THE MEASURABLE BOUNDARIES OF AN ARBITRARY FUNCTION.

BY

HENRY BLUMBERG.

of COLUMBUS, Ohio, U. S. A.

Introduction.

The present paper communicates a number of new properties of arbitrary real functions, of which the most important is the theorem on the measurable boundaries (§ 2). This theorem associates with every function $f(x)$ two measurable functions — its »measurable boundaries» — between which the given function lies almost everywhere, in the sense of measure; moreover, the clustering of the points of $y = f(x)$ is maximal (exterior metric density 1) at almost every point of these measurable functions. A general function, in a certain sense, is essentially represented by its measurable boundaries, as is shown by the diverse applications (§ 3) of the central theorem of the present paper. This central result, it seems, is the most informative that has appeared concerning the essential internal structure of an arbitrary function.

Section 1 gives a useful decomposition of a general set into two components, the one measurable and the other »homogeneously non-measurable.» This property, though easily derived, seems not to have been explicitly remarked. Section 2 proves various theorems on general functions, including the theorem on the measurable boundaries. Section 3 gives various applications of this theorem, and section 4 briefly discusses several related questions.

Section 1.

A Theorem on General Sets.

The following theorem, easily derived, does not seem to have been explicitly remarked:[1]

Theorem I. *Every non-measurable set S of Euclidean n-space admits a decomposition $S = M + N$ into a measurable subset M and a non-measurable subset N such that at every point of N the metric density*[2] *of both N and $\overline{N}$, the complement of N, is 1; this decomposition is unique if sets of measure zero are regarded as negligible.*

Proof. We suppose, as we may, that S is bounded and lies in the sphere P. Let m be the maximum measure of measurable subsets of S; m is attained as the measure of some subset M of S, for if M_r, $r = 1, 2, \ldots \infty$, is a sequence of measurable subsets of S of measure m_r with $\lim m_r = m$, the set $\sum_1^\infty M_r$ is a measurable subset of S of measure m. Let $S = M + N$; N therefore contains no measurable subset of positive measure. We shall call a set »*homeogeneously non-measurable*» if, like N, it contains no measurable subsets of positive measure. By a »*measurable envelope*» of a set A we understand a measurable set containing A and of measure equal to the exterior measure of A[3]. Let T be a measurable envelope of N. It follows that $m_e(\overline{N}T) = m(T)$ where $\overline{N} = I - N$, for otherwise $NT = N$ would contain a measurable subset of positive measure. T is thus also a measurable envelope of $\overline{N}T$. Since the metric density of a measurable set is 1 at »almost» all of its points — i. e., with the possible exception of a set of measure zero — and the metric density of a set is at every point equal to the metric density of any of its measurable envelopes, it follows that the metric

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[1] Cf., for example, the theorem of Kamke: Eine beschränkte lineare Menge die fast in jedem ihrer Punkte eine positive innere Dichte hat ist messbar, Fund. Math. vol. X (1927) p. 433, which is an immediate consequence of our Theorem I.

[2] The metric density of a set S (= exterior metric density — but for brevity we discard the adjective »exterior») at a point x is the limit (if it exists) of the relative exterior measure of S in a sphere P enclosing x and of infinitesimal radius — this relative exterior measure meaning the ratio of $m_e(SP)$, the exterior measure of SP, to the volume of P. If the limit of this ratio does not exist, we have, at any rate, its lim sup (lim inf) = upper (lower) metric density of S at x.

[3] Cf., for example, Carathéodory, *Vorlesungen über reelle Funktionen*, Teubner Verlag, 1927, p. 260.

density of both N and $\bar{N}T$ is 1 at almost every point of T; hence the metric density of N and $\bar{N}$ is 1 at almost every point of N. This is tantamount to the statement of the theorem, since, as far as the argument is concerned, N may be replaced by the subset of its points where the metric density of both N and $\bar{N}$ is 1.

To show the uniqueness of the decomposition, on the assumption that sets of measure 0 may be neglected, it is sufficient to note that if $S = M_1 + N_1$ is another decomposition of the type considered, MN_1 being equal to $M - MM_1$ is measurable. According to hypothesis, the metric density of $\bar{N}_1$ is 1 at every point of N_1, hence N_1 cannot contain a measurable subset of positive measure. It follows that $m(MN_1) = 0$. Likewise $m(M_1 N) = 0$, so that M and M_1, on the one hand, and N and N_1, on the other, are identical if sets of zero measure are negligible.

It may be remarked that if *A and B are any two given sets, the metric density of A is either 0 or 1 at almost every point of B.* For if T is a measurable envelope of A, then at almost every point of BT, the metric density of T and therefore of A is 1; and at almost every point of $B\bar{T}$, the metric density of T and therefore of A is 0.

Section 2.

The Theorem on the Measurable Boundaries of an Arbitrary Function.

Let now $y = f(x)$ be a given real function,[1] unconditioned except that to every x there corresponds at least one value of y, the number of y's associated with x permissibly varying with x. For convenience, we assume also that $f(x)$ is bounded; the case of f unbounded is not substantially different and can be treated by such a transformation as $\bar{f} = \frac{f}{1 + |f|}$. We shall say that the point (ξ, η) of the xy-plane is »*fully approached*» by the curve $y = f(x)$, if for every $\varepsilon > 0$, the set $E_{\eta\varepsilon} = E_{|f(x)-\eta|<\varepsilon}$, which signifies the set of x's for which there is at least one value of $f(x)$ such that $|f(x) - \eta| < \varepsilon$ is of metric density 1 at ξ. If an ε exists such that $E_{\eta\varepsilon}$ is of metric density 0 at ξ we shall say that the point (ξ, η) is »*vanishingly approached*» by $y = f(x)$. If (ξ, η) is not vanish-

[1] For greater simplicity of exposition, we deal with functions of one variable, though the argument of this section applies equally well to functions of n variables.

ingly approached, we shall say that it is »*positively approached*»; in this case, if ε is a given positive number, the metric density of $E_{\eta\varepsilon}$ either does not exist or is positive, that is to say, in both cases, the upper metric density is positive at ξ.

In connection with the modes of approach just defined, we have the following two theorems, the first of which is known but included for expository completeness.

Theorem II.[1] *If $f(x)$ is an arbitrary, one- or many-valued function, the set of numbers ξ such that there is a point $(\xi, f(\xi))$ not fully approached by $y=f(x)$ is of measure* o.

