ON THE REDUCTION OF A GROUP OF HOMOGENEOUS LINEAR SUBSTITUTIONS OF FINITE ORDER

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Although the conception of a group does not occur explicitly in ABEL's published writings it is incontestible that, from the point of view of the present time, the idea underlies the whole of his wonderful investigations into the theory of algebraically soluble equations. More than one passage in these investigations suggests strongly that the idea was present in the writer's mind though it has not found direct expression in his mode of presenting his results. It will not then appear improper that a memoir dealing with some of the recent results obtained in the theory of groups of linear substitutions of finite order should appear in this volume which commemorates the great mathematician.

In the course of the last five or six years great advances have been made in this theory. The appearance of two memoirs by Herr FROBENIUS *Über Gruppencharactere* and *Über die Primfactoren der Gruppendeterminante* (Berliner Sitzungsberichte, 1896, pp. 985—1021 and pp. 1343—1382), which have been followed by a series of others developing and extending the same ideas, marks a new departure of great importance in this connection. Later in date than Herr FROBENIUS, but independently as regards method, I have considered the theory of the factors of the group-determinant and the corresponding theory of the representation of a group of finite order as an irreducible group of linear substitutions; basing my investigation on a certain continuous group which is completely defined by any given abstract group of finite order.[1]

So far as I am aware the only proof hitherto given of what is defined below as the »complete reducibility» of a reducible group of finite order of homogeneous linear substitutions, other than that due to Herr FROBENIUS, is contained in a memoir by Herr MASCHKE.[2] The number of distinct representations of a group of finite order as an irreducible group of homogeneous linear substitutions has hitherto been determined only by the processes, both of them indirect, of which Herr FROBENIUS and I have made use.

My principal object in the present memoir is to establish these two results by direct and comparatively simple methods, based on a repeated use of the theorem that, for every group of homogeneous linear substitutions of finite order, there is at least one invariant Hermitian form.

As the phraseology of the subject has not yet become uniform, I define here the sense in which certain phrases will be used.

A group of homogeneous linear substitutions is spoken of as reducible or irreducible according as it is or is not possible to find a set of linear functions of the variables, less in number than the variables, which are transformed among themselves by every operation of the group.

A reducible group of homogeneous linear substitution s is called »completely reducible» when it is possible to choose the variables in such a way that (I) they fall into sets, each set of variables being transformed among themselves by every operation of the group, while (II) the group in each separate set is irreducible. In this sense the first result to be proved is that a group of linear homogeneous substitutions of finite order is either irreducible or completely reducible.

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[1] Proceedings of the London Mathematical Society, Vol. 29, pp. 207—224; pp. 546—565, (1898); Vol. 35, pp. 206—220, (1902).

[2] *Beweis des Satzes, dass diejenigen endlichen linearen Substitutionsgruppen, in welchen einige durchgehends verschwindende Coefficienten auftreten, intransitiv sind.* Math. Ann., Vol. 52, pp. 363—368, 1899.

Since this paper was written Herr LOEWY in a memoir *Über die Reducibilität der Gruppen linearer homogener Substitutionen* (Transactions of the American Mathematical Society, Vol. 4, pp. 44—64, 1903) has obtained a more general result of which the theorem in question is a particular case.

If

$$S_1\,,\ S_2\,,\ \ldots,\ S_N$$

are the operations of an abstract group G of finite order N; and

$$x'_i = \sum_{j=1}^{j=n} \alpha_{ijk} x_j\,, \qquad (i=1,2,\ldots,n)$$

$$(k = 1\,,\ 2\,,\ \ldots,\ N)$$

a set of linear homogeneous substitutions

$$s_1\,,\ s_2\,,\ \ldots,\ s_N;$$

such that if

$$S_p S_q = S_r\,,$$

then

$$s_p s_q = s_r\,,$$

for all sets of suffixes, the group of linear substitutions is said to give a »representation» of the abstract group G. The one-to-one correspondence of the operations of the group and the substitutions is an essential part of the representation. Thus a second representation in the same number of variables

