ON A SYSTEM OF DIFFERENTIAL EQUATIONS LEADING TO PERIODIC FUNCTIONS

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The present paper contains an elementary algebraic deduction of a system of differential equations satisfied by all the hyperelliptic sigma functions which, as is believed, were first stated, but without demonstration, in the Proceedings of the Cambridge Philosophical Society, Vol. IX, Part IX, 1898, p. 513. In that note will be found indications of a method of solution of the equations in connexion with the theory, considered by PICARD, of integrals of total differentials, and of a method of obtaining from them the expansion of any sigma function, and of their use, in case $p = 2$, for expressing the geometry of KUMMER's sixteen nodal quartic surface. The establishment of a theory of the sigma functions directly from these differential equations would appear likely to be of the greatest suggestiveness for the development of the theory of functions of several variables. It is from this general point of view that the equations appear to the present writer to be of peculiar interest; though their simplicity would also recommend them merely as a contribution to the theory of the hyperelliptic functions.

I.

Let $(x_1 y_1) \dots (x_p y_p)$ be pairs satisfying the equation

$$y^2 = f(x) = 4P(x)Q(x),$$

where

$$P(x) = (x - a_1)\dots(x - a_p), \qquad Q(x) = (x - c_1)\dots(x - c_p)(x - c);$$

let

$$F(x) = (x - x_1)\dots(x - x_p), \qquad F'(x) = \frac{d}{dx}F(x),$$

and, $e_1, e_2, e_3, \dots$ being undetermined quantities, let

$$\Delta_i = \sum_{r=1}^{p} \frac{y_r}{(e_i - x_r)F'(x_r)}, \qquad \Delta_{ij} = -\frac{\Delta_i - \Delta_j}{e_i - e_j},$$

so that

$$(e_2 - e_3)\Delta_{23} + (e_3 - e_1)\Delta_{31} + (e_1 - e_2)\Delta_{12} = 0$$

$$(e_2 - e_3)(e_4 - e_1)\Delta_{23}\Delta_{41} + (e_3 - e_1)(e_4 - e_2)\Delta_{31}\Delta_{42}$$
$$+ (e_1 - e_2)(e_4 - e_3)\Delta_{12}\Delta_{43} = 0;$$

put further

$$\frac{f(e_i)}{[F(e_i)]^2} = \varphi_i$$

and

$$\Omega_{ij} = (e_i - e_j)^2 \Delta_{ij}^2 - \varphi_i - \varphi_j;$$

also let

$$\chi_{p-i}(x) = x^{p-i} - h_1 x^{p-i-1} + h_2 x^{p-i-2} - \dots + (-1)^{p-i} h_{p-i},$$

so that

$$\frac{F(x)}{x - x_i} = x^{p-1} + x^{p-2}\chi_1(x_i) + x^{p-3}\chi_2(x_i) + \dots + \chi_{p-1}(x_i),$$

h_r being the sum of the homogeneous products of $x_1 \dots x_p$, without repetitions, r together.

We assume in this paper that $u_1 \dots u_p$ are arbitrary variables, and that the pairs $(x_1 y_1) \dots (x_p y_p)$ are determined from them by the p equations

$$\int_{m_1}^{x_1} \frac{x^{r-1} dx}{y} + \dots + \int_{m_p}^{x_p} \frac{x^{r-1} dx}{y} = u_r, \qquad r=1,\dots,p$$

where the lower limits denote p pairs satisfying the equation $y^2 = f(x)$, to be chosen arbitrarily and kept the same throughout the following investigation. It is further assumed that any rational symmetric function of the pairs $(x_1 y_1) \dots (x_p y_p)$ is a single valued analytic function of $u_1 \dots u_p$. Such a function has in fact no essential singularities for finite values of $u_1 \dots u_p$.

It is proved at once that

$$dx_i = \sum_{r=1}^{p} \left[\frac{y_i}{F'(x_i)} \chi_{p-r}(x_i) \right] du_r,$$

and therefore

$$\frac{\partial x_i}{\partial u_r} = \frac{y_i}{F'(x_i)} \chi_{p-r}(x_i), \qquad \frac{\partial y_i}{\partial u_r} = \frac{1}{2} \frac{f'(x_i)}{F'(x_i)} \chi_{p-r}(x_i);$$

we put further

$$\sum_{r=1}^{p} e_i^{r-1} \frac{\partial}{\partial u_r} = \delta_i.$$

Now consider the expression

$$H = \frac{1}{4} F^2(e_1) F^2(e_2) \Delta_{12}^2 - \frac{F(e_1)P(e_2) - F(e_2)P(e_1)}{e_1 - e_2} \cdot \frac{F(e_1)Q(e_2) - F(e_2)Q(e_1)}{e_1 - e_2};$$

it is easily seen to vanish when e_1 is replaced by x_1; it is therefore an integral polynomial in e_1 and e_2 dividing identically by $F(e_1)F(e_2)$.

Take a symmetrical system of $\frac{1}{2} p(p+1)$ constants $c_{\lambda\mu}$, of arbitrary values, and put

$$f(e_1, e_2) = 4[P(e_1)Q(e_2) + P(e_2)Q(e_1)] - 4(e_1 - e_2)^2 \sum_{\lambda=1}^{p} \sum_{\mu=1}^{p} c_{\lambda\mu} e_1^{\lambda-1} e_2^{\mu-1},$$

so that the expression

$$f(e_1, e_2) + 4(e_1 - e_2)^2 \sum_{\lambda=1}^{p} \sum_{\mu=1}^{p} c_{\lambda\mu} e_1^{\lambda-1} e_2^{\mu-1} - \frac{f(e_1)F(e_2)}{F(e_1)} - \frac{f(e_2)F(e_1)}{F(e_2)}$$

is equal to

$$-\frac{4[F(e_1)P(e_2)-F(e_2)P(e_1)][F(e_1)Q(e_2)-F(e_2)Q(e_1)]}{F(e_1)F(e_2)};$$

then the quantity

$$\Phi=\frac{H}{F(e_1)F(e_2)}-\sum_{\lambda=1}^{p}\sum_{\mu=1}^{p}c_{\lambda\mu}e_1^{\lambda-1}e_2^{\mu-1}$$

is equal to

$$\frac{1}{4}F(e_1)F(e_2)\Delta_{12}^2+\frac{1}{4(e_1-e_2)^2}\left[f(e_1\,,\,e_2)-f(e_1)\frac{F(e_2)}{F(e_1)}-f(e_2)\frac{F(e_1)}{F(e_2)}\right],$$

which is therefore a rational symmetric polynomial in e_1 and e_2, of degree $(p-1)$ in each, of which the coefficients are rational symmetric functions of the p pairs $(x_1y_1)\ldots(x_py_p)$.

We may therefore define $\frac{1}{2}p(p+1)$ single-valued analytic functions of the variables $u_1\ldots u_p$, without essential singularity for finite values of these variables, by putting

$$\Phi=\sum_{\lambda=1}^{p}\sum_{\mu=1}^{p}\wp_{\lambda\mu}(u)e_1^{\lambda-1}e_2^{\mu-1}.$$

These functions depend on the $\frac{1}{2}p(p+1)$ arbitrary constants $c_{\lambda\mu}$, but only additively; and they depend on the p arbitrary fixed places denoted above by $m_1\ldots m_p$, of which the alteration is equivalent only to the addition of constants to the arguments $u_1\ldots u_p$; moreover they satisfy the equations

$$\wp_{\lambda\mu}(u)=\wp_{\mu\lambda}(u).$$

We shall put

$$\wp_{\lambda\mu\nu}(u)=\frac{\partial\wp_{\lambda\mu}(u)}{\partial u_\nu},\qquad \wp_{\lambda\mu\nu\rho}(u)=\frac{\partial\wp_{\lambda\mu\nu}(u)}{\partial u_\rho},$$

and it will be found to be an incidental consequence of the following work that in all the functions $\wp_{\lambda\mu\nu}(u)$, $\wp_{\lambda\mu\nu\rho}(u)$, the order of the suffixes is indifferent, or $\wp_{\lambda\mu\nu}(u)=\wp_{\lambda\nu\mu}(u)$; etc.