Proof. If r and s are two real numbers and $r<s$, let $E_{rs}=E_{r<f(x)<s}$ be the set of x's such that there is at least one value of $f(x)$ satisfying $r<f(x)<s$. According to the theorem on the decomposability of an arbitrary set into a set of measure o and a set which has metric density 1 at each of its points,[2] we may write $E_{rs}=Z_{rs}-H_{rs}$, where Z_{rs} is of measure o and H_{rs} has metric density 1 at each of its points. Let Z be the sum of all Z_{rs} as r and s, $r<s$, range independently over the set of rational numbers; Z is of measure o. Let $(\xi, f(\xi))$ be a point of $y=f(x)$ such that ξ does not belong to Z; ξ therefore belongs to H_{rs} for all rational r, s satisfying $r<f(\xi)<s$. Since H_{rs} has metric density 1 at ξ and is a subset of E_{rs} the metric density of E_{rs} is 1 at ξ. It follows that $(\xi, f(\xi))$ is fully approached by $y=f(x)$.

Theorem III. *The set of points ξ for which an η exists such that (ξ, η) is positively but not fully approached by the curve $y=f(x)$ is of measure* o.

Proof. We define the function $f^*(x)$ as the one which, for every x, is to take on all the values $f(x)$ and, in addition, all the values η such that (x, η) is positively approached by $y=f(x)$. Now suppose (ξ, η) is a point of the (x, y) plane not fully approached by $y=f(x)$; then it cannot be fully approached by $y=f^*(x)$. For by the definition of full approach, there exists an $\varepsilon>0$ such that the lower metric density of $E_{\eta\varepsilon}$ equals a number k less than 1, hence there exists a sequence of intervals I_ν of infinitesimal length l_ν such that $\lim m_e(E_{\eta\varepsilon} I_\nu)/l_\nu=k$, and since we may neglect a finite number of ν's we may assume that $m_e(E_{\eta\varepsilon} I_\nu)/l_\nu<k'$, a number between k and 1, for all ν. Let M_ν be

[1] Cf. Blumberg, *New Properties of All Real Functions,* Trans. Am. Math. Soc., vol. 24 (1922), Theorem IX in conjunction with the first sentence of Concluding Remarks.

[2] Cf. for example, Blumberg, Bull. Am. Math. Soc., vol. 25 (1919), p. 352.

a measurable envelope of $E_\nu = E_{\eta\varepsilon} I_\nu$, and $\overline{M}_\nu = I_\nu - M_\nu$. The relative measure of $\overline{M}_\nu$ in I_ν is greater than $1 - k'$. Let $\overline{M}_\nu = \overline{M}^*_\nu + Z_\nu$ where Z_ν is of measure 0 and $\overline{M}^*_\nu$ is such that at each of its points the metric density of $\overline{M}_\nu$ is 1. The metric density of E_ν is therefore 0 at every point of $\overline{M}^*_\nu$ and hence no point (x, y) with x in $\overline{M}^*_\nu$ and $\eta - \varepsilon < y < \eta + \varepsilon$ is positively approached by the curve $y = f(x)$; no such point is, therefore, on the curve $y = f^*(x)$. Consequently, if $E^*_{\eta\varepsilon}$ has the same meaning for f^* as $E_{\eta\varepsilon}$ for f, we conclude that $m_e(E^*_{\eta\varepsilon} I_\nu)/l_\nu < k'$ and therefore (ξ, η) is not fully approached by $y = f^*(x)$.

Now let A be the set of ξ's for which there is at least one point (ξ, η) positively but not fully approached by $y = f(x)$. Then (ξ, η) is on the curve $y = f^*(x)$, and as we have just seen, it is not fully approached by $y = f^*(x)$. Therefore according to Theorem II, A is of measure 0.

Definition. The »*metrical upper boundary*» $u(\xi)$ of a one- or many-valued function $f(x)$ at ξ is the lower boundary (= greatest lower bound) of all numbers k such that the set $E_{f(x) > k}$ is of metric density 0 at ξ.[1] Likewise, the metrical lower boundary $l(\xi)$ of $f(x)$ at ξ is the upper boundary of all numbers k such that the metric density of $E_{f(x) < k}$ is 0 at ξ. The »*metrical saltus*» of $f(x)$ at ξ is defined by the equation $s(\xi) = u(\xi) - l(\xi)$.

The points $(\xi; u(\xi))$, $(\xi, l(\xi))$ are respectively the highest and lowest points on $x = \xi$ positively approached by $y = f(x)$. For since $E_{f > u(\xi) + \varepsilon}$ is, for every positive ε, of metric density 0 at ξ, the point (ξ, k) is vanishingly approached by $y = f(x)$ for every $k > u(\xi)$. On the other hand, $(\xi, u(\xi))$ is positively approached by $y = f(x)$. For if this were not so, there would exist a positive δ such that $E_{u(\xi) - \delta < f < u(\xi) + \delta}$ is of metric density 0 at ξ; but this, together with the fact that $E_{f > k}$ is of metric density 0 at ξ for every $k > u(\xi)$ would imply that $E_{f > u(\xi) - \delta}$ is of metric density 0 at ξ, in contradiction with the definitional property of $u(\xi)$. Likewise, $(l(\xi), \xi)$ is the lowest point on $x = \xi$ positively approached by $y = f(x)$.

Definition. The one-valued function $f(x)$ is said to be »*metrically upper-semi-continuous*» at the point ξ, if for every positive ε, the set $E_{f > f(\xi) + \varepsilon}$ has metric density 0 at, ξ; in other words, if $u(\xi) \leqq f(\xi)$. Similarly, f is metrically lower-semi-continuous at ξ if $l(\xi) \geqq f(\xi)$. If f is both metrically upper-semi-

[1] Cf. Kempisty, *Sur les fonctions approximativement descontinues*, Fund. Math., vol. VI (1924), p. 6.

continuous and lower-semi-continuous at ξ, we have $u(\xi) = f(\xi) = l(\xi)$; we then say that f is »*metrically continuous at* ξ».[1]

We have the following

Theorem IV. *$u(x)$ is everywhere metrically upper-semi-continuous.*

Proof. If ξ is a particular value of x and $\varepsilon > 0$, there exists for every positive η — on account of the definitional property of $u(x)$ — a positive number h such that $m_e(IE_{f > u(\xi)+\varepsilon}) < l\eta$ for every interval I enclosing ξ and of length $l < h$. If $\bar{E}$ is the complement of $E_{f > u(\xi)+\varepsilon}$ with respect to the interval I, we have $m_i(\bar{E}) > (1-\eta)l$, where $m_i(E)$ stands for the interior Lebesgue measure of $\bar{E}$. Let M be a measurable subset of $\bar{E}$ of measure $> (1-\eta)l$ and such that the metric density of M is 1 at each of its points. It follows that for every point x of M, we have $u(x) \leqq u(\xi) + \varepsilon$. Therefore the relative exterior measure in I of $IE_{u > u(\xi)+\varepsilon}$ is $< \eta$, and this holds for all I's of length $< h$. Since η is arbitrarily small, the metric density of $E_{u > u(\xi)+\varepsilon}$ is 0 at ξ, and therefore $u(x)$ is metrically upper-semi-continuous at ξ.