$$y'_i = \sum_{j=1}^{j=n} \beta_{ijk} y_j\,, \qquad (i=1,2,\ldots,n)$$

$$(k = 1\,,\ 2\,,\ \ldots,\ N)$$

is spoken of as »distinct» or not distinct from the former according as it is not or is possible to find a linear substitution

$$y_i = \sum_{j=1}^{j=n} r_{ij} x_j\,, \qquad (i=1,2,\ldots,n)$$

which, *for each k,* will transform

$$x'_i = \sum_j \alpha_{ijk} x_j\,,$$

into $\qquad (i=1,2,\ldots,n)$

$$y'_i = \sum_j \beta_{ijk} y_j\,.$$

It is thus to be noticed that it may very well be possible to transform the one group of substitutions into the other while at the same time they give distinct representations of G. In particular the two groups may consist of the same set of substitutions and yet may give distinct representations of G. Two representations which are not distinct will be called *equivalent*.

When the word »distinct representation» is used in this sense, the second result proved here is that the number of distinct irreducible representations of a group of finite order is equal to the number of separate conjugate sets of operations which the group contains.

1. A group of homogeneous linear substitutions in n variables, if of finite order, has at least one invariant Hermitian form of non-vanishing determinant in the n variables and their conjugates; and by a suitable transformation of the variables one such form may always be taken to be

$$x_1\bar{x}_1 + x_2\bar{x}_2 + \ldots + x_n\bar{x}_n.$$

This theorem, due to Prof. A. Loewy[1] and to Prof. E. H. Moore,[2] is of fundamental importance in the theory of groups of finite order.

The step-by-step process, by which any Hermitian form of non-vanishing determinant is brought to the form quoted, must break off at some step before the last when the determinant of the form vanishes. Hence a form in the n variables and their conjugates, whose determinant vanishes can always be reduced to the form

$$y_1\bar{y}_1 + y_2\bar{y}_2 + \cdots + y_s\bar{y}_s, \qquad (s < n),$$

where $y_1, y_2, \ldots, y_s$ are s linearly independent functions of the original variables.

Suppose now that for a group G of linear substitutions in the variables

$$x_1, x_2, \ldots, x_n,$$

an Hermitian form f or

$$y_1y_1 + y_2\bar{y}_2 + \cdots + y_s\bar{y}_s$$

of vanishing determinant is invariant. Choose new variables of which

[1] Comptes Rendus, Vol. 123, pp. 168—171 (1896).

[2] Mathematische Annalen, Vol. 50, pp. 213—219 (1898).

$y_1, y_2, \ldots, y_s$ are the first s; and in these variables, let the substitutions of the group be

$$y'_i = \sum_{j=1}^{j=n} a_{ijk} y_j, \qquad (i=1,2,\ldots,n)$$
$$(k = 1, 2, \ldots, N),$$

where N is the order of the group, and the different operations correspond to different values of the suffix k.

For any substitution of the group f becomes

$$\sum_{i=1}^{i=s} \Big(\sum_j a_{ijk} y_j\Big)\Big(\sum_j \bar{a}_{ijk} \bar{y}_j\Big).$$

The coefficient of $y_j \bar{y}_j$ in this is

$$\sum_{i=1}^{i=s} a_{ijk} \bar{a}_{ijk},$$

and if $j > s$, this is zero. Hence

$$a_{ijk} = 0,$$

if

$$j > s.$$

Every operation of the group therefore transforms $y_1, y_2, \ldots, y_s$ among themselves. If then a group of linear substitutions in n variables, of finite order, has an invariant form of zero determinant, the group is reducible.

Suppose now that the operations of a group G of finite order in $r + s$ variables are of the form

$$x'_u = \sum_{v=1}^{v=r} \alpha_{uvk} x_v, \qquad (u=1,2,\ldots,r)$$

$$x'_{r+u} = \sum_{v=1}^{v=r} \beta_{uvk} x_v + \sum_{w=1}^{w=s} \gamma_{uwk} x_{r+w}, \qquad (u=1,2,\ldots,s)$$

so that the symbols $x_1, x_2, \ldots, x_r$ are transformed among themselves by every operation of the group. The equations

$$x'_{r+u} = \sum_{w=1}^{w=s} \gamma_{uwk} x_{r+w}, \qquad (u=1,2,\ldots,s)$$

constitute a group of finite order, with which the given group is isomorphic; as also do the equations

$$x'_u = \sum_{v=1}^{v=r} \alpha_{uvk} x_v . \qquad (u=1,2,\ldots,r)$$

Suppose further that both these groups are irreducible; and that the latter has been so transformed, if necessary, that

$$x_1\bar{x}_1 + x_2\bar{x}_2 + \ldots + x_r\bar{x}_r = f'$$

is an invariant Hermitian form for it; the same transformation of the first r x's being carried out also in the last s equations of G.