The definition of the functions $\wp_{\lambda\mu}(u)$ is equivalent with

$$4(e_1-e_2)^2\sum_{\lambda=1}^{p}\sum_{\mu=1}^{p}\wp_{\lambda\mu}(u)\,.\,e_1^{\lambda-1}e_2^{\mu-1}-f(e_1\,,\,e_2)=F(e_1)F(e_2)\Omega_{12},$$

where, as before,

$$\Omega_{12} = (e_1 - e_2)^2 \Delta_{12}^2 - \varphi_1 - \varphi_2 .$$

To this equation we apply the operator

$$\partial_3 = \sum_{\nu=1}^{p} e_3^{\nu-1} \frac{\partial}{\partial u_\nu} .$$

Recalling the values of $\partial x_i | \partial u_r$ and $\partial y_i | \partial u_r$ we find easily

$$\frac{1}{F(e_3)} \partial_3 F(e_1) = - F(e_1) \Delta_{13} , \qquad \frac{1}{F(e_3)} \partial_3 \varphi_1 = 2 \varphi_1 \Delta_{13} ;$$

with some calculation, of which the details are given below, we find

$$\frac{1}{F(e_3)} \partial_3 \Delta_{13} = \frac{1}{2} \frac{(e_1 - e_3) \Delta_{13}^2 - (e_2 - e_3) \Delta_{23}^2}{e_1 - e_2}$$

$$+ \frac{\varphi_1}{2(e_1 - e_2)(e_1 - e_3)} + \frac{\varphi_2}{2(e_2 - e_3)(e_2 - e_1)} + \frac{\varphi_3}{2(e_3 - e_1)(e_3 - e_2)} ,$$

which gives

$$\frac{1}{F(e_1) F(e_2) F(e_3)} \partial_3 [\Omega_{12} F(e_1) F(e_2)]$$

$$= (e_1 - e_2) \Delta_{12} \left[(e_1 - e_3) \Delta_{13}^2 - (e_2 - e_3) \Delta_{23}^2 + (e_1 - e_2) \sum^{1,2,3} \frac{\varphi_1}{2(e_1 - e_2)(e_1 - e_3)} \right]$$

$$- 2\varphi_1 \Delta_{13} - 2\varphi_2 \Delta_{23} - (\Delta_{13} + \Delta_{23}) [(e_1 - e_2)^2 \Delta_{12}^2 - \varphi_1 - \varphi_2] ,$$

and in virtue of

$$(e_2 - e_3) \Delta_{23} + (e_3 - e_1) \Delta_{31} + (e_1 - e_2) \Delta_{12} = 0$$

this reduces to

$$- \frac{\partial_3 [\Omega_{12} F(e_1) F(e_2)]}{(e_1 - e_2)^2 F(e_1) F(e_2) F(e_3)}$$

$$= \Delta_{23} \Delta_{31} \Delta_{12} + \frac{(e_2 - e_3) \varphi_1 \Delta_{23} + (e_3 - e_1) \varphi_2 \Delta_{31} + (e_1 - e_2) \varphi_3 \Delta_{12}}{\varpi_{123}}$$

where

$$\varpi_{123} = (e_2 - e_3)(e_3 - e_1)(e_1 - e_2) .$$

We thus deduce that the expression

$$- \frac{4}{F(e_1) F(e_2) F(e_3)} \sum_{\lambda=1}^{p} \sum_{\mu=1}^{p} \sum_{\mu=1}^{p} \wp_{\lambda\mu\nu}(u) e_1^{\lambda-1} e_2^{\mu-1} e_3^{\nu-1}$$

is, for all values of e_1, e_2, e_3, equal to the expression on the right side of the last written equation. As this is symmetrical in e_1, e_2, e_3 it follows that in $\wp_{\lambda\mu\nu}(u)$ the order of the suffixes is indifferent. It is not possible to express the functions $\wp_{\lambda\mu\nu}(u)$ rationally in terms of the functions $\wp_{\lambda\mu}(u)$; it is a consequence of what follows that the squares and products

$$\wp^2_{\lambda\mu\nu}(u)\,,\ \wp_{\lambda\mu\nu}(u)\wp_{\rho\sigma\tau}(u)$$

can be so expressed. We proceed therefore to further apply the operator

$$\partial_4 = \sum_{\rho=1}^{p} e_4^{\rho-1}\frac{\partial}{\partial u_\rho}$$

to obtain the expressions for $\wp_{\lambda\mu\nu\rho}(u)$.

Before doing this we give the calculation referred to above to find the expression for

$$\frac{1}{F(e_3)}\partial_3\Delta_{13}$$

we have

$$\frac{\partial}{\partial u_r}\left[\frac{y_k}{F'(x_k)}\right] = \frac{1}{2}\frac{f'(x_k)}{[F'(x_k)]^2}\chi_{p-r}(x_k) - \frac{y_k}{[F'(x_k)]^2}\frac{\partial}{\partial u_r}[F'(x_k)]$$

$$= \frac{1}{2}\frac{f'(x_k)}{[F'(x_k)]^2}\chi_{p-r}(x_k) - \frac{y_k}{F'(x_k)}\left\{\frac{y_k}{F'(x_k)}\chi_{p-r}(x_k)\sum_{i=1}^{p}{}^{(k)}\frac{1}{x_k-x_i}\right.$$

$$\left.-\sum_{i=1}^{p}{}^{(k)}\frac{y_i}{F'(x_i)}\chi_{p-r}(x_i)\frac{1}{x_k-x_i}\right\},$$

where $\sum_{i=1}^{p}{}^{(k)}$ is a summation from which the term for $i = k$ is omitted, so that

$$\sum_{i=1}^{p}{}^{(k)}\frac{1}{x_k-x_i} = \frac{1}{2}\frac{F''(x_k)}{F'(x_k)};$$

therefore

$$\frac{\partial}{\partial u_r}\left[\frac{y_k}{F'(x_k)}\right] = \frac{1}{2}\frac{f'(x_k)}{[F'(x_k)]^2}\chi_{p-r}(x_k) - \frac{1}{2}\frac{f(x_k)F''(x_k)}{[F'(x_k)]^3}\chi_{p-r}(x_k)$$

$$+\frac{y_k}{F'(x_k)}\sum_{i=1}^{p}{}^{(k)}\frac{y_i}{F'(x_i)}\frac{\chi_{p-r}(x_i)}{x_k-x_i};$$

hence

$$\frac{1}{F(e_3)}\delta_3\left[\frac{y_k}{F'(x_k)}\right] = \frac{1}{2}\frac{f'(x_k)F'(x_k)-f(x_k)F''(x_k)}{(e_3-x_k)[F'(x_k)]^3} + \frac{y_k}{F'(x_k)}\sum_{i=1}^{p}{}^{(k)}\frac{y_i}{(e_3-x_i)(x_k-x_i)F'(x_i)},$$

while

$$\delta_3\frac{y_k}{(e_1-x_k)F'(x_k)} = \frac{f(x_k)F(e_3)}{(e_1-x_k)^2(e_3-x_k)[F'(x_k)]^2} + \frac{1}{e_1-x_k}\delta_3\left[\frac{y_k}{F'(x_k)}\right];$$

wherefore

$$\frac{1}{F(e_3)}\delta_3\Delta_1 = \frac{1}{F(e_3)}\sum_{k=1}^{p}\delta_3\left[\frac{y_k}{(e_1-x_k)F'(x_k)}\right]$$

$$= \sum_{k=1}^{p}\left\{\frac{1}{2}\frac{f'(x_k)F''(x_k)-f(x_k)F''(x_k)}{(e_3-x_k)(e_1-x_k)[F'(x_k)]^3} + \frac{f(x_k)}{(e-x_k)^2(e_3-x_k)[F'(x_k)]^2}\right\}$$

