Ark. Mat., **42** (2004), 31–59

Projections in the space H^∞ and the corona theorem for subdomains of coverings of finite bordered Riemann surfaces

Alexander Brudnyi(1)

Abstract. Let M be a non-compact connected Riemann surface of a finite type, and $R\Subset M$ be a relatively compact domain such that $H_1(M,\mathbf{Z})=H_1(R,\mathbf{Z})$. Let $\widetilde{R}\to R$ be a covering. We study the algebra $H^\infty(U)$ of bounded holomorphic functions defined in certain subdomains $U\subset\widetilde{R}$. Our main result is a Forelli type theorem on projections in $H^\infty(\mathbf{D})$.

1. Introduction

1.1. Let X be a connected complex manifold and $H^\infty(X)$ be the algebra of bounded holomorphic functions on X with pointwise multiplication and with norm

$$\|f\|=\sup_{x\in X}|f(x)|.$$

Let $r:\widetilde{X}\to X$ be the universal covering of X. The fundamental group $\pi_1(X)$ acts discretely on $\widetilde{X}$ by biholomorphic maps. By $r^*(H^\infty(X))\subset H^\infty(\widetilde{X})$ we denote the Banach subspace of functions invariant with respect to the action of $\pi_1(X)$.

In this paper we describe a class of manifolds X for which there is a linear continuous projector $P: H^\infty(\widetilde{X})\to r^*(H^\infty(X))$ satisfying

$$P(fg)=P(f)g \quad \text{for any } f\in H^\infty(\widetilde{X}) \text{ and } g\in r^*(H^\infty(X)). \tag{1.1}$$

Forelli [F] was the first to discover that such projectors P exist for X being a finite bordered Riemann surface. (In this case $\widetilde{X}$ is the open unit disk $\mathbf{D}\subset\mathbf{C}$.) Subsequently, existence of a projection operator satisfying (1.1) for certain infinitely connected Riemann surfaces was established by Carleson [Ca3] and Jones and Marshall [JM]. In all these results the projector can be constructed explicitly.

(1) Research supported in part by NSERC.

In the present paper we prove existence of such projectors for a wide class of (not necessarily one-dimensional) complex manifolds. Our construction is more abstract and uses some techniques of the theory of coherent Banach sheaves over Stein manifolds.

In order to formulate our results let us introduce some definitions.

Let $N \Subset M$ be a relatively compact domain (i.e. an open connected subset) in a connected Stein manifold M such that

$$\pi_1(N) \cong \pi_1(M). \tag{1.2}$$

By $\mathcal{F}_c(N)$ we denote the class of unbranched coverings of N. Any covering from $\mathcal{F}_c(N)$ corresponds to a subgroup of $\pi_1(N)$. Assume that the complex connected manifold U admits a holomorphic embedding $i: U \hookrightarrow R$ for some $R \in \mathcal{F}_c(N)$. Let $i_*: \pi_1(U) \to \pi_1(R)$ be the induced homomorphism of fundamental groups. We set $K(U) := \operatorname{Ker} i_* \subset \pi_1(U)$. Consider the regular covering $p_U: \widetilde{U} \to U$ of U corresponding to the group $K(U)$, so that, $\pi_1(\widetilde{U}) = K(U)$ and $\pi_1(U)/K(U)$ acts on $\widetilde{U}$ as the group of deck transformations. Further, by $p_U^*(H^\infty(U)) \subset H^\infty(\widetilde{U})$ we denote the subspace of holomorphic functions invariant with respect to the action of $\pi_1(U)/K(U)$ (i.e. the pullback by p_U of $H^\infty(U)$ to $\widetilde{U}$). Let $F_z := p_U^{-1}(z)$, $z \in U$, and let $l^\infty(F_z)$ be the Banach space of bounded complex-valued functions on F_z with the supremum norm. By $c(F_z) \subset l^\infty(F_z)$ we denote the subspace of constant functions.

Theorem 1.1. *There is a linear continuous projector* $P: H^\infty(\widetilde{U}) \to p_U^*(H^\infty(U))$ *satisfying the following properties*:

(1) *there exists a family of linear continuous projectors* $P_z: l^\infty(F_z) \to c(F_z)$ *holomorphically depending on* $z \in U$ *such that* $P[f]|_{p_U^{-1}(z)} := P_z[f|_{p_U^{-1}(z)}]$ *for any* $f \in H^\infty(\widetilde{U})$;

(2) $P(fg) = P(f)g$ *for any* $f \in H^\infty(\widetilde{U})$ *and* $g \in p_U^*(H^\infty(U))$;

(3) *if* $f \in H^\infty(\widetilde{U})$ *is such that* $f|_{F_z}$ *is constant, then* $P(f)|_{F_z} = f|_{F_z}$;

(4) *each* P_z *is continuous in the weak* topology of* $l^\infty(F_z)$;

(5) *the norm* $\|P\| \le C < \infty$, *where* $C = C(N)$(2).

Remark 1.2. From (1)–(5) it follows that there exists an $h \in H^\infty(\widetilde{U})$ such that

(a) $\hat{h}(z) := \sum_{w \in F_z} |h(w)|$ is continuous on U, and $\sup_U \hat{h} \le C < \infty$;

(b) $\sum_{w \in F_z} h(w) = 1$ for any $z \in U$;

(c) the projector P is defined as

$$P(f)(y) := \sum_{w \in F_z} f(w)h(w), \quad y \in F_z.$$

(2) Here and below the notation $C = C(\alpha, \beta, \gamma, \ldots)$ means that the constant depends only on the parameters $\alpha, \beta, \gamma, \ldots$.

Also, from (c) it follows that $P: H^\infty(\widetilde{U}) \to p_U^*(H^\infty(U))$ is weak* continuous.

Let $\widetilde{R}$ be a covering of a finite bordered Riemann surface R. The fundamental group $\pi_1(\widetilde{R})$ is a free group with a finite or countable family of generators J. Let $U \subset \widetilde{R}$ be a domain such that $\pi_1(U)$ is generated by a subfamily of J. Let $r: \mathbf{D} \to U$ be the universal covering map. Then Theorem 1.1 implies the following result.

Corollary 1.3. *There exists a linear continuous projector* $P: H^\infty(\mathbf{D}) \to r^*(H^\infty(U))$ *satisfying the properties of Theorem* 1.1.

Remark 1.4. The remarkable class of Riemann surfaces U for which the Forelli type theorem (like Corollary 1.3) is valid was introduced by Jones and Marshall [JM]. The definition is in terms of an interpolating property for the critical points of the Green function on U. We conjecture that any $R \in \mathcal{F}_c(N)$, where N is the Riemann surface satisfying (1.2), belongs to this class.

Example 1.5. Let $r: \mathbf{D} \to X$ be the universal covering of a compact complex Riemann surface of genus $g \geq 2$. Let $K \subset \mathbf{D}$ be the fundamental compact region with respect to the action of the deck transformation group $\pi_1(X)$. By definition, the boundary of K is the union of $2g$ analytic curves. Let $D_1, \dots, D_k$ be a family of mutually disjoint closed disks situated in the interior of K. We set

$$S := \bigcup_{i=1}^{k} D_i, \quad K' := K \setminus S \quad \text{and} \quad \widetilde{R} := \bigcup_{g \in \pi_1(X)} g(K').$$

Then $R := r(K') \subset X$ is a finite bordered Riemann surface, and $r: \widetilde{R} \to R$ is a regular covering corresponding to the quotient group $\pi_1(X)$ of $\pi_1(R)$. Here $\pi_1(\widetilde{R})$ is generated by the family of simple closed curves in $\widetilde{R}$ with the origin at a fixed point $x_0 \in \widetilde{R}$ so that each such curve goes around only one of $g(D_i)$, $g \in \pi_1(X)$, $i = 1, \dots, k$. Let $Y \subset \mathbf{D}$ be a simply connected domain with the property that there is a subset $L \subset \pi_1(X)$ so that

$$Y \bigcap \Big(\bigcup_{g \in \pi_1(X)} g(S) \Big) = \bigcup_{g \in L} g(S).$$

Clearly $U := Y \setminus \bigcup_{g \in L} g(S)$ satisfies the hypotheses of Corollary 1.3. Therefore the projector P, described above, exists for U.

One of the possible applications of Forelli's theorem is to the solution of the *corona problem* (for results and references related to the corona problem we refer to Garnett [Ga2], Jones and Marshall [JM] and Slodkowski [S]). Let us recall the corresponding definitions.

Let X be a Riemann surface such that $H^\infty(X)$ separates points of X. By $M(H^\infty(X))$ we denote the maximal ideal space of $H^\infty(X)$, i.e. the set of non-trivial multiplicative linear functionals on $H^\infty(X)$ with the weak* topology (which is called the *Gelfand topology*). It is a compact Hausdorff space. Each point $x \in X$ corresponds in a natural way (point evaluation) to an element of $M(H^\infty(X))$. So X is naturally embedded into $M(H^\infty(X))$. Then the corona problem for $H^\infty(X)$ asks: Is $M(H^\infty(X))$ the closure (in the Gelfand topology) of X? (The complement of the closure of X in $M(H^\infty(X))$ is called the *corona*.)

For example, according to Carleson's celebrated corona theorem [Ca2] this is true for X being the open unit disk $\mathbf{D}$. Also, there are non-planar Riemann surfaces for which the corona is non-trivial (see e.g. [G], [JM], [BD], [L] and references therein). The general problem for planar domains is still open, as is the problem in several variables for the ball and polydisk.

It is well known that the corona problem has the following analytic reformulation.

A collection $f_1, \dots, f_n$ of functions from $H^\infty(X)$ satisfies the *corona condition* if there exists $\delta > 0$ such that

$$|f_1(x)| + |f_2(x)| + \dots + |f_n(x)| \geq \delta \quad \text{for all } x \in X. \tag{1.3}$$

The corona problem being solvable means that the Bezout equation

$$f_1 g_1 + f_2 g_2 + \dots + f_n g_n \equiv 1$$

has a solution $g_1, \dots, g_n \in H^\infty(X)$ for any $f_1, \dots, f_n$ satisfying the corona condition. We refer to $\max_j \|g_j\|$ as a "bound on the corona solutions". Using Carleson's solution [Ca2] of the corona problem for $H^\infty(\mathbf{D})$ and property (2) for the projector P constructed in Theorem 1.1 we obtain the following corollary.

Corollary 1.6. *Let $N \Subset M$, $R \in \mathcal{F}_c(N)$ and $i: U \hookrightarrow R$ be open Riemann surfaces satisfying the hypotheses of Theorem* 1.1. *Assume that $K(U) := \operatorname{Ker} i_*$ is trivial. Let $f_1, \dots, f_n \in H^\infty(U)$ satisfy* (1.3). *Then the corona problem has a solution $g_1, \dots, g_n \in H^\infty(U)$ with the bound $\max_j \|g_j\| \leq C(N, n, \delta / \max_j \|f_j\|)$.*

Remark 1.7. (1) We cannot avoid the restriction that N be an open bordered Riemann surface: It follows from the results of Lárusson [L] and the author [Br2] that for any integer $n \geq 2$ there are a compact Riemann surface S_n and its regular covering $p_n: \widetilde{S}_n \to S_n$ such that

(a) $\widetilde{S}_n$ is a complex submanifold of an open Euclidean ball $\mathbf{B}_n \subset \mathbf{C}^n$;

(b) the embedding $i: \widetilde{S}_n \hookrightarrow \mathbf{B}_n$ induces an isometry $i^*: H^\infty(\mathbf{B}_n) \to H^\infty(\widetilde{S}_n)$.

In particular, (b) implies that the maximal ideal spaces of $H^\infty(\widetilde{S}_n)$ and $H^\infty(\mathbf{B}_n)$ coincide. Thus the corona problem is not solvable for $H^\infty(\widetilde{S}_n)$.

(2) In [Br1, Theorem 1.1] we proved the following matrix version of Corollary 1.6.

Theorem 1.8. *Let U be a Riemann surface satisfying the conditions of Corollary* 1.6. *Let $A=(a_{ij})$ be an $n\times k$ matrix, $k<n$, with entries in $H^\infty(U)$. Assume that the family of determinants of submatrices of A of order k satisfies the corona condition. Then there exists an $n\times n$ matrix $\tilde{A}=(\tilde{a}_{ij})$, $\tilde{a}_{ij}\in H^\infty(U)$, so that $\tilde{a}_{ij}=a_{ij}$ for $1\le j\le k$, $1\le i\le n$, and* $\det\tilde{A}=1$.

