fibration is homotopy equivalent to $S^k(X_+)$. This means that the top class in $S^k(X_+)$ is spherical, and a transversality argument, making the map $S^{n+k} \to S^k(X_+)$ transverse to X, sets up a surgery problem

$$\begin{tikzcd} \nu_M \arrow[r] \arrow[d] & \varepsilon \arrow[d] \\ M \arrow[r] & X, \end{tikzcd}$$

where ε denotes the trivial bundle. In the case $\pi_1(X)=0$, Browder now proceeded to show that X is homotopy equivalent to a smooth manifold except possibly in dimensions $4k+2$. In odd dimensions this is because the surgery obstruction groups vanish, and in dimension $4k$ the argument is that the rational cohomology of X is an exterior algebra. Hence the index of X is trivial. Hirzebruch's index formula shows that the index of M is trivial since stably, the normal bundle is trivial, and it follows that the surgery obstruction is trivial, being the difference. In the non-simply-connected case these surgery obstruction groups can be very complicated even for finitely generated abelian groups.

Notation 1.1. We denote by $\mathbf{Z}_p$ and $\mathbf{Q}_p$ the p-adic integers and rationals, respectively. For a nilpotent space X and a commutative ring R, we denote by $L_R X$ the localization of X with respect to $H_*(-;R)$. We abbreviate $L_{\mathbf{Z}/p}$ by L_p and $L_{\mathbf{Z}_{(p)}}$ by $L_{(p)}$.

Denote the $\mathbf{Q}_p$-algebra $H^*(X;\mathbf{Z}_p)\otimes\mathbf{Q}$ by $H^*_{\mathbf{Q}_p}(X)$. On finite nilpotent CW-complexes, this agrees with $H^*(X;\mathbf{Q}_p)$, but whereas the latter functor is not invariant under $\mathbf{Z}/p$-localization, the former one is. In algebra, this is mirrored by the fact that $\hom_{\mathbf{Z}}(\mathbf{Z}_p,\mathbf{Q}_p)\not\cong\mathbf{Q}_p$, but $\hom_{\mathbf{Z}}(\mathbf{Z}_p,\mathbf{Z}_p)\cong\mathbf{Z}_p$.

For a quasifinite loop space X, define the *dimension* $\dim X$ to be the homological dimension, the *rank* $\mathrm{rk}(X)$ to be the number of exterior generators of $H^*(X;\mathbf{Q})$, and the *type* to be the multi-set of dimensions of those generators.

2. Basic considerations and outline of the proof

Let $X=\Omega B$ be a quasifinite connected loop space. Since $H^2(B;\mathbf{Z})$ is free abelian, there is a map $B\to K(\mathbf{Z}^r,2)=BT^r$ inducing an isomorphism on $H^2(-;\mathbf{Z})$. Let B' be the homotopy fiber. We now have a fibration $\Omega B'\to X\to T^r$, and arguing as in the introduction, we have $X\simeq\Omega B'\times T^r$. Since T^r is a smooth parallelizable manifold, we may without loss of generality assume that X has finite fundamental group. We shall do so for the rest of this paper.

Surgery arguments are only valid in dimensions $\geqslant 5$. Clark's theorem [10] states that $H^3(X;\mathbf{Q})\neq 0$, and assuming a finite fundamental group, the only instances of quasifinite loop spaces of dimension $\leqslant 4$ are rational homology 3-spheres. By [7, Theorem 5.2], this means that X is homotopy equivalent to S^3 or $\mathrm{SO}(3)$, which are smooth parallelizable manifolds. We may thus assume that the dimension of X is $\geqslant 5$.

Our method of proof is to construct an orientable fibration $S^1\to X\to Y$ of quasifinite simple spaces. This suffices to prove that X is homotopy equivalent to a finite CW-complex using the theory of finiteness obstructions [37]. The finiteness obstruction is a generalized Euler characteristic defined by considering the chains of the universal cover of X as a $\mathbf{Z}[\pi]$-module chain complex, which turns out to be chain homotopy equivalent to a finite-length chain complex of finitely generated projective $\mathbf{Z}[\pi]$-modules. This allows for the definition of an Euler characteristic in $\widetilde{K}_0(\mathbf{Z}[\pi])$. The vanishing of this obstruction ensures that the space X is of the homotopy type of a finite complex.

To deal with the surgery obstructions, the fibration has to have some additional properties. We will discuss two slightly different concepts, *double* 1-*tori* and *special* 1-*tori*. In both cases the constructions rely on the theory of p-compact groups and on arithmetic square arguments. The construction of a double 1-torus is more elementary and needs less input from the theory of p-compact groups. Special 1-tori, on the other hand, reveal much more internal structure of finite loop spaces. For this reason, we have included both versions of the proof in this paper.

Our general arguments break down in some special cases, namely when the type of X is $(3^k,7^\varepsilon)$, $\varepsilon=0,1$. These cases have to be dealt with by special arguments.

3. The surgery arguments

The arguments in this section are modeled on the arguments in [29] and [30]. In those papers the fourth author (of this paper) studied conditions on $\mathbf{Z}_{(p)}$-local S^1-fibrations making it possible to produce integral S^1-fibrations by gluing. The concept of spaces *admitting a* 1-*torus* and a *special* 1-*torus*, respectively, were used. In this paper, we propose a concept somewhere in between:

Let R be a commutative ring. Recall that a nilpotent space X is called R-finite if $\bigoplus_i H_i(X;R)$ is finitely generated, and R-local if $[Y,X]=0$ for every HR-acyclic space Y. Note that $\mathbf{Z}$-finite is the same as quasifinite. For simple spaces, $\mathbf{Z}$-locality is an empty condition and $\mathbf{Z}/p$-locality is the same as p-completeness.

Definition 3.1. An R-finite R-local space X is called *stably reducible* if there is a stable map from an R-local sphere to X inducing an isomorphism in the top-dimensional homology.

Definition 3.2. Let R be a ring and X be an R-finite, R-local, nilpotent, connected space. We call a fibration of nilpotent spaces $X\xrightarrow{p}Y\to L_RBS^1$ an *R-local* 1-*torus* if it satisfies that

(1) Y is R-finite, R-local and stably reducible;

(2) $\pi_1(p)$ is an isomorphism.

An R-local 1-torus is an *R-local double* 1-*torus* if this fibration is the pullback of a fibration of nilpotent spaces $X\xrightarrow{p}Z\to L_RBS^1\times L_RB\mathbf{Z}/2$ satisfying that

(1) Z is R-finite, R-local and nilpotent;

(2) the induced map $\pi_1(Z)\to\mathbf{Z}/2$ is a split epimorphism.

We call the R-local 1-torus *rationally splitting* if the map p rationally has a retract of the form $h\colon L_{\mathbf{Q}}L_RS^3\to L_{\mathbf{Q}}L_RS^2$, where h is $L_{\mathbf{Q}}L_R$ applied to the Hopf map.

Notice that when $\frac{1}{2}\in R$ there is no difference between a 1-torus and a double 1-torus. When $\frac{1}{2}\notin R$, a double 1-torus leads to a diagram of fibrations

$$\begin{tikzcd}
L_RS^1 \arrow[r] \arrow[d] & L_RS^1 \arrow[r] \arrow[d] & * \arrow[d] \\
L_RS^1\times\mathbf{Z}/2 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & Z \arrow[d] \\
\mathbf{Z}/2 \arrow[r] & Y \arrow[r] & Z,
\end{tikzcd}$$

where $S^1\to X\to Y$ is an orientable fibration of R-finite simple spaces and $\pi_1(Y)\cong\pi_1(Z)\times\mathbf{Z}/2$.

PROPOSITION 3.3. *Let X be a Poincaré duality space of dimension $n \geqslant 5$ admitting an integral double 1-torus. Then X is homotopy equivalent to a compact, stably parallelizable, smooth manifold.*

Proof. Denote the double 1-torus by $X \xrightarrow{p} Y \to Z$. A quasifinite, simple space is finitely dominated by [22]. We first need to deal with the finiteness obstruction $\sigma(X) \in \widetilde{K}_0(\mathbf{Z}\pi_1(X))$. The formula of [32] tells us that $p_*(\sigma(X)) = \chi(S^1)\sigma(Y)$, where $\chi(S^1)$ is the Euler characteristic, and hence $p_*(\sigma(X)) = 0$. But p_* is an isomorphism so $\sigma(X) = 0$, and X is thus homotopy equivalent to a finite complex. We now let E be the total space of the corresponding $\mathbf{D}^2$-fibration. It follows from [17] that Y is a Poincaré duality space, and hence (E, X) is a Poincaré duality pair. We consider the classifying map of the Spivak normal fibration $\nu_E: E \to BG$. We have the equation

$$\nu_E = p^*(\nu_Y) \oplus p^*(p)^{-1}.$$

Now ν_Y is trivial since Y was assumed to be stably reducible. Also p is an S^1-fibration classified by $G(2)$. But $O(2) \subseteq G(2)$ is a homotopy equivalence, so p is fiber homotopy equivalent to an $O(2)$-bundle, actually an S^1-bundle since the fibration was assumed orientable. We thus get a linear reduction ζ of ν_E, and the reduction is trivial when restricted to X since the pullback of an S^1-bundle to its own total space is trivial. The procedure of surgery (see, e.g., Browder [8, p. 38]) sets up a degree-1 normal map

$$(M, \partial M) \xrightarrow{\phi} (E, X), \quad \hat{\phi}: \nu_M \longrightarrow \zeta,$$

with $\pi = \pi_1(E) \cong \pi_1(Y) \cong \pi_1(X)$. However, E is possibly not finite, only finitely dominated. Since $Y \simeq E$, $\sigma(E) = \sigma(Y)$. This situation was studied in [31], where it was shown that the surgery obstruction of $\partial M \to X$ is $\delta([\sigma(E)])$, where δ is the boundary in the Ranicki–Rothenberg exact sequence

$$\ldots \longrightarrow H^{n+1}(\mathbf{Z}/2; \widetilde{K}_0(\mathbf{Z}\pi)) \xrightarrow{\delta} L_n^h(\mathbf{Z}\pi) \longrightarrow L_n^p(\mathbf{Z}\pi) \longrightarrow \ldots.$$