$$+\sum_{k=1}^{p}\frac{y_k}{(e_1-x_k)F'(x_k)}\sum_{i=1}^{p}{}^{(k)}\frac{y_i}{(e_3-x_i)(x_k-x_i)F'(x_i)};$$

herein the second term of the right side, arising in a form consisting of $p(p-1)$ terms, is in fact a sum of $\frac{1}{2}p(p-1)$ terms, namely equal to

$$\sum_{k,i}^{1..p}\sum\frac{y_ky_i}{(e_1-x_k)(e_3-x_k)F'(x_k)(e_1-x_i)(e_3-x_i)F'(x_i)}\cdot\frac{(e_3-x_k)(e_1-x_i)-(e_3-x_i)(e_1-x_k)}{x_k-x_i}$$

wherein $k \neq i$, and therefore equal to

$$\sum_{k,i}^{1...p}\sum\frac{(e_3-e_1)y_ky_i}{(e_3-x_k)(e_1-x_k)F'(x_k)(e_3-x_i)(e_1-x_i)F'(x_i)}$$

or

$$\frac{1}{2}(e_3-e_1)\left\{\left[\sum_{k=1}^{p}\frac{y_k}{(e_3-x_k)(e_1-x_k)F'(x_k)}\right]^2 - \sum_{k=1}^{p}\frac{f(x_k)}{(e_3-x_k)^2(e_1-x_k)^2[F'(x_k)]^2}\right\}.$$

Thus

$$\frac{1}{F(e_3)}\delta_3\Delta_1 = \frac{1}{2}(e_3-e_1)\Delta_{13}^2$$

$$+\frac{1}{2}\sum_{k=1}^{p}\left\{\frac{f'(x_k)F'(x_k)-f(x_k)F''(x_k)}{(e_3-x_k)(e_1-x_k)[F'(x_k)]^3} + \frac{(e_3+e_1-2x_k)f(x_k)}{(e_3-x_k)^2(e_1-x_k)^2[F'(x_k)]^2}\right\}$$

which is the same as

$$\frac{1}{2}(e_3 - e_1)\Delta_{13}^2 + \frac{1}{2}\sum_{k=1}^{p} \frac{1}{F'(x_k)}\frac{\partial}{\partial x_k}\left[\frac{f(x_k)}{(e_3 - x_k)(e_1 - x_k)F'(x_k)}\right].$$

This gives

$$\frac{1}{F(e_3)}\partial_3 \Delta_{12} = \frac{1}{2}\frac{(e_1 - e_3)\Delta_{13}^2 - (e_2 - e_3)\Delta_{23}^2}{e_1 - e_2}$$

$$+ \frac{1}{2}\sum_{k=1}^{p} \frac{1}{F'(x_k)}\frac{\partial}{\partial x_k}\left[\frac{f(x_k)}{(e_1 - x_k)(e_2 - x_k)(e_3 - x_k)F'(x_k)}\right];$$

now if $R(x)$ be a rational function of x not becoming infinite or zero for $x = x_k$, it is easy to prove that the coefficient of $(x - x_k)^{-1}$ in the expansion of $R(x) | [F(x)]^2$ is equal to

$$\frac{1}{F'(x_k)}\frac{\partial}{\partial x_k}\left[\frac{R(x_k)}{F'(x_k)}\right];$$

thus, applying the well known partial fraction theorem

$$\left[\frac{f(x)}{(e_1 - x)(e_2 - x)(e_3 - x)[F(x)]^2}\frac{dx}{dt}\right]_{t^{-1}} = 0,$$

we find, finally, as stated above, that

$$\frac{1}{F(e_3)}\partial_3 \Delta_{12} = \frac{1}{2}\frac{(e_1 - e_3)\Delta_{13}^2 - (e_2 - e_3)\Delta_{23}^2}{e_1 - e_2} + \frac{1}{2}\sum^{1,2,3} \frac{\varphi_1}{(e_1 - e_2)(e_1 - e_3)}.$$

Proceeding now to apply the operator

$$\partial_4 = \sum_{\rho=1}^{p} e_4^{\rho-1}\frac{\partial}{\partial u_\rho}$$

to the equation before proved

$$-\frac{4}{F(e_1)F(e_2)F(e_3)}\sum_{\lambda=1}^{p}\sum_{\mu=1}^{p}\sum_{\nu=1}^{p} \wp_{\lambda\mu\nu}(u) \, . \, e_1^{\lambda-1}e_2^{\mu-1}e_3^{\nu-1}$$

$$= \Delta_{23}\Delta_{31}\Delta_{12} + \frac{1}{\varpi_{123}}\sum^{1,2,3} \varphi_1(e_2 - e_3)\Delta_{23},$$

we have at once, by use of the equations

$$\partial_4 F(e_i) = - F(e_4)F(e_i)\Delta_{i4}, \qquad \partial_4\varphi_i = 2F(e_4)\varphi_i\Delta_{i4}$$

$$\partial_4 \Delta_{12} = \frac{1}{2}F(e_4)\left\{\frac{(e_1 - e_4)\Delta_{14}^2 - (e_2 - e_4)\Delta_{24}^2}{e_1 - e_2} - \frac{1}{\varpi_{124}}\sum^{1,2,4} \varphi_1(e_2 - e_4)\right\}$$

the result that

$$-\frac{4}{F(e_1)F(e_2)F(e_3)F(e_4)}\sum_{\lambda=1}^{p}\sum_{\mu=1}^{p}\sum_{\nu=1}^{p}\sum_{\rho=1}^{p}\wp_{\lambda\mu\nu\rho}(u)e_1^{\lambda-1}e_2^{\mu-1}e_3^{\nu-1}e_4^{\rho-1}$$

is equal to

$$-(\Delta_{14}+\Delta_{24}+\Delta_{34})\Big[\Delta_{23}\Delta_{31}\Delta_{12}+\frac{1}{\varpi_{123}}\sum^{1,2,3}\varphi_1(e_2-e_3)\Delta_{23}\Big]+\frac{2}{\varpi_{123}}\sum^{1,2,3}\varphi_1(e_2-e_2)\Delta_{23}\Delta_{14}$$

$$+\frac{1}{2}\Big[\Delta_{12}\Delta_{13}+\frac{1}{\varpi_{123}}\varphi_1(e_2-e_3)\Big]\Big[\frac{(e_2-e_4)\Delta_{24}^2-(e_3-e_4)\Delta_{34}^2}{e_2-e_3}-\frac{1}{\varpi_{234}}\sum^{2,3,4}(e_2-e_3)\varphi_4\Big]$$

$$+\frac{1}{2}\Big[\Delta_{23}\Delta_{21}+\frac{1}{\varpi_{123}}\varphi_2(e_3-e_1)\Big]\Big[\frac{(e_3-e_4)\Delta_{34}^2-(e_1-e_4)\Delta_{14}^2}{e_3-e_1}-\frac{1}{\varpi_{314}}\sum^{3,1,4}(e_3-e_1)\varphi_4\Big]$$

$$+\frac{1}{2}\Big[\Delta_{31}\Delta_{32}+\frac{1}{\varpi_{123}}\varphi_3(e_1-e_2)\Big]\Big[\frac{(e_1-e_4)\Delta_{14}^2-(e_2-e_4)\Delta_{24}^2}{e_1-e_2}-\frac{1}{\varpi_{124}}\sum^{1,2,4}(e_1-e_2)\varphi_4\Big];$$

by means of the identity

$$(e_j-e_k)\Delta_{jk}+(e_k-e_i)\Delta_{ki}+(e_i-e_j)\Delta_{ij}=0$$

the right side, multiplied by -2, reduces to

$$\Delta_{12}\Delta_{13}\Delta_{42}\Delta_{43}+\Delta_{23}\Delta_{21}\Delta_{43}\Delta_{41}+\Delta_{31}\Delta_{32}\Delta_{41}\Delta_{43}$$

$$-\sum^{1,2,3,4}\varphi_h\Big[\frac{\Delta_{ij}\Delta_{ik}}{(e_h-e_j)(e_h-e_k)}+\frac{\Delta_{jk}\Delta_{ji}}{(e_h-e_k)(e_h-e_i)}+\frac{\Delta_{ki}\Delta_{kj}}{(e_h-e_i)(e_h-e_j)}\Big]$$

$$+\sum^{1,2,3,4}\frac{\varphi_h\varphi_i+\varphi_j\varphi_k}{(e_i-e_j)(e_i-e_k)(e_h-e_j)(e_h-e_k)}.$$