We shall now prove that $u(x)$ is metrically lower-semi-continuous almost everywhere. Let ξ be a point where $u(x)$ is not metrically lower-semi-continuous; hence, for some positive ε, the metric density of $E_{u(x) < u(\xi)-\varepsilon}$ is not zero, in other words, there exists a positive number p and a sequence of intervals I_n containing ξ and of infinitesimal length l_n such that $m_e(I_n E_{u(x) < u(\xi)-\varepsilon}) > pl_n$. Let γ be a point of I_n such that $u(\gamma) < u(\xi) - \varepsilon$. According to Theorem III, $u(x)$ is metrically upper-semi-continuous at γ, and we may therefore enclose γ in an interval I_γ lying in I_n and of arbitrarily small length l_γ, such that $m_e(I_\gamma E_{u(x) \geqq u(\xi)-\varepsilon}) <$ $< \eta l_\gamma$ where η is infinitesimal with l_γ. Since the exterior measure of the set of available γ's is greater than pl_n, we may, according to the Vitali Covering Theorem, select a finite number of non-overlapping intervals of type I_γ such that the sum of their lengths is grater than pl_n. Hence $m_i(I_n E_{u(x) < u(\xi)-\varepsilon})$ is at least $(1-\eta)pl_n$, and therefore the lower metric density at ξ of $E_{u(x) \geqq u(\xi)-\varepsilon}$ is at most $1 - p + \eta p$. It follows, since η is infinitesimal, that ξ is one of the ex-

[1] The notion of metrical continuity as here defined is identical with that of »approximate continuity» according to Denjoy, Bull. de la soc. math. de France, vol. 43 (1915) p. 165. Since many types of approximate continuity are possible, — see, for example, Blumberg, *On the Characterization of the Set of Points of λ-Continuity*, Annals of Mathematics, 2d ser. vol. 25 (1923), p. 118 — the term »metrical continuity» designates the particular property in question more definitely than »approximate continuity».

ceptional points of the curve $y = u(x)$ at which the approach is not full, the totality of such ξ's constituting, according to Theorem II a set of measure 0. If ξ is not one of these exceptional points, $u(x)$ is metrically lower-semi-continuous at ξ.

Having now proved that $u(x)$ is metrically upper-semi-continuous everywhere and lower-semi-continuous almost everywhere, we conclude that $u(x)$ is metrically continuous almost everywhere.[1] Likewise $l(x)$, the metrical lower boundary function, and $s(x)$, the metrical saltus, are almost everywhere metrically continuous. Now *a function which is almost everywhere metrically continuous is measurable.* For let $g(x)$ be metrically continuous almost everywhere. For a given number k, we may write $E_{g(x)>k} = M + N$, where M is measurable and N, if not the null set, homogeneously non-measurable. If ξ belongs to N, the metric density of both $E_{g>k}$ and $E_{g \leqq k}$ is 1 at ξ; therefore $g(x)$ is not metrically continuous at ξ, and since $g(x)$ is metrically continuous almost everywhere, N is of measure 0. $E_{g(x)>k}$ is thus measurable for every k, and hence $g(x)$ is measurable. We may therefore state

Theorem V. *The metrical upper boundary, lower boundary and saltus are measurable functions.*

It is known of a measurable function that it is metrically continuous almost everywhere. But obviously not every measurable function can be a metrical upper boundary, since a metrical upper boundary is metrically upper-semi-continuous everywhere without exception; an arbitrary change of functional values at the points of a set of measure 0 would preserve the measurability of a measurable function, but by means of such a change we can eliminate the property of metrical upper semi-continuity at various points. It is also evident — at any rate, this follows from the next theorem — that not every function which is metrically upper-semi-continuous at every point can be a metrical upper boundary. One way of fully characterizing a metrical upper boundary is as follows:

[1] In fact, every function which is metrically upper-semi continuous almost everywhere is metrically lower-semi-continuous almost everywhere, because, if $f(x)$ is metrically upper-semi-continuous almost everywhere, it is necessarily measurable, as the following brief argument shows: Let $E = E_{f(x)<k}$, where k is a given constant. By Theorem I, $E = M + N$, M measurable, N homogeneously non-measurable. If ξ belongs to E, $f(x)$ is metrically upper-semi-continuous at ε; and ε is positive, then $E_{f>f(\xi)+\varepsilon}$ is of metric density 0 at ξ. If ε is chosen so that $f(\xi) + \varepsilon < k$, $E_{f \geqq k}$ is of metric density 0 at ξ. The metric clustering of the complement of E is thus infinitesimal at almost every point of N; therefore N is of measure 0, and hence f is measurable.

Theorem VI. *A necessary and sufficient condition that a function $f(x)$ be a metrical upper boundary is that $u(x)$, the metrical upper boundary of $f(x)$, be identical with $f(x)$.*

All that requires proof is that the condition is necessary, namely that the metrical upper boundary $v(x)$ of a metrical upper boundary $u(x)$ is identical with $u(x)$. Since $u(x)$ is metrically upper-semi-continuous, it follows that $v(x) \leqq u(x)$. Suppose, contrary to our assertion, we have $v(\xi) < u(\xi)$ for some point ξ. Then $E_{u \geqq k}$, where k is a number between $v(\xi)$ and $u(\xi)$, is measurable, according to Theorem V, and of zero metric density at ξ, so that for a variable interval I enclosing ξ and of infinitesimal length l_I, we have

$$m(IE_{u \geqq k}) = \varepsilon_I l_I,$$

where ε_I is infinitesimal with l_I. If t is a point of $IE_{u<k}$, the metric density of $E_{f \geqq k}$ is o at t, so that we can enclose t in an interval I_t of length l_t as small as we please and lying in I such that $m_e(I_t E_{f \geqq k}) < \zeta_I l_t$, where ζ_t is infinitesimal with I. According to the Vitali Covering Theorem, we may select a set σ of non-overlapping intervals I_t containing all of $IE_{u<k}$ except a set of measure o. The set $IE_{f(x) \geqq k}$ is partly inside and partly outside the intervals of σ; the former part is of exterior measure $< \zeta_I l_I$, and the latter part of exterior measure $\leqq \varepsilon_I l_I$, so that $m_e(IE_{f \geqq k}) < (\varepsilon_I + \zeta_I) l_I$, i. e., the relative measure of $E_{f \geqq k}$ in I is infinitesimal with l_I. Therefore $u(\xi) \leqq k$, which is a contradiction, and hence $v(\xi) = u(\xi)$.