The proof of the theorem is based on Theorem 1.1 and a Grauert type theorem for "holomorphic" vector bundles on maximal ideal spaces (which are not usual manifolds) of certain Banach algebras.

1.2. Another application of Theorem 1.1 is related to the classification of interpolating sequences in U (cf. [St] and [JM]).

Recall that a sequence $\{z_j\}_{j=1}^\infty\subset U$ is *interpolating* for $H^\infty(U)$ if for every bounded sequence of complex numbers $\{a_j\}_{j=1}^\infty$, there is an $f\in H^\infty(U)$ so that $f(z_j)=a_j$. The *constant of interpolation* for $\{z_j\}_{j=1}^\infty$ is defined as

$$\sup_{\|a_j\|_{l^\infty}\le 1}\inf\{\|f\|: f\in H^\infty(U),\ f(z_j)=a_j,\ j=1,2,\dots\}.$$

Theorem 1.9. *Let $N\Subset M$, $R\in\mathcal{F}_c(N)$, $i:U\hookrightarrow R$ and $\widetilde{U}$ be complex manifolds satisfying the conditions of Theorem* 1.1. *A sequence $\{z_j\}_{j=1}^\infty\subset U$ is interpolating for $H^\infty(U)$ if and only if $r^{-1}(\{z_j\}_{j=1}^\infty)$ is interpolating for $H^\infty(\widetilde{U})$.*

Example 1.10. (1) Let $M\subset\mathbf{D}$ be a bounded domain, whose boundary B consists of k simple closed continuous curves $B_1,\dots,B_k$, with B_1 forming the outer boundary. Let D_1 be the interior of B_1, and $D_2,\dots,D_k$ the exteriors of $B_2,\dots,B_k$, including the point at infinity. Then each D_i is biholomorphic to $\mathbf{D}$. Let $\{z_{ji}\}_{j\in J}$ be interpolating sequences for $H^\infty(D_i)$, $i=1,\dots,k$, such that the Euclidean distance between any two distinct sequences is bounded from below by a positive number. Then for any covering $p:R\to M$, the sequence $p^{-1}(\{z_{ji}\}_{j,i})$ is interpolating for $H^\infty(R)$.

(2) Let $N\subset\mathbf{C}^n$ be a strongly pseudoconvex domain. Then Theorem 1.9 is valid for any $i:U\hookrightarrow R$, $R\in\mathcal{F}_c(N)$, and $\widetilde{U}$ satisfying the hypotheses of Theorem 1.1.

Let $N\Subset M$, $R\in\mathcal{F}_c(N)$ and $i:U\hookrightarrow R$ be complex manifolds satisfying the conditions of Theorem 1.1. Let $r:\widetilde{U}\to U$ be the universal covering. The group $\pi_1(U)$ acts discretely on $\widetilde{U}$ by biholomorphic maps.

A *character* of $\pi_1(U)$ is a complex-valued function $\varrho\colon \pi_1(U)\to\mathbf{C}^*$ satisfying

$$\varrho(\phi\gamma)=\varrho(\phi)\varrho(\gamma) \text{ and } |\varrho(\phi)|=1, \quad \phi,\gamma\in\pi_1(U).$$

A holomorphic function $f\in H^\infty(\widetilde{U})$ is called *character-automorphic* if

$$f(\gamma(z))=\varrho(\gamma)f(z), \quad \gamma\in\pi_1(U),\ z\in\widetilde{U}. \tag{1.4}$$

By $H^\infty(\pi_1(U),\varrho)$ we denote the Banach space of bounded holomorphic functions satisfying (1.4).

Theorem 1.11. *Under the above assumptions there is a (non-trivial) linear continuous operator $H^\infty(\widetilde{U})\to H^\infty(\pi_1(U),\varrho)$ whose norm is bounded by a constant $A=A(N)$.*

Remark 1.12. Let us recall that a non-parabolic Riemann surface X with a Green function G_o is of *Widom type* if

$$\int_0^\infty b(t)\,dt<\infty,$$

where $b(t)$ is the first Betti number of the set $\{x\in X:G_o(x)>t\}$. This means that the topology of X grows slowly as measured by the Green function. Widom type surfaces are the only infinitely connected ones for which Hardy theory has been developed to any extent. They have many bounded holomorphic functions. In particular, such functions separate points and directions. We refer to [Ha] for an exposition.

Let U be a Riemann surface satisfying the hypotheses of Theorem 1.11. Then this theorem and the remarkable results of Widom [W] imply that U is of Widom type. In fact, it was first noted by Jones and Marshall [JM, p. 295] that if the projection operator $P\colon H^\infty(\mathbf{D})\to r^*(H^\infty(U))$ of the form constructed in Theorem 1.1 exists then U must be of Widom type.

1.3. In this section we formulate some results on interpolating sequences in U for U being a Riemann surface satisfying the hypotheses of Corollary 1.6. Our results have much in common with similar properties of interpolating sequences for $H^\infty(\mathbf{D})$.

Let $r\colon\mathbf{D}\to U$ be the universal covering map. From Theorem 1.9 we know that for any $z\in U$ the sequence $r^{-1}(z)\subset\mathbf{D}$ is interpolating for $H^\infty(\mathbf{D})$. Then we can define a Blaschke product $B_z\in H^\infty(\mathbf{D})$, $z\in U$, with simple zeros at all points of $r^{-1}(z)$. If B_z' is another Blaschke product with the same property then we have $B_z'=\alpha B_z$ for some $\alpha\in\mathbf{C}$, $|\alpha|=1$. In particular, the subharmonic function $|B_z|$ is invariant with respect to the action on $\mathbf{D}$ of the deck transformation group $\pi_1(U)$. Thus there is a non-negative subharmonic function P_z on U with the only zero at z, such that $r^*(P_z)=|B_z|$. It is also clear that $P_z(y)=P_y(z)$ for any $y,z\in U$, and $\sup_U P_z=1$.

Proposition 1.13. *A sequence $\{z_i\}_{i=1}^\infty \subset U$ is interpolating for $H^\infty(U)$ if and only if*

$$\inf_j \left\{ \prod_{k:k\neq j} P_{z_k}(z_j) \right\} =: \delta > 0. \tag{1.5}$$

The number δ is *the characteristic* of the interpolating sequence $\{z_j\}_{j=1}^\infty$.
Using Proposition 1.13 we prove the following result.

Corollary 1.14. *Let $\{z_j\}_{j=1}^\infty \subset U$ be an interpolating sequence with characteristic δ. Let K be the constant of interpolation for $\{z_j\}_{j=1}^\infty$. Then there is a constant $A=A(N)$ (depending on the Riemann surface N from Corollary 1.6) such that*

$$K \leq \frac{A}{\delta}\left(1+\log\frac{1}{\delta}\right).$$

Let

$$\varrho(z,w) := \left|\frac{z-w}{1-\bar{z}w}\right|, \quad z,w\in \mathbf{D},$$

be the pseudohyperbolic metric on $\mathbf{D}$. Let $x,y\in U$ and $x_0\in \mathbf{D}$ be such that $r(x_0)=x$. We define the distance $\varrho^*(x,y)$ by the formula

$$\varrho^*(x,y) := \inf_{w\in r^{-1}(y)} \varrho(x_0,w).$$

It is easy to see that this definition does not depend of the choice of x_0 and determines a metric on U compatible with its topology.

The following result shows that interpolating sequences are stable under small perturbations.

Proposition 1.15. *Let $\{z_j\}_{j=1}^\infty \subset U$ be an interpolating sequence with characteristic δ. Assume that $0<\lambda<2\lambda/(1+\lambda^2)<\delta<1$. If $\{\xi_j\}_{j=1}^\infty \subset U$ satisfies $\varrho^*(\xi_j, z_j)\leq \lambda$, $j=1,2,\ldots$, then for any k,*

$$\prod_{j:j\neq k} P_{\xi_j}(\xi_k) \geq \frac{\delta-2\lambda/(1+\lambda^2)}{1-2\lambda\delta/(1+\lambda^2)}.$$

Remark 1.16. This proposition is similar to [Ga1, Chapter VII, Lemma 5.3] used in the proof of Earl's theorem on interpolation. We will show how to modify the proof of this lemma to obtain our result.

Proposition 1.17. *Let* $\{z_i\}_{i=1}^{\infty}$ *and* $\{y_i\}_{i=1}^{\infty}$ *be interpolating sequences in* U. *Assume that there is a constant* $c>0$ *such that for any* i *and* j,

$$\varrho^*(z_j, y_i) \geq c.$$

Then the sequence $\{z_i\}_{i=1}^{\infty} \cup \{y_i\}_{i=1}^{\infty} \subset U$ *is interpolating.*

Finally we formulate an analog of Corollary 1.6 from [Ga1, Chapter X].

Proposition 1.18. *Let* $\{z_j\}_{j=1}^{\infty} \subset U$ *be an interpolating sequence with characteristic* δ. *Then* $\{z_j\}_{j=1}^{\infty}$ *can be represented as a disjoint union* $\{z_{1j}\}_{j=1}^{\infty} \sqcup \{z_{2j}\}_{j=1}^{\infty}$ *of two subsequences such that the characteristic of* $\{z_{sj}\}_{j=1}^{\infty}$ *is* $\geq \sqrt{\delta}$, $s=1,2$.

Remark 1.19. Using the above properties of interpolating sequences in U it is possible to define non-trivial analytic maps of $\mathbf{D}$ to the maximal ideal space of $H^{\infty}(U)$ related to limit points of interpolating sequences. The construction is similar to the classical one given in the case of $H^{\infty}(\mathbf{D})$ by Hoffman [H].

Acknowledgement. I would like to thank L. Carleson for some valuable remarks on the original version of this paper and D. Marshall for useful discussions.

2. Construction of bundles

In this section we formulate and prove some preliminary results used in the proofs of our main theorems.

2.1. Definitions and examples

(For standard facts about bundles see e.g. Hirzebruch's book [Hi].) In what follows all topological spaces are allowed to be finite- or infinite-dimensional.

Let X be a complex analytic space and S be a complex analytic Lie group with the unit $e \in S$. Consider an effective holomorphic action of S on a complex analytic space F. Here *holomorphic action* means a holomorphic map $S \times F \to F$ sending $s \times f \in S \times F$ to $sf \in F$ such that $s_1(s_2 f) = (s_1 s_2) f$ and $ef = f$ for any $f \in F$. *Efficiency* means that the condition $sf = f$ for some s and any f implies that $s = e$.

Definition 2.1. A complex analytic space W together with a holomorphic map (projection) $\pi: W \to X$ is a *holomorphic bundle* on X with the *structure group* S and the *fibre* F, if there exists a system of coordinate transformations, i.e., if

(1) there is an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of X and a family of biholomorphisms $h_i: \pi^{-1}(U_i) \to U_i \times F$ that map "fibres" $\pi^{-1}(u)$ onto $u \times F$;

(2) for any $i,j\in I$ there are elements $s_{ij}\in\mathcal{O}(U_i\cap U_j,S)$ such that

$$(h_i h_j^{-1})(u\times f)=u\times s_{ij}(u)f \quad \text{for any } u\in U_i\cap U_j \text{ and } f\in F.$$

In particular, a holomorphic bundle $\pi: W\to X$ whose fibre is a Banach space F and the structure group is $\mathrm{GL}(F)$ (the group of linear invertible transformations of F) is a *holomorphic Banach vector bundle.*

A *holomorphic section* of a holomorphic bundle $\pi: W\to X$ is a holomorphic map $s: X\to W$ satisfying $\pi\circ s=\mathrm{id}$. Let $\pi_i: W_i\to X$, $i=1,2$, be holomorphic Banach vector bundles. A holomorphic map $f: W_1\to W_2$ satisfying

(a) $f(\pi_1^{-1}(x))\subset\pi_2^{-1}(x)$ for any $x\in X$;

(b) $f|_{\pi_1^{-1}(x)}$ is a linear continuous map of the corresponding Banach spaces, is a *homomorphism.* If, in addition, f is a homeomorphism, then f is an *isomorphism.*

We also use the following construction of holomorphic bundles (see, e.g. [Hi, Chapter 1]):

Let S be a complex analytic Lie group and $\mathcal{U}=\{U_i\}_{i\in I}$ be an open cover of X. By $Z^1_{\mathcal{O}}(\mathcal{U},S)$ we denote the set of holomorphic S-valued $\mathcal{U}$-cocycles. By definition, $s=\{s_{ij}\}\in Z^1_{\mathcal{O}}(\mathcal{U},S)$, where $s_{ij}\in\mathcal{O}(U_i\cap U_j,S)$ and $s_{ij}s_{jk}=s_{ik}$ on $U_i\cap U_j\cap U_k$. Consider the disjoint union $\bigsqcup_{i\in I}U_i\times F$ and for any $u\in U_i\cap U_j$ identify the point $u\times f\in U_j\times F$ with $u\times s_{ij}(u)f\in U_i\times F$. We obtain a holomorphic bundle W_s on X whose projection is induced by the projection $U_i\times F\to U_i$. Moreover, any holomorphic bundle on X with the structure group S and the fibre F is isomorphic (in the category of holomorphic bundles) to a bundle W_s.