If now we put

$$M=(e_2-e_3)(e_3-e_1)(e_1-e_2)(e_4-e_1)(e_4-e_2)(e_4-e_3),\qquad \lambda_{ij}=(e_i-e_j)^2\Delta_{ij}^2$$

and use the identities

$$\sum^{1,2,3,4}(e_2-e_3)(e_4-e_1)\Delta_{23}\Delta_{41}=0,\qquad \sum^{1,2,3}(e_2-e_3)\Delta_{23}=0,$$

we find that the expression above, multiplied by M, can be written as the sum of three expressions of the form

$$(e_2-e_3)(e_4-e_1)[\lambda_{23}\lambda_{41}-\lambda_{23}(\varphi_1+\varphi_4)-\lambda_{41}(\varphi_2+\varphi_3)-(\varphi_2\varphi_3+\varphi_4\varphi_1)]$$

which is equal to

$$(e_2-e_3)(e_4-e_1)\Big[(\lambda_{23}-\varphi_2-\varphi_3)(\lambda_{41}-\varphi_4-\varphi_1)-\sum^{1,2,3}(\varphi_2\varphi_3+\varphi_4\varphi_1)\Big];$$

thus, with

$$\Omega_{ij}=\lambda_{ij}-\varphi_i-\varphi_j,$$

we finally have the formula

$$8(e_2-e_3)(e_3-e_1)(e_1-e_2)(e_4-e_1)(e_4-e_2)(e_4-e_3)\sum_{\lambda=1}^{p}\sum_{\mu=1}^{p}\sum_{\nu=1}^{p}\sum_{\rho=1}^{p}\wp_{\lambda\mu\nu\rho}(u)\,.\,e_1^{\lambda-1}e_2^{\mu-1}e_3^{\nu-1}e_4^{\rho-1}$$

$$=(e_2-e_3)(e_4-e_1)\Big[f(e_2\,,\,e_3)-4(e_2-e_3)^2\sum_{\lambda=1}^{p}\sum_{\mu=1}^{p}\wp_{\lambda\mu}(u)e_2^{\lambda-1}e_3^{\mu-1}\Big]$$

$$\Big[f(e_4\,,\,e_1)-4(e_4-e_1)^2\sum_{\lambda=1}^{p}\sum_{\mu=1}^{p}\wp_{\lambda\mu}(u)e_4^{\lambda-1}e_1^{\mu-1}\Big]$$

$$+(e_3-e_1)(e_4-e_2)\Big[f(e_3\,,\,e_1)-4(e_3-e_1)^2\sum_{\lambda=1}^{p}\sum_{\mu=1}^{p}\wp_{\lambda\mu}(u)e_3^{\lambda-1}e_1^{\mu-1}\Big]$$

$$\Big[f(e_4\,,\,e_2)-4(e_4-e_2)^2\sum_{\lambda=1}^{p}\sum_{\mu=1}^{p}\wp_{\lambda\mu}(u)e_4^{\lambda-1}e_2^{\mu-1}\Big]$$

$$+(e_1-e_2)(e_4-e_3)\Big[f(e_1\,,\,e_2)-4(e_1-e_2)^2\sum_{\lambda=1}^{p}\sum_{\mu=1}^{p}\wp_{\lambda\mu}(u)e_1^{\lambda-1}e_1^{\mu-1}\Big]$$

$$\Big[f(e_4\,,\,e_3)-4(e_4-e_3)^2\sum_{\lambda=1}^{p}\sum_{\mu=1}^{p}\wp_{\lambda\mu}(u)e_4^{\lambda-1}e_3^{\mu-1}\Big]$$

which, to save repetitions, we shall refer to as the fundamental formula. It is clear from it that the functions $\wp_{\lambda\mu\nu\rho}(u)$ have values independent of the order of the suffixes λ,μ,ν,ρ. It is also clear that the arbitrariness in the lower limits of the integrals by which $x_1\ldots x_p$ were initially determined from $u_1\ldots u_p$, equivalent as it is only to arbitrary additive constants for the arguments $u_1\ldots u_p$, is of no importance, and that, similarly, the arbitrariness of the coefficients $c_{\lambda\mu}$ in the definition of the polynomial $f(x\,,\,z)$, cancelled as it is by corresponding arbitrary additive constants for the functions $\wp_{\lambda\mu}(u)$, is of no importance.

The above work has been carried out on the hypothesis that the hyperelliptic equation $y^2=f(x)$ has no term in x^{2p+2}. By putting

$$x=\frac{A}{\xi-a},\qquad x_i=\frac{A}{\xi_i-a},\qquad e_i=\frac{A}{\varepsilon_i-a},\qquad \eta=\frac{(\xi-a)^{p+1}}{H}y,$$

where A and α are arbitrary, and, with λ_{2p+2} arbitrary,

$$H^2 = -4a_1 \ldots a_p c_1 \ldots c_p c \,|\, \lambda_{2p+2},$$

we easily find the corresponding results for an equation

$$\eta^2 = \lambda_{2p+2}(\xi - \alpha)(\xi - \alpha_1) \ldots (\xi - \alpha_p)(\xi - \gamma)(\xi - \gamma_1) \ldots (\xi - \gamma_p);$$

I have carried through the work, which, though long, is not difficult. It will be sufficient to state the result, which may therefore be reckoned equivalent with the former, or can be directly proved in the same way.

Let

$$y^2 = \lambda_{2p+2} P(x) Q(x) = f(x)$$

where

$$P(x) = (x-a)(x-a_1) \ldots (x-a_p), \qquad Q(x) = (x-c)(x-c_1) \ldots (x-c_p);$$

let $u_1 \ldots u_p$ be arbitrary variables, and $x_1 \ldots x_p$ be thence determined by means of

$$\sum_{k=1}^{p} \int_{m_k}^{x_k} \frac{x^{r-1}dx}{y} = u_r, \qquad {\scriptstyle r=1\ldots p}$$

and put

$$R(x) = (x-a)(x-x_1) \ldots (x-x_p), \qquad \Phi(x) = f(x) \,|\, [R(x)]^2,$$

$$\nabla_{12} = \sum_{k=1}^{p} \frac{y_k}{(e_1 - x_k)(e_2 - x_k) R'(x_k)};$$

further, taking $\frac{1}{2}p(p+1)$ arbitrary constant coefficients $c_{\lambda\mu}$, define, for undetermined quantities e_1, e_2, the function $f(e_1, e_2)$ by means of

$$f(e_1, e_2) = \lambda_{2p+2}[P(e_1)Q(e_2) + P(e_2)Q(e_1)] - 4(e_1 - e_2)^2 \sum_{\lambda=1}^{p} \sum_{\mu=1}^{p} c_{\lambda\mu} e_1^{\lambda-1} e_2^{\mu-1};$$

then, if we define $\frac{1}{2}p(p+1)$ functions $\wp_{\lambda\mu}(u)$ by means of the equation

$$\frac{4(e_1-e_2)^2 \sum\limits_{\lambda=1}^{p} \sum\limits_{\mu=1}^{p} \wp_{\lambda\mu}(u) e_1^{\lambda-1} e_2^{\mu-1} - f(e_1, e_2)}{R(e_1)R(e_2)} = (e_1 - e_2)^2 \nabla_{12}^2 - \Phi(e_1) - \Phi(e_2),$$

we shall arrive at an equation having precisely the same form as the previously deduced fundamental formula.