Since Z is an $(n-1)$-dimensional Poincaré duality space, the finiteness obstruction satisfies the formula $\sigma(Z) = (-1)^{n-1}\sigma(Z)^*$. Obviously, $\sigma(Y)$ is just the restriction $\operatorname{Res}\sigma(Z)$. It now follows from [32] or just general covering space theory that $(p_1)_* \operatorname{Res}\sigma(Z) = [(\mathbf{Z} \oplus \mathbf{Z}) \otimes P]$, where P is a projective module representing $\sigma(Z)$, and $\pi_1(Z)$ acts on $\mathbf{Z} \oplus \mathbf{Z}$ through its $\mathbf{Z}/2$-quotient by permuting the two factors. There is an exact sequence of $\pi_1(Z)$-modules $0 \to \mathbf{Z} \to \mathbf{Z} \oplus \mathbf{Z} \to \mathbf{Z}^- \to 0$ with trivial action on the first term, and nontrivial action on the last. This implies that $p_* \operatorname{Res}\sigma(Z) = [P] + [\mathbf{Z}^- \otimes P]$. Let $r: \pi_1(Z) \to \pi_1(Y)$ be a splitting. We then get

$$\begin{aligned} \sigma(Y) &= \operatorname{Res}\sigma(Z) = r_*(p_1)_* \operatorname{Res}\sigma(Z) = 2r_*(\sigma(Z)) \\ &= r_*(\sigma(Z)) + (-1)^{n-1} r_*(\sigma(Z)^*) = r_*(\sigma(Z)) + (-1)^{n-1} r_*(\sigma(Z))^*, \end{aligned}$$

from which it follows that $[\sigma(Y)] = 0$ in $H^{n+1}(\mathbf{Z}/2; \widetilde{K}_0(\mathbf{Z}\pi))$. □

4. The reduction to a $\mathbf{Z}/p$-local problem

PROPOSITION 4.1. *Let X be a quasifinite loop space such that for every p, $L_{(p)}X$ admits a rationally splitting double 1-torus. Then so does X.*

Proof. This was shown for ordinary rationally splitting 1-tori in [29, Proposition 3.2]. The extension to double 1-tori is immediate since $B\mathbf{Z}/2$ is rationally trivial, so if X admits a 1-torus, and $X_{(2)}$ admits a 2-local double 1-torus, then X admits a double 1-torus. □

To reduce the problem further to constructing double 1-tori in $\mathbf{Z}/p$-local loop spaces, we will make use of an easy fact about p-adic squares:

LEMMA 4.2. *Every p-adic rational number is the product of a rational number and the square of a p-adic integral unit.*

Proof. It is enough to show that every p-adic unit $a\in\mathbf{Z}_p^\times$ can be written as a product of a rational integer and the square of a p-adic unit. Since the Legendre symbol is a group homomorphism, it suffices to exhibit an $n\in\mathbf{Z}$ whose image in $\mathbf{Z}_p$ is a unit with no square root. Any lift of a generator of $(\mathbf{Z}/p)^\times$ (or $(\mathbf{Z}/8)^\times$) to $\mathbf{Z}$ will do. □

PROPOSITION 4.3. *Let X be a $\mathbf{Z}_{(p)}$-local, $\mathbf{Z}_{(p)}$-finite loop space such that L_pX admits a $\mathbf{Z}/p$-local rationally splitting double 1-torus. Then X admits a $\mathbf{Z}_{(p)}$-local double 1-torus.*

Proof. Let $L_pX\to Y_p\to L_pBS^1$ be the rationally splitting 1-torus. Let $\{e_1,\dots,e_k\}$ be a basis of the free part of $\pi_3(X)$, thus inducing a basis in $\pi_3(L_{\mathbf{Q}}X)$ and in $\pi_3(L_{\mathbf{Q}}L_pX)$, and elements in $\pi_3(L_pX)$. We also denote the induced elements by $\{e_1,\dots,e_k\}$. Since the fibration is rationally splitting we may produce a diagram

$$\begin{tikzcd} L_{\mathbf{Q}}S^3 \arrow[r] \arrow[d] & L_pL_{\mathbf{Q}}S^3 \arrow[r] \arrow[d] & L_{\mathbf{Q}}L_pX \arrow[d] \\ L_{\mathbf{Q}}S^2 \arrow[r] & L_pL_{\mathbf{Q}}S^2 \arrow[r] & L_{\mathbf{Q}}Y_p. \end{tikzcd}$$

Let a be the image of a generator of $\pi_3(L_{\mathbf{Q}}S^3)$ in $\pi_3(L_{\mathbf{Q}}L_pX)$. We have $a=\alpha_1e_1+\alpha_2e_2+\dots+\alpha_ke_k$ with $\alpha_i\in\mathbf{Q}_p$, and we choose to order the basis so that $\alpha_1\neq 0$.

We first want to show that we can change the problem so that the α_i are rational. The Hopf fibration h admits an automorphism of the form

$$\begin{tikzcd} L_{\mathbf{Q}}L_pS^3 \arrow[r, "u^2"] \arrow[d, "h"] & L_{\mathbf{Q}}L_pS^3 \arrow[d, "h"] \\ L_{\mathbf{Q}}L_pS^2 \arrow[r, "u"] & L_{\mathbf{Q}}L_pS^2 \end{tikzcd}$$

for any $u \in \mathbf{Q}_p^\times$, so by Lemma 4.2 this means that we may assume that α_1 is rational.

We may, however, compose the splitting by any homotopy equivalence that can be lifted to a homotopy equivalence of L_pX. This still gives a rationally splitting 1-torus, and it does not change X in its local genus, see Definition 4.5. We now show that we can find such a homotopy equivalence to change $\alpha_2, \dots, \alpha_k$ to be rational.

Using the H-space structure on X we may produce a rational equivalence $B \xrightarrow{b} X$, where B is a product of p-local odd-dimensional spheres. The lifting problem

$$\begin{tikzcd} & B \arrow[d, "b"] \\ X \arrow[ur, dashed, "c"] \arrow[r, "p^l"'] & X \end{tikzcd}$$

may be solved using obstruction theory for sufficiently large l: First try to lift the identity map, and use the fact that the homotopy groups of the fiber are finite p-groups. Whenever an obstruction is encountered, we may precompose with a map of degree p^i, to kill the obstruction. Given any map $L_pS^3 \to L_pX$ sending e_j to βe_i, expressed in the basis chosen above, we now consider a map of the type

$$L_pX \xrightarrow{(L_pc,1)} L_pB \times L_pX \longrightarrow L_pS^3 \times L_pX \longrightarrow L_pX \times L_pX \longrightarrow L_pX,$$

where $L_pB \to L_pS^3$ is the projection on the ith 3-sphere. This map realizes the elementary matrix on π_3, where the (i,j)th off-diagonal element is of type $p^l\beta$. To see that we may choose all the α_i to be rational, we observe that the element $a = \alpha_1 e_1 + \dots + \alpha_k e_k$ already has α_1 rational, and denoting α_1 by q_1, the equations

$$\alpha_i + q_1 p^l \beta_i = q_i \in \mathbf{Q}$$

are solvable with $\beta_i \in \mathbf{Z}_p$, but the left-hand side of the equation above is precisely the effect of applying an elementary operation.

We now extend the element $a \in \pi_3(L_{\mathbf{Q}}X)$ to a basis $\{a, a_2, \dots, a_k\}$, and denote the images of a_i in $\pi_3(L_{\mathbf{Q}}Y_p)$ for $i>1$ by b_i. Choose a splitting

$$(L_{\mathbf{Q}}L_pX \longrightarrow L_{\mathbf{Q}}Y_p) \longrightarrow (L_{\mathbf{Q}}L_pS^3 \longrightarrow L_{\mathbf{Q}}L_pS^2).$$

Since $S^3 \to S^2 \to BS^1$ is a principal fibration, we see that we can vary this splitting by any compatible pair $L_{\mathbf{Q}}L_pX \to L_{\mathbf{Q}}L_pS^3$ and $L_{\mathbf{Q}}Y_p \to L_{\mathbf{Q}}L_pS^2$, and after such a variation we may assume that the b_i map trivially on homotopy groups, without changing the image of a.