Example 2.2(a). Let M be a complex manifold. For any subgroup $G\subset\pi_1(M)$ consider the unbranched covering $g: M_G\to M$ corresponding to G. We will describe M_G as a holomorphic bundle on M.

First, assume that $G\subset\pi_1(M)$ is a normal subgroup. Then M_G is a regular covering of M and the quotient group $Q:=\pi_1(M)/G$ acts holomorphically on M_G by deck transformations. It is well known that M_G in this case can be thought of as a principle fibre bundle on M with fibre Q (here Q is equipped with the discrete topology). Namely, let us consider the map $R_Q(g): Q\to Q$ defined by the formula

$$R_Q(g)(h)=hg^{-1}, \quad h\in Q.$$

Then there is an open cover $\mathcal{U}=\{U_i\}_{i\in I}$ of M by sets biholomorphic to open Euclidean balls in some $\mathbf{C}^n$ and a locally constant cocycle $c=\{c_{ij}\}\in Z^1_{\mathcal{O}}(\mathcal{U},Q)$ such that M_G is biholomorphic to the quotient space of the disjoint union $V=\bigsqcup_{i\in I}U_i\times Q$ by the equivalence relation $U_i\times Q\ni x\times R_Q(c_{ij})(h)\sim x\times h\in U_j\times Q$. The identification space is a holomorphic bundle with projection $p: M_G \to M$ induced by the projections $U_i \times Q \to U_i$. In particular, when $G=e$ we obtain the definition of the universal covering M_e of M.

Assume now that $G \subset \pi_1(M)$ is not necessarily normal. Let $X_G = \pi_1(M)/G$ be the set of cosets with respect to the (left) action of G on $\pi_1(M)$ defined by left multiplications. By $[Gq] \in X_G$ we denote the coset containing $q \in \pi_1(M)$. Let $H(X_G)$ be the group of all homeomorphisms of X_G (equipped with the discrete topology). We define the homomorphism $\tau: \pi_1(M) \to H(X_G)$ by

$$\tau(g)([Gq]) := [Gqg^{-1}], \quad q \in \pi_1(M).$$

Set $Q(G) := \pi_1(M)/\operatorname{Ker}\tau$ and let $\tilde{g}$ be the image of $g \in \pi_1(M)$ in $Q(G)$. We denote the unique homomorphism whose pullback to $\pi_1(M)$ coincides with τ by $\tau_{Q(G)}: Q(G) \to H(X_G)$. Consider the action of G on $V = \bigsqcup_{i \in I} U_i \times \pi_1(M)$ induced by the left action of G on $\pi_1(M)$ and let $V_G = \bigsqcup_{i \in I} U_i \times X_G$ be the corresponding quotient set. Define the equivalence relation $U_i \times X_G \ni x \times \tau_{Q(G)}(\tilde{c}_{ij})(h) \sim x \times h \in U_j \times X_G$ with the same $\{c_{ij}\}$ as in the definition of M_e. The corresponding quotient space is a holomorphic bundle with fibre X_G biholomorphic to M_G.

Example 2.2(b). We retain the notation of Example 2.2(a). Let B be a complex Banach space with norm $|\cdot|$. Let $\operatorname{Iso}(B) \subset \operatorname{GL}(B)$ be the group of linear isometries of B. Consider a homomorphism $\varrho: Q \to \operatorname{Iso}(B)$. Without loss of generality we assume that $\operatorname{Ker}\varrho = e$, for otherwise we can pass to the corresponding quotient group. The *holomorphic Banach vector bundle* $E_\varrho \to M$ *associated with* ϱ is defined as the quotient of $\bigsqcup_{i \in I} U_i \times B$ by the equivalence relation $U_i \times B \ni x \times \varrho(c_{ij})(w) \sim x \times w \in U_j \times B$ for any $x \in U_i \cap U_j$. Further, we can define a function $E_\varrho \to \mathbf{R}_+$ which will be called the *norm* on E_ϱ (and denoted by the same symbol $|\cdot|$). The construction is as follows. For any $x \times w \in U_i \times B$ we set $|x \times w| := |w|$. Since the image of ϱ belongs to $\operatorname{Iso}(B)$, the above definition is invariant with respect to the equivalence relation determining E_ϱ and so it determines a "norm" on E_ϱ. Let us consider some examples.

Let $l_1(Q)$ be the Banach space of complex-valued sequences on Q with l_1-norm. The action R_Q from (a) induces the homomorphism $\varrho: Q \to \operatorname{Iso}(l_1(Q))$,

$$\varrho(g)(w)[x] := w(R_Q(g)(x)), \quad g, x \in Q, \ w \in l_1(Q).$$

By $E_1^M(Q)$ we denote the holomorphic Banach vector bundle associated with ϱ.

Let $l_\infty(Q)$ be the Banach space of bounded complex-valued sequences on Q with l_∞-norm. The homomorphism $\varrho^*: Q \to \operatorname{Iso}(l_\infty(Q))$, dual to ϱ is defined as

$$\varrho^*(g)(v)[x] := v(xg^{-1}), \quad g, x \in Q, \ v \in l_\infty(Q).$$

(It coincides with the homomorphism $(\varrho^t)^{-1}$: $((\varrho^t)^{-1}(g)[v])(w):=v(\varrho(g^{-1})[w])$, $g\in Q$, $v\in l_\infty(Q)$, $w\in l_1(Q)$.) The holomorphic Banach vector bundle associated with ϱ^* will be denoted by $E_\infty^M(Q)$. By definition it is dual to $E_1^M(Q)$.

2.2. Main construction

Let B be a complex Banach space with norm $|\cdot|$ and let $\|\cdot\|$ denote the corresponding norm on $\mathrm{GL}(B)$. For a discrete set X, denote by $B_\infty(X)$ the Banach space of "sequences" $b:=\{(x,b(x))\}_{x\in X}$, $b(x)\in B$, with norm

$$|b|_\infty:=\sup_{x\in X}|b(x)|.$$

By definition, for $b_i=\{(x,b_i(x))\}_{x\in X}$, $\alpha_i\in\mathbf{C}$, $i=1,2$, we have

$$\alpha_1b_1+\alpha_2b_2=\{(x,\alpha_1b_1(x)+\alpha_2b_2(x))\}_{x\in X}.$$

Further, recall that a B-valued function $f:U\to B$ defined in an open set $U\subset\mathbf{C}^n$ is said to be *holomorphic* if f satisfies the B-valued Cauchy integral formula in any polydisk contained in U. Equivalently, if locally f can be represented as the sum of absolutely convergent holomorphic power series with coefficients in B. Now any family $\{(x,f_x)\}_{x\in X}$, where f_x is a B-valued function holomorphic on U satisfying $|f_x(z)|<A$ for any $z\in U$ and $x\in X$, can be considered as a $B_\infty(X)$-valued holomorphic function on U. In fact, the local Taylor expansion in this case follows from the Cauchy estimates of the coefficients in the Taylor expansion of each f_x.

Let $t:X\to X$ be a bijection and $h:X\times X\to\mathrm{GL}(B)$ be such that

$$h(t(x),x)\in\mathrm{GL}(B)\quad\text{and}\quad\max\Big\{\sup_{x\in X}\|h(t(x),x)\|,\sup_{x\in X}\|h^{-1}(x,t(x))\|\Big\}<\infty.$$

Then we can define $a(h,t)\in\mathrm{GL}(B_\infty(X))$ by the formula

$$a(h,t)[(x,b(x))]:=(t(x),h(t(x),x)[b(x)]),\quad b=\{(x,b(x))\}_{x\in X}\in B_\infty(X).$$

We retain the notation of Example 2.2. For the acyclic cover $\mathcal{U}=\{U_i\}_{i\in I}$ of M we have $g^{-1}(U_i)=\bigsqcup_{s\in X_G}V_{is}\subset M_G$ where $g|_{V_{is}}:V_{is}\to U_i$ is biholomorphic. Consider a holomorphic Banach vector bundle $\pi:E\to M_G$ with fibre B defined by coordinate transformations subordinate to the cover $\{V_{is}\}_{i\in I,s\in X_G}$ of M_G, i.e. by a holomorphic cocycle $h=\{h_{is,jk}\}\in Z^1_{\mathcal{O}}(g^{-1}(\mathcal{U}),\mathrm{GL}(B))$, $h_{is,jk}\in\mathcal{O}(V_{is}\cap V_{jk},\mathrm{GL}(B))$, such that E is biholomorphic to the quotient space of the disjoint union $\bigsqcup_{i,s}V_{is}\times B$ by the equivalence relation $V_{is}\times B\ni x\times h_{is,jk}(x)[v]\sim x\times v\in V_{jk}\times B$, $s:=\tau_{Q(G)}(\tilde{c}_{ij})(k)$. The projection π is induced by the coordinate projections $V_{is}\times B\to V_{is}$. Assume also that for any x,

$$\sup_{i,j,s,k}\max\{\|h_{is,jk}(x)\|,\|h^{-1}_{is,jk}(x)\|\}\le A<\infty. \tag{2.1}$$

Further, define $\tilde{\pi}:=g\circ\pi:E\to M$.

Proposition 2.3. *The triple $(E, M, \tilde{\pi})$ determines a holomorphic Banach vector bundle on M with fibre $B_\infty(X_G)$. (We denote this bundle by E_M.)*

Proof. Let $\phi_{is}: U_i \to V_{is}$ be the map inverse to $g|_{V_{is}}$. We identify $V_{is}\times B$ with $U_i\times s\times B$ by ϕ_{is}, and $\{s\times B\}_{s\in X_G}$ with $B_\infty(X_G)$. Further, for any $x\in U_i\cap U_j$, we set $\tilde{h}_{is,jk}(x):=h_{is,jk}(\phi_{is}(x))$. Then E can be defined as the quotient space of $\bigsqcup_{i\in I} U_i\times B_\infty(X_G)$ by the equivalence relation $U_j\times B_\infty(X_G)\ni x\times\{(k,b(k))\}_{k\in X_G}\sim x\times\{(\tau_{Q(G)}(\tilde{c}_{ij})(k), \tilde{h}_{i\tau_{Q(G)}(\tilde{c}_{ij})(k),jk}(x)[b(k)])\}_{k\in X_G}\in U_i\times B_\infty(X_G)$.

Define

$$\tilde{h}_{ij}(x): X_G\times X_G\to \mathrm{GL}(B), \quad x\in U_i\cap U_j,$$

and

$$d_{ij}\in \mathcal{O}(U_i\cap U_j, \mathrm{GL}(B_\infty(X_G)))$$

by the formulas

$$\tilde{h}_{ij}(x)(s,k):=\tilde{h}_{is,jk}(x)$$

and

$$d_{ij}(x)[b]:=a(\tilde{h}_{ij}(x), \tau_{Q(G)}(\tilde{c}_{ij}))[b], \quad b\in B_\infty(X_G).$$

Here holomorphy of d_{ij} follows from (2.1). Clearly, $d=\{d_{ij}\}$ is a holomorphic cocycle with values in $\mathrm{GL}(B_\infty(X_G))$, because $\{h_{is,jk}\}$ and $\{\tau_{Q(G)}(\tilde{c}_{ij})\}$ are cocycles. Now E can be considered as a holomorphic Banach vector bundle on M with fibre $B_\infty(X_G)$ obtained by identification in $\bigsqcup_{i\in I} U_i\times B_\infty(X_G)$ of $x\times d_{ij}(x)[b]\in U_i\times B_\infty(X_G)$ with $x\times b\in U_j\times B_\infty(X_G)$, $x\in U_i\cap U_j$. Moreover, according to our construction the projection $E\to M$ coincides with $\tilde{\pi}$. $\square$

Let W be a holomorphic Banach vector bundle on a complex analytic space X. In what follows, by $\mathcal{O}(U,W)$ we denote the vector space of holomorphic sections of W defined in an open set $U\subset X$.