This second equation being regarded as deducible from the former by the transformation suggested, the functions $\wp_{\lambda\mu}(u)$ occurring in it are not identical with but linear functions of the former.

It is easy to see, as is well known, that the polynomial $f(x, z)$ satisfies the two conditions (1) of being a rational polynomial in x and z, of degree $p+1$ in each, and symmetrical in regard to them, (2) of reducing to $2f(x)$ when $z = x$, (3) of being such that

$$\left[\frac{\partial f(x, z)}{\partial z}\right]_{z=x} = \frac{df(x)}{dx},$$

the condition (3) being a consequence of (1) and (2); and that conversely any expression satisfying these is included in our form above by suitably choosing the constants $c_{\lambda\mu}$. This is so whether $f(x)$ is of order $2p+2$ or $2p+1$. If we write $f(x)$ symbolically in the form a_x^{2p+2}, one possible form for $f(x, z)$, considered by Prof. KLEIN, is $2a_x^{p+1}a_z^{p+1}$. Another form (suggested by an identity due to ABEL, see the present writer's *Abelian Functions*, p. 195) though not invariantive, appears to possess great simplicity for purposes of calculation, namely putting $f(x) = \sum_0^{2p+2} \lambda_\gamma x^\gamma$ we may take $f(x, z) = \sum_{i=0}^{p+1} x^i z^i [2\lambda_{2i} + \lambda_{2i+1}(x+z)]$, with $\lambda_{2p+3} = 0$. It will save repetitions to refer to this as ABEL's form for $f(x, z)$.

If we suppose $\lambda_{2p+2} = 0$, $\lambda_{2p+1} = 4$, and take this form for $f(x, z)$, the equations which express $(x_1 y_1) \ldots (x_p y_p)$ in terms of $u_1 \ldots u_p$ are given at once in a simple form by the formulae above. From the definition formula for the functions $\wp_{\lambda\mu}(u)$, dividing by e_2^{p+1}, putting $e_2 = \infty$, and then $e_1 = x_i$, we find that $x_1 \ldots x_p$ are the roots of the equation

$$x^p - x^{p-1}\wp_{pp}(u) - x^{p-2}\wp_{p,p-1}(u) - \ldots - \wp_{p,1}(u) = 0;$$

while, taking the formula

$$-4 \sum_{\lambda=1}^{p} \sum_{\mu=1}^{p} \sum_{\nu=1}^{p} \wp_{\lambda\mu\nu}(u) e_1^{\lambda-1} e_2^{\mu-1} e_3^{\nu-1}$$

$$= F(e_1)F(e_2)F(e_3)\left[\Delta_{23}\Delta_{31}\Delta_{12} + \frac{\sum^{1,2,3} \varphi_1(e_2 - e_3)\Delta_{23}}{\varpi_{123}}\right],$$

we obtain, for the right side, after dividing by e_3^{p-1} and putting $e_3 = \infty$, the value

$$-4F(e_1)F(e_2)\Delta_{12};$$

if we now divide by e_2^{p-1} and put $e_2 = \infty$, and afterwards put $e_1 = x_i$, we find that

$$y_i = x_i^{p-1}\wp_{ppp}(u) + x_i^{p-2}\wp_{p,p,p-1}(u) + \dots + \wp_{pp1}(u).$$

The fact we have proved, that $\wp_{\lambda\mu\nu}(u) = \wp_{\lambda\nu\mu}(u)$, shews that

$$\wp_{\lambda 1}(u)du_1 + \dots + \wp_{\lambda p}(u)du_p, \quad = -d\zeta_\lambda(u), \text{ say,}$$

is a perfect differential; in the present order of development the study of the character of the functions $\zeta_\lambda(u)$ is subsequent to that of the differential equations. From

$$\frac{\partial\zeta_\lambda(u)}{\partial u_\mu} = -\wp_{\lambda\mu}(u) = \frac{\partial\zeta_\mu(u)}{\partial u_\lambda}$$

follows that

$$\zeta_1(u)du_1 + \dots + \zeta_p(u)du_p$$

is also a perfect differential. If we write it equal to $d\log\sigma(u)$ it will be found that the differential equations naturally suggest the consideration of $\sigma(u)$ as a dependent variable, and that they are satisfied by the hypothesis that $\sigma(u)$ is an integral function.

Note. The formula for the functions $\wp_{\lambda\mu}(u)$ which is made the basis of this paper was first given by Bolza, Gött. Nachr., 1894, p. 270. A deduction from the theory of algebraic integrals was given by him, Amer. J. of Math., XVII (1895), and, independently, by the present writer (*Abel. Functions,* Cambridge, 1897, p. 329); see also Baker, *On the hyperelliptic sigma functions,* Amer. J. of Math., XX, 1898, p. 378, and Math. Annal., L, 1898, p. 462. For the equations of this paper, without demonstration, but with indications of their application, see Camb. Phil. Proc., Vol. IX, Pt. IX, p. 513, September 1898. The expression for the functions $\zeta_\lambda(u)$ in terms of algebraic integrals are given in the writers *Abelian Functions* (pp. 321 and 195). The present development is complete in itself, and requires no previous study of the associated Riemann surface, if the simple case of Jacobi's theorem of inversion which is utilised be assumed. But, if we allow the formula which expresses a theta function of any characteristic, not necessarily half-integral, by the addition of certain constants (parts of the period system) to the arguments of a theta function with zero characteristic, we see that the equations are satisfied by sigma functions of quite arbitrary characteristic.

II.

We consider now, as next in logical order, the algebraic problem of forming the explicit differential equations from the fundamental formula above established, obtaining them by way of example for $p = 2$ and $p = 3$. The method followed can be regarded only as provisional. Not only is the question how far some of these equations are deducible from the others left unconsidered; but the isobaric character of the equations, remarked below, which promises a general rule for writing down the equations for any value of p, remains not utilised. The present deduction has however great simplicity and some algebraic interest.

The following notation is employed:

The quantities before denoted by e_1, e_2, e_3, e_4 are denoted respectively by x, y, z, t, and so

$$M = (y-z)(z-x)(x-y)(t-x)(t-y)(t-z);$$

a summation extending to these four letters is denoted by S; so that for instance

$$S(y-z)^2(t-x)^2 = (y-z)^2(t-x)^2 + (z-x)^2(t-y)^2 + (x-y)^2(t-z)^2;$$

further we denote the symmetric function $S(x^\alpha y^\beta z^\gamma t^\delta)$ by $(\alpha\beta\gamma\delta)$, and the sum of the homogeneous products of x, y, z, t, including repetitions, α together, by H_α, so that for instance $H_2 = Sx^2 + Syz$ or $H_2 = (2000) + (1100)$; and we denote by $|\alpha\beta\gamma\delta|$ the determinant

$$|\alpha\beta\gamma\delta| = \begin{vmatrix} H_\alpha & H_\beta & H_\gamma & H_\delta \\ H_{\alpha-1} & H_{\beta-1} & H_{\gamma-1} & H_{\delta-1} \\ H_{\alpha-2} & H_{\beta-2} & H_{\gamma-2} & H_{\delta-2} \\ H_{\alpha-3} & H_{\beta-3} & H_{\gamma-3} & H_{\delta-3} \end{vmatrix},$$

where $H_0 = 1$ and, when n is negative, $H_n = 0$; similarly in what follows quantities usually arising with positive suffixes are to be put zero when the general rules would give negative suffixes;

we shall need to consider the coefficients $(\alpha\beta)$ arising in the product

$$\Phi_1(x, y) = (x-y)\,\Phi(x, y) = (x-y)\sum_{\alpha=0}^{N}\sum_{\beta=0}^{N} a_{\alpha\beta}x^\alpha y^\beta = -\sum_{\alpha=0}^{N+1}\sum_{\beta=0}^{N+1}(\alpha\beta)x^\alpha y^\beta,$$

wherein $\Phi(x, y)$ is any rational polynomial symmetric in x and y so that $a_{\alpha\beta} = a_{\beta\alpha}$, and