Let now

$$f = \sum a_{ij} x_i \bar{x}_j$$

be an invariant positive Hermitian form, of non-vanishing determinant, of G. If α and β are arbitrary constants, each of the set of forms

$$\alpha f + \beta f'$$

is invariant for G. If D is the determinant of f, the determinant of $\alpha f + \beta f'$ is

$$\alpha^n D + \alpha^{n-1}\beta\left(\frac{\partial D}{\partial a_{11}} + \frac{\partial D}{\partial a_{22}} + \ldots + \frac{\partial D}{\partial a_{rr}}\right) + \ldots .$$

Now $\dfrac{\partial D}{\partial a_{11}}$ is the determinant of the form that results from f on making x_1 zero. This is a positive form of non-vanishing determinant in the remaining $n - 1$ symbols and their conjugates, and its determinant therefore is a positive (non-zero) number. Hence the coefficient of $\alpha^{n-1}\beta$ in the determinant of $\alpha f + \beta f'$ is different from zero, and therefore the determinant must vanish, when β is suitably chosen, for some finite value of α.

It follows that f' is not the only Hermitian form of vanishing determinant which is invariant for G; or in order words, the set of symbols $x_1, x_2, \ldots, x_r$ is not the only set, less than $r + s$, which are transformed among themselves by every operation of the group. By hypothesis the substitutions on the first r x's form an irreducible group, and therefore the other set of symbols which are transformed among themselves cannot be functions of the first r x's alone. Let

$$y_1 = \sum_{i=1}^{i=r+s} b_i x_i$$

be one symbol of the set. Since by hypothesis the equations

$$x'_{r+u} = \sum_{w=1}^{w=s} \gamma_{uwk} x_{r+w}, \qquad (u=1,2,\ldots,s)$$

constitute an irreducible group, the functions that arise from y_1 by the substitutions of G, when considered as functions of the last s x's alone, must be s linearly independant functions. If, on the other hand, more than s linearly independent functions of all the x's so arise, the last s x's could be eliminated among them, and a linear function of the first r x's expressed in terms of the y's. Since the substitutions on the first r x's form an irreducible group, this would mean that the set of y's contained $r+s$ independent functions, which is not the case. Hence just s linearly independent functions

$$y_1, y_2, \ldots, y_s$$

arise from y_1 by the substitutions of the group; and this set of functions are transformed among themselves by every operation of the group. Moreover the last s x's can be expressed in terms of the y's and the first r x's.

By a suitable choice of new variables for the last s x's, the equations of G can therefore be given a form in which the variables are divided into two sets, of r and s, those of each set being transformed among themselves by the group.

Let G now be any group of linear substitutions, of finite order, in n variables. If G is reducible it must be possible to find a set of n' $(< n)$ linear functions of the variables which are transformed among themselves by every operation of G. If the group in the n' variables is reducible the process may be repeated. At last a set of, say n_1, linear functions of the original variables must be arrived at such that the group in these variables is irreducible. Take these n_1 functions for the first n_1 of a set of new variables. Then every operation of the group has the form

$$x'_u = \sum_{v=1}^{v=n_1} \alpha_{uvk} x_v, \qquad (u=1,2,\ldots,n_1)$$

$$x'_{n_1+u} = \sum_{v=1}^{v=n_1} \beta_{uvk} x_v + \sum_{w=1}^{w=n-n_1} \gamma_{uwk} x_{n_1+w}. \qquad (u=1,2,\ldots,n-n_1)$$