We retain the notation of Proposition 2.3. By the construction of Proposition 2.3, a fibre $(\tilde{\pi})^{-1}(z)$, $z\in U_i$, of E_M can be identified with $\prod_{s\in X_G}\pi^{-1}(\phi_{is}(z))$ such that if also $z\in U_j$ then

$$(2.2) \qquad \prod_{s\in X_G}\pi^{-1}(\phi_{is}(z))=\prod_{s\in X_G}\pi^{-1}(\phi_{j\tau_{Q(G)}(\tilde{c}_{ji})(s)}(z)).$$

We recall the following definitions.

Let J_q be the set of sequences $(i_0s_0,\dots,i_qs_q)$ with $i_t\in I$, $s_t\in X_G$ for $t=0,\dots,q$. A family

$$f=\{f_{i_0s_0,\dots,i_qs_q}\}_{(i_0s_0,\dots,i_qs_q)\in J_q}, \quad f_{i_0s_0,\dots,i_qs_q}\in\mathcal{O}(V_{i_0s_0}\cap\dots\cap V_{i_qs_q}, E)$$

is a *q-cochain* on the cover $g^{-1}(\mathcal{U}):=\bigcup_{i\in I,s\in X_G} V_{is}$ of M_G with coefficients in the sheaf of germs of holomorphic sections of E. These cochains generate a complex vector space $C^q(g^{-1}(\mathcal{U}),E)$. In the trivialization which identifies $\pi^{-1}(V_{i_0s_0})$ with $V_{i_0s_0}\times B$ any $f_{i_0s_0,\dots,i_qs_q}$ is represented by $b_{i_0s_0,\dots,i_qs_q}\in\mathcal{O}(V_{i_0s_0}\cap\dots\cap V_{i_qs_q},B)$. Assume that for any $(i_0s_0,\dots,i_qs_q)\in J_q$ and any compact $K\subset U_{i_0}\cap\dots\cap U_{i_q}$ there is a constant $C=C(K)$ such that

$$\sup_{\substack{s_0,\dots,s_q\\ z\in K}} |(b_{i_0s_0,\dots,i_qs_q}\circ\phi_{i_0s_0})(z)|<C. \tag{2.3}$$

The set of cochains f satisfying (2.3) is a vector subspace of $C^q(g^{-1}(\mathcal{U}),E)$ which will be denoted by $C^q_b(g^{-1}(\mathcal{U}),E)$. Further, the formula

$$(\delta^q f)_{i_0s_0,\dots,i_{q+1}s_{q+1}}=\sum_{k=0}^{q+1}(-1)^k r_W^{W_k}(f_{i_0s_0,\dots,\widehat{i_ks_k},\dots,i_{q+1}s_{q+1}}), \tag{2.4}$$

where $f\in C^q(g^{-1}(\mathcal{U}),E)$, determines a homomorphism

$$\delta^q: C^q(g^{-1}(\mathcal{U}),E)\longrightarrow C^{q+1}(g^{-1}(\mathcal{U}),E).$$

Here $\widehat{}$ over a symbol means that this symbol is omitted. Moreover, we set $W=V_{i_0s_0}\cap\dots\cap V_{i_{q+1}s_{q+1}}$, $W_k=V_{i_0s_0}\cap\dots\cap\widehat{V}_{i_ks_k}\cap\dots\cap V_{i_{q+1}s_{q+1}}$ and $r_W^{W_k}$ is the restriction map from W to W_k. Also, condition (2.1) implies that δ^q maps $C^q_b(g^{-1}(\mathcal{U}),E)$ into $C^{q+1}_b(g^{-1}(\mathcal{U}),E)$. We will denote $\delta^q|_{C^q_b(g^{-1}(\mathcal{U}),E)}$ by δ^q_b. As usual, $\delta^{q+1}\circ\delta^q=0$ and $\delta^{q+1}_b\circ\delta^q_b=0$. Thus one can define cohomology groups on the cover $g^{-1}(\mathcal{U})$ by

$$H^q(g^{-1}(\mathcal{U}),E):=\operatorname{Ker}\delta^q/\operatorname{Im}\delta^{q-1}\quad\text{and}\quad H^q_b(g^{-1}(\mathcal{U}),E):=\operatorname{Ker}\delta^q_b/\operatorname{Im}\delta^{q-1}_b.$$

In what follows the cohomology group $H^q(\mathcal{U},E_M)$ on the cover $\mathcal{U}$ of M with coefficients in the sheaf of germs of holomorphic sections of E_M is defined similarly to $H^q(g^{-1}(\mathcal{U}),E)$. Elements of $\operatorname{Ker}\delta^q$ and $\operatorname{Ker}\delta^q_b$ will be called q-cocycles and of $\operatorname{Im}\delta^{q-1}$ and $\operatorname{Im}\delta^{q-1}_b$ q-coboundaries.

Proposition 2.4. *There is a linear isomorphism*

$$\Phi^q: H^q_b(g^{-1}(\mathcal{U}),E)\longrightarrow H^q(\mathcal{U},E_M).$$

Proof. Let $f=\{f_{i_0s_0,\dots,i_qs_q}\}\in C^q_b(g^{-1}(\mathcal{U}),E)$. Let furthermore $b_{i_0s_0,\dots,i_qs_q}\in\mathcal{O}(V_{i_0s_0}\cap\dots\cap V_{i_qs_q},B)$ be the representation of $f_{i_0s_0,\dots,i_qs_q}$ in the trivialization identifying $\pi^{-1}(V_{i_0s_0})$ with $V_{i_0s_0}\times B$. If $V_{i_0s_0}\cap\dots\cap V_{i_qs_q}\neq\emptyset$ then $s_k=\tau_{Q(G)(\tilde{c}_{i_ki_0})}(s_0)$,

$k=0,\dots,q$, and $U_{i_0}\cap\dots\cap U_{i_q}\neq\emptyset$. For otherwise, $b_{i_0s_0,\dots,i_qs_q}=0$. Thus for $s_0,\dots,s_q$ satisfying the above identities we can define

$$\tilde{b}_{i_0,\dots,i_q}:=\{b_{i_0s_0,\dots,i_qs_q}\circ\varphi_{i_0s_0}\}_{s_0\in X_G}.$$

For $U_{i_0}\cap\dots\cap U_{i_q}=\emptyset$ we set $\tilde{b}_{i_0,\dots,i_q}=0$. Further, by (2.3), $\tilde{b}_{i_0,\dots,i_q}\in\mathcal{O}(U_{i_0}\cap\dots\cap U_{i_q}, B_\infty(X_G))$. This implies that

$$\tilde{f}_{i_0,\dots,i_q}:=\{f_{i_0s_0,\dots,i_qs_q}\circ\varphi_{i_0s_0}\}_{s_0\in X_G},$$

defined similarly to $\tilde{b}_{i_0,\dots,i_q}$, belongs to $\mathcal{O}(U_{i_0}\cap\dots\cap U_{i_q}, E_M)$, because $\tilde{b}_{i_0,\dots,i_q}$ is just another representation of $\tilde{f}_{i_0,\dots,i_q}$ under identification of $\pi^{-1}(U_{i_0})$ with $U_{i_0}\times B_\infty(X_G)$. For $\tilde{f}=\{\tilde{f}_{i_0,\dots,i_q}\}$ we set $\widetilde{\Phi}^q(f)=\tilde{f}$. Then, clearly, $\widetilde{\Phi}^q: C_b^q(g^{-1}(\mathcal{U}), E)\to C^q(\mathcal{U}, E_M)$ is linear and injective. Now for a cochain $\tilde{f}\in C^q(\mathcal{U}, E_M)$ we can convert the construction for $\widetilde{\Phi}^q$ to find a cochain $f\in C_b^q(g^{-1}(\mathcal{U}), E)$ such that $\widetilde{\Phi}^q(f)=\tilde{f}$. Thus $\widetilde{\Phi}^q$ is an isomorphism. Moreover, a simple calculation based on (2.2) shows that

$$\delta^q\circ\widetilde{\Phi}^q=\widetilde{\Phi}^{q+1}\circ\delta_b^q, \tag{2.5}$$

where δ^q in the left-hand side is the operator for E_M defined similarly to (2.4). Hence $\widetilde{\Phi}^q$ determines a linear isomorphism $\Phi^q: H_b^q(g^{-1}(\mathcal{U}), E)\to H^q(\mathcal{U}, E_M)$. □

We close this section by the following result.

Proposition 2.5. *Let $\varrho: G\to\mathrm{Iso}(B)$ be a homomorphism and $E_\varrho\to M_G$ be the holomorphic Banach vector bundle associated with ϱ. Then E_ϱ satisfies the conditions of Proposition* 2.3.

Proof. Let $M_e\to M_G$ be the universal covering (recall that $G=\pi_1(M_G)$). Since the open cover $g^{-1}(\mathcal{U})=\{V_{is}\}_{i\in I, s\in X_G}$ of M_G is acyclic, M_e can be defined with respect to $g^{-1}(\mathcal{U})$. Namely, there is a cocycle $h=\{h_{is,jk}\}\in Z_{\mathcal{O}}^1(g^{-1}(\mathcal{U}), G)$ such that M_e is biholomorphic to the quotient space of $\bigsqcup_{i,s}V_{is}\times G$ by the equivalence relation $V_{is}\times G\ni x\times R_G(h_{is,jk})(f)\sim x\times f\in V_{jk}\times G$, $s=\tau_{Q(G)}(\tilde{c}_{ij})(k)$; here $R_G(q)(f):=fq^{-1}$, $f,q\in G$. Now E_ϱ is biholomorphic to the quotient space of $\bigsqcup_{i,s}V_{is}\times B$ by the equivalence relation $V_{is}\times B\ni x\times\varrho(h_{is,jk})(v)\sim x\times v\in V_{jk}\times B$. Clearly, the family $\{\varrho(h_{is,jk})\}$ satisfies the estimate (2.1). □

3. Proofs of Theorem 1.1 and Corollaries 1.3 and 1.6

Proof of Theorem 1.1

Let us briefly describe the basic idea of the proof.

First we construct the required projectors locally over simply connected sets. The differences of these local projectors form a holomorphic 1-cocycle with values in a certain holomorphic Banach vector bundle. Then we will prove that this cocycle is a coboundary satisfying some boundedness condition. This will do the job.

Let $N \Subset M$ be an open connected subset of a connected Stein manifold M satisfying (1.2). Let $G \subset \pi_1(M)$ be a subgroup. As before, by M_G and N_G we denote the covering spaces of M and N corresponding to G. Then by the covering homotopy theorem (see e.g. [Hu, Chapter III, Section 16]), there is a holomorphic embedding $N_G \hookrightarrow M_G$. Without loss of generality we regard N_G as an open subset of M_G. Denote also by $g_{MG}: M_G \to M$ and $g_{NG}: N_G \to N$ the corresponding projections such that $g_{MG}|_{N_G} = g_{NG}$. Let $i: U \hookrightarrow N_G$ be a holomorphic embedding of a complex connected manifold U.