Using the basis $\{a, a_2, \ldots, a_k\}$ we may produce a diagram of fibrations with the horizontal maps being homotopy equivalences:

$$\begin{tikzcd}
L_{\mathbf{Q}}L_pS^1 \arrow[d] & L_{\mathbf{Q}}L_pS^1 \arrow[d] \\
L_{\mathbf{Q}}L_pX \arrow[r] \arrow[d] & L_{\mathbf{Q}}L_pS^3 \times \prod L_{\mathbf{Q}}L_pS^3 \times L_{\mathbf{Q}}L_pA \arrow[d, "{(h,1,1)}"] \\
L_{\mathbf{Q}}Y_p \arrow[r] & L_{\mathbf{Q}}L_pS^2 \times \prod L_{\mathbf{Q}}L_pS^3 \times L_{\mathbf{Q}}L_pA,
\end{tikzcd}$$

where h is the Hopf fibration and A is a product of S^{2n+1}, $n>1$.

We complete this diagram to the diagram

$$\begin{tikzcd}
L_{\mathbf{Q}}L_pX \arrow[d] \arrow[dr, "\simeq"] & & L_{\mathbf{Q}}X \arrow[ll] \arrow[dl] \arrow[d, "\simeq"] \\
& L_{\mathbf{Q}}L_pS^3 \times \prod L_{\mathbf{Q}}L_pS^3 \times L_{\mathbf{Q}}L_pA \arrow[d, "{(h,1)}"] & \\
L_{\mathbf{Q}}Y_p \arrow[r, "\simeq"] & L_{\mathbf{Q}}L_pS^2 \times \prod L_{\mathbf{Q}}L_pS^3 \times L_{\mathbf{Q}}L_pA. & L_{\mathbf{Q}}S^3 \times \prod L_{\mathbf{Q}}S^3 \times L_{\mathbf{Q}}A \arrow[l]
\end{tikzcd}$$

The desired fibration $X \to Y$ is now obtained by presenting X as a pullback mapping to a pullback diagram defining Y:

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & L_{\mathbf{Q}}X \arrow[d] & & Y \arrow[r] \arrow[d] & L_{\mathbf{Q}}S^2 \times \prod L_{\mathbf{Q}}S^3 \times L_{\mathbf{Q}}A \arrow[d] \\
L_pX \arrow[r] & L_pL_{\mathbf{Q}}X & \longrightarrow & Y_p \arrow[r] & L_{\mathbf{Q}}Y_p.
\end{tikzcd}$$

The extension to a double 1-torus is obtained by just noting that the lifting of the map to $B\mathbf{Z}/2$ is trivial rationally. □

The proof of the main theorem is divided up into some special cases and the general case. The special case is when the type of X is $(3^k, 7^\varepsilon)$, $\varepsilon=0,1$.

We first state the general case in the next theorem. Its proof will be given in the following section.

THEOREM 4.4. *Let X be a quasifinite loop space. Then for any p, L_pX admits a rationally splitting 1-torus except possibly when $p=2$ and X is of type 3^k. The 1-torus can be extended to a rationally splitting double 1-torus unless $p=2$ and X is of type $(3^k, 7^\varepsilon)$, $\varepsilon=0,1$.*

We now turn to the special cases.

Definition 4.5. Let X be a $\mathbf{Z}/p$-local space. The *p-genus* $G_p(X)$ of X is the set of all $\mathbf{Z}_{(p)}$-local homotopy types Y such that $L_pY\simeq X$.

LEMMA 4.6. *If G is a center-free p-compact group which is a rational homology 3-sphere, then $G\simeq L_p\,\mathrm{SO}(3)$. If X is a quasifinite rank-1 loop space, with L_pX center-free for all p, then $X\simeq \mathrm{SO}(3)$.*

Proof. Mixing with a rationalized sphere produces a $\mathbf{Z}_{(p)}$-local loop space. Mixing this with a sphere at the other primes produces an H-space which is a rational homology sphere. By Browder's theorem [7, Theorem 5.2] the only possibilities are S^3, SO(3), S^7, and $\mathbf{R}P^7$, but L_2S^3 is not center-free, and L_2S^7 is not a loop space. This at the same time proves the statement about X by [10]. □

LEMMA 4.7. (1) *There is only one element in $G_p(L_p\,\mathrm{SO}(3)^k)$.*
(2) *There is only one element in $G_p(L_p\,\mathrm{SO}(5))$.*
(3) *There is only one element in $G_p(L_p\,\mathrm{SO}(3)^k\times \mathrm{SO}(5))$.*

Proof. If $Y\in G_p(X)$, then Y is obtained as a pullback

$$\begin{tikzcd} Y \arrow[rr] \arrow[d] & & L_{\mathbf{Q}}X \arrow[d] \\ L_pX \arrow[r] & L_{\mathbf{Q}}L_pX \arrow[r, "f"] & L_{\mathbf{Q}}L_pX \end{tikzcd}$$

from a self-equivalence $f\in \mathrm{Aut}(L_{\mathbf{Q}}L_pX)$. Precomposing f with an element of $\mathrm{Aut}(L_pX)$ leaves Y unchanged up to homotopy, as does postcomposing with an element of $\mathrm{Aut}(L_{\mathbf{Q}}X)$. Thus there is a bijection

$$G_p(L_pX)\cong \mathrm{Aut}(L_pX)\backslash \mathrm{Aut}(L_{\mathbf{Q}}L_pX)/\mathrm{Aut}(L_{\mathbf{Q}}X).$$

In the case $X=\mathrm{SO}(3)^k$, $\mathrm{Aut}(L_{\mathbf{Q}}L_pX)\cong \mathrm{GL}_k(\mathbf{Q}_p)$, and since every p-adic integer can be realized as the degree of a self-map of $L_p\,\mathrm{SO}(3)$, we have that $\mathrm{Aut}(L_pX)\cong \mathrm{GL}_k(\mathbf{Z}_p)$. Thus,

$$G_p(L_p\,\mathrm{SO}(3)^k)\cong \mathrm{GL}_k(\mathbf{Z}_p)\backslash \mathrm{GL}_k(\mathbf{Q}_p)/\mathrm{GL}_k(\mathbf{Q})\cong *.$$

For $X=\mathrm{SO}(5)$ the gluing map is given by two p-adic rationals α_3 and α_7, which describe the induced map on the homotopy groups in dimensions 3 and 7. Looping down unstable Adams operations $\psi^r: L_pB\mathrm{SO}(5)\to L_pB\mathrm{SO}(5)$ shows that for any p-adic unit r, the pair (r^2, r^4) can be realized by a self-equivalence of $L_p\,\mathrm{SO}(5)$. We may also realize self-maps of $L_p\,\mathrm{SO}(5)$ of degree (s,s) for any p-adic unit s, and the proof is now completed by noting that it is possible to choose r and s so that $(r^2s\alpha_3, r^4s\alpha_7)$ is a pair of rational numbers. This shows that $G_p(L_p\,\mathrm{SO}(5))$ contains only one element.

In the case of $X=\mathrm{SO}(3)^k\times\mathrm{SO}(5)$, the situation is slightly more complicated. We have to show that $H_3(f)\in\mathrm{GL}_{k+1}(\mathbf{Q}_p)$ can be turned into the identity matrix by pre- and postcompositions as above. We argue similarly to the proof of Proposition 4.3. We have that $\mathrm{Aut}(L_{\mathbf{Q}}X)$ surjects onto $\mathrm{GL}_{k+1}(\mathbf{Q})$.

As noted above, any p-adic integer can be realized as the H_3-degree of a map $\mathrm{SO}(3)\to\mathrm{SO}(3)$, $\mathrm{SO}(3)\to\mathrm{SO}(5)$ or $\mathrm{SO}(5)\to\mathrm{SO}(5)$. Similarly, there is an integer $N>0$ and a map $\mathrm{SO}(5)\to\mathrm{SO}(3)$ inducing multiplication by N on H_3 ($N=48$ is possible, but that is irrelevant to the argument). To see this, represent the 3-dimensional generator of $H^3(\mathrm{SO}(5))$ by a map $g:\mathrm{SO}(5)\to K(\mathbf{Z},3)$ and consider the obstruction classes for lifting this map to $\mathrm{SO}(3)\to K(\mathbf{Z},3)$. They lie in finitely many torsion groups, so precomposing g with the product of their orders yields a map that lifts to $\mathrm{SO}(3)$.

This implies that in H_3 any invertible matrix of the following form can be realized as an automorphism of L_pX:

$$\left(\begin{array}{c|c} & Nb_1 \\ * & \vdots \\ & Nb_k \\ \hline * & * \end{array}\right),$$

where $*, b_i\in\mathbf{Z}_p$. It remains to show that a matrix of the form

$$\begin{pmatrix} I & * \\ 0 & 1 \end{pmatrix}$$

can be written as a product of a matrix as above and a rational matrix. This follows from the easy fact that every p-adic rational can be written as a sum of a p-adic integer multiple of N and a rational. $\square$

THEOREM 4.8. *Let X be a quasifinite loop space of type $(3^k, 7^\varepsilon)$, $\varepsilon=0,1$. Then X is homotopy equivalent to a compact, smooth, stably parallelizable manifold.*

Proof. In [24] it is proved that L_pX is center-free for large p, and that the center is finite when $\pi_1(X)$ is finite (our standing assumption). We may then construct a new space from $L_p(X)/Z(L_p(X))$ so that the original space is a finite covering space of this new space. Hence we may as well assume that L_pX is center-free for all p.