The last $n-n_1$ equations still define a group of finite order G', isomorphic with G, when in them $x_1, x_2, \ldots, x_{n_1}$ are made zero. If this group is reducible, a set of n_2 linear functions of $x_{n_1+1}, x_{n_1+2}, \ldots, x_n$ may be found such that they are transformed among themselves by every operation of G', while the group of substitutions in these n_2 variables is irreducible. If these linear functions are represented by

$$y_1, y_2, \ldots, y_{n_2}$$

and are taken for new variables, the substitutions of G may be written in the form

$$x'_u = \sum_{v=1}^{v=n_1} \alpha_{uvk} x_v, \qquad (u=1,2,\ldots,n_1)$$

$$y'_u = \sum_{v=1}^{v=n_1} \beta_{uvk} x_v + \sum_{v=1}^{v=n_2} \gamma_{uvk} y_v, \qquad (u=1,2,\ldots,n_2)$$

$$x'_{n_1+n_2+u} = f(x, y) + \sum_{w=1}^{w=n-n_1-n_2} \delta_{uvk} x_{u_1+u_2+w}, \qquad (u=1,2,\ldots,n-n_1-n_2)$$

where $f(x, y)$ represents a linear function of $x_1, x_2, \ldots, x_{n_1}, y_1, y_2, \ldots, y_{n_2}$.

Here again the last $n - n_1 - n_2$ equations still define a group of finite order G'', isomorphic with G, when in them $x_1, x_2, \ldots, x_{n_1}, y_1, y_2, \ldots, y_{n_2}$ are made zero. If G'' is reducible the same process may be repeated, till an irreducible group is arrived at for the group that remains when all preceding sets of variables are made zero. Let the third set of variables thus introduced be denoted by

$$z_1, z_2, \ldots, z_{n_3},$$

and so on till all the variables are accounted for.

Consider now the group that has been called G', so far as it affects the y's and the z's. (This is equivalent to supposing that the variables are divided by the above process into three sets, but it will be seen that the argument will apply equally well whatever the number of sets.) By the result of the previous paragraph the z's may be replaced by linear functions of themselves and the y's, so that the equations of G' have the form

$$y'_u = \sum_{v=1}^{v=n_2} \gamma_{uvk} y_v, \qquad (u=1,2,\ldots,n_2)$$

$$\zeta'_u = \sum_{v=1}^{v=n_3} \delta'_{uvk} \zeta_v. \qquad (u=1,2,\ldots,n_3)$$

With the x's, y's and ζ's as variables the equations of G take the form

$$x'_u = \sum_{v=1}^{v=n_1} \alpha_{uvk} x_v, \qquad (u=1,2,\ldots,n_1)$$

$$y'_u = \sum_{v=1}^{v=n_1} \beta_{uvk} x_v + \sum_{v=1}^{v=n_2} \gamma_{uvk} y_v, \qquad (u=1,2,\ldots,n_2)$$

$$\zeta'_u = \sum_{v=1}^{v=n_1} \varepsilon_{uvk} x_v + \sum_{v=1}^{v=n_3} \delta'_{uvk} \zeta_v. \qquad (u=1,2,\ldots,n_3)$$

A second precisely similar application of the result of the previous paragraph, enables us to replace the y's by n_2 linear functions of themselves and the x's, and the ζ's by n_3 linear functions of themselves and the x's, so that with these new variables, the variables of each set are transformed among themselves by every operation of the group. Hence:

Theorem. If a group of homogeneous linear substitutions, of finite order, is reducible, new variables may be chosen so that (I) the variables fall into sets, those of each set being transformed among themselves by every operation of the group, while (II) the group of linear substitutions in each separate set is irreducible.

2. If a group of linear substitutions of finite order has two distinct invariant Hermitian forms f and f' then every form of the set $\alpha f + \beta f'$ is invariant. Now α and β may be chosen so that the determinant of $\alpha f + \beta f'$ is zero without the form being identically zero; and the group is then, as shewn in § 1, reducible. An irreducible group has therefore only one invariant Hermitian form.