Lemma 3.1. *It suffices to prove the theorem under the assumption that the homomorphism $i_*: \pi_1(U) \to G (= \pi_1(N_G))$ is surjective.*

Proof. Assume that $G' := \operatorname{Im} i_*$ is a proper subgroup of G. By $t: N_{G'} \to N_G$ we denote the covering of N_G corresponding to $G' \subset G$. By definition, $g_{NG} \circ t = g_{NG'}: N_{G'} \to N$ is the covering of N corresponding to $G' \subset \pi_1(N)$. Further, by the covering homotopy theorem there is a holomorphic embedding $i': U \hookrightarrow N_{G'}$ such that $t \circ i' = i$, $\operatorname{Ker} i'_* = \operatorname{Ker} i_*$, and $i'_*: \pi_1(U) \to G' (= \pi_1(N_{G'}))$ is surjective. Clearly, it suffices to prove the theorem for $i'(U) \subset N_{G'}$. □

In what follows we assume that i_* is surjective. By $p_U: \widetilde{U} \to U$ we denote the regular covering of U corresponding to $K(U) := \operatorname{Ker} i_*$, where $\pi_1(\widetilde{U}) = K(U)$. Consider the holomorphic Banach vector bundle $E_1^{M_G}(G) \to M_G$ associated with the homomorphism $\varrho_G: G \to \operatorname{Iso}(l_1(G))$, $[\varrho_G(g)(v)](x) := v(xg^{-1})$, $v \in l_1(G)$, $x, g \in G$ (see Example 2.2(b)). Since i_* is surjective, $E_1^{M_G}(G)|_U = E_1^U(G)$.

Let $K_G \subset l_1(G)$ be the kernel of the linear functional $l_1(G) \ni \{v_g\}_{g \in G} \mapsto \sum_{g \in G} v_g$. Then K_G is invariant with respect to any $\varrho_G(g)$, $g \in G$. In particular, ϱ_G determines a homomorphism $h_G: G \to \operatorname{Iso}(K_G)$, $h_G(g) = \varrho_G(g)|_{K_G}$. Here we consider K_G with the norm induced by the norm of $l_1(G)$. Let $F_G \to M_G$ be the holomorphic Banach vector bundle associated with h_G. Clearly, F_G is a subbundle of $E_1^{M_G}(G)$. Further, the quotient bundle $C_G := E_1^{M_G}(G)/F_G \to M_G$ is the trivial flat vector bundle of complex rank 1. Indeed, it is associated with the quotient homomorphism $\tilde{h}_G: G \to \mathbf{C}^*$, $\tilde{h}_G(g)(v + K_G) := \varrho_G(g)(v) + K_G$, $g \in G$, $v \in l_1(G)$, where $w + K_G$ is the image of $w \in l_1(G)$ in the factor space $l_1(G)/K_G = \mathbf{C}$. This homomorphism is trivial because

$\varrho_G(g)(v)-v\in K_G$ by definition. Thus we have the short exact sequence

$$0\longrightarrow F_G\longrightarrow E_1^{M_G}(G)\xrightarrow{k_G} C_G\longrightarrow 0. \tag{3.1}$$

Our goal is to construct a holomorphic section $I_G: C_G\to E_1^{M_G}(G)$ (linear on the fibres) such that $k_G\circ I_G=\mathrm{id}$. Then we obtain the bundle decomposition $E_1^{M_G}(G)=I_G(C_G)\oplus F_G$.

Let $\{t_s\}_{s\in G}$ be a standard basis of unit vectors in $l_1(G)$, $t_s(g)=\delta_{sg}$, $s,g\in G$. Define $A:\mathbf{C}\to l_1(G)$ by $A(c)=ct_e$, where $e\in G$ is the unit. Then A is a linear operator of norm 1. Now let us recall the construction of $E_1^{M_G}(G)$ given in Proposition 2.5.

Let $M_e\to M_G$ be the universal covering. Consider the open cover $g_{MG}^{-1}(\mathcal{U})=\{V_{G,is}\}_{i\in I,s\in X_G}$ of M_G, where $\mathcal{U}:=\{U_i\}_{i\in I}$ is an open cover of M by complex balls, and $\bigsqcup_{s\in X_G} V_{G,is}=g^{-1}(U_i)$. Then there is a cocycle $c_G=\{c_{G,is,jk}\}\in Z_{\mathcal{O}}^1(g_{MG}^{-1}(\mathcal{U}),G)$ such that $E_1^{M_G}(G)$ is biholomorphic to the quotient space of $\bigsqcup_{i,s} V_{G,is}\times l_1(G)$ by the equivalence relation $V_{G,is}\times l_1(G)\ni x\times\varrho_G(c_{G,is,jk})(v)\sim x\times v\in V_{G,jk}\times l_1(G)$. The construction of F_G is similar, the only difference is that in the above formula we take h_G instead of ϱ_G. These constructions restricted to $V_{G,is}$ determine isomorphisms of holomorphic Banach vector bundles: $e_{G,is}: E_1^{M_G}(G)|_{V_{G,is}}\to V_{G,is}\times l_1(G)$, $f_{G,is}: F_G|_{V_{G,is}}\to V_{G,is}\times K_G$ and $c_{G,is}: C_G|_{V_{G,is}}\to V_{G,is}\times\mathbf{C}$. Then we define $A_{G,is}: C_G\to E_1^{M_G}(G)$ on $V_{G,is}$ as $e_{G,is}^{-1}\circ A'\circ c_{G,is}$, where $A'(x\times c):=x\times A(c)$, $x\in V_{G,is}$, $c\in\mathbf{C}$. Clearly, $k_G\circ A_{G,is}=\mathrm{id}$ on $V_{G,is}$. Thus

$$B_{G,is,jk}:=A_{G,is}-A_{G,jk}: C_G|_{V_{G,is}\cap V_{G,jk}}\longrightarrow F_G|_{V_{G,is}\cap V_{G,jk}}$$

is a homomorphism of bundles of norm ≤ 2 on each fibre (here the norms on F_G, C_G and $E_1^{M_G}(G)$ are defined as in Example 2.2(b)). We also use the identification $\mathrm{Hom}(C_G,F_G)\cong F_G$ (this is because C_G is trivial and $\mathrm{Hom}(\mathbf{C},K_G)\cong\mathbf{C}^*\otimes K_G=K_G$). Further, according to Proposition 2.5, the holomorphic Banach vector bundle $\mathrm{Hom}(C_G,F_G)$ associated with the homomorphism $\tilde{h}_G\otimes h_G: G\to\mathrm{Iso}(\mathrm{Hom}(\mathbf{C},K_G))$ satisfies the conditions of Proposition 2.3. Therefore, by definition, $B_G=\{B_{G,is,jk}\}$ is a holomorphic 1-cocycle with respect to δ_b^1 defined on the cover $g_{MG}^{-1}(\mathcal{U})$. By $\phi_{G,is}: U_i\to V_{G,is}$ we denote the map inverse to $g_{MG}|_{V_{G,is}}$. Next we will prove the following lemma.

Lemma 3.2. *There is $\widetilde{B}_G=\{\widetilde{B}_{G,is}\}\in C_b^0(g_{MG}^{-1}(\mathcal{U}),F_G)$, $\widetilde{B}_{G,is}\in\mathcal{O}(V_{is},F_G)$, so that $\delta_b^0(\widetilde{B}_G)=B_G$. Moreover, for any $i\in I$ there is a continuous non-negative function $F_i: U_i\to\mathbf{R}_+$ such that for any G,*

$$\sup_{\substack{s\in X_G\\ z\in U_i}} |(\widetilde{B}_{G,is}\circ\phi_{G,is})(z)|\le F_i(z). \tag{3.2}$$

Here $|\cdot|$ denotes the norm on F_G.

Proof. According to Proposition 2.3, we can construct the holomorphic Banach vector bundle $(F_G)_M$. It is defined on the cover $\mathcal{U}$ of M by a cocycle $d_G:=\{d_{G,ij}\}\in Z^1_{\mathcal{O}}(\mathcal{U},\mathrm{Iso}((K_G)_\infty(X_G)))$, where $d_{G,ij}\in\mathcal{O}(U_i\cap U_j,\mathrm{Iso}((K_G)_\infty(X_G)))$. Let $\widetilde{\Phi}^q_G: C^q_b(g^{-1}_{MG}(\mathcal{U}),F_G)\to C^q(\mathcal{U},(F_G)_M)$ be the isomorphism defined in the proof of Proposition 2.4. Then $\widetilde{\Phi}^1_G(B_G):=b_G=\{b_{G,ij}\}$ is a holomorphic 1-cocycle with respect to δ^1 defined on $\mathcal{U}$. Here $b_{G,ij}\in\mathcal{O}(U_i\cap U_j,(F_G)_M)$, and

$$\sup_{\substack{i,j\in I\\ z\in M}} |b_{G,ij}(z)|_{(F_G)_M}\le 2,$$

where $|\cdot|_{(F_G)_M}$ stands for the norm on $(F_G)_M$.

Let $\mathcal{G}$ be the set of all subgroups $G\subset\pi_1(M)$. We define the Banach space $K=\bigoplus_{G\in\mathcal{G}}(K_G)_\infty(X_G)$ such that $x=\{x_G\}_{G\in\mathcal{G}}$ belongs to K if $x_G\in(K_G)_\infty(X_G)$ and

$$|x|:=\sup_{G\in\mathcal{G}}|x_G|_{(K_G)_\infty(X_G)}<\infty,$$

where $|\cdot|_{(K_G)_\infty(X_G)}$ is the norm on $(K_G)_\infty(X_G)$. Further, let us define $d:=\{d_{ij}\}\in Z^1_{\mathcal{O}}(\mathcal{U},\mathrm{Iso}(K))$ as $d:=\bigoplus_{G\in\mathcal{G}}d_G$. Here

$$d_{ij}:=\bigoplus_{G\in\mathcal{G}}d_{G,ij},\quad [d_{ij}(z)](\{v_G\}_{G\in\mathcal{G}}):=\{[d_{G,ij}(z)](v_G)\}_{G\in\mathcal{G}},\ z\in U_i\cap U_j.$$

Clearly $d_{ij}\in\mathcal{O}(U_i\cap U_j,\mathrm{Iso}(K))$. Now we define the holomorphic Banach vector bundle F on M by the identification $U_i\times K\ni x\times d_{ij}(x)[v]\sim x\times v\in U_j\times K$ for any $x\in U_i\cap U_j$. In fact, this bundle coincides with $\bigoplus_{G\in\mathcal{G}}(F_G)_M$. A vector f of F over $z\in M$ can be identified with a family $\{f_G\}_{G\in\mathcal{G}}$ so that $f_G\in(F_G)_M$ is a vector over z. Moreover, the norm $|f|_F:=\sup_{G\in\mathcal{G}}|f_G|_{(F_G)_M}$ of f is finite. Now we can define a holomorphic 1-cocycle $b=\{b_{ij}\}$ of F on the cover $\mathcal{U}$ as

$$b:=\{b_G\}_{G\in\mathcal{G}},\quad b_{ij}:=\{b_{G,ij}\}_{G\in\mathcal{G}}\in\mathcal{O}(U_i\cap U_j,F).$$

Here holomorphy of b_{ij} follows from the uniform estimate of the norms of $b_{G,ij}$.

Next, we use the fact that M is a Stein manifold. According to the theorem of Bungart [B, Section 4] (i.e. the version of the classical Cartan Theorem B for cohomology of sheaves of germs of holomorphic sections of holomorphic Banach vector bundles), a cocycle b represents 0 in the corresponding cohomology group $H^1(M,F)$. Further, the cover $\{U_i\}_{i\in I}$ of M consists of Stein manifolds (and so it is acyclic). Therefore by the classical Leray theorem (on calculation of cohomology groups by acyclic covers),

$$H^1(M,F)=H^1(\mathcal{U},F).$$

Thus b represents 0 in $H^1(\mathcal{U},F)$, that is, b is a coboundary. In particular, there are holomorphic sections $b_i\in\mathcal{O}(U_i,F)$ such that

$$b_i(z)-b_j(z)=b_{ij}(z) \quad \text{for any } z\in U_i\cap U_j.$$

We also set

$$F_i(z):=|b_i(z)|_F.$$

Then F_i is a continuous non-negative function on U_i. Further, by definition each b_i can be represented as a family $\{b_{G,i}\}_{G\in\mathcal{G}}$, where $b_{G,i}\in\mathcal{O}(U_i,(F_G)_M)$. The family $\tilde b_G=\{b_{G,i}\}_{i\in I}$ belongs to $C^0(\mathcal{U},(F_G)_M)$. Using the isomorphism $\widetilde\Phi^0_G$ from Proposition 2.4 we obtain a cochain $\widetilde B_G:=[\widetilde\Phi^0_G]^{-1}(\tilde b_G)\in C^0_b(g^{-1}_{MG}(\mathcal{U}),F_G)$. Now if $\widetilde B_G:=\{\widetilde B_{G,is}\}$, $\widetilde B_{G,is}\in\mathcal{O}(V_{is},F_G)$, it follows from identity (2.5) that

$$\widetilde B_{G,is}(z)-\widetilde B_{G,jk}(z)=B_{G,is,jk}(z) \quad \text{for any } z\in V_{is}\cap V_{jk}.$$

Finally, inequality (3.2) is the consequence of the definitions of F_i and $\widetilde\Phi^0_G$. $\square$

Let us consider now the family $\{A_{G,is}-B_{G,is}\}_{i,s}$. By definition, it determines a holomorphic linear section $I_G\colon C_G\to E_1^{M_G}(G)$, $k_G\circ I_G=\mathrm{id}$. Thus we have $E_1^{M_G}(G)=I_G(C_G)\oplus F_G$. In the next result the norm $\|\cdot\|$ of I_G is defined with respect to the norms $|\cdot|_{C_G}$ and $|\cdot|_{E_1^{M_G}(G)}$.