By [16], $L_p(X)$ can be written as a product of simple p-compact groups which by the classification of reflection groups will be of rank 1 in the case where there are only 3-dimensional generators. In this case it now follows from Lemma 4.6 that $L_p(X)$ is a product of $L_p\,\mathrm{SO}(3)$, and from Lemma 4.7 (1) that $L_{(p)}(X)$ is a product of $L_{(p)}\,\mathrm{SO}(3)$, and finally it follows from [30] that X is homotopy equivalent to a product of SO(3). In case there is also a 7-dimensional generator, we similarly get that L_pX is a product of rank-1 p-compact groups, and one of rank 2. It now follows from Theorem 6.1 that at the prime 2, the 2-compact group of rank 2 is $L_2Y = L_2\,\mathrm{SO}(5)$, and by the classification at odd primes that the p-compact group of rank 2 is also $L_p\,\mathrm{SO}(5)$ (ignoring the loop structure). As above, we now use Lemma 4.7 (3) to show that X is in the Mislin genus of $\mathrm{SO}(3)^k \times \mathrm{SO}(5)$. Hence by [30], X is homotopy equivalent to a stably parallelizable manifold. □

Proof of the main theorem. Theorems 4.4 and 4.8 together with the reduction steps in this section and the surgery arguments in §3 imply the main theorem, if we also show that the manifolds obtained are parallelizable, not only stably parallelizable.

To see this we use the criterion of Dupont [12], [34]. If $\dim X$ is even, the difference between parallelizability and stable parallelizability is determined by the Euler characteristic, which is obviously 0 for X being of the homotopy type of a loop space. In odd dimensions parallelizability is automatic in dimensions 1, 3 and 7, and in other dimensions it is determined by the mod 2 Kervaire semi-characteristic

$$\varkappa(X;2) = \sum_i \dim H_{2i}(X;\mathbf{F}_2) \in \mathbf{Z}/2.$$

But the cohomology of a loop space with $\mathbf{F}_2$-coefficients is a tensor product of truncated polynomial algebras $\mathbf{F}_2[z]/(z^{2^k})$, so this number is obviously zero. □

5. Constructing 1-tori in p-compact groups

The p-complete analog of a finite loop space is called a *p-compact group*. This is by definition a connected, pointed, $\mathbf{Z}/p$-local space BG such that $G := \Omega BG$ is $\mathbf{Z}/p$-finite. We can (and will) choose a topological group model for G and call G itself a p-compact group. For a compact Lie group G, we will also write G for the associated p-compact group obtained by p-completion.

Recall [14] that every p-compact group G has a maximal torus T, a maximal torus normalizer $N_G(T)$, and a Weyl group W acting on T. These loop spaces fit into a diagram

$$\begin{tikzcd} BT \arrow[r] \arrow[dr] & BN_G(T) \arrow[r] \arrow[d] & BW \\ & BG. & \end{tikzcd}$$

Here, $BT\simeq K(\mathbf{Z}_p^n, 2)$ is homotopy equivalent to an Eilenberg–Mac Lane space of degree 2. We call n the *rank* of G. Let $L=\pi_1(T)\cong\mathbf{Z}_p^n$ be the associated lattice. The top row of the diagram is a fibration and determines the action of W on T, or equivalently, on L. If G is connected, this representation is faithful and gives W the structure of a p-adic pseudo-reflection group.

We call a connected G *semisimple* if $\pi_1(G)$ is finite, and *simple* if the associated representation $W\to \mathrm{GL}(L\otimes\mathbf{Q})$ is irreducible.

The center of a p-compact group G is denoted by $Z(G)$. If G is either simply-connected or center-free then it splits uniquely into a product of simple p-compact groups of the same sort.

For details and further notions we refer the reader to the survey articles [23] and [26] and the references mentioned there.

The main new ingredient in this section comes from the first author's thesis [3]. For any connected p-compact group, define $S_G=(\Sigma_+^\infty G)^{hG}$ to be the homotopy fixed-point spectrum of G, acting on its suspension spectrum by multiplication from the right. If G is the p-completion of a connected compact Lie group with Lie algebra $\mathfrak{g}$, then S_G, equipped with the remaining left G-action, is equivariantly homotopy equivalent to the p-completion of $\mathfrak{g}\cup\{\infty\}$ by results of Klein [21].

THEOREM 5.1. ([3]) *For every p-compact group G, S_G is a p-complete sphere. For an inclusion of p-compact groups $H<G$, we have a homotopy equivalence in the p-complete category*

$$G_+\wedge_H S_H \simeq D(G/H)_+\wedge S_G.$$

In particular, if $T<G$ is a torus, the action of T on S_T is trivial, and we have

$$G/T_+ \simeq D(G/T)_+\wedge S^{d-t},$$

where $d=\dim G$ and $t=\dim T$. Hence G/T is a p-complete self-dual space of dimension $d-t$. In particular, it is finitely dominated and stably reducible.

LEMMA 5.2. *Let G be a nontrivial connected p-compact group with finite fundamental group, such that if $p=2$, G does not have type 3^k. Then* $\mathrm{rk}_p(Z(G))<\mathrm{rk}(G)$. *Moreover, if $p=2$ and* $\mathrm{rk}_2(Z(G))=\mathrm{rk}(G)-1$, *then G has type $(3^k,7)$.*

Here rk_p is the p-rank of the abelian p-group $Z(G)$.

Proof. Let $T<G$ be a maximal torus with Weyl group G and lattice $L=\pi_1(T)$. Dwyer and Wilkerson show in [15] that the p-discrete center of G is always contained in the center of $\breve{\mathbf{N}}(T)$, the p-discrete normalizer of the maximal torus. This in turn is the same as the fixed points of $\breve{T}$ under the W-action. Thus the claim follows for $p>2$ from the classification of odd pseudo-reflection groups.

Now let $p=2$. In terms of L, the order-2 elements of the center of G are contained in the invariants $(L/2L)^W$.

Assume that $\mathrm{rk}_2(Z(G))\geqslant \mathrm{rk}(G)-1$. This means that $(L/2L)^W$ has codimension 1 or 0. We then know that the image of the representation $W\to \mathrm{GL}(L/2L)$ consists entirely of 2-torsion. Let K be the kernel and consider the diagram of short exact sequences

$$\begin{tikzcd}
1 \arrow[r] & K \arrow[r, hook] \arrow[d] & W \arrow[r] \arrow[d, hook] & W/K \arrow[r] \arrow[d, hook] & 1 \\
1 \arrow[r] & \mathrm{Id}+2\,\mathrm{End}(L) \arrow[r] & \mathrm{GL}(L) \arrow[r] & \mathrm{GL}(L/2L) \arrow[r] & 1.
\end{tikzcd}$$

Since $\mathrm{Id}+2\,\mathrm{End}(L)$ is a 2-group, so is K, and since W/K is also an (elementary abelian) 2-group, W must be a 2-group. By inspection of the Clark–Ewing list of 2-adic pseudo-reflection groups, we see that the only possibilities are

$$W=(\mathbf{Z}/2)^k\times((\mathbf{Z}/2)^2\wr\mathbf{Z}/2)^l=W(\mathrm{SU}(2)^k\times\mathrm{Spin}(5)^l).$$

It follows from the classification of 2-compact groups up to rank 2 given in §6 that, indeed, $G/Z(G)\simeq L_2(\mathrm{SO}(3)^k\times\mathrm{SO}(5)^l)$. Since $Z(\mathrm{Spin}(5))=\mathbf{Z}/2$ and $\mathrm{rk}(\mathrm{SO}(5))=2$, only $l=0$ or 1 can occur, and $l=1$ if and only if $\mathrm{rk}_2(Z(G))=\mathrm{rk}(G)-1$. This concludes the proof since in this case the type is $(3^k,7)$. □

PROPOSITION 5.3. *Let G be as in Lemma 5.2 and assume that $H^3_{\mathbf{Q}_p}(G)\neq 0$. Then G has a circle subgroup S, not meeting the center, such that $H^*_{\mathbf{Q}_p}(G/S)\cong\bigwedge(t)\otimes R$ for some ring R and a 2-dimensional class t.*

Proof. Let T'' be a subtorus of G which is minimal containing $Z(G)$. Extend to a maximal torus $T=T'\times T''$. Let r' be the dimension of T'.

By Lemma 5.2, $r'\geqslant 1$. Choose coordinates $t_i: S^1\to T$ in such a way that $\{t_i\,|\,1\leqslant i\leqslant r'\}$ span T'. Let W be the Weyl group of G with respect to T. Rationally,

$$H^*_{\mathbf{Q}_p}(BG)\cong H^*_{\mathbf{Q}_p}(BT)^W \quad\text{and}\quad H^*_{\mathbf{Q}_p}(G/T)\cong H^*_{\mathbf{Q}_p}(BT)/\!\!/H^*_{\mathbf{Q}_p}(BG).$$

Since $H^2_{\mathbf{Q}_p}(BG)=0$, we know that $H^2_{\mathbf{Q}_p}(BT)\xrightarrow{\sim}H^2_{\mathbf{Q}_p}(G/T)\cong\mathbf{Q}_p\{t_1,\dots,t_r\}$, and since $H^4_{\mathbf{Q}_p}(BG)\neq 0$, we know that there is a nontrivial quadratic polynomial $f(t_1,\dots,t_r)\in\mathbf{Q}_p[t_1,\dots,t_r]$ such that $f\equiv 0$ in $H^4_{\mathbf{Q}_p}(G/T)$.