If $f(x)$ is a metrical upper boundary, we have just proved that $u(x) = f(x)$; and since $(\xi, u(\xi))$ is positively approached by $y = f(x)$ for every ξ, it follows that a metrical upper boundary function $f(x)$ is positively approached at every point of $y = f(x)$. A metrical upper boundary function $f(x)$ thus possesses the properties of metrical upper semi-continuity and of positive approach for every x. Conversely, if $f(x)$ has these two properties for every x, it is a metrical upper boundary. For on account of the metrical upper semi-continuity of $f(x)$ at x, we have $u(x) \leqq f(x)$; and on account of the positive approach of $(x, f(x))$ by $y = f(x)$, we have $u(x) \geqq f(x)$. Therefore $u(x) = f(x)$ is metrical upper boundary function. We thus have

Theorem VII. *A necessary and sufficient condition that $f(x)$ be a metrical upper boundary is that f be everywhere metrically upper-semi-continuous and that $y = f(x)$ approach all of its points positively. Likewise, a metrical lower boundary* *is completely characterized by the properties of metrical lower-semi-continuity and positive approach.*

Consider a point ξ such that $(\xi, u(\xi))$ is not fully approached by the curve $y = f(x)$. Therefore, for some positive ε, the metric density at ξ of $E = E_{u(\xi)-\varepsilon<f(x)<u(\xi)+\varepsilon}$ is not 1, so that there is a positive number h and a sequence of intervals I_n enclosing ξ and of infinitesimal length l_n, such that $m_e(E I_n) < (1 - h) l_n$. Hence there is a measurable subset M_n of $I_n - E I_n$ of measure $> h l_n$ and of metric density 1 at each of its points. Since $f(x) \leqq u(\xi) - \varepsilon$ or $\geqq u(\xi) + \varepsilon$ at every point of M_n, $u(x)$ must also be $\leqq u(\xi) - \varepsilon$ or $\geqq u(\xi) + \varepsilon$ at every point of M_n. Therefore

$$m_e(I_n E_{u(\xi)-\varepsilon<u(x)<u(\xi)+\varepsilon}) < (1 - h) l_n,$$

and hence $E_{u(\xi)-\varepsilon<u(x)<u(\xi)+\varepsilon}$ is not of metric density 1 at ξ; ξ is thus, according to Theorem II, one of the exceptional points aggregating a set of measure 0. We conclude that *for almost every x the point $(x, u(x))$ is fully approached by the* curve $y = f(x)$. Similarly for $l(x)$.

This shows, on the one hand, how rich, from the point of view of measure, is the clustering of the points of $y = f(x)$ at the points of $y = u(x)$, $y = l(x)$. On the other hand, the set of x's for which $f(x) > u(x)$ or $< l(x)$ is of measure 0. For let ξ be a point at which $u(x)$ is metrically continuous. According to the definition of metrical upper boundary, we can enclose ξ in an interval I_ξ of length l_ξ such that $m_e\left(I_\xi E_{f(x)>u(\xi)+\frac{\eta}{2}}\right) < \frac{\varepsilon}{2} l_\xi$, where ε and η are arbitrary positive numbers. We may furthermore, on account of the metric continuity of $u(x)$ at ξ, choose I_ξ with the additional property that

$$m_e\left(I_\xi E_{u(x)<u(\xi)-\frac{\eta}{2}}\right) < \frac{\varepsilon}{2} l_\xi.$$

Therefore, for every point x in a measurable subset M_ξ of I_ξ of measure $> (1 - \varepsilon) l_\xi$, we have both $f(x) \leqq u(\xi) + \eta/2$ and $u(x) \geqq u(\xi) - \eta/2$, hence $f(x) \leqq u(x) + \eta$. Since the ξ's at which u is metrically continuous constitute a set of measure $b - a$, — (a, b) being, as we suppose, the interval of definition of $f(x)$ — we may, according to the Vitali Covering Theorem, select a set of non-overlapping I_ξ's with length sum equal to $b - a$, hence the sum of the measures of the corresponding M_ξ's is $> (1 - \varepsilon)(b - a)$. Consequently the interior measure of the set of points where $f(x) \leqq u(x) + \eta$ is $> (1 - \varepsilon)(b - a)$, and since this holds for all positive ε and η, the set of points at which $f(x) > u(x)$ is of measure o.

Summarizing, we may now state the following

Theorem VIII. *With every given one- or many-valued function $f(x)$, there are uniquely associated two measurable functions $u(x)$ and $l(x)$, the respective metric upper and lower boundaries of $f(x)$ at x; these functions are respectively metrically upper-semi-continuous and lower-semi-continuous. Moreover, for every x, the points $(x, u(x))$, $(x, l(x))$ are positively approached by $y = u(x)$, $y = l(x)$ respectively; and for almost every x, these points are fully approached by $y = f(x)$; on the other hand, the set of points x for which $f(x) > u(x)$ or $f(x) < l(x)$ is of measure* o. *Conversely, if $u(x)$, $l(x)$, with $u(x) \geqq l(x)$, are 2 given functions which are metrically upper- and lower-semi-continuous respectively, and $y = u(x)$, $l(x)$ respectively approach every point $(x, u(x))$, $(x, l(x))$ positively, there exists a function $f(x)$ having $u(x)$ and $l(x)$ respectively as metrical upper and lower boundaries.*

To prove the converse part of this theorem, suppose that the interval (a, b) is decomposed into two sets M_1 and M_2 — necessarily non-measurable — such that each has metric density I at every point of (a, b); let $f(x) = u(x)$ or $l(x)$ according to whether x belongs to M_1 or M_2. We shall show that the metrical upper boundary of $f(x)$, which we now denote by $v(x)$ is identical with $u(x)$. For, since $u(x)$ is, by hypothesis, metrically upper-semi-continuous at every point ξ, the set $E_{u > u(\xi) + \varepsilon}$ has metric density o at ξ for every $\varepsilon >$o; therefore, since $f(x) \leqq u(x)$ for all x, the set $E_{f > u(\xi) + \varepsilon}$ has metric density o at ξ. Hence $v(\xi) \leqq u(\xi)$. On the other hand, since every point on $y = u(x)$ is positively approached by this curve, $E_1 = E_{u(x) > u(\xi) - \varepsilon}$ has positive upper metric density at ξ for every positve ε; and since $u(x)$ is metrically upper-semi-continuous, it is, according to the last footnote, measurable, and E_1 is therefore measurable. Since M_1 and M_2 are each of metric density I at every point of (a, b) it follows that $m_e(E_1 M_1) = m(E_1)$, and we may conclude, because of the identity of $f(x)$ and $u(x)$ on $E_1 M_1$, that $E_{f(x) > u(\xi) - \varepsilon}$ has positive metric approach at ξ. Therefore $v(\xi) \geqq u(\xi)$, and hence $v(\xi) = u(\xi)$.

Theorem VIII shows with a certain explicitness the degree of arbitrariness possessed by an arbitrary function. Except with the aid of recent developments in the Theory of Point Sets, one cannot see how mathematicians could have surmised that every function $f(x)$ is necessarily built, so to speak, on the scaffolding of two functions $u(x)$, $l(x)$ of a relatively restricted nature — belonging to a certain subset of the set of measurable functions, — built in the sense that on the one hand, $y = f(x)$ has only a negligible set of points above $y = u(x)$ or below $y = l(x)$, while on the other hand, almost every point of these curves is as »richly» approached as possible — in a truly significant sense.