Lemma 3.3. *There is a constant $C=C(N)$ such that for any $G\in\mathcal{G}$,*

$$\sup_{z\in N}\|I_G(z)\|\le C.$$

Proof. Let $\mathcal{V}=\{V_i\}_{i\in I}$ be a refinement of the cover $\mathcal{U}$ of M such that each V_i is relatively compact in some $U_{k(i)}$. Then from Lemma 3.2 it follows that

$$\sup_{\substack{s\in X_G\\ z\in V_i}}|(\widetilde B_{G,k(i)s}\circ\phi_{G,k(i)s})(z)|\le\sup_{z\in V_i}F_{k(i)}(z)=C_i<\infty.$$

Now for any $z\in g^{-1}_{MG}(V_i)$ we have

$$\|I_G(z)\|\le\sup_{\substack{s\in X_G\\ y\in V_i}}(\|(A_{G,k(i)s}\circ\phi_{G,k(i)s})(y)\|+\|(\widetilde B_{G,k(i)s}\circ\phi_{G,k(i)s})(y)\|)\le 1+C_i.$$

Since $\overline N\subset M$ is compact, we can find a finite number of sets $V_{i_1},\dots,V_{i_l}$ which cover $\overline N$. Then

$$\sup_{z\in N}\|I_G(z)\|\le\max_{1\le t\le l}\{1+C_{i_t}\}:=C<\infty. \quad \square$$

Consider now the restriction of the exact sequence (3.1) to U. Using the identification $E_1^{M_G}(G)|_U \cong E_1^U(G)$ we obtain

$$0 \longrightarrow F_G|_U \longrightarrow E_1^U(G) \longrightarrow C_G|_U \longrightarrow 0.$$

Similarly, we have the dual sequence obtained by taken the dual bundles in the above sequence

$$0 \longrightarrow [C_G|_U]^* \longrightarrow E_\infty^U(G) \longrightarrow [F_G|_U]^* \longrightarrow 0.$$

Let $c(G)$ be the space of constant functions in $l_\infty(G)$. By definition, $[C_G|_U]^*$ is a subbundle of $E_\infty^U(G)$ of complex rank 1 with fibre $c(G)$ associated with the trivial homomorphism $G \mapsto \mathrm{Iso}(c(G))$. Let $P_U := [I_G|_U]^* : E_\infty^U(G) \to [C_G|_U]^*$ be the homomorphism of bundles dual to $I_G|_U$. Then for any $z \in U$, $P_U(z)$ projects the fibre of $E_\infty^U(G)$ over z onto the fibre of $[C_G|_U]^*$ over z. Moreover, we have

$$\sup_{z\in U} \|P_U(z)\| \le C, \tag{3.3}$$

where $\|\cdot\|$ is the dual norm defined with respect to $|\cdot|_{E_\infty^U(G)}$ and $|\cdot|_{[C_G|_U]^*}$. The operator P_U induces also a linear map $P_U' : \mathcal{O}(U, E_\infty^U(G)) \to \mathcal{O}(U, [C_G|_U]^*)$,

$$[P_U'(f)](z) := [P_U(z)](f(z)), \quad f \in \mathcal{O}(U, E_\infty^U(G)).$$

Further, any $f \in H^\infty(\widetilde{U})$ can be considered in a natural way as a bounded holomorphic section of the trivial bundle $\widetilde{U} \times \mathbf{C} \to \widetilde{U}$. This bundle satisfies the assumptions of Proposition 2.5 (for U instead of M). Furthermore, it easy to see that in this case the bundle $(\widetilde{U} \times \mathbf{C})_U$ defined in Proposition 2.3 coincides with $E_\infty^U(G)$. Let $\Phi_U^0 : H_b^0(g^{-1}(\mathcal{U}), \widetilde{U} \times \mathbf{C}) \to H^0(\mathcal{U}, E_\infty^U(G))$ be the isomorphism of Proposition 2.4. (This is just the direct image map with respect to $p_U : \widetilde{U} \to U$.) We define the Banach subspace $S_\infty(U) \subset H^0(\mathcal{U}, E_\infty^U(G))$ with norm $|\cdot|_U$ by the formula

$$f \in S_\infty(U) \quad \Longleftrightarrow \quad |f|_U := \sup_{z\in U} |f(z)|_{E_\infty^U(G)} < \infty.$$

Clearly Φ_U^0 maps $H^\infty(\widetilde{U})$ isomorphically onto $S_\infty(U)$. Moreover, $s_U := \Phi_U^0|_{H^\infty(\widetilde{U})}$ is a linear isometry of Banach spaces. By definition, the space $s_U(p_U^*(H^\infty(U)))$ coincides with $\mathcal{O}(U, [C_U|_U]^*) \cap S_\infty(U)$. Then according to the definition of s_U and the inequality (3.3), the linear operator $P := s_U^{-1} \circ P_U' \circ s_U$ maps $H^\infty(\widetilde{U})$ onto $p_U^*(H^\infty(U))$. By our construction P is a bounded projector satisfying (1). Here the required projector $P_z : l^\infty(F_z) \to c(F_z)$ can be naturally identified with $P_U(z)$. Let now $f \in H^\infty(\widetilde{U})$ and $g \in p_U^*(H^\infty(U))$. Then by definition we have

$$P[fg]|_{p_U^{-1}(z)} = P_z[(fg)|_{p_U^{-1}(z)}] = P_z[f|_{p_U^{-1}(z)}]g|_{p_U^{-1}(z)} = (P[f]g)|_{p_U^{-1}(z)}.$$

Here we used that $g|_{p_U^{-1}(z)}$ is a constant and P_z is a linear operator. This implies (2). Property (3) follows from the fact that P_z is a projector onto $c(F_z)$. Further, (4) is a consequence of the fact that $P_U(z)$ is dual to $(I_G)(z)$ and so P_z is continuous in the weak* topology of $l_\infty(F_z)$. Finally, the norm of P coincides with $\sup_{z\in U}\|P_U(z)\|$. Thus $\|P\|\le C$ for C as in (3.3). This completes the proof of (5). □

Proof of Corollary 1.3. First note that any finite bordered Riemann surface N admits an embedding to a Riemann surface M so that the pair $N\Subset M$ satisfies condition (1.2). Let $\widetilde{R}$ be a covering of N and $i: U\hookrightarrow\widetilde{R}$ be such that $\pi_1(U)$ is generated by a subfamily of generators of the free group $\pi_1(\widetilde{R})$. Then the homomorphism $i_*:\pi_1(U)\to\pi_1(\widetilde{R})$ is injective. In particular, $K(U):=\operatorname{Ker} i_*=\{1\}$ and $p_U:\widetilde{U}\to U$ is the universal covering. Since $\widetilde{U}$ is biholomorphic to $\mathbf{D}$, the existence of the projector $P: H^\infty(\mathbf{D})\to p_U^*(H^\infty(U))$ follows from Theorem 1.1. □

Proof of Corollary 1.6. Let $N\Subset M$, $R\subset\mathcal{F}_c(N)$ and $i: U\hookrightarrow R$ be open Riemann surfaces satisfying the hypotheses of Theorem 1.1. Assume also that $K(U):=\operatorname{Ker} i_*=\{1\}$. Let $p_U:\mathbf{D}\to U$ be the universal covering map. Then there is a projector $P: H^\infty(\mathbf{D})\to p_U^*(H^\infty(U))$ with properties (1)–(5) of Theorem 1.1. Let $f_1,\dots,f_n\in H^\infty(U)$ satisfy the corona condition (1.3) with $\delta>0$. Without loss of generality we will assume also that $\max_i\|f_i\|_{H^\infty(U)}\le 1$. For $1\le i\le n$ we set $h_i:=p_U^*(f_i)$. Then $h_1,\dots,h_n\in H^\infty(\mathbf{D})$ satisfy the corona condition in $\mathbf{D}$ (with the same δ). Also $\max_i\|h_i\|_{H^\infty(\mathbf{D})}\le 1$. Now according to the solution of the Carleson Corona theorem [Ca2], there are a constant $C(n,\delta)$ and $g_1,\dots,g_n\in H^\infty(\mathbf{D})$ satisfying $\max_i\|g_i\|_{H^\infty(\mathbf{D})}\le C(n,\delta)$ such that $\sum_{i=1}^n g_ih_i\equiv 1$. Let us define $d_i\in H^\infty(U)$ by the formula

$$p_U^*(d_i):=P[g_i],\quad 1\le i\le n.$$

Then property (2) for P implies that $\sum_{i=1}^n d_if_i\equiv 1$. Moreover, $\max_i\|d_i\|_{H^\infty(U)}\le C(N)C(n,\delta)$, where $C(N)$ is the constant from Lemma 3.3. □

4. Proof of Theorem 1.9

Let $N\Subset M$ be a relatively compact domain of a connected Stein manifold M satisfying (1.2). For a subgroup $G\subset\pi_1(M)$ we denote by $g_{NG}: N_G\to N$ and $g_{MG}: M_G\to M$ the covering spaces of M and N corresponding to the group G with $N_G\subset M_G$. Further, assume that $i: U\hookrightarrow N_G$ is a holomorphic embedding of a complex connected manifold U, $K(U):=\operatorname{Ker} i_*\subset\pi_1(U)$, and $p_U:\widetilde{U}\to U$ is the regular covering of U corresponding to $K(U)$. As before, without loss of generality we may assume that homomorphism $i_*:\pi_1(U)\to G(=\pi_1(N_G))$ is surjective (see the arguments of Lemma 3.1). Thus the deck transformation group of $\widetilde{U}$ is G. We begin the proof of the theorem with the following result.

Proposition 4.1. *For any $z \in U$, the sequence $p_U^{-1}(z) := \{w_s\}_{s\in G} \subset \widetilde{U}$ is interpolating with respect to $H^\infty(\widetilde{U})$. Moreover, let*

$$M(z) = \sup_{\|a_s\|_{l^\infty(G)} \le 1} \inf\{\|g\|_{H^\infty(\widetilde{U})} : g \in H^\infty(\widetilde{U}),\ g(w_s) = a_s,\ j = 1, 2, \ldots\}$$

be the constant of interpolation for $p_U^{-1}(z)$. Then there is a constant $C = C(N)$ such that

$$\sup_{z\in U} M(z) \le C.$$

Proof. Consider the homomorphism $\varrho_G^*: G \to \mathrm{Iso}(l_\infty(G))$,

$$[\varrho_G^*(g)(w)](x) := w(xg^{-1}), \quad w \in l_\infty(G),\ x, g \in G.$$

Let $E_\infty^{M_G}(G) \to M_G$ be the holomorphic Banach vector bundle associated with ϱ_G^*. Then $E_\infty^{M_G}(G)|_U = E_\infty^U(G)$ (see Example 2.2(b)). According to Proposition 2.5, we can define the holomorphic Banach vector bundle $[E_\infty^{M_G}(G)]_M \to M$ with the fibre $[l^\infty(G)]_\infty(X_G)$. Let $\mathcal{G}$ be the set of all subgroups $G \subset \pi_1(M)$. We define the Banach space $L = \bigoplus_{G\in\mathcal{G}} [l^\infty(G)]_\infty(X_G)$ such that $x = \{x_G\}_{G\in\mathcal{G}}$ belongs to L if $x_G \in [l^\infty(G)]_\infty(X_G)$ and

$$|x|_L := \sup_{G\in\mathcal{G}} |x_G|_{[l^\infty(G)]_\infty(X_G)} < \infty,$$

where $|\cdot|_{[l^\infty(G)]_\infty(X_G)}$ is the norm on $[l^\infty(G)]_\infty(X_G)$. Then similarly to the construction of Lemma 3.2, we can define the holomorphic Banach vector bundle B on M with the fibre L by the formula

$$B := \bigoplus_{G\in\mathcal{G}} [E_\infty^{M_G}(G)]_M.$$

Note that the structure group of B is $\mathrm{Iso}(L)$. Therefore the norm $|\cdot|_L$ induces a norm $|\cdot|_B$ on B (see Example 2.2.(b)). Let $O \Subset M$ be a relatively compact domain containing $\overline{N}$. Denote by $H^\infty(O, B)$ the Banach space of bounded holomorphic sections from $\mathcal{O}(O, B)$, that is,

$$f \in H^\infty(O, B) \iff \|f\| := \sup_{z\in O} |f(z)|_B < \infty.$$

For any $z\in O$ consider the restriction operator $r(z): H^\infty(O,B)\to L$,

$$r(z)[f]:=f(z), \quad f\in H^\infty(O,B).$$

Then $r(z)$ is a continuous linear operator with the norm $\|r(z)\|\le 1$. Moreover, by a theorem of Bungart (see [B, Section 4]), for any $v\in L$ there is a section $f\in\mathcal{O}(M,B)$ such that $f(z)=v$. Since O is relatively compact in M, the restriction $f|_O$ belongs to $H^\infty(O,B)$. This shows that $r(z)$ is surjective. For any $v\in L$ we set $K_v(z):=r(z)^{-1}(v)\subset H^\infty(O,B)$. The constant

$$h(z):=\sup_{|v|_L\le 1}\inf_{t\in K_v(z)}\|t\|$$

will be called *the constant of interpolation* for $r(z)$. We will prove the following result.