Define $S \subseteq K(\mathbf{Z}_p^r, 1) = T$ by the coprime coordinates $\alpha_1, ..., \alpha_r \in \mathbf{Z}_p$. Note that S will intersect $Z(G)$ trivially if $\alpha_1, ..., \alpha_{r'}$ are also coprime. Let $t \in H^2_{\mathbf{Q}_p}(G/S)$ be a generator. Under the map $H^*_{\mathbf{Q}_p}(G/T) \to H^*_{\mathbf{Q}_p}(G/S)$, t_i is mapped to $\alpha_i t$. Therefore, the polynomial $0 = f(t_1, ..., t_r) \in H^4_{\mathbf{Q}_p}(G/T)$ is mapped to $f(\alpha_1 t, ..., \alpha_r t) = f(\alpha_1, ..., \alpha_r) t^2$, which then must be 0. Since f is nonzero, we can choose α_i such that $f(\alpha_1, ..., \alpha_r) \neq 0$ and $\alpha_1, ..., \alpha_{r'}$ are coprime, whence $t^2 = 0$. Thus S satisfies the rational cohomology condition. □

Proof of Theorem 4.4. Let $G = L_p X$. Then G is a p-compact group, and by [10], $H^3_{\mathbf{Q}_p}(G) \neq 0$. By Proposition 5.3, there is a circle subgroup $S^1 \to G$ not meeting the center. By Theorem 5.1, G/S^1 is stably parallelizable. The quasifiniteness condition on Y and the orientability of $X \to Y$ are clearly satisfied.

The 1-torus is rationally splitting because the 2-dimensional class in $H^*_{\mathbf{Q}_p}(G/S^1)$ produces a split map to $L_{\mathbf{Q}} L_p S^2$ making the following diagram commute:

$$\begin{tikzcd}
L_{\mathbf{Q}}G \arrow[rr] \arrow[d] & & L_{\mathbf{Q}}L_pS^3 \arrow[ll, bend right] \arrow[d] \\
L_{\mathbf{Q}}G/S^1 \arrow[rr] \arrow[dr] & & L_{\mathbf{Q}}L_pS^2 \arrow[ll, bend right] \arrow[dl] \\
& K(\mathbf{Q}_p, 2). &
\end{tikzcd}$$

Now let $p = 2$. By Lemma 5.2, the dimension of a maximal torus T' not meeting the center is at least 2. We may choose T' in such a way that the constructed S^1 is contained in T'. This can be extended to an $S^1 \times \mathbf{Z}/2 < T'$, thus giving a rationally splitting double 1-torus. □

6. 2-compact groups of rank 2

In this section we will classify all simple 2-compact groups of rank 2. For the purpose of this paper, we really only need the uniqueness of Spin(5), but little extra work will also deal with the general case.

THEOREM 6.1. *Any simple* 2-*compact group* G *of rank* 2 *is isomorphic to the* 2-*adic completion of* SU(3), Spin(5)=Sp(2), SO(5) *or* G_2.

The rest of this section is devoted to the proof of this statement.

Let U be a finite-dimensional $\mathbf{Q}_2$-representation of a finite group W. A W-lattice L of U is a $\mathbf{Z}_2$-lattice $L \subset U$ of maximal rank fixed under the action of W; i.e. L is a $\mathbf{Z}_2[W]$-module and $L \otimes \mathbf{Q} \cong U$. Two W-lattices L and L' of U are called *isomorphic* if $L \cong L'$ as

$\mathbf{Z}_2[W]$-modules. A W_1-lattice L_1 and a W_2-lattice L_2 are called *weakly isomorphic* if there exists an isomorphism $W_1 \xrightarrow{\sim} W_2$ such that L_1 and L_2 are isomorphic as W_1-lattices.

We say that two p-compact groups G and G' *have the same Weyl group data* if the representations $W_G \to \mathrm{GL}(L_G)$ and $W_{G'} \to \mathrm{GL}(L_{G'})$ are weakly isomorphic. Renaming the elements of $W_{G'}$, we can always identify $W_{G'}$ with W_G and assume that the two lattices are actually isomorphic.

From the Clark–Ewing list [11] we get a complete list of all irreducible reflection groups of rank 2 defined over $\mathbf{Q}_2$. They are the dihedral groups D_6, D_8 and D_{12} with their standard representations as reflection groups. In fact, these are the only dihedral groups which can be represented as reflection groups over $\mathbf{Q}_2$. They correspond to the rational Weyl group representations of SU(3), Spin(5) or SO(5), and G_2, respectively. The classification of Clark and Ewing only works up to weak equivalence.

The Lie groups Spin(5) and Sp(2) are isomorphic, hence they have the same Weyl group data. In the following, we will always use the one of these two which seems more natural.

If a p-compact group G has finite fundamental group then the universal cover $\widetilde{G}$ is again a p-compact group, and G and $\widetilde{G}$ have the same rational Weyl group data. In that case, $G \cong \widetilde{G}/Z$, where $Z \subset \widetilde{G}$ is a central subgroup [24]. Simple p-compact groups have finite fundamental groups [24]. Therefore, Theorem 6.1 is a consequence of the following classification result for simply-connected simple 2-compact groups.

THEOREM 6.2. *Let H be* SU(3), Sp(2) *or* G_2. *A simply-connected* 2*-compact group G has the same rational Weyl group data as H if and only if G and H are isomorphic as* 2*-compact groups.*

For the proof of this theorem we first have to classify all 2-adic lattices of the representation $W_H \to \mathrm{GL}(L_H \otimes \mathbf{Q})$.

LEMMA 6.3. *Let $W \to \mathrm{GL}(U)$ be a reflection group, where $U = \mathbf{Q}_2^2$.*

(1) *If $W = D_6$ or $W = D_{12}$ then, up to isomorphism, there exists exactly one W-lattice of U.*

(2) *If $W = D_8$, each W-lattice of U is isomorphic either to $L_{\mathrm{SO}(5)}$ or to $L_{\mathrm{Spin}(5)}$. Both lattices are weakly isomorphic.*

Proof. Case (1) follows from [2, Proposition 4.3 and Theorem 6.2].

Now let $W = D_8$, and let L be a W-lattice of U. For r large enough, the lattice $2^r L$, i.e. the submodule of all elements divisible by 2^r, is a submodule of $L_{\mathrm{SO}(5)} \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2$. We choose r minimal with this property, i.e. $2^r L \subset L_{\mathrm{SO}(5)}$ but $2^r L \not\subset 2L_{\mathrm{SO}(5)}$. Since $L \otimes \mathbf{Q} \cong L_{\mathrm{SO}(5)} \otimes \mathbf{Q}$, we get a short exact sequence of $\mathbf{Z}_2[W]$-modules

$$0 \longrightarrow 2^r L \longrightarrow L_{\mathrm{SO}(5)} \xrightarrow{\varrho} Q \longrightarrow 0.$$

The minimality of r implies that Q is a finite cyclic group; i.e. $Q \cong \mathbf{Z}/2^s$ generated by $\varrho(1,0)$ or $\varrho(0,1)$. The dihedral group D_8 is generated by the three elements σ_1, σ_2 and τ, where σ_i multiplies the ith coordinate by -1 and τ exchanges the two coordinates. Since the automorphism group of Q is abelian, the action of W on Q factors through the abelianization of W. It follows that the element $\sigma_1\sigma_2 = \sigma_1\tau\sigma_1\tau$ acts trivially on Q. Hence the elements $(1,0),(0,1) \in L_{\mathrm{SO}(5)}$ are mapped onto elements of order 2 in Q. Thus, either $Q=0$ or $Q=\mathbf{Z}/2$. In the first case, we have $L \cong L_{\mathrm{SO}(5)}$. In the second case, D_8 acts trivially on Q with $\varrho(1,0)=\varrho(0,1)\neq 0$ in $\mathbf{Z}/2$, and consequently $L \cong L_{\mathrm{Spin}(5)}$. This proves the first part of (2).

The second part follows from the facts that $L_{\mathrm{Sp}(2)}$ and $L_{\mathrm{Spin}(5)}$ are weakly isomorphic and that $L_{\mathrm{Sp}(2)}$ and $L_{\mathrm{SO}(5)}$ are isomorphic. $\square$

Proof of Theorem 6.2. If G and H have the same rational Weyl group data, the above lemma shows that they also have the same 2-adic Weyl group data. We can assume that $W := W_G = W_H$ and that $L := L_G = L_H$. We can also identify the maximal tori $T := T_G \cong T_H$.

This implies that $H \cong G$ for $H = \mathrm{SU}(3)$ [25] and for $H = G_2$ [36].

For $H = \mathrm{Sp}(2)$, uniqueness results are only known in terms of the maximal torus normalizer [28], [35]. We have to show that $BN_G \simeq BN_{\mathrm{Sp}(2)}$.

Since G and $\mathrm{Sp}(2)$ have the same rational Weyl group data, they have isomorphic rational cohomology. Hence, $H^*_{\mathbf{Q}_2}(X)$ is an exterior algebra with generators in dimensions 3 and 7. If $H^*(G;\mathbf{Z}_2)$ has 2-torsion, then G and G_2 have isomorphic mod 2 cohomology [20]. The Bockstein spectral sequence then shows that G does not have the correct rational cohomology. Therefore, G has no 2-torsion, and $H^*(G;\mathbf{Z}_2)$ is an exterior algebra with generators in dimensions 3 and 7. Hence $H^*(BG;\mathbf{F}_2) \cong \mathbf{F}_2[x_4, x_8]$. Since $H^*(BG;\mathbf{F}_2)$ is a finitely generated module over $H^*(BT;\mathbf{F}_2)$, the composition

$$\begin{aligned} H^*(BG;\mathbf{F}_2) \cong H^*(BG;\mathbf{Z}_2)\otimes\mathbf{F}_2 &\longrightarrow H^*(BT;\mathbf{Z}_2)^W \otimes \mathbf{F}_2 \\ &\cong H^*(B\mathrm{Sp}(2);\mathbf{F}_2) \longrightarrow H^*(BT;\mathbf{F}_2) \end{aligned}$$

is a monomorphism.