Since, according to a theorem of Borel[1], every measurable function equals, for every positve ε, a polynomial function, with a possible error $< \varepsilon$, and according to a theorem of Vitali[2], every measurable function equals a function of class $\leqq 2$, according to the Baire classification, except in a set of measure 0, we have the following theorem as a consequence of Theorem VIII:

Theorem IX. *a) With every function $f(x)$ we may, for every $\varepsilon > 0$, associate two polynomials $p_1(x)$ and $p_2(x)$ such that the set of points at which $f(x) > p_1(x) + \varepsilon$ or $< p_2(x) - \varepsilon$ is of exterior measure less than ε, and the exterior measure of each of the sets $E_{|f(x)-p_1(x)| > \varepsilon}$ and $E_{|f(x)-p_2(x)| < \varepsilon}$ is at least $b-a-\varepsilon$. b) With every function $f(x)$ we may associate two functions $g_1(x)$ and $g_2(x)$, each of Baire's second class at most, such that $g_1(x) \geqq g_2(x)$ everywhere, $g_1(x) \geqq f(x)$ and $g_2(x) \leqq f(x)$ almost everywhere, and the two points $(x, g_1(x))$, and $(x, g_2(x))$ are, for almost every x, fully approached by $y = f(x)$.*

These formulations serve to indicate the degree of smoothness an arbitrary function necessarily possesses.

We prove additionally the following

Theorem X. *For every function $f(x)$, defined in the interval (a, b), there exists, for every positive ε, a measurable subset M of (a, b) of measure $> b - a - \varepsilon$ such that, for every x of M, $u(x)$ and $l(x)$ are equal respectively to $u_0(x)$, $l_0(x)$, the ordinary upper, lower boundary (= least upper bound, greatest lower bound) of $f(x)$ at x on the understanding that these four numbers are computed with respect to M, i. e.. that the values of $f(x)$ outside of M are neglected.*

Proof. According to Theorem VII, $f(x) \leqq u(x)$ almost everywhere. In virtue of a theorem of Lusin[3], $u(x)$ being measurable, is continuous with respect to some measurable subset M of (a, b) of measure $> b - a - \varepsilon$, ε being a preassigned positive number. We assume, as we may, that the metric density of M is 1 at each of its points. Furthermore, according to Theorem VIII, the

[1] See, for example, Sierpiński, Fund. Math., vol. III (1922), p. 316.

[2] Rend. Lomb., vol. 38 (1905), p. 599.

[3] See, for example, Sierpiński, l. c. p. 320.

approach of $y = f(x)$ is full at almost every point of $y = u(x)$, and we may therefore assume, there being nothing more involved than the discarding of a set of measure 0, that for every ξ of M the point $(\xi, u(\xi))$ is fully approached by $y = f(x)$; and since the metric density of M is 1 at each of its points this approach is also full via M. Therefore, for ξ in M, $u(\xi)$ computed with reference to M, equals $u(\xi)$ computed with reference to the entire interval (a, b). Moreover, since we may assume for our present purposes that $f(x) \leqq u(x)$ in M, and since $u(x)$ is continuous at ξ with respect to M, it follows that the ordinary upper boundary $u_0(x)$ of $f(x)$ at ξ, computed with respect to M, is $\leqq u(\xi)$. But, of course, $u_0(\xi) \geqq u(\xi)$ with respect to M, so we conclude that $u(\xi) = u_0(\xi)$ with respect to M. Similarly $l(\xi) = l_0(\xi)$ with respect to M.

Remark. If ε is allowed to be 0, this theorem becomes false, as the following example shows: Let S be a nowhere dense, perfect subset of (a, b) of positive measure. Define $f(x)$ to be 0 in S and 1 in its complement $\bar{S}$. Since the metric density of S is 1 at almost all of its points, we have $u(\xi) = 0$ for almost every ξ of S. If M is a set of measure $b - a$ such that, for every x of M, $u(x) = u_0(x)$ with respect to M, then at almost every point x of SM we have $u_0(x) = 0$, hence a neighbourhood exists for such an x in which there are no points of $\bar{S}M$, whence $m(M) < b - a$ contrary to our supposition.

Section 3.

Applications of the Theorem on the Measurable Boundaries.

If $f(x)$ is an arbitrarily given, bounded, one-valued function, defined in the interval (a, b) we difine the upper Lesbegue integral of f in $I = (a, b)$ as follows: Let $I = M_1 + M_2 + \cdots + M_n$ be a decomposition (δ) of I into a finite number[1] of non-overlapping measurable sets $M_1, \ldots M_n$; and μ_ν, $\nu = 1, \ldots n$, the upper boundary of $f(x)$ in M_ν. Then the upper Lesbegue integral of f in I, in symbols $\overline{\int_I} f$, is the lower boundary of $\Sigma\, m(M_\nu)\mu_\nu$ for all possible decompositions of type (δ). Likewise $\underline{\int_I} f$ (= lower Lesbegue integral of f in I) is the upper

[1] There is no advantage in using an infinite number.

boundary of $\Sigma m(M_\nu) l_\nu$ for all possible decompositions (δ), where l_ν stands for the lower boundary of f in M_ν. Then we have

Theorem X. *The Lebesgue upper (lower) integral of a bounded one-valued function equals the Lebesgue integral of the metrical upper (lower) boundary function associated with* $f(x)$.

For if $\varepsilon > 0$ is given, there exists, on account of the measurability of $u(x)$, the metrical upper boundary of the given function $f(x)$, a decomposition of the interval I of definition of f into n non-overlapping measurable sets $I = M_1 + \cdots + M_n$ such that the saltus of u in M_ν, $\nu = 1, 2, \ldots n$ is less than ε. Therefore the Lebesgue integral of u over I differs from $\sum_{\nu=1}^{n} m(M_\nu)\bar{\mu}_\nu$ by less than $(b-a)\varepsilon$, where $\bar{\mu}_\nu$ represents the upper boundary of u in M_ν. Let M_0' be the set of points where $f(x) > u(x)$; according to Theorem VIII, M_0' is of measure zero. Let $I = \sum_{0}^{n} M_\nu'$ be a new decomposition of I, where $M_\nu' = M_\nu - M_0$, $\nu = 1, 2, \ldots n$. If μ_ν', $\nu = 1, \ldots n$, represents the upper boundary of f in M_ν', we have $\mu_\nu' \leqq \mu_\nu$ and therefore, since $m(M_0') = 0$, we have

$$\sum_{0}^{n} m(M_\nu')\mu_\nu' \leqq \sum m(M_\nu)\bar{\mu}_\nu \leqq \int_I u + (b-a)\varepsilon.$$