Lemma 4.2. *It is true that*

$$\sup_{z\in\overline{N}} h(z)\le C<\infty,$$

where C depends on N only.

Proof. In fact it suffices to cover $\overline{N}$ by a finite number of open balls and prove the required inequality for z varying in each of these balls. Moreover, since $\overline{N}$ is compact, for any $w\in\overline{N}$ it suffices to find an open neighbourhood $U_w\subset O$ of w such that $\{h(z)\}_{z\in U_w}$ is bounded from above by an absolute constant.

Let $w\in\overline{N}$. Without loss of generality we may identify a small open neighbourhood of w in O with the open unit ball $B_c(0,1)\subset\mathbf{C}^n$, $n=\dim O$, such that w corresponds to 0 in this identification. It is easy to see that $r(z)$, $z\in B_c(0,1)$, is the family of linear continuous operators holomorphic in z. Let $R:=1/4h(w)$. Since $h(w)\ge 1$, $B_c(0,1)$ contains $B_c(0,R)$. For a $y\in B_c(0,R)$ consider the one-dimensional complex subspace l_y of $\mathbf{C}^n$ containing y. Without loss of generality we may identify $l_y\cap B_c(0,1)$ with the open unit disk $\mathbf{D}\subset\mathbf{C}$. With this identification, let $r(z):=\sum_{i=0}^\infty r_i z^i$ be the Taylor expansion of $r(z)$ in $\mathbf{D}$. Here $r_i: H^\infty(O,B)\to L$ is a linear operator with the norm $\|r_i\|\le 1$. The last estimate follows from the Cauchy estimates for derivatives of holomorphic functions. We also have $r_0:=r(0)$ (recall that $w=0$). Let $v\in L$, $|v|_L\le 1$. For $z<R$ we will construct $v(z)\in H^\infty(O,B)$ which depends holomorphically on z, such that $\|v(z)\|\le 8h(w)$ and $r(z)[v(z)]=v$.

Let $v(z)=\sum_{i=0}^\infty v_i z^i$. Then we have the formal decomposition

$$v=r(z)[v(z)]=\sum_{i=0}^\infty z^i\sum_{j=0}^\infty r_i(v_j z^j)=\sum_{k=0}^\infty z^k\sum_{i+j=k} r_i(v_j).$$

Let us define v_i from the equations

$$r_0(v_0)=v \quad \text{and} \quad \sum_{i+j=k} r_i(v_j)=0 \quad \text{for } k\geq 1.$$

Since the constant of interpolation for $r(0)$ is $h(w)$, we can find $v_0\in H^\infty(O,B)$, $\|v_0\|<2h(w)$, satisfying the first equation. Substituting this v_0 into the second equation we obtain $r_0(v_1)=-r_1(v_0)$. Here $\|r_1(v_0)\|\leq 2h(w)$ because $\|r_1\|\leq 1$. Thus again we can find $v_1\in H^\infty(O,B)$ satisfying the second equation such that $\|v_1\|\leq(2h(w))^2$. Continuing step by step to solve the above equations we obtain $v_n\in H^\infty(O,B)$ satisfying the nth equation such that $\|v_n\|\leq\sum_{i=1}^n(2h(w))^{i+1}<n(2h(v))^{n+1}$ (because $h(w)\geq 1$). Thus we have

$$\|v(z)\|\leq\sum_{n=0}^\infty n(2h(w))^{n+1}R^n<\frac{2h(w)}{(1-2h(w)R)^2}=8h(w).$$

The above arguments show that $h(z)\leq 8h(w)$ for any $z\in B_c(0,1/4h(w))$. □

Now let us prove Proposition 4.1. Consider the fibre $p_U^{-1}(z)\subset\widetilde{U}$ for $z\in U$. Using the isometric isomorphism between $H^\infty(\widetilde{U})$ and the space $H^\infty(U,E_\infty^U(G))$ of bounded holomorphic sections of $E_\infty^U(G)$ (which is defined by taking the direct image of each function from $H^\infty(\widetilde{U})$ with respect to p_U: see the construction of Proposition 2.4), we can reformulate the required interpolation problem as follows:

Given $h\in l_\infty(G)$ find $v\in H^\infty(U,E_\infty^U(G))$ of least norm $\|v\|$ such that $v(z)=h$.

Let us consider $y=g_{NG}(z)\in N$ and its preimage $g_{NG}^{-1}(y)\subset N_G$. Further, consider the bundle $E_\infty^{M_G}(G)\to M_G$. We define a new function $\tilde{h}\in[l_\infty(G)]_\infty(X_G)$ by the formula

$$\tilde{h}(z)=h \text{ and } \tilde{h}(x)=0 \quad \text{for any } x\in g_{NG}^{-1}(y),\ x\neq z.$$

Then $|\tilde{h}|_{[l_\infty(G)]_\infty(X_G)}=|h|_{l_\infty(G)}$. Let us now consider the bundle $[E_\infty^{M_G}(G)]_M$ on M. Taking the direct image with respect to g_{MG}, we can identify $\tilde{h}$ with a section of $[E_\infty^{M_G}(G)]_M$ over y. Since $[E_\infty^{M_G}(G)]_M$ is a component of the bundle B, we can extend $\tilde{h}$ by 0 to obtain a section h' of B over y whose norm equals $|h|_{l_\infty(G)}$. Therefore according to Lemma 4.2, there is a holomorphic section $v'\in H^\infty(O,B)$ such that $\sup_{w\in N}|v'(w)|_B\leq C|h|_{l_\infty(G)}$ and $v'(y)=h'$. Now consider the natural projection π of B onto the component $[E_\infty^{M_G}(G)]_M$ in the direct decomposition of B. Then $\tilde{v}:=\pi(v')$ satisfies

$$\sup_{w\in N}|\tilde{v}(w)|_{[E_\infty^{M_G}(G)]_M}\leq C|h|_{l_\infty(G)} \quad \text{and} \quad \tilde{v}(y)=\tilde{h}.$$

Using identification of $\tilde{v}|_N$ with a bounded holomorphic section v of $E_\infty^{N_G}(G)$ (see the construction of Proposition 2.4), we obtain that $v(z)=h$ and $\sup_{w\in U}|v|_{E_\infty^{N_G}(G)}\le C|h|_{l_\infty(G)}$. It remains to note that $E_\infty^{N_G}(G)|_U=E_\infty^U(G)$ and so $v|_U\in H^\infty(U,E_\infty^U(G))$. In particular, $\sup_{z\in U}M(z)\le C$. $\square$

Proof of Theorem 1.9. Assume that $\{z_j\}_{j=1}^\infty\subset U$ is an interpolating sequence with the constant of interpolation

$$M=\sup_{\|a_j\|_\infty\le 1}\inf\{\|g\|:\ g\in H^\infty(U),\ g(z_j)=a_j,\ j=1,2,\dots\}.$$

We will prove that $p_U^{-1}(\{z_j\}_{j=1}^\infty)\subset\widetilde{U}$ is also interpolating. According to [Ga1, Chapter VII, Theorem 2.2], there are functions $f_n\in H^\infty(U)$ such that

$$f_n(z_n)=1,\quad f_n(z_k)=0,\ k\ne n,\quad \text{and}\quad \sum_{n=1}^\infty|f_n(z)|\le M^2.$$

Further, according to Proposition 4.1, for any $x\in U$, $p_U^{-1}(x)$ is an interpolating sequence with the constant of interpolation $\le C$. Let $p_U^{-1}(z_n)=\{z_{ng}\}_{g\in G}$. Then [Ga1, Chapter VII, Theorem 2.2] implies that there are functions $f_{ng}\in H^\infty(\widetilde{U})$ such that for any n,

$$f_{ng}(z_{ng})=1,\quad f_{ng}(z_{ns})=0,\ s\ne g,\quad \text{and}\quad \sum_g|f_{ng}(z)|\le C^2.$$

Define now $b_{ng}\in H^\infty(\widetilde{U})$ by the formula

$$b_{ng}(z):=f_{ng}(z)(p_U^*(f_n))(z).$$

Then we have

$$b_{ng}(z_{ng})=1,\quad b_{ng}(z_{ks})=0,\ k\ne n \text{ or } g\ne s,$$
$$\sum_{n,g}|b_{ng}(z)|=\sum_{n=1}^\infty\left(|(p_U^*(f_n))(z)|\sum_g|f_{ng}(z)|\right)\le(MC)^2.$$

Now we have the linear interpolation operator $S:l^\infty\to H^\infty(\widetilde{U})$ defined by $S(\{a_{ng}\})=\sum_{n,g}a_{ng}b_{ng}(z)$ for any $\{a_{ng}\}\in l^\infty$. This shows that $\{p_U^{-1}(z_n)\}$ is interpolating.

Conversely, assume that $\{z_n\}_{n=1}^\infty\subset U$ is such that $\{p_U^{-1}(z_n)\}$ is interpolating for $H^\infty(\widetilde{U})$. Let $\{a_n\}_{n=1}^\infty\in l^\infty$, and consider the function $t\in l^\infty(\{p_U^{-1}(z_n)\})$ defined by $t|_{p_U^{-1}(z_n)}=a_n$ for $n=1,2,\dots$. Then there is $f\in H^\infty(\widetilde{U})$ such that $f|_{\{p_U^{-1}(z_n)\}}=t$.

Applying the projector P constructed in Theorem 1.1 to f, we obtain a function $k\in H^\infty(U)$ with $p_U^*(k)=P(f)$ which solves the required interpolation problem. □

Proof of Theorem 1.11. Let $r:\widetilde{U}\to U$ be the universal covering and $h\in H^\infty(\widetilde{U})$ be the function defining the projector $P:H^\infty(\widetilde{U})\to r^*(H^\infty(U))$, see Remark 1.2. For a character ϱ we define the map L_ϱ by the formula

$$L_\varrho[g](z):=\sum_{\gamma\in\pi_1(U)} h(\gamma(z))g(\gamma(z))\varrho(\gamma^{-1}),\quad g\in H^\infty(\widetilde{U}),\ z\in\widetilde{U}.$$

It is readily seen that L_ϱ maps $H^\infty(\widetilde{U})$ in $H^\infty(\pi_1(U),\varrho)$ and its norm is bounded by the norm of P (i.e. it depends on N only). Moreover, from Theorem 1.9 it follows that for any $o\in\widetilde{U}$ there is a function $f\in H^\infty(\widetilde{U})$ such that

$$f(\gamma(o))=\varrho(\gamma),\quad \gamma\in\pi_1(U).$$

Thus $L_\varrho[f](o)=1$ showing that L_ϱ is non-trivial. □

5. Interpolating sequences on Riemann surfaces

In this section we prove Propositions 1.13, 1.15, 1.17, 1.18 and Corollary 1.14.