The isomorphism $H^*(BT;\mathbf{Z}_2)^W \otimes \mathbf{F}_2 \cong H^*(B\mathrm{Sp}(2);\mathbf{F}_2)$ follows from the fact that G and $\mathrm{Sp}(2)$ have the same 2-adic Weyl group data (Lemma 6.3). Since the first and third terms are both polynomial algebras of the same type,

$$H^*(BG;\mathbf{F}_2) \longrightarrow H^*(BT;\mathbf{Z}_2)^W \otimes \mathbf{F}_2 \cong H^*(B\mathrm{Sp}(2);\mathbf{F}_2)$$

is an isomorphism.

Let $t \subset T$ denote the elements of order 2 and $K := \mathrm{Sp}(1) \times \mathrm{Sp}(1) \subset \mathrm{Sp}(2)$ the subgroup of diagonal quaternionic matrices. We have a chain of inclusions $t \subset T \subset K \subset \mathrm{Sp}(2)$ and $K = C_{\mathrm{Sp}(2)}(t)$. The action of D_8 on t factors through the $\mathbf{Z}/2$-action on t given by switching the coordinates.

Now we use Lannes' T-functor theory (see e.g. [33]). We get a map $f: Bt \to BG$ which looks in mod 2 cohomology like the map $Bt \to B\mathrm{Sp}(2)$. This map is $\mathbf{Z}/2$-equivariant up to homotopy. The mod 2 cohomology of the classifying space $BC_G(t) := \mathrm{map}(Bt, BG)_f$ of the centralizer $C_G(t)$ can be calculated with the help of Lannes' T-functor and

$$H^*(BC_H(t); \mathbf{F}_2) \cong H^*(BC_{\mathrm{Sp}(2)}(t); \mathbf{F}_2) \cong H^*(BG; \mathbf{F}_2).$$

Moreover, the Weyl group of $C_G(t)$ is given by the elements of D_8 acting trivially on t. Hence $W_{C_G(t)} \cong \mathbf{Z}/2 \times \mathbf{Z}/2$. By [13, Theorem 0.5B], this implies that $BC_G(t) \simeq BK$. We will identify $C_G(t)$ with K. The $\mathbf{Z}/2$-action on t induces a $\mathbf{Z}/2$-action on K. Since $Bt \to BG$ was $\mathbf{Z}/2$-equivariant up to homotopy, the inclusion $BC_G(t) \to BG$ extends to a map $BL := BK_{h\mathbf{Z}/2} \to BG$. In this case, the homotopy orbit space BL happens to be a 2-compact group and has the same Weyl group as G. That is, $N_L = N_G$. Moreover, the space BL is part of a fibration

$$BK \longrightarrow BL \longrightarrow B\mathbf{Z}/2,$$

which is classified by obstructions in $H^*(B\mathbf{Z}/2; \pi_*(B\,\mathrm{aut}_1(BK)))$.

Here, $\mathrm{aut}_1(BK)$ is the monoid of self-equivalences of BK homotopic to the identity. Since $\mathrm{aut}_1(BK) \simeq (B\mathbf{Z}/2)^2$ [15] and since $\mathbf{Z}/2$ acts on $\pi_2(B^2(\mathbf{Z}/2)^2) \cong (\mathbf{Z}/2)^2$ by switching the coordinates, all obstruction groups vanish and the above fibration splits. This shows that $BL \simeq B(K \rtimes \mathbf{Z}/2) =: BK'$ and that $BN_G = BN_L \simeq BN_{K'} = BN_{\mathrm{Sp}(2)}$. That is, G and $\mathrm{Sp}(2)$ have isomorphic maximal torus normalizer, and hence $G \cong \mathrm{Sp}(2)$. □

Remark. The only simply-connected 2-compact group of rank 1 is S^3. Hence we get the following complete list (up to isomorphism) of connected 2-compact groups of rank 2:

$$S^1 \times S^1, \quad S^1 \times S^3, \quad U(2), \quad S^1 \times \mathrm{SO}(3), \quad S^3 \times S^3, \quad S^3 \times \mathrm{SO}(3),$$
$$\mathrm{SO}(3) \times \mathrm{SO}(3), \quad \mathrm{SO}(4), \quad \mathrm{SU}(3), \quad \mathrm{Sp}(2), \quad \mathrm{SO}(5), \quad G_2.$$

COROLLARY 6.4. *For any simple, connected 2-compact group G of rank 2, there exists a homomorphism $S^3 \to G$ such that the composition $S^3 \to G \to \overline{G}$ is a monomorphism and such that $H^4_{\mathbf{Q}_2}(BG) \cong H^4_{\mathbf{Q}_2}(BS^3)$.*

Proof. Because of Theorem 6.1 we only have to check this for the compact connected Lie groups $\mathrm{SU}(3)$, $\mathrm{Sp}(2)$, $\mathrm{SO}(5)$ and G_2. There exists a chain of monomorphisms

$S^3 = \mathrm{SU}(2) \subset \mathrm{SU}(3) \subset G_2$. Both groups, SU(3) and G_2, are 2-adically center-free. This proves the claim in these two cases. Let $S^3 \subset \mathrm{Sp}(2)$ denote the inclusion into the first coordinate. Since the intersection of S^3 and the center of Sp(2) is trivial, the composition $S^3 \subset \mathrm{Sp}(2) \to \mathrm{SO}(5)$ is also a monomorphism. This proves the claim in the other cases. The condition on the rational cohomology is obvious. □

7. Geometric properties of loop spaces

In the final sections we describe a different proof of our main theorem, which is based on the concept of special 1-tori. This concept was exploited by the fourth author to prove that finite loop spaces in the genus of a compact connected Lie group are homotopy equivalent to stably parallelizable manifolds [29], [30].

Definition 7.1. For a subring of the rationals R, a nilpotent R-local space X admits an R-local *special* 1-*torus* if, up to homotopy, there exists a diagram of orientable fibrations of nilpotent spaces

$$\begin{array}{ccccc}
L_R S^1 & \longrightarrow & L_R S^3 & \longrightarrow & L_R S^2 \\
\| & & \downarrow & & \downarrow \\
L_R S^1 & \longrightarrow & X & \longrightarrow & Y \\
\downarrow & & \downarrow & & \downarrow \\
* & \longrightarrow & Z & = & Z
\end{array}$$

such that

(1) Z is R-finite;

(2) Y is R-finite and stably reducible;

(3) localized at 0, the diagram is homotopy equivalent to

$$\begin{array}{ccccc}
L_{\mathbf{Q}} S^1 & \longrightarrow & L_{\mathbf{Q}} S^3 & \longrightarrow & L_{\mathbf{Q}} S^2 \\
\| & & \downarrow & & \downarrow \\
L_{\mathbf{Q}} S^1 & \longrightarrow & L_{\mathbf{Q}} Z \times L_{\mathbf{Q}} S^3 & \longrightarrow & L_{\mathbf{Q}} Z \times S^2_0 \\
\downarrow & & \downarrow & & \downarrow \\
* & \longrightarrow & L_{\mathbf{Q}} Z & = & L_{\mathbf{Q}} Z,
\end{array}$$

where all vertical fibrations are trivial.

In [30] the fourth author showed that for a quasifinite Poincaré complex X, the existence of rationally splitting $\mathbf{Z}_{(p)}$-local special 1-tori implies the existence of a global special 1-torus and that, as a consequence, X is homotopy equivalent to a compact, smooth, stably parallelizable manifold. The next proposition, which will be proved in §8, allows us to establish $\mathbf{Z}_{(p)}$-local special 1-tori.

PROPOSITION 7.2. *Let X be a connected quasifinite loop space which is not of type 3^k. Then there exists a loop space Y and a fibration $A \to L_{(p)}BS^3 \xrightarrow{f} L_{(p)}BY$ such that A is simple, $\mathbf{Z}_{(p)}$-finite and such that Y and X are homotopy equivalent spaces. Moreover, localized at 0, there exists a left inverse $s: L_{\mathbf{Q}}BY \to L_{\mathbf{Q}}BS^3$ of f, i.e. $sf_0 = \mathrm{id}_{L_{\mathbf{Q}}BS^3}$.*

The proof of this proposition will be given in §9.

COROLLARY 7.3. *Under the above assumption, the localization $L_{(p)}X$ admits a $\mathbf{Z}_{(p)}$-local special 1-torus.*

Proof. Since the loop space Y of the last proposition is equivalent to X, we only have to prove the claim for Y, or equivalently, we may assume that there exists a fibration $L_{(p)}BS^3 \to L_{(p)}BX$ with the desired properties.