$$(\alpha\beta) = a_{\alpha,\beta-1} - a_{\alpha-1,\beta},$$

for which $(\alpha\beta) = -(\beta\alpha)$, $(\alpha\beta) = 0$; and shall meet with the Pfaffian forms

$$\{\alpha\beta\gamma\delta\} = (\alpha\beta)(\gamma\delta) - (\alpha\gamma)(\beta\delta) + (\alpha\delta)(\beta\gamma);$$

it is easy to see that when the polynomial $\Phi(x, y)$ is the Abelian form

$$\sum_{i=0}^{p+1} x^i y^i[2\lambda_{2i} + \lambda_{2i+1}(x+y)]$$

all the quantities $(\alpha\beta)$ are zero in which the difference of α and β is not 1 or 2, and that

$$(\alpha, \alpha+1) = 2\lambda_{2\alpha}, \qquad (\alpha, \alpha+2) = \lambda_{2\alpha+1};$$

similarly from two such rational symmetric polynomials

$$\Phi(xy) = \sum_{\alpha=0}^{N}\sum_{\beta=0}^{N} a_{\alpha\beta}x^\alpha y^\beta, \qquad \Phi'(x, y) = \sum_{\alpha=0}^{N}\sum_{\beta=0}^{N} a'_{\alpha\beta}x^\alpha y^\beta$$

we shall form the quantities

$$\{\alpha\beta\gamma'\delta'\} = (\alpha\beta)(\gamma'\delta') - (\alpha\gamma)(\beta'\delta') + (\alpha\delta)(\beta'\gamma') + (\gamma\delta)(\alpha'\beta') - (\beta\delta)(\alpha'\gamma') + (\beta\gamma)(\alpha'\delta')$$

reducing, when $a'_{\alpha\beta} = a_{\alpha\beta}$, to $2\{\alpha\beta\gamma\delta\}$; in particular when the first polynomial is the Abelian form above and the second is

$$(x-y)^2\sum_{\alpha=0}^{p-1}\sum_{\beta=0}^{p-1}\wp_{\alpha+1,\beta+1}x^\alpha y^\beta,$$

that is

$$\sum_{\alpha=0}^{p+1}\sum_{\beta=0}^{p+1}(\wp_{\alpha-1,\beta+1} - 2\wp_{\alpha,\beta} + \wp_{\alpha+1,\beta-1})x^\alpha y^\beta,$$

then $(\alpha\beta)$ is as before and

$$(\alpha'\beta') = -(\wp_{\alpha-2,\beta+1} - 3\wp_{\alpha-1,\beta} + 3\wp_{\alpha,\beta-1} - \wp_{\alpha+1,\beta-2}),$$

functions $\wp_{\lambda\mu}$ with negative suffixes being, as explained above, put zero.

The forms just explained arise naturally in the problem of expressing the quotient

$$-\frac{1}{M} S(y-z)(t-x)\,\Phi(y\,,\,z)\,\Phi(t\,,\,x),$$

which is an integral symmetric polynomial in $x\,,\,y\,,\,z\,,\,t$; it is equal to

$$\frac{1}{M}\left[\Phi_1(x\,,y)\Phi_1(z\,,\,t)-\Phi_1(x\,,\,z)\Phi_1(y\,,\,t)+\Phi_1(x\,,\,t)\Phi_1(y\,,\,z)\right],$$

and contains the term

$$\frac{1}{M} x^\alpha y^\beta z^\gamma t^\delta \{\alpha\beta\gamma\delta\},$$

and is therefore equal to the sum, for all combinations four together of the unequal numbers $\alpha\,,\,\beta\,,\,\gamma\,,\,\delta$ chosen from the set $0 \ldots (N+1)$, of the expressions

$$\frac{1}{M}\begin{vmatrix} x^\alpha & x^\beta & x^\gamma & x^\delta \\ y^\alpha & y^\beta & y^\gamma & y^\delta \\ z^\alpha & z^\beta & z^\gamma & z^\delta \\ t^\alpha & t^\beta & t^\gamma & t^\delta \end{vmatrix}\{\alpha\beta\gamma\delta\},$$

that is, as is well known, of the expressions

$$|\alpha\beta\gamma\delta|\{\alpha\beta\gamma\delta\}.$$

In precisely the same way the expression

$$-\frac{1}{M} S(y-z)(t-x)\left[\Phi(y\,,\,z)\,\Phi'(t\,,\,x)+\Phi(t\,,\,x)\,\Phi'(y\,,\,z)\right]$$

is equal to the sum of all possible expressions arising of the form

$$|\alpha\beta\gamma\delta|\{\alpha\beta\gamma'\delta'\}.$$

Returning now to our differential equations, and writing for brevity $f_{12}=f(x\,,\,y)$, etc., the suffixes $1\,,\,2\,,\,3\,,\,4$ being respectively associated with $x\,,\,y\,,\,z\,,\,t$, and $f(x\,,\,y)$ denoting as before a rational polynomial symmetrical in $x\,,\,y$, of degree $p+1$ in each, for which $f(x\,,\,x)=2f(x)$, and writing further

$$P_{12}=\sum_{\alpha=0}^{p-1}\sum_{\beta=0}^{p-1}\wp_{\alpha+1,\beta+1}x^\alpha y^\beta,$$

the differential equations can be put into the form

$$8 \sum_{\lambda,\mu,\nu,\rho}^{1\ldots p} (\lambda - 1\,,\ \mu - 1\,,\ \nu - 1\,,\ \rho - 1)\left[\wp_{\lambda\mu\nu\rho} - 2\,(\wp_{\mu\lambda}\wp_{\rho\lambda} + \wp_{\nu\lambda}\wp_{\rho\mu} + \wp_{\lambda\mu}\wp_{\rho\nu})\right]$$

$$= \frac{1}{M} S(y-z)(t-x) f_{23} f_{41} - \frac{4}{M} S(y-z)(t-x)\left[f_{23}(t-x)^2 P_{41} + f_{41}(y-z)^2 P_{23}\right]$$

$$+ \frac{16}{M} H S(y-z)(t-x) P_{23} P_{41}$$

wherein

$$H = \frac{1}{2} S(y-z)^2 (t-x)^2 = (2200) - (2110) + 6(1111)$$

and the summation on the left extends to every combination of four of the numbers $\lambda - 1\,,\ \mu - 1\,,\ \nu - 1\,,\ \rho - 1$ from the set $0 \ldots (p-1)$. We are to express the right side in terms of the symmetric functions $(\alpha\beta\gamma\delta)$ and equate coefficients of these on the two sides. The form of the fundamental formula here taken is recommended, not only by the simplicity of the right side, but also by the fact that if we put

$$\wp_{\lambda\mu} = -\frac{\partial^2}{\partial u_\lambda \partial u_\mu} \log \sigma(u), \qquad \sigma_i = \frac{\partial \sigma(u)}{\partial u_i}, \qquad \sigma_{ij} = \frac{\partial^2 \sigma(u)}{\partial u_i \partial u_j}, \quad \text{etc.,}$$

the expression

$$Q_{\lambda\mu\nu\rho} = \wp_{\lambda\mu\nu\rho} - 2(\wp_{\mu\nu}\wp_{\lambda\rho} + \wp_{\nu\lambda}\wp_{\rho\mu} + \wp_{\lambda\mu}\wp_{\rho\nu}) = -\frac{1}{\sigma^2}\left\{\sigma\sigma_{\lambda\mu\nu\rho} - \Sigma\sigma_\rho\sigma_{\lambda\mu\nu} + \Sigma\sigma_{\mu\nu}\sigma_{\rho\lambda}\right\}$$

involves only σ^2 in its denominator; when it is proved, as indeed follows from the differential equations, that $\sigma(u)$ is an integral function, it will be permissable to say that $Q_{\lambda\mu\nu\rho}$ is a function whose (unessential) singularities are such that $\sigma^2(u)\,Q_{\lambda\mu\nu\rho}(u)$ is an integral function. We remark moreover that if

$$\Delta_\lambda = \frac{\partial}{\partial u_\lambda} - \frac{\partial}{\partial u'_\lambda},$$

then

$$\wp_{\lambda\mu}(u) = -\frac{1}{2\sigma^2(u)} \Delta_\lambda \Delta_\mu \sigma(u)\sigma(u'),$$

$$Q_{\lambda\mu\nu\rho}(u) = -\frac{1}{2\sigma^2(u)} \Delta_\lambda \Delta_\mu \Delta_\nu \Delta_\rho \sigma(u)\sigma(u'),$$

where, after differentiation, u'_λ is to be replaced by u_λ.