Proof of Proposition 1.13. Assume that $\{z_j\}_{j=1}^\infty\subset U$ is an interpolating sequence. Then by Theorem 1.9, $r^{-1}(\{z_j\}_{j=1}^\infty)$ is interpolating for $H^\infty(\mathbf{D})$. Let $r^{-1}(z_j)=\{z_{jg}\}_{g\in\pi_1(U)}$. Then by the Carleson theorem [Ca1] on the characterization of interpolating sequences we have (for any j and g)

$$\biggl(\prod_{k:k\neq j}|B_{z_k}(z_{jg})|\biggr)\biggl(\prod_{h:h\neq g}\left|\frac{z_{jh}-z_{jg}}{1-\bar{z}_{jh}z_{jg}}\right|\biggr)\geq c>0.$$

Further, since

$$\prod_{h:h\neq g}\left|\frac{z_{jh}-z_{jg}}{1-\bar{z}_{jh}z_{jg}}\right|\leq 1,$$

from the above inequality it follows that for any j,

$$\prod_{k:k\neq j}P_{z_k}(z_j):=\prod_{k:k\neq j}|B_{z_k}(z_{jg})|\geq c>0.$$

Conversely, assume that for any j we have

$$\prod_{k:k\neq j}P_{z_k}(z_j)\geq c>0.$$

From the proof of Theorem 1.9 we know that the constant of interpolation for $r^{-1}(z)$ with an arbitrary $z\in U$ is bounded from above by some $C=C(N)<\infty$. Thus according to the inequality which connects the constant of interpolation with the characteristic of an interpolating sequence (see [Ca1]) we obtain for any j and any $g\in\pi_1(U)$,

$$\prod_{h:h\neq g}\left|\frac{z_{jh}-z_{jg}}{1-\bar{z}_{jh}z_{jg}}\right|\geq\frac{1}{C}>0.$$

Combining these two inequalities we have (for any j and g)

$$\left(\prod_{k:k\neq j}P_{z_k}(z_j)\right)\left(\prod_{h:h\neq g}\left|\frac{z_{jh}-z_{jg}}{1-\bar{z}_{jh}z_{jg}}\right|\right)=\left(\prod_{k:k\neq j}|B_{z_k}(z_{jg})|\right)\left(\prod_{h:h\neq g}\left|\frac{z_{jh}-z_{jg}}{1-\bar{z}_{jh}z_{jg}}\right|\right)$$
$$\geq\frac{c}{C}>0.$$

This inequality implies that the sequence $r^{-1}(\{z_j\}_{j=1}^{\infty})$ is interpolating (see [Ca1]). Hence by Theorem 1.9, $\{z_j\}_{j=1}^{\infty}$ is interpolating for $H^{\infty}(U)$. □

Proof of Corollary 1.14. From the proof of Proposition 1.13 and Theorem 1.9 it follows that the characteristic δ' of the interpolating sequence $r^{-1}(\{z_j\}_{j=1}^{\infty})$ is $\geq\delta/C$, where $C\geq1$ depends on N only. Then according to the Carleson theorem [Ca1], the constant of interpolation K' of $r^{-1}(\{z_j\}_{j=1}^{\infty})$ is

$$\leq\frac{cC}{\delta}\left(1+\log\frac{C}{\delta}\right)<\frac{C_1}{\delta}\left(1+\log\frac{1}{\delta}\right).$$

Here c is an absolute constant and $C_1=C_1(N)$. Thus applying the projector P of Theorem 1.1 to functions $f\in H^{\infty}(\mathbf{D})$ which are constant on each fibre $r^{-1}(z_j)$, $j=1,2,\dots$, and using that $\|P\|\leq C_2=C_2(N)<\infty$ we obtain that

$$K\leq C_2K'\leq\frac{C_1C_2}{\delta}\left(1+\log\frac{1}{\delta}\right).\quad\square$$

Proof of Proposition 1.15. We start by letting $r^{-1}(z_j)=\{z_{jg}\}_{g\in\pi_1(U)}$ and $r^{-1}(\xi_j)=\{\xi_{jg}\}_{g\in\pi_1(U)}$. By the definition of ϱ^* and because $\pi_1(U)$ acts discretely on $\mathbf{D}$, we can choose the above indices such that $\varrho(\xi_{jg},z_{jg})\leq\lambda$ for any g. Let us fix some $h\in\pi_1(U)$. Then by the definition, for $j\neq k$ we have

$$P_{\xi_j}(\xi_k)=\prod_{g\in\pi_1(U)}\varrho(\xi_{kh},\xi_{jg}).$$

Using an inequality from the proof of Lemma 5.3 in [Ga1, Chapter VII] gives

$$\varrho(\xi_{jg},\xi_{kh}) \geq \frac{\varrho(z_{jg},z_{kh})-\alpha}{1-\alpha\varrho(z_{jg},z_{kh})}$$

for $\alpha := 2\lambda/(1+\lambda^2)$. According to our assumption we have

$$\prod_{j:j\neq k} P_{z_j}(z_k) := \prod_{j:j\neq k}\prod_{g\in\pi_1(U)} \varrho(z_{kh},z_{jg}) \geq \delta.$$

Therefore $\varrho(z_{kh},z_{jg})\geq\delta$ for any $j\neq k$ and any $g\in\pi_1(U)$. Hence we can apply the inequality of [Ga1, Chapter VII, Lemma 5.2] to obtain

$$\begin{aligned}\prod_{j:j\neq k} P_{\xi_j}(\xi_k) &:= \prod_{j:j\neq k}\prod_{g\in\pi_1(U)} \varrho(\xi_{jg},\xi_{kh}) \geq \prod_{j:j\neq k}\prod_{g\in\pi_1(U)} \frac{\varrho(z_{jg},z_{kh})-\alpha}{1-\alpha\varrho(z_{jg},z_{kh})}\\ &\geq \frac{\prod_{j:j\neq k}\prod_{g\in\pi_1(U)}\varrho(z_{jg},z_{kh})-\alpha}{1-\alpha\prod_{j:j\neq k}\prod_{g\in\pi_1(U)}\varrho(z_{jg},z_{kh})} = \frac{\prod_{j:j\neq k}P_{z_j}(z_k)-\alpha}{1-\alpha\prod_{j:j\neq k}P_{z_j}(z_k)} \geq \frac{\delta-\alpha}{1-\alpha\delta}.\end{aligned}$$

This gives the required inequality. □

Proof of Proposition 1.17. From the condition of the proposition it follows that the distance in the pseudohyperbolic metric on $\mathbf{D}$ between interpolating sequences $r^{-1}(\{z_i\}_{i=1}^\infty)$ and $r^{-1}(\{y_i\}_{i=1}^\infty)$ is $\geq c$. This implies that $r^{-1}(\{z_i\}_{i=1}^\infty)\cup r^{-1}(\{y_i\}_{i=1}^\infty)$ is interpolating for $H^\infty(\mathbf{D})$ (see e.g. [Ga1, Chapter VII, Problem 2]). Therefore by Theorem 1.9 $\{z_i\}_{i=1}^\infty\cup\{y_i\}_{i=1}^\infty\subset U$ is interpolating for $H^\infty(U)$. □

Proof of Proposition 1.18. Consider the function $F(z):=\prod_j P_{z_j}(z)$. Then we have a decomposition $F(z)=F_1(z)F_2(z)$ with $F_s(z):=\prod_j P_{z_{sj}}(z)$, $s=1,2$. It suffices to choose the required decomposition such that

$$\prod_{j:j\neq n} P_{z_{1j}}(z_n)\geq F_2(z_n),\quad \text{if } F_1(z_n)=0,$$
$$\prod_{j:j\neq n} P_{z_{2j}}(z_n)\geq F_1(z_n),\quad \text{if } F_2(z_n)=0.$$

The proof of the above inequalities repeats word-by-word the combinatorial proof of Lemma 1.5 in [Ga1, Chapter X] given by Mills, where we must define the matrix $[a_{kn}]$ by the formula

$$a_{kn}=\log P_{z_k}(z_n),\quad k\neq n,\ a_{nn}=0.$$

We leave the details to the reader. Now from the above inequalities for $F_1(z_n)=0$ we have

$$\delta \le \prod_{j:j\neq n} P_{z_j}(z_n) = \Big(\prod_{j:j\neq n} P_{z_{1j}}(z_n)\Big) F_2(z_n) \le \Big(\prod_{j:j\neq n} P_{z_{1j}}(z_n)\Big)^2,$$

which gives the required estimate of the characteristic for $\{z_{1j}\}_{j=1}^{\infty}$. The same is valid for $\{z_{2j}\}_{j=1}^{\infty}$. □

References

[BD] Barrett, D. E. and Diller, J., A new construction of Riemann surfaces with corona, *J. Geom. Anal.* **8** (1998), 341–347.

[Br1] Brudnyi, A., Grauert and Lax–Halmos type theorems and extension of matrices with entries in H^∞, to appear in *J. Funct. Anal.*

[Br2] Brudnyi, A., A uniqueness property for H^∞ on coverings of projective manifolds, to appear in *Michigan Math. J.*

[B] Bungart, L., On analytic fibre bundles I. Holomorphic fibre bundles with infinite dimensional fibres, *Topology* **7** (1968), 55–68.

[Ca1] Carleson, L., An interpolation problem for bounded analytic functions, *Amer. J. Math.* **80** (1958), 921–930.

[Ca2] Carleson, L., Interpolation of bounded analytic functions and the corona problem, *Ann. of Math.* **76** (1962), 547–559.

[Ca3] Carleson, L., On H^∞ in multiply connected domains, in *Conference on Harmonic Analysis in Honor of Antoni Zygmund*, Vol. II (Beckman, W., Calderón, A. P., Fefferman, R. and Jones, P. W., eds.), pp. 349–372, Wadsworth, Belmont, Calif., 1983.

[F] Forelli, F., Bounded holomorphic functions and projections, *Illinois J. Math.* **10** (1966), 367–380.

[G] Gamelin, T. W., *Uniform Algebras and Jensen Measures*, London Math. Soc. Lecture Note Series **32**, Cambridge Univ. Press, Cambridge–New York, 1978.

[Ga1] Garnett, J., *Bounded Analytic Functions*, Academic Press, Orlando, Fla., 1980.

[Ga2] Garnett, J., Corona problems, interpolation problems and inhomogeneous Cauchy–Riemann equations, in *Proceedings of the International Congress of Mathematicians, Vol.* **2** *(Berkeley, Calif. 1986)* (Gleason, A. M., ed.), pp. 917–923, Amer. Math. Soc., Providence, R. I., 1987.

[Ha] Hasumi, M., *Hardy Classes on Infinitely Connected Riemann Surfaces*. Lecture Notes in Math. **1027**, Springer-Verlag, Berlin–Heidelberg, 1983.

[Hi] Hirzebruch, F., *Topological Methods in Algebraic Geometry*, Springer-Verlag, New York, 1966.

[H] Hoffman, K., Bounded analytic functions and Gleason parts, *Ann. of Math.* **86** (1967), 74–111.

[Hu] Hu, S.-T., *Homotopy Theory*, Academic Press, New York, 1959.

[JM] JONES, P. W. and MARSHALL, D., Critical points of Green's functions, harmonic measure and the corona theorem, *Ark. Mat.* **23** (1985), 281–314.

[L] LÁRUSSON, F., Holomorphic functions of slow growth on nested covering spaces of compact manifolds, *Canad. J. Math.* **52** (2000), 982–998.

[S] SLODKOWSKI, Z., On bounded analytic functions in finitely connected domains, *Trans. Amer. Math. Soc.* **300** (1987), 721–736.

[St] STOUT, E. L., Bounded holomorphic functions on finite Riemann surfaces, *Trans. Amer. Math. Soc.* **120** (1965), 255–285.

[W] WIDOM, H., H_p sections of vector bundles over Riemann surfaces, *Ann. of Math.* **94** (1971), 304–324.

Received December 2, 2002

Alexander Brudnyi
Department of Mathematics and Statistics
University of Calgary
Calgary, AB
Canada T2N 1N4
email: albru@math.ucalgary.ca