Let $S^1 \subset S^3$ be the maximal torus of S^3. Passing to classifying spaces and localizations, and taking homotopy fibers, we get a commutative diagram of fibration sequences:

$$\begin{tikzcd}
 & & & & Z \arrow[d] \\
L_{(p)}S^1 \arrow[r] \arrow[d, equal] & L_{(p)}S^3 \arrow[r] \arrow[d] & L_{(p)}S^2 \arrow[r] \arrow[d] & L_{(p)}BS^1 \arrow[r, "i"] \arrow[d, equal] & L_{(p)}BS^3 \arrow[d, "f"] \\
L_{(p)}S^1 \arrow[r] \arrow[d, equal] & L_{(p)}X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & L_{(p)}BS^1 \arrow[r, "g"] & L_{(p)}BX \\
* \arrow[r] & Z \arrow[r, equal] & Z. & &
\end{tikzcd} \qquad (*)$$

Here Y is the homotopy fiber of the composition $L_{(p)}BS^1 \to L_{(p)}BS^3 \to L_{(p)}BX$. As the homotopy fiber of maps between simply-connected spaces, Z and Y are simple.

The three left columns of diagram $(*)$ will establish a $\mathbf{Z}_{(p)}$-local special 1-torus for $L_{(p)}X$. All rows of this 3×3-diagram are given by principal fibrations and are therefore orientable. The same holds for the two left columns. For the right column we have a pullback diagram

$$\begin{tikzcd}
L_{(p)}S^2 \arrow[r] \arrow[d, equal] & Y \arrow[r] \arrow[d] & Z \arrow[d] \\
L_{(p)}S^2 \arrow[r] & L_{(p)}BS^1 \arrow[r] & L_{(p)}BS^3.
\end{tikzcd}$$

The bottom row is an orientable fibration. Hence, this also holds for the top row. This shows that the 3×3-part of diagram $(*)$ consists of orientable fibrations.

Since Z is $\mathbf{Z}_{(p)}$-finite, a Serre spectral sequence argument shows that the same holds for Y.

Localized at 0, there exists a left inverse $s: L_{\mathbf{Q}}BX \to L_{\mathbf{Q}}BS^3$. Since $sL_{\mathbf{Q}}g = sL_{\mathbf{Q}}fL_{\mathbf{Q}}i = L_{\mathbf{Q}}i$, this left inverse establishes rationally compatible left inverses for all vertical arrows between the second and third row of $(*)$. In particular, this shows that, localized at 0, the vertical fibrations of the 3×3-diagram are trivial and that this diagram satisfies the third condition of special 1-tori.

To complete the proof it remains to show that Y is $\mathbf{Z}_{(p)}$-stably reducible. We pass to completions. Then L_pX becomes a p-compact group. We get a fibration $L_pY \to L_pBS^1 \to L_pBX$. Since Y was $\mathbf{Z}_{(p)}$-finite and simple, Y and L_pY have isomorphic mod p homology. This shows that L_pY is $\mathbf{Z}/p$-finite, that $L_pS^1 \to L_pX$ is a monomorphism of p-compact groups and that Y is equivalent to the homogeneous space L_pX/L_pS^1. By Theorem 5.1, Y is $\mathbf{Z}_{(p)}$-stably reducible. This completes the proof and shows that $L_{(p)}X$ admits a $\mathbf{Z}_{(p)}$-local special 1-torus. □

Second proof of the main theorem. The passage from stably parallelizable to parallelizable is already discussed in §4. If L_2X is not of type 3^k, then the statement follows from Corollary 7.3 and [30, Theorem 1.4]. The exceptional cases were already discussed in §4. □

8. Particular subgroups of p-compact groups

In this section we will construct particular subgroups of p-compact groups whose center-free quotient contains one simple factor of rank at least 2.

PROPOSITION 8.1. *Let G be a semisimple connected p-compact group such that $r = \dim H^4_{\mathbf{Q}_p}(BX) \neq 0$. If $p=2$, assume that G is not of type 3^k. Then there exists a compact Lie group H and a map $f: BH \to BG$ such that the following hold:*

(1) *The Lie group $H \cong S^3 \times H'$, with H' semisimple and its universal cover $\widetilde{H}'$ isomorphic to $(S^3)^{r-1}$. If p is odd, we can choose $H = (S^3)^r$.*

(2) *The induced map $H^4_{\mathbf{Q}_p}(BG) \to H^4_{\mathbf{Q}_p}(BH)$ is an isomorphism.*

(3) *The homotopy fiber G/H of f is simple and $\mathbf{Z}/p$-finite.*

For the proof we need the following lemma.

LEMMA 8.2. *Let G be a simple simply-connected 2-compact group satisfying that $H^4_{\mathbf{Q}_2}(BG) \neq 0$. Then there exists a map $BS^3 \to BG$ such that*

(1) *$H^4_{\mathbf{Q}_2}(BG)\to H^4_{\mathbf{Q}_2}(BS^3)$ is an isomorphism;*
(2) *if $\mathrm{rk}(G)\geqslant 2$, then $BS^3\to BG\to B\overline{G}:=B(G/Z(G))$ is a monomorphism.*

Proof. Since G is simple and since $H^4_{\mathbf{Q}_2}(BG)\neq 0$, the Weyl group W_G is an honest reflection group and already defined over $\mathbf{Z}$. This follows from the classification of irreducible pseudo-reflection groups. Actually, the Weyl group is isomorphic to the Weyl group of a compact Lie group.

If W_G is abelian then BG is either $B\mathrm{SO}(3)$ or BS^3, and the first part is obvious. Hence we can assume that W_G is nonabelian. Let $W'\subset W_G$ be a subgroup of the Weyl group of G generated by two noncommuting reflections. Let $T\subset T_G^{W'}\subset T_G$ denote the connected component of the fixed-point set of the W_G-action on T_G, which has codimension 2. The centralizer $C=C_G(T)$ is a connected 2-compact group whose Weyl group W_C contains W' [24]. There exists a finite covering of C which splits into a product $K\times T$, where K is a simply-connected 2-compact group of rank 2 with Weyl group isomorphic to W_C. The action of W' on the maximal torus T_K of K gives rise to an irreducible representation over $\mathbf{Q}_2$. Otherwise, W' would split into a product, and the two chosen reflections would commute. Hence, the 2-compact group K is simple and of rank 2.

Let G' be the simple simply-connected compact Lie group with the same Weyl group. The above construction is only based on the Weyl group action of W_G on T_G. Hence, applying the construction to G' establishes a map $BK'\to BG'$, which, as a map between classifying spaces of compact Lie groups, is defined globally, and which in rational cohomology induces the same map as the composition $BK\to BC\to BG$, which is only defined $\mathbf{Z}/p$-locally. By [1], $H^4_{\mathbf{Q}_p}(BG')\to H^4_{\mathbf{Q}}(BK')$ is nontrivial, in fact an isomorphism. And the same holds for $BK\to BG$.

Let $S^3\subset K$ be the subgroup constructed in Proposition 6.4.

The subgroup $S^3\subset G$ is constructed via the composition $S^3\to K\to G$, where K is a 2-compact group of rank 2. We get a diagram

$$\begin{tikzcd} S^3 \arrow[r] \arrow[d, equal] & K \arrow[r] \arrow[d] & G \arrow[d] \\ S^3 \arrow[r] & K/A \arrow[r] & \overline{G}, \end{tikzcd}$$

where A denotes the kernel of the composition $K\to G\to\overline{G}$. The bottom right arrow is a monomorphism. And since $S^3\to K\to\overline{K}$ is a monomorphism, the same holds for the bottom left arrow as well as for the composition in the bottom row. This proves the second part. Part (1) also follows from Proposition 6.4. □

Remark. The same statement holds for odd primes. In this case, the proof for the second claim is even simpler. Since S^3 is center-free at odd primes, we do not need to construct a subgroup of rank 2. Choosing a subgroup of W_G generated by one reflection will produce a monomorphism $S^3 \to G$ with all the desired properties. This clarifies a detail overlooked in [27]. The argument for the first part given there is not complete.

Proof of Proposition 8.1. We can compare the statement with [27, Proposition 3.1]. For odd primes there is no difference. Hence we have to prove the statement only for $p=2$. In this case we have an extra assumption on the rational type, and the additional output is that H contains a factor S^3.

Since $H^*_{\mathbf{Q}_p}(G)$ has a generator of degree greater than 3, the Weyl group W_G is nonabelian [13, Theorem 0.5B]. The universal cover $\widetilde{G}$ of G splits into a direct product $\widetilde{G} \cong \prod_i G_i$ of simple simply-connected pieces [16]. Since G and $\widetilde{G}$ have isomorphic Weyl groups, we can assume that G_1 has a nonabelian Weyl group W_1. If $H^4_{\mathbf{Q}_p}(BG_1) \neq 0$, Lemma 8.2 will produce a monomorphism $f\colon BS^3 \to BG_1$ such that $H^4_{\mathbf{Q}_p}(f)$ is an isomorphism and such that $BS^3 \to BG_1 \to B\overline{G}_1$ is a monomorphism.

If for all factors with nonabelian Weyl group this rational cohomology group vanishes, there exists a factor G_2 of rank 1 with $H^*_{\mathbf{Q}_p}(BG_2) \cong H^*_{\mathbf{Q}_p}(BS^3)$. This implies that $G_2 \cong S^3$. We define $G_1' := G_1 \times G_2$. Since the Weyl group W_{G_1} is defined over $\mathbf{Q}_2$, it is an honest reflection group, and the arguments of Lemma 8.2 show that the second part does hold in this case. Hence, we can then define a map $BS^3 \to BG_1' = BG_1 \times BG_2$ which is the identity on the second factor and satisfies the claims of Lemma 8.2.