Suppose now that when a group G has been completely reduced, the two sets of variables

$$\begin{matrix} x_1, x_2, \ldots, x_r, \\ y_1, y_2, \ldots, y_s, \end{matrix} \qquad (r \not> s),$$

are transformed, each among themselves, irreducibly. Let f be an invariant Hermitian form in these $r+s$ variables of non-vanishing determinant. When in f we make $y_1 = y_2 = \ldots = y_s = 0$, f must reduce to a Hermitian form f_1 in the x's, invariant for the transformation of the x's; and therefore of non-vanishing determinant in the r variables and their conjugates. Hence f may be expressed in the form

$$\xi_1\bar{\xi}_1 + \xi_2\bar{\xi}_2 + \ldots + \xi_r\bar{\xi}_r + f',$$

where

$$\xi_i = X_i + Y_i;$$

$X_1, X_2, \ldots, X_r$ are r linearly independent functions of the x's; Y_1, $Y_2, \ldots, Y_r$ are r linear functions of the y's; and f' is a form in the y's alone, of non-vanishing determinant as regards them. Since the y's are transformed among themselves by the group, there must be a Hermitian form f'' in the y's alone which is invariant. Hence

$$\alpha(\xi_1\bar{\xi}_1 + \xi_2\bar{\xi}_2 + \ldots + \xi_r\bar{\xi}_r) + \alpha f' + \beta f''$$

is invariant for the group. Now, since the determinant of f'', regarded as a form in the y's alone, is not zero, a non-zero value of α may be found so that the determinant of $\alpha f' + \beta f''$, regarded as a form in the y's alone, and therefore of $\alpha f + \beta f''$, regarded as a forme in the x's and y's, vanishes. For this value $\alpha f' + \beta f''$ must vanish identically; since $\xi_1, \xi_2, \ldots, \xi_r$ are linearly independent as regards the x's, while the y's are transformed irreducibly among themselves. Hence $\xi_1, \xi_2, \ldots, \xi_r$ are transformed among themselves by every operation of the group. It follows that

$$X_1, X_2, \ldots, X_r$$

and

$$Y_1, Y_2, \ldots, Y_r$$

undergo, each set among themselves, the same substitution for every operation of the group. If $r < s$, this is impossible since the group in the y's is irreducible. If $r = s$, it must be possible to transform the group of the y's, so that for each operation of the group the x's and y's undergo the same substitution.

The form f can therefore only have terms containing the product of an x by a y, when the number of x's and y's are equal, while the group in one set can be so transformed that the substitutions in the two sets, corresponding to each operation of the group, are identical.

Suppose next that in the completely reduced form of G, there are just s sets of r variables each

$$x_{i1}, x_{i2}, \ldots, x_{ir},$$
$$(i = 1, 2, \ldots, s)$$

such that (I) the variables of each set are transformed irreducibly among themselves, and (II) the group in each set can be so transformed that the substitution on its variables, corresponding to each operation of G_1 is identical with the corresponding substitution on the variables of the first set.

Let these transformations be carried ont, and further transform all the sets, if necessary, so that for each the invariant Hermitian form is

$$x_{i1}\bar{x}_{i1} + x_{i2}\bar{x}_{i2} + \ldots + x_{ir}\bar{x}_{ir}.$$

When thus transformed the operations of the group will give for each set the substitutions

$$x'_{ip} = \sum_{q=1}^{q=r} \alpha_{pqk} x_{iq}, \qquad (p=1,2,\ldots,r)$$
$$(k = 1, 2, \ldots, N).$$

Let

$$f = \sum a_{ip,jq} x_{ip} \bar{x}_{jq}$$

be an invariant form for the group. On transformation by any operation of the group f becomes

$$f' = \sum a_{ip,jq} \sum_{u=1}^{u=r} \alpha_{puk} x_{iu} \sum_{v=1}^{v=r} \bar{\alpha}_{qvk} \bar{x}_{jv}.$$

Hence

$$a_{iu,jv} = \sum_{p,q} a_{ip,jq} \alpha_{puk} \bar{\alpha}_{qvk}$$

for each k. These relations express that

$$\sum_{u,v} a_{iu,jv} x_u \bar{x}_v$$

is an invariant Hermitian form for the group

$$x'_p = \sum_{q=1}^{q=r} \alpha_{pqk} x_q. \qquad (p=1,2,\ldots,r)$$

But, by supposition, the only invariant form for this group is

$$x_1\bar{x}_1 + x_2\bar{x}_2 + \ldots + x_r\bar{x}_r.$$

Hence

$$a_{ip,jq} = 0, \quad \text{if} \quad p \neq q$$

and

$$a_{ip,jp} = a_{iq,jq},$$

for all suffixes p and q.