It follows from the definition of $\overline{\int_I} f$ that $\overline{\int_I} f \leqq \int_I u$. On the other hand, suppose $I = \sum_{1}^{n} M_\nu$ is a decomposition of I into a finite number of non-overlapping measurable sets. Let Z_ν be the set of points of M_ν at which the metric density of M_ν is not equal to 1, i. e., either does not exist or is less than 1; then Z_ν is of measure 0. Let $M_\nu' = M_\nu - Z_\nu$, $\nu = 1, 2, \ldots n$, and $I = M_0' + M_1' + \cdots + M_n'$ a second decomposition of I, where $M_0' = \sum_{\nu=1}^{n} Z_\nu$. We have $\sum_{\nu=1}^{n} \bar{\mu}_\nu m(M_\nu') \geqq \int_I u$, where $\bar{\mu}_\nu$ is the upper boundary of u in M_ν'. Since M_ν', $\nu = 1, 2, \ldots n$, has density 1 at each of its points, and the point $(x, u(x))$ is positively approached by the curve $y = f(x)$, it follows that $\mu_\nu' \geqq \bar{\mu}_\nu$, where μ_ν' is the upper boundary of f in M_ν', $\nu = 1, 2, \ldots n$. If μ_ν is the upper boundary of f in M_ν, we have

$$\sum_{\nu=1}^{n} \mu_\nu\, m(M_\nu) \geqq \sum \mu'_\nu\, m(M'_\nu) \geqq \sum \mu_\nu\, m(M'_\nu) \geqq \underline{\int_I} u.$$

Therefore $\overline{\int_I} f \geqq \underline{\int_I} u$, and hence $\overline{\int_I} f = \underline{\int_I} u$. Similarly $\underline{\int_I} f = \int_I l(x)$.

It is known that one method for deriving properties of continuous functions is to approximate them by means of »step-functions», i. e. functions constant except possibly at a finite number of points. Again, there are well known general procedures for deriving properties of measurable functions from properties of continuous functions, or we may transform properties of step-functions or of linear functions into properties of measurable functions without the mediation of continuous functions. Now Thorem VIII gives us a means, as we shall see more definitely in the illustrations to follow, of deriving properties of general functions from known properties of measurable functions, and thus constitutes a last link in a general procedure for converting theorems on functions $y=$ constant to theorems on general functions. This essentially amounts to saying that this theorem, in conjunction with methods already known, may enable us to derive properties of general functions from properties of intervals. We shall now give illustrations of this passage, by means of Theorem VIII, from theorems on measurable functions to theorems on general functions.

Ex. 1. *Extension of the theorem of Borel on the approximation of a measurable function by means of a polynomial function.* This extension has already been mentioned above.

Ex. 2. *Extension of a theorem due to Vitali.* The theorem in question concerning measurable functions asserts that for every measurable function $m(x)$ there exists a function $g(x)$, of Baire's second class at most, such that $g(x)=m(x)$ almost everywhere. Since $u(x)$, the metrical upper boundary function of a given function $f(x)$ is measurable, there exists a function $g(x)$ of Baire's second or lower class, such that $g(x)=u(x)$ almost everywhere. Suppose, for simplicity, that $f(x)$ is defined in the interval (0, 1), and let $T=$ the set of points x such that $(x, u(x))$ is fully approached by both $y=u(x)$ and $y=f(x)$. According to Theorems II and VIII, T is of measure 1. If ε is a given positive number, and ξ a point of T, we may enclose ξ in an interval I_ξ such that the exterior measure of the points x of I_ξ for which the inequalities $|f(x)-u(\xi)|<\frac{\varepsilon}{2}$,

$|u(x) - u(\xi)| < \frac{\varepsilon}{2}$ hold is greater than $(1 - \eta) l_\xi$, where l_ξ is the length of I_ξ and η is, independently of ε, as small as we please. Let S_ε be the set of points x such that $|f(x) - u(x)| < \varepsilon$; therefore $m_e(S_\varepsilon T_\xi) > (1 - 2\eta) l_\xi$. Since a set of non-overlapping intervals I_ξ can be chosen with sum of lengths equal to 1, we conclude that $m_e(S_\varepsilon) > 1 - 2\eta$, and hence $m_e(S_\varepsilon) = 1$; the set of points x where $|f(x) - g(x)| < \varepsilon$ is therefore of exterior measure 1. This conclusion amounts to an extension of Vitali's theorem to an unconditioned function, and we may state the theorem:

If $f(x)$ is a given function, there exists a function $g(x)$ of Baire's second class at most such that for every positive ε the set of points where $g(x)$ differs from $f(x)$ by more than ε is of interior measure 0.[1]

Ex. 3. *Extension of a theorem of Denjoy.* Denjoy has shown[2] that if $f(x)$ is a continuous function, the »directional angle» of the curve $y = f(x)$ is, for almost every x, either 0°, — when the derivative exists; or 180°, — when the lower Dini derivative on one side equals the upper Dini derivative on the other, and the other two Dini derivatives are $\pm\infty$ respectively; or 360°, — when the upper derivatives are both $+\infty$, and the lower derivatives, both $-\infty$. G. C. Young has shown[3] that if in the hypotheses of this theorem we substitute for the condition of continuity of f that of measurability, the conclusion remains valid. More recently, Saks and Banach have shown[4] that the same conclusion holds for an entirely unrestricted $f(x)$. Now for us the question comes up whether, by means of Theorem VIII, we can prove the theorem of Denjoy for an unrestricted function, assuming its truth for measurable functions. This turns out to be really so, the necessary adjustments requiring no invention to speak of, as the following argument shows:

Let $f(x)$ be any function whatsoever, and $u(x)$, $l(x)$ its metric upper, lower boundary functions. Since a set of measure 0 is negligible in our present considerations, and the set of points x where $f(x) > u(x)$ or $< l(x)$ is, according to Theorem VIII, of measure 0, we need to discuss the directional angle only at those points ξ for which $u(\xi) \geqq f(\xi) \geqq l(\xi)$. If $u(\xi) > f(\xi) > l(\xi)$ it follows, that

[1] A proof of some length of this theorem was published by Saks and Serpiński, Fund. Math. vol. XI (1928), p. 105.

[2] Jour. de Math. (7) vol. I (1915), p. 105.

[3] Proc. Lond. Math. Soc. (2) vol. 19 (1917), p. 360.