Now we can proceed similarly as in [27]. For all other pieces with $H^4_{\mathbf{Q}_p}(BG_i) \neq 0$ there exist monomorphisms $BH_i \to BG_i$ inducing an isomorphism on $H^4(-;\mathbf{Z}_p) \otimes \mathbf{Q}$ such that H_i is isomorphic to S^3 or to SO(3) (see [27]). This produces a homomorphism $\prod_i H_i \to \prod_i G_i \cong \widetilde{X} \to X$ of p-compact groups. The kernel K of this homomorphism, which might be nontrivial, is a central subgroup of $H_1 \times \prod_{i>1} H_i$. Since the center-free quotient $\overline{G}$ is isomorphic to $\prod_i \overline{G}_i$ we have a homomorphism $G \to \overline{G}_1$. By construction the composition $S^3 \to G_1 \to \overline{G}_1$ is a monomorphism. We get a commutative diagram

$$\begin{tikzcd}
K \arrow[r] \arrow[d, equal] & S^3 \times \prod_{i>1} H_i \arrow[r] \arrow[d] & G \arrow[d] \\
K \arrow[r] & S^3 \arrow[r] & \overline{G}_1,
\end{tikzcd}$$

where the right arrow in the bottom row is a monomorphism. Since $\overline{G}_1$ is center-free, the composition $K \to S^3 \times \prod_{i>1} H_i \to S^3$ is trivial. Therefore, K is a subgroup of $\prod_{i>1} H_i$ and the map $S^3 \times \prod_{i>1} H_i \to G$ factors through a monomorphism $H := S^3 \times ((\prod_{i>1} H_i)/K) \to G$ with all the desired properties. □

9. Proof of Proposition 7.2

The proof of Proposition 7.2 is based on an arithmetic square argument. First we need a statement about the existence of a particular sub-loop space. Actually, a $\mathbf{Z}_{(p)}$-local version of the next proposition would be sufficient for our purpose, but with no extra effort we can prove a global result.

PROPOSITION 9.1. *Let X be a semisimple $\mathbf{Z}$-finite loop space not of type 3^k. Then there exist a semisimple compact Lie group $H \cong S^3 \times H'$, loop spaces U and Y, and a fibration*

$$A \to BU \to BY$$

such that the following hold:

(1) *the universal cover of H is isomorphic to a product of S^3's;*

(2) *$H^4(BY;\mathbf{Q}) \to H^4(BU;\mathbf{Q})$ is an isomorphism;*

(3) *A is simple and $\mathbf{Z}$-finite;*

(4) *the spaces H and U as well as X and Y are homotopy equivalent;*

(5) *for each prime p there exists a commutative diagram*

$$\begin{tikzcd} L_pBU \arrow[r] \arrow[d] & L_pBY \arrow[d] \\ L_pBH \arrow[r] & L_pBX, \end{tikzcd}$$

where the vertical maps are equivalences. The same holds for the rationalizations of the classifying spaces.

Proof. This statement is a refinement of [27, Proposition 1.4]. The proof of that statement is an arithmetic square argument which uses its p-completed version as input [27, Proposition 3.1]. The proof carries over word by word. We only have to replace that proposition by a p-completed version of the above claim, namely by Proposition 8.1. Claim (5), which is not part of [27, Proposition 1.4], follows from the arithmetic square argument and Proposition 8.1. □

Remark. The above proposition establishes an oriented fibration $H \to X \to A$. The existence of such an oriented fibration is sufficient to show that the finiteness obstruction vanishes and that every quasifinite loop space is actually finite (see [27]). The existence of a special 1-tori is needed for the vanishing of the appropriate surgery obstruction.

For the proof of Proposition 7.2 we need a higher-dimensional version of Lemma 4.2.

LEMMA 9.2. *Let $A\in\mathrm{GL}(n,\mathbf{Z}_p)$. Then, there exists a vector $v=(v_1,...,v_n)\in\mathbf{Z}_p^n$ such that v_i is a square of a p-adic unit for all i and such that Av is a vector whose components are given by elements of $\mathbf{Z}_{(p)}$.*

Proof. Let $B=A^{-1}$. We have to solve the following problem: Find a vector $w\in\mathbf{Z}_{(p)}^n$ such that $Bw\neq 0$ has squares of p-adic units as components. The question whether a p-adic unit is a square can be decided by reducing to $\mathbf{Z}/p$ for p odd or to $\mathbf{Z}/8$ for $p=2$. In both cases the reduction $\overline{B}$ of B is an invertible matrix and therefore induces an epimorphism on $V=(\mathbf{Z}/p)^n$. In particular, if $\bar{v}\in V$ is a vector with components given by squares mod p such that all entries are units in $\mathbf{Z}/p$, there exists a vector $w\in\mathbf{Z}^n$ such that $\overline{B}w=v$. Hence, Bw is a vector whose components are squares of nontrivial p-adic units. For $p=2$, the same argument works, we only have to replace $\mathbf{Z}/p$ by $\mathbf{Z}/8$. □

Proof of Proposition 7.2. Let U and Y denote the loop spaces and H the Lie group constructed in Proposition 9.1. Since $L_pBU\simeq L_pBH$ and $L_{\mathbf{Q}}BU\simeq L_{\mathbf{Q}}BH$ we have a pullback diagram

$$\begin{tikzcd} L_{(p)}BU \arrow[rr] \arrow[d] & & L_pBH \arrow[d] \\ L_{\mathbf{Q}}BH \arrow[r] & L_{\mathbf{Q}}L_pBH \arrow[r, "\alpha"] & L_{\mathbf{Q}}L_pBH. \end{tikzcd}$$

The map α is an equivalence and induces a continuous map in homotopy. The homotopy groups $\pi_*(L_{\mathbf{Q}}L_pBH)$ carry a natural topology since $\pi_*(L_pBH)\cong\pi_*(BH)\otimes\mathbf{Z}_p$ (details may be found in [39]). The space $L_{\mathbf{Q}}L_pBH\cong K(\mathbf{Q}_p^r,4)$ is a rational Eilenberg–Mac Lane space. Since self-maps of rational Eilenberg–Mac Lane spaces are determined by the induced maps in homotopy, and since α induces a continuous map in homotopy, we can think of α as a matrix in $\mathrm{GL}(n,\mathbf{Q}_p)$ inducing a continuous self-equivalence of $\mathbf{Q}_p^n$. Such matrices can be written as a product $\gamma\varrho$, where $\gamma\in\mathrm{GL}(n,\mathbf{Z}_p)$ and $\varrho\in\mathrm{GL}(n;\mathbf{Q})$.

Since ϱ can be realized as a self-equivalence of $L_{\mathbf{Q}}HG$, replacing α by γ does not change the homotopy type of the pullback. Hence we may assume that $\alpha\in\mathrm{GL}(n,\mathbf{Z}_p)$.

Every square unit of $\mathbf{Z}_p$, considered as a self-map of $\pi_4(L_pBS^3)$ can be realized by a self-equivalence $L_pBS^3\to L_pBS^3$. Since $H\cong S^3\times H'$ and since the universal cover of H' is a product of S^3's, Lemma 9.2 shows that there exists a map $BS^3\to L_pBS^3\times L_pB\widetilde{H}'$ such that the composition $L_pBS^3\to L_pBS^3\times L_pB\widetilde{H}'\to L_pBH$ is a monomorphism and such that the composition

$$\begin{aligned} L_{(p)}BS^3 &\longrightarrow L_pBS^3\longrightarrow L_p(BS^3\times B\widetilde{H}') \\ &\longrightarrow L_{\mathbf{Q}}L_p(BS^3\times B\widetilde{H}')\simeq L_{\mathbf{Q}}L_pBH \xleftarrow{\alpha^{-1}} L_{\mathbf{Q}}L_pBH \end{aligned}$$

lifts to a map $L_{(p)}BS^3\to L_{\mathbf{Q}}BH\simeq L_{\mathbf{Q}}(BS^3\times BH')$. Moreover, localized at 0, composition with the projection on the first factor is an equivalence. This establishes a map

$L_{(p)}BS^3 \to L_{(p)}BU$ such that the completion of $L_{(p)}BS^3 \to L_{(p)}BU$ is induced by the monomorphism $L_pS^3 \to L_pS^3 \times L_pH' \cong L_pH$ of p-compact groups. This shows that the homotopy fiber of $L_{(p)}BS^3 \to L_{(p)}BU$ is simple and $\mathbf{Z}_{(p)}$-finite, as is the homotopy fiber of the composition $f: L_{(p)}BS^3 \to L_{(p)}BU \to L_{(p)}BY$. Since $H^4(BY;\mathbf{Q}) \cong H^4(BU;\mathbf{Q})$, there exists a left inverse $s: L_{\mathbf{Q}}BY \to L_{\mathbf{Q}}BU$ for $L_{\mathbf{Q}}g$. Projection onto the first factor gives a left inverse of $L_{\mathbf{Q}}BS^3 \to L_{\mathbf{Q}}BU \simeq L_{\mathbf{Q}}BS^3 \times L_{\mathbf{Q}}BH'$. This shows that, localized at 0, the map $f: L_{(p)}BS^3 \to L_{(p)}BY$ has a left inverse and finishes the proof of the proposition. □

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