If then

$$a_{ip,jp} = b_{i,j},$$

the most general invariant Hermitian form in the rs variables is

$$\sum_{i,j,p} b_{i,j}\, x_{ip}\, \bar{x}_{jp}.$$

This form contains just s^2 arbitrary coefficients; it is in fact a linear combination of the s^2 forms

$$\sum_p x_{ip}\, \bar{x}_{ip}, \qquad (i=1,2,\ldots,s)$$

$$\left.\begin{aligned} &\sum_p (x_{ip}\,\bar{x}_{jp} + \bar{x}_{ip}\, x_{jp}), \\ &\sum_p \sqrt{-1}\,(x_{ip}\,\bar{x}_{jp} - \bar{x}_{ip}\, x_{jp}), \end{aligned}\right\} \quad \begin{aligned} i, j &= 1, 2, \ldots, s \\ i &\neq j \end{aligned}$$

Combining the last two results, the number of linearly independant invariant Hermitian forms which a group possesses is given by the following statement.

Theorem. If, when a group of finite order has been completely reduced, the variables are divided into ν_1 sets of n_1 each, ν_2 sets of n_2 each, ... such that the groups transforming each of the ν_i sets of n_i variables are equivalent to each other, and are distinct from those transforming each of the ν_j sets of n_j variables $(j \neq i)$, then the number of linearly independent invariant Hermitian forms for the group is

$$\nu_1^2 + \nu_2^2 + \ldots + \nu_i^2 + \ldots,$$

3. The nature of the complete reduction of a group G, of finite order N, when represented as a group of regular permutations of N symbols

$$x_1, x_2, \ldots, x_N$$

will next be investigated.

Suppose that when the reduction has been completely effected, the variables fall into ν_1 sets of n_1 each, ν_2 sets of n_2 each, ..., ν_m sets of n_m each, such that (I) the groups transforming each of the ν_i sets of n_i are equivalent to each other, while (II), if $j \neq i$ the group of substitutions of one of the ν_j sets is distinct from that of one of the ν_i sets. The irreducible substitution group in any one of the sets will be spoken of as an *irreducible component* of G; and the condition (II) of the preceding sentence will be expressed by saying that the irreducible component given by one of the ν_i sets is *distinct* from that given by one of the ν_j sets. The number of distinct irreducible components of G, when represented as a regular permutation group in N symbols is then denoted by m.

The only linear function of the x's which is invariant for every operation of G is their sum. This necessarily occurs as one of the sets of variables transformed among themselves in the completely reduced form. Hence we may and shall take

$$n_1 = \nu_1 = 1,$$

the corresponding reduced variable being the sum of the x's. Further since the x's can be expressed in terms of the new variables

$$\sum_{i=1}^{i=m} n_i \nu_i = N.$$

When x_1 is expressed in terms of the new variables which effect the complete reduction of G, it will, in respect of the ν sets of n each

$$x_{i1}, x_{i2}, \ldots, x_{in}$$
$$(i = 1, 2, \ldots, \nu)$$

which all undergo the same substitutions, contain the terms

$$\sum_{i=1}^{i=\nu} (\alpha_1^{(i)} x_{i1} + \alpha_2^{(i)} x_{i2} + \ldots + \alpha_n^{(i)} x_{in}).$$

If ν is greater than n, not more than n of the linear functions

$$\alpha_1^{(i)} x_1 + \alpha_2^{(i)} x_2 + \ldots + \alpha_n^{(i)} x_n$$
$$(i = 1, 2, \ldots, \nu)$$

can be linearly independent; and therefore the terms in question can be expressed as the sum of not more than n linear functions of the form

$$\alpha_1^{(i)} \xi_{i1} + \alpha_2^{(i)} \xi_{i2} + \ldots + \alpha_n^{(i)} \xi_{in},$$

where

$$\xi_{i1} = \sum_j \beta_j^{(i)} x_{j1}, \qquad \xi_{i2} = \sum_j \beta_j^{(i)} x_{j2}, \ \ldots, \ \xi_{in} = \sum_j \beta_j^{(i)} x_{jn}.$$