On consideration of the forms arising in the fundamental formula it is immediately clear that if we reckon $\wp_{\lambda\mu}(u)$ as of *weight* $\lambda + \mu$, $\wp_{\lambda\mu\nu\rho}(u)$ as of weight $\lambda + \mu + \nu + \rho$, and, in

$$f(x\,,\,y) = \sum_0^{p+1} \sum_0^{p+1} a_{\alpha\beta} x^\alpha y^\beta,$$

reckon $a_{\alpha\beta}$ as of weight $\alpha + \beta$, then the coefficient of the symmetric function $(\alpha\beta\gamma\delta)$ on each side of the formula is isobarically of weight $\alpha + \beta + \gamma + \delta + 4$. Thus the expression to be obtained for $\wp_{\lambda\mu\nu\rho}(u)$ is isobarically of weight $\lambda + \mu + \nu + \rho$; for instance the function $\wp_{1111}(u)$ can only contain terms of weight 4, and therefore, however great p may be, cannot have more than a limited number of terms. While further, the form of $\wp_{\lambda\mu\nu\rho}(u)$ being obtained for any value of p, its form for any lower value, p_1, of p, is obtainable by the mere omission of coefficients $a_{\alpha\beta}$ which contain suffixes α or β greater then $p_1 + 1$ and of functions $\wp_{\lambda\mu}(u)$ which contain suffixes λ or μ greater then p_1. As before terms to which the general rules give negative suffixes are throughout to be omitted.

We content ourselves here with forming the equations for $p = 3$. In every form $|\alpha\beta\gamma\delta|$, or $\{\alpha\beta\gamma\delta\}$, we suppose $\alpha < \beta < \gamma < \delta$; the only forms $|\alpha\beta\gamma\delta|$ arising for $p = 3$, with their values in terms of the symmetric functions $(\alpha\beta\gamma\delta)$, are

$$|0123| = 1; \quad |0124| = (1000), \quad |0134| = (1100),$$
$$|0234| = (1110), \quad |1234| = (1111);$$
$$|0125| = (2000) + (1100), \quad |0135| = (2100) + 2(1110),$$
$$|0235| = (2110) + 3(1111), \quad |0145| = (2200) + (2110) + 2(1111)$$
$$|0245| = (2210) + 2(2111), \quad |0345| = (2220) + (2211),$$
$$|1235| = (2111), \quad |1245| = (2211), \quad |1345| = (2221), \quad |2345| = (2222).$$

With the help of these equations we can arrange the expression

$$-\frac{1}{M} S(y - z)(t - x) f(y\,,\,z) f(t\,,\,x) = \sum |\alpha\beta\gamma\delta| \{\alpha\beta\gamma\delta\}$$

where

$$f(x\,,\,y) = \sum_0^4 \sum_0^4 a_{\alpha\beta} x^\alpha y^\beta,$$

in terms of the symmetric functions (0000) ... (2222); for the expression

$$-\frac{1}{M} S(y-z)(t-x) P_{23} P_{41} = \sum |\alpha\beta\gamma\delta| \{\alpha\beta\gamma\delta\}',$$

where

$$P_{12} = \sum_0^2 \sum_0^2 \wp_{\alpha+1,\beta+1} x^\alpha y^\beta,$$

only one term arises, namely

$$|0123|\{0123\}' = (01)'(23)' - (02)'(13)' + (03)'(12)',$$

wherein

$$(\alpha\beta)' = \wp_{\alpha+1,\beta} - \wp_{\alpha,\beta+1},$$

so that the term is equal to

$$-(\wp_{32}\wp_{12} - \wp_{31}\wp_{22} + \wp_{31}^2 - \wp_{33}\wp_{11})$$

which we shall denote by $-\Delta$.

For instance by equating the coefficients of (0112) on the two sides of the fundamental formula we obtain the equation

$$8\{\wp_{1223} - 4\wp_{12}\wp_{23} - 2\wp_{22}\wp_{13}\} = -\{0235\} - \{0145\}$$
$$+ 4\{023'5'\} + 4\{014'5'\} + 16\{0123\}';$$

it will be sufficient to denote the right side of the equation by

$$-\{0235\} - \{0145\} + 4\{..''\} - 16\Delta,$$

and so for the others, and the left side by [1223]. With these notations the set of equations is as follows, the left column giving the symmetrical function $(\alpha\beta\gamma\delta)$ of which the other terms in the same horizontal line are the coefficients: —

(2222); $[3333] = -\{2345\} + 4\{..''\}$

(2221); $[3332] = -\{1345\} + 4\{..''\}$

(2220); $[3331] = -\{0345\} + 4\{..''\}$

(2211); $[3322] = -\{0345\} - \{1245\} + 4\{..''\}$

(2210); $[3321] = -\{0245\} + 4\{..''\}$

(2200); $[3311] = -\{0145\} + 4\{..''\} + 16\Delta$

(2111); $[3222] = -2\{0245\} - \{1235\} + 4\{..''\}$

(2110); $[3221] = -\{0235\} - \{0145\} + 4\{..''\} - 16\Delta$

(2100); $[3211] = -\{0135\} + 4\{..''\}$

(2000); $[3111] = -\{0125\} + 4\{..''\}$

(1111); $[2222] = -\{1234\} - 3\{0235\} - 2\{0145\} + 4\{..''\} + 96\Delta$

(1110); $[2221] = -\{0234\} - 2\{0135\} + 4\{..''\}$

(1100); $[2211] = -\{0134\} - \{0125\} + 4\{..''\}$

(1000); $[2111] = -\{0124\} + 4\{..''\}$

(0000); $[1111] = -\{0123\} + 4\{..''\}$.

To calculate now explicit values for the quantities $\{\alpha\beta\gamma\delta\}$ we limit ourselves to the hypothesis that $f(x, y)$ is of the socalled Abelian form

$$f(x, y) = \sum_0^4 \sum_0^4 x^i y^i [2\lambda_{2i} + \lambda_{2i+1}(x+y)],$$

where $\lambda_9 = 0$, the corresponding results for other forms of $f(x, y)$ being obtainable by adding a suitable constant to each of the functions $\wp_{\lambda\mu}(u)$. Then with the equations, remarked before, $(\alpha, \alpha+1) = 2\lambda_{2\alpha}$, $(\alpha, \alpha+2) = \lambda_{2\alpha+1}$, we obtain, for the forms $\{\alpha\beta\gamma\delta\}$ which arise when $p = 3$,

$$\{0123\} = 4\lambda_0\lambda_4 - \lambda_1\lambda_3; \quad \{0124\} = 2\lambda_0\lambda_5, \quad \{0134\} = 4\lambda_0\lambda_6, \quad \{0234\} = 2\lambda_1\lambda_6,$$
$$\{1234\} = 4\lambda_2\lambda_6 - \lambda_3\lambda_5;$$

$$\{0125\} = 0, \quad \{0135\} = 2\lambda_0\lambda_7, \quad \{0235\} = \lambda_1\lambda_7, \quad \{0145\} = 4\lambda_0\lambda_8,$$
$$\{0245\} = 2\lambda_1\lambda_8, \quad \{0345\} = 0, \quad \{1235\} = 2\lambda_2\lambda_7, \quad \{1245\} = 4\lambda_2\lambda_8,$$
$$\{1345\} = 2\lambda_3\lambda_8, \quad \{2345\} = 4\lambda_4\lambda_8 - \lambda_5\lambda_7.$$