[4] Fund. Math. vol. IV (1923), p. 205, and vol. V (1924), p. 98.

the directional angle at $(\xi, f(\xi))$ is 360° — at least for almost every ξ satisfying these inequalities — because, according to Theorem VIII, for almost every ξ both $(\xi, u(\xi))$ and $(\xi, l(\xi))$ are fully approached by $y = f(x)$. We may thus restrict the discussion to the case where $f(\xi) = u(\xi)$ or $f(\xi) = l(\xi)$; and since these two cases are similar, we may assume that $f(\xi) = u(\xi)$. Let the function $\bar{u}(x)$ be defined as equal to $u(x)$ if $u(x) \geqq f(x)$, and equal to $f(x)$ if $u(x) < f(x)$. Since $\bar{u}(x)$ equals the measurable function $u(x)$ almost everywhere, it is itself measurable, and since we are here assuming that the Denjoy theorem holds for measurable functions, we can use the fact that the directional angle is 0° or 180° or 360° at almost every point of $y = \bar{u}(x)$. We distinguish the two cases: (a) $u(\xi) > l(\xi)$, (b) $u(\xi) = l(\xi)$, at the same time assuming in both cases, as we may, that the directional angle of $y = u(x)$ is 0° or 180° or 360° at $(\xi, u(\xi))$ and that $(\xi, u(\xi))$ and $(\xi, l(\xi))$ are fully approached by $y = f(x)$.

Case (a) $u(\xi) > l(\xi)$. If the directional angle of $y = \bar{u}(x)$ is 0° at $(\xi, \bar{u}(\xi))$, we conclude, since $f(x) \leqq \bar{u}(x)$ for every x and $(x, u(x))$ and $(x, l(x))$ are positively approached for every x, that the directional angle at $(\xi, f(\xi)) = (\xi, u(\xi))$ of the curve $y = f(x)$ is 180°. The conclusion is the same and the reasoning similar if the directional angle of $y = \bar{u}(x)$ is 180° at $(\xi, \bar{u}(\xi))$ provided D^- and D_+, the upper left and lower right Dini derivatives of $y = \bar{u}(x)$ at ξ are equal to $\pm\infty$ respectively; if, however, the directional angle is 180° but D_-, $D^+ = \pm\infty$ respectively, it follows similarly that the directional angle of $y = f(x)$ is 360° at ξ. Finally, it follows in the same way that if the directional angle of $y = \bar{u}(x)$ is 360° at ξ, then the directional angle of $y = f(x)$ is also 360° at ξ.

Case (b) $u(\xi) = l(\xi)$. As before, we consider the measurable function $\bar{u}(x)$, and likewise the measurable function $\bar{l}(x)$ equal to $l(x)$ if $f(x) \geqq l(x)$, and to $f(x)$ if $f(x) < l(x)$. Since sets of measure 0 are negligible for our present purpose, we may assume that the directional angles at ξ, both for $y = \bar{u}(x)$ and for $y = \bar{l}(x)$ are 0° or 180° or 360°, and also that $(\xi, \bar{u}(\xi))$ is fully approached by $y = \bar{u}(x)$, and $(\xi, \bar{l}(\xi))$ fully approached by $y = \bar{l}(x)$. If the directional angles at ξ are 0° for both $y = \bar{u}(x)$ and $y = \bar{l}(x)$, it follows, since $\bar{u}(x) \geqq u(x) \geqq l(x) \geqq \bar{l}(x)$ for every x, that the curves $y = \bar{u}(x)$, $y = \bar{l}(x)$ have the same directions at ξ; and furthermore, since $\bar{u}(x) \geqq f(x) \geqq \bar{l}(x)$, that the curve $y = f(x)$ has this same direction at ξ. It follows similarly, if $y = \bar{u}(x)$ has a direction at ξ, and the directional angle at $y = \bar{l}(x)$ is 180° at ξ, that the sides of this 180° angle lie along the direction of $y = \bar{u}(x)$ at ξ; consequently, since $(x, u(x))$ and $(x, l(x))$ are, for every x, positively approached by $y = f(x)$, and $\bar{u}(x) > f(x) \geqq \bar{l}(x)$, the directional angle at $(\xi, u(\xi)) = (\xi, f(\xi))$ is 180° for the curve $y = f(x)$. Likewise, we see that the directional angle for $y = f(x)$ is 180° at ξ, if the directional angle of $y = \bar{l}(x)$ is 0° and the directional angle of $y = \bar{u}(x)$ is 180° at this point. If the directional angles at $(\xi, \bar{u}(\xi))$ are 180° for both $y = \bar{u}(x)$ and $y = \bar{l}(x)$, then if the infinite derivative on one and the same side of ξ is of different sign for these two curves, it follows that the directional angle for $y = f(x)$ is 360°; but if the infinite derivative on the same side of ξ is of the same sign for these two curves, it follows, because $\bar{u}(x) \geqq \bar{l}(x)$ for every x, that the sides of these two 180° angles lie along the same straight line, and therefore, since $(x, u(x))$ and $(x, l(x))$ are, for every x, positively approached by $y = f(x)$, the directional angle of $y = f(x)$ is also 180° at ξ. Finally, it follows readily with the aid of the considerations used above, that if for at least one of the curves $y = \bar{u}(x)$, $y = \bar{l}(x)$ the directional angle is 360° at ξ, then the directional angle of $y = f(x)$ is 360° at ξ.

Ex. 4. *Extension of a theorem of Arzela.* The theorem we have in mind is the one that gives a necessary and sufficient condition for the compactness[1] of an infinite set T of continuous functions, lying in a given interval, namely, that T be »equibounded» and »equicontinuous».[2] This theorem can be extended to measurable functions, as Fréchet has shown.[3] Now since, in one sense, the essential nature of a function is determined by its metrical boundaries, we shall, for our present purpose, regard two functions as identical if their metrical boundaries are respectively identical. This convention amounts to replacing a function by 2 measurable functions. We then define the distance $d(f_1, f_2)$ (=écart) between 2 functions f_1 and f_2 as the sum of the distances between their respective metric boundaries, understanding by the distance between two measurable functions $m_1(x)$ and $m_2(x)$ the greatest lower bound of all numbers k such that $|m_1(x) - m_2(x)| < k$ except in a set of measure $< k$. With this definition of distance, it follows that the set of all functions is made a metric space. It then

[1] A set T lying in an abstract space S in which convergence of a sequence of elements has meaning is »compact» if for every infinite sequence of elements of T there is a subsequence converging to an element of S.

[2] T is »equibounded» if there is a constant M such that $|f(x)| < M$ for all x's of the interval and all functions $f(x)$ of T. T is »equicontinuous» if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|x_1 - x_2| < \delta$ implies $|f(x_1) - f(x_2)| < \varepsilon$ for all f's of T.

[3] Fund. Math. vol. IX (1927) p. 25. In a note by E. H. Hansen, Bull. Am. Math. Soc. vol. 39 (1933) p. 397, the result of Fréchet is deduced directly from a general criterion for compactness in metric, complete spaces.

follows — and the proof may be made as in Hansen's note — *that a necessary and sufficient condition that a sequence $\{f_n\}$ of functions be compact is that they be »almost equibounded»*[1] *and that the set of their metric boundaries be »almost equicontinuous»*.[2]

Section 4.