But for each i,

$$\xi_{i1}, \xi_{i2}, \ldots, \xi_{in}$$

undergo the same substitution as

$$x_{i1}, x_{i2}, \ldots, x_{in}.$$

Hence the reduced variables may be chosen so that of the ν sets, the n sets of ξ's form a part. When so chosen, the remaining $\nu - n$ sets do not appear at all in x_1; and therefore do not appear at all in the expressions of any of the original variables. But this is impossible since the N original variables, by supposition independent, would then be expressed in terms of $N - n(\nu - n)$ reduced variables. Hence no ν can be greater than the corresponding n.

The invariant Hermitian forms of G are next to be considered. Their number is N. In fact every invariant Hermitian form for G will arise on carrying out the permutations of G in

$$\sum_{i=1}^{i=N} \alpha_i x_i \sum_{i=1}^{i=n} \bar{\alpha}_i \bar{x}_i,$$

where the α's are arbitrary coefficients and summing the resulting expressions. There can therefore be no forms linearly independent of those that arise from

$$x_1 \bar{x}_1, \ x_1 \bar{x}_i + \bar{x}_1 x_i, \ \sqrt{-1}(x_1 \bar{x}_i - \bar{x}_1 x_i)$$
$$(i = 2, 3 \ldots, N)$$

as leading terms. If G contains a permutation which changes x_1 into x_i and x_i into x_1, the form that arises from $x_1\bar{x}_i + \bar{x}_1 x_i$ is distinct from all the rest, while that which arises from $\sqrt{-1}(x_1\bar{x}_i - \bar{x}_1 x_i)$ is identically zero. If the permutation of G, which changes x_i into x_1, changes x_1 into x_j, then $x_1\bar{x}_i + \bar{x}_1 x_i$ and $\sqrt{-1}(x_1\bar{x}_i - \bar{x}_1 x_i)$ give rise to the same pair of forms as $x_1\bar{x}_j + \bar{x}_1 x_j$ and $\sqrt{-1}(x_1\bar{x}_j - \bar{x}_1 x_j)$. The total number of linearly independant Hermitian forms for G is therefore N. Now by considering the completely reduced form of G, it has been shown in § 2, that this number is $\sum_{i=1}^{i=m} \nu_i^2$.

Hence

$$\sum_{i=1}^{i=m} \nu_i^2 = N;$$

and combining this with

$$\sum_{i=1}^{i=m} n_i \nu_i = N,$$

and

$$\nu_i \not> n_i,$$

it follows that

$$\nu_i = n_i$$

for each i. Hence:

Theorem. In the completely reduced form of a regular permutation group, the number of times that each distinct irreducible component of the group occurs is equal to the number of variables which it transforms among themselves.

4. Any linear substitution on the original variables which is permutable with every operation of the regular permutation group G must, when expressed in terms of thereduced variables, transform among themselves for each i, the n_i^2 variables contained in the n_i sets of n_i each. This is the consequence of the groups in the different sets being »distinct». Suppose now that it were possible to form n independent linear functions

$$\xi_1, \xi_2, \ldots, \xi_n$$

of

$$x_{i1}, x_{i2}, \ldots, x_{in}$$

such that the symbols in these two lines undergo identical substitutions for each operation of G. Then

$$x'_{i1} = \xi_1, \qquad x'_{i2} = \xi_2, \ \ldots, \ x'_{in} = \xi_n,$$

would be permutable with every operation of the group of linear substitution in

$$x_{i1}, x_{i2}, \ldots, x_{in}.$$

Since this group is irreducible there can be no substitution permutable with every one of its substitutions except

$$x'_{i1} = \alpha x_{i1}, \qquad x'_{i2} = \alpha x_{i2}, \ \ldots, \ x'_{in} = \alpha x_{in}.$$

Hence the only linear functions of the n^2 variables, of which the set considered is one set of n, which undergo for every operation of G, the same substitution as

$$x_{i1}, x_{i2}, \ldots, x_{in}$$

are those given by

$$\sum_i \alpha_i x_{i1}, \ \sum_i \alpha_i x_{i2}, \ \ldots, \ \sum_i \alpha_i x_{in}.$$