To calculate the quantities $\{\alpha'\beta'\gamma'\delta'\}$ we require the values of the quantities

$$(\alpha'\beta') = -(\wp_{\alpha-2,\beta+1} - 3\wp_{\alpha-1,\beta} + 3\wp_{\alpha,\beta-1} - \wp_{\alpha+1,\beta-2});$$

those which enter are found to be given by

$$\begin{array}{lllll}
(0'1')=0 & (0'2')=0 & (0'3')=\wp_{11} & (0'4')=\wp_{12} & (0'5')=\wp_{13} \\
 & (1'2')=-3\wp_{11} & (1'3')=-2\wp_{12} & (1'4')=\wp_{22}-3\wp_{13} & (1'5')=\wp_{23} \\
 & & (2'3')=4\wp_{13}-3\wp_{22} & (2'4')=-2\wp_{23} & (2'5')=\wp_{33} \\
 & & & (3'4')=-3\wp_{33} & (3'5')=0 \\
 & & & & (4'5')=0
\end{array}$$

From these we easily calculate the fifteen quantities $\{012'3'\}\ldots\{234'5'\}$; for instance

$$\{012'3'\}=(01)(2'3')-(02)(1'3')+(03)(1'2')+(23)(0'1')-(13)(0'2')+(12)(0'3')$$
$$=2\lambda_0(4\wp_{13}-3\wp_{22})+2\lambda_1\wp_{12}+2\lambda_2\wp_{11}.$$

When all these are substituted we find the following differential equations

$$\wp_{3333}-6\wp_{33}^2=-\frac{1}{2}\lambda_4\lambda_8+\frac{1}{8}\lambda_5\lambda_7+\lambda_6\wp_{33}+\lambda_7\wp_{32}+\lambda_8(4\wp_{31}-3\wp_{22})$$

$$\wp_{3332}-6\wp_{23}\wp_{33}=-\frac{1}{4}\lambda_3\lambda_8+\lambda_6\wp_{32}+\frac{1}{2}\lambda_7(3\wp_{31}-\wp_{22})+2\lambda_8\wp_{21}$$

$$\wp_{3331}-6\wp_{31}\wp_{33}=\lambda_6\wp_{31}-\frac{1}{2}\lambda_7\wp_{21}+\lambda_8\wp_{11}$$

$$\wp_{3322}-4\wp_{32}^2-2\wp_{22}\wp_{33}=-\frac{1}{2}\lambda_2\lambda_8+\frac{1}{2}\lambda_5\wp_{32}+\lambda_6\wp_{31}-\frac{1}{2}\lambda_7\wp_{21}-2\lambda_8\wp_{11}$$

$$\wp_{3321}-2\wp_{12}\wp_{33}-4\wp_{23}\wp_{13}=-\frac{1}{4}\lambda_1\lambda_8+\frac{1}{2}\lambda_5\wp_{31}$$

$$\wp_{3311}-4\wp_{31}^2-2\wp_{11}\wp_{33}=-\frac{1}{2}\lambda_0\lambda_8+2\Delta$$

$$\wp_{3222}-6\wp_{22}\wp_{23}=-\frac{1}{2}\lambda_1\lambda_8-\frac{1}{4}\lambda_2\lambda_7-\frac{1}{2}\lambda_3\wp_{33}+\lambda_4\wp_{32}+\lambda_5\wp_{31}-\frac{3}{2}\lambda_7\wp_{11}$$

$$\wp_{3221}-4\wp_{12}\wp_{23}-2\wp_{22}\wp_{13}=-\frac{1}{8}\lambda_1\lambda_7-\frac{1}{2}\lambda_0\lambda_8+\lambda_4\wp_{31}-2\Delta$$

$$\wp_{3211}-4\wp_{12}\wp_{13}-2\wp_{11}\wp_{23}=-\frac{1}{4}\lambda_0\lambda_7+\frac{1}{2}\lambda_3\wp_{31}$$

$$\wp_{3111}-6\wp_{11}\wp_{31}=\lambda_0\wp_{33}-\frac{1}{2}\lambda_1\wp_{32}+\lambda_2\wp_{31}$$

$$\wp_{2222}-6\wp_{22}^2 = -\frac{1}{2}\lambda_2\lambda_6+\frac{1}{8}\lambda_3\lambda_5-\frac{3}{8}\lambda_1\lambda_7-\lambda_0\lambda_8-3\lambda_2\wp_{33}+\lambda_3\wp_{32} + \lambda_4\wp_{22}+\lambda_5\wp_{21}-3\lambda_6\wp_{11}+12\Delta$$

$$\wp_{2221}-6\wp_{21}\wp_{22} = -\frac{1}{4}\lambda_1\lambda_6-\frac{1}{2}\lambda_0\lambda_7-\frac{3}{2}\lambda_1\wp_{33}+\lambda_3\wp_{31}+\lambda_4\wp_{21}-\frac{1}{2}\lambda_5\wp_{11}$$

$$\wp_{2211}-4\wp_{12}^2-2\wp_{22}\wp_{11} = -\frac{1}{2}\lambda_0\lambda_6-2\lambda_0\wp_{33}-\frac{1}{2}\lambda_1\wp_{32}+\lambda_2\wp_{31}+\frac{1}{2}\lambda_3\wp_{21}$$

$$\wp_{2111}-6\wp_{11}\wp_{12} = -\frac{1}{4}\lambda_0\lambda_5-2\lambda_0\wp_{32}+\frac{1}{2}\lambda_1(3\wp_{31}-\wp_{22})+\lambda_2\wp_{21}$$

$$\wp_{1111}-6\wp_{11}^2 = -\frac{1}{2}\lambda_0\lambda_4+\frac{1}{8}\lambda_1\lambda_3+\lambda_0(4\wp_{31}-3\wp_{22})+\lambda_1\wp_{21}+\lambda_2\wp_{11}$$

wherein

$$\Delta = \wp_{32}\wp_{21}-\wp_{31}\wp_{22}+\wp_{31}^2-\wp_{33}\wp_{11}.$$

Of these the last five equations give the proper equations for $p=2$, by putting therein $\lambda_7=\lambda_8=0$ and $\wp_{33}=\wp_{32}=\wp_{31}=0$; while the last equation gives the proper equation for $p=1$.

These equations put a *problem:* To obtain a theory of differential equations which shall shew from them why, if we assume

$$\wp_{\lambda\mu}(u) = -\partial^2 \log \sigma(u) \,|\, \partial u_\lambda \partial u_\mu,$$

the function $\sigma(u)$ has the properties which *a priori* we know it to possess, and how far the forms of the equations are essential to these properties. It must suffice for the present to have stated the problem.

Cambridge (Engl.), 14 February, 1902.

[15 August. In illustration of the remarks as to weight (p. 152), it may be added that the equation given above for $\wp_{1111}$ is true for any value of p, and that the equations for the preceding four functions $\wp_{2111}, \wp_{2211}, \wp_{2221}, \wp_{2222}$ are true for any value of p if we add to the right sides the respective terms,

for $\wp_{2111}$ the term $3\lambda_0\wp_{14}$, for $\wp_{2211}$ the terms $\lambda_0(2\wp_{15}+\wp_{24})+\frac{3}{2}\lambda_1\wp_{14}$,
for $\wp_{2221}$ the terms

$$\lambda_0(\wp_{16}+3\wp_{25}-3\wp_{34})+\lambda_1\left(\wp_{15}+\frac{3}{2}\wp_{24}\right)+\lambda_2\wp_{14},$$

and for $\wp_{2222}$ the terms

$$\lambda_0(4\wp_{26}-3\wp_{44})+\lambda_1(4\wp_{35}-3\wp_{34})+4\lambda_2\wp_{24}+12(\wp_{11}\wp_{24}-\wp_{12}\wp_{14}).]$$