Additional Remarks.

Other boundary functions. For brevity we shall speak only of *upper* boundaries. Among those that immediately suggest themselves are:

a) The ordinary upper boundary function $u(x)$, defined as the upper boundary (maximum) of $f(x)$ at x. $u(x)$ is an upper semi-continuous function, and conversely, every upper semi-continuous function is an upper boundary function, namely of itself.

b) The d-upper boundary $u_d(x)$, defined as the upper boundary of $f(x)$ at x when denumerable sets are regarded as negligible.[3] $u_d(x)$ is again upper-semi-continuous, and non-denumerably approached at each of its points. Conversely, if f is upper semi-continuous and non-denumerably approached et each of its points, it is a d-upper boundary, namely of itself. Furthermore the number of points of $y = f(x)$ above its d-upper boundary is at most $\aleph_0$; for every such point is enclosable in a »rational» rectangle — i. e., one bounded by $x = r_1$, $x = r_2$, $x = r_3$, $x = r_4$ with the r's rational — containing at most $\aleph_0$ such points. A similar result holds in the case of the f-upper boundary, »f» denoting here that finite sets are negligible.

The e-upper boundary, »e» denoting that exhaustible sets are regarded as negligible. It is easily seen that the e-upper boundary is upper-semi-continuous, and inexhaustibly approached at each of its points. Conversely, an upper-semi-continuous function with the latter property is an e-upper boundary — of itself. The set of abscissas corresponding to points of $y = f(x)$ above its e-upper

[1] $\{f_n\}$ is said to be »almost equibounded» if for every positive ε there exists a constant k such that $|f_n(x)| < k$ for every x and every n except for the x's belonging to a set E_n, dependent on n, and of exterior measure $< \varepsilon$.

[2] A set $\{f\}$ of measurable functions is almost equicontinuous if for every positive ε there exists a positive number δ independent of the f's, and a set E_f varying with f and of measure $< \varepsilon$, such that for every f and every pair (x_1, x_2) of x's not in E_f and such that $|x_1 - x_2| < \delta$, we have $|f(x_1) - f(x_2)| < \varepsilon$.

[3] Cf., for example, Blumberg, *Certain General Properties of Functions,* Annals of Math., 2d ser., vol. XVIII (1917) p. 147.

boundary is exhaustible. Similar results hold in the case of neglect of sets of measure zero.

We finally add a few remarks concerning the structure of a general, real function between its measurable boundaries. If such a bounded function f, lying in (a, b), is given, and $y_0 \leqq f < y_2$, we divide the interval (y_0, y_2) into 2 equal parts, each of these 2 intervals into 2 equal parts, and so on, designating the 2 intervals of the first stage by $I_0 = (y_0, y_1)$, $I_1 = (y_1, y_2)$; the four intervals of the second stage by $I_{00} = (y_{00} = y_0, y_{01})$, $I_{01} = (y_{01}, y_1)$, $I_{10} = (y_{10} = y_1, y_{11})$; $I_{11} = (y_{11}, y_2)$; and so on. Let $E_0 = E_{y_0 \leqq f \leqq y_1}$, $E_1 = E_{y_1 \leqq f < y_2}$; $E_{00} = E_{y_{00} \leqq f < y_{01}}$, $E_{01} = E_{y_{01} \leqq f < y_{10}}$, $E_{10} = E_{y_{10} \leqq f < y_{11}}$, $E_{11} = E_{y_{11} \leqq f < y_2}$; and so on. With each of these $E_{\alpha_1, \alpha_2, \ldots \alpha_n}$ we associate $\Gamma_{\alpha_1, \alpha_2, \ldots \alpha_n}$, one of its measurable envelopes; $\Gamma_{\alpha_1, \alpha_2, \ldots \alpha_n}$ may be taken to be a G_π, i. e., a product of a sequence of open sets. Every Γ is, except for a set of measure zero, the sum of the 2 Γ's of the next stage whose subscripts, except for the last, are identical with those of the given Γ — (and in accordance with this property, we have $(a, b) = \Gamma_0 + \Gamma_1$ except for a set of measure 0). Thus the pattern of the metric clustering about the curve points of an arbitrary function is completely given by a sequence of G_π's: Γ_0, Γ_1; $\Gamma_{00}, \Gamma_{01}, \Gamma_{10}, \Gamma_{11}$; ... Two functions with the same system of associated Γ's are such that if a point (ξ, η) is fully approached by the one function it is fully approached by the other. Since there are in all c G_π's, there are for the totality of real functions, of cardinal 2^c, only c possibilities for the pattern of metric clustering. This reduction from 2^c to c in itself indicates a procedure for associating with general functions properties relating to metric clustering.

Suppose, conversely, we have given an infinite system of Γ's, each a G_π: $\Gamma = (a, b)$; Γ_0, Γ_1; $\Gamma_{00}, \Gamma_{01}, \Gamma_{10}, \Gamma_{11}; \ldots$, with the property that every Γ is, except for a set of measure 0, the sum of the 2 Γ's of the next stage having one additional subscript. We show that a one-valued function exists having the given Γ's for its associated set of Γ's, — always, of course, with the possible neglect of sets of measure 0. For let E_0 and E_1 be two non-overlapping sets such that they lie respectively in Γ_0 and Γ_1 and $E_0 + E_1 = \Gamma$ $(= (a, b))$ except for a set of measure 0, both E_0 and E_1 having respectively metric density 1 at each of their points. Next let E_{00}, E_{01} be two non-overlapping sets lying respectively in Γ_{00} and Γ_{01}, such that $E_{00} + E_{01} = E_0$, except for a set of measure 0, both E_{00} and E_{01} having respectively metric density 1 at each of their points. In the same way a pair of sets E_{10}, E_{11} is associated with Γ_{10}, Γ_{11}; and in the same way we define $E_{\alpha_1, \ldots \alpha_k}$ for every sequence of subscripts consisting of 0's or 1's. If x is a point of (a, b) and does not belong to an exceptional set of measure 0, it lies, for a given n, in one and just one E with n subscipts, say in $E_{\alpha_1, \dots \alpha_n}$; we then set $f(x) = \lim y_{\alpha_1, \dots \alpha_n}$. It is clear that this f has associated with it, except for sets of measure 0, precisely the original sequence of I's.

Again, if f is given, let us associate with it, as above, the sets $E_{\alpha_1, \dots \alpha_n}$. $E_{\alpha_1, \dots \alpha_n}$ is the sum of 2 non-overlapping sets $M_{\alpha_1, \dots \alpha_n}$ and $N_{\alpha_1, \dots \alpha_n}$, the former measurable and the latter completely non-measurable. If T is the set of points common to an infinite number of M's, it can be shown without difficulty, that, except for a set of measure 0, T is precisely the set of points where the measurable boundaries of f coincide.