A substitution which is permutable with every operation of G must therefore, so far as it affects these n^2 variables, be of the form

$$x'_{ij} = \sum_{k=1}^{k=n} \alpha_{ik} x_{kj}$$

$$(i, j = 1, 2, \ldots, n).$$

Now it is well known that there is a group G', of order N, of regular permutations in the N symbols

$$x_1, x_2, \ldots, x_N,$$

which is simply isomorphic with G, while every one of its operations is permutable with every operation of G. Combining this fact with the previously determined form of any linear substitution which is permutable with every operation of G, it follows that for the variables in the scheme,

$$\begin{array}{l} x_{11}, x_{12}, \ldots, x_{1n}, \\ x_{21}, x_{22}, \ldots, x_{2n}, \\ \ldots\ldots\ldots\ldots \\ x_{n1}, x_{n2}, \ldots, x_{nn}, \end{array}$$

(I) every operation of G gives the same transformation of the set of variables in each line; (II) every operation of G' gives the same transformation of the set of variables in each column, and (III) the group of transformations of the variables in each line corresponding to the operations of G and of the variables in each column corresponding to G' are both irreducible. From the last result it is an immediate corollary that for the group $\{G, G'\}$ the n^2 variables undergo an irreducible group of linear substitutions.

The group of permutations $\{G, G'\}$ therefore, when completely reduced, transforms the N variables among themselves in m sets of $n_1^2, n_2^2, \dots, n_m^2$ each such that the group in each separate set is irreducible and distinct from all the others. Hence there are just m linearly independent invariant Hermitian forms for $\{G, G'\}$.

The number of such forms can again be determined directly. Suppose that when $\{G, G'\}$ is represented as a permutation group in the N original variables, the subgroup which leaves x_1 unchanged permutes $x_i, x_j, \dots, x_k$ transitively among themselves. Then of the N invariant Hermitian forms of G, those containing the terms $x_1\bar{x}_i, x_1\bar{x}_j, \dots, x_1\bar{x}_k$ will be permuted among themselves by $\{G, G'\}$, and their sum only will be invariant for the latter group. The total number of independent invariant forms for $\{G, G'\}$ is therefore equal to the number of transitive sets in which the subgroup of $\{G, G'\}$, which leaves x_1 unchanged, permutes the symbols; including of course x_1 as one of the sets. This number is known [1] to be equal to the number of distinct sets of conjugate operations contained in G. Hence:

Theorem. When a group G of finite order N, containing m distinct conjugate sets of operations, and represented as a regular permutation group in N symbols is completely reduced, the number of its distinct irreducible components is m.

5. If there are one or more linear relations among the variables

$$x_1, x_2, \dots, x_n$$

affected by a group of linear homogeneous substitutions, the group must be reducible. Suppose in fact that the variables are connected by just t independent linear relations

$$\sum a_i^{(k)} x_i = 0,$$
$$(k = 1, 2, \dots, t).$$

[1] *Theory of groups of finite order*, p. 146. (Cambridge 1897.)

Then if

$$x'_1, x'_2, \ldots, x'_n$$

are the transformed variables for any operation of the group

$$\sum a_i^{(k)} x'_i = 0$$

is true, for each k, in virtue of the preceding relations. Hence if new variables are chosen of which the first t are defined by

$$y_k = \sum a_i^{(k)} x_i,$$

$$(k = 1, 2, \ldots, t)$$

the variables

$$y_1, y_2, \ldots, y_t$$

are transformed among themselves by every operation of the group. For an irreducible group of linear homogeneous substitutions the variables are therefore necessarily independent; the only non-independent set which undergo formally the operations of the group being a set of zeroes.

Suppose now that

$$x'_i = \sum_{j=1}^{j=n} a_{ijk} x_j, \qquad (i=1, 2, \ldots, n)$$

$$(k = 1, 2, \ldots, N)$$

is any representation of a group G of finite order N as a group of linear substitutions. Let y_1 be any arbitrarily chosen linear function of the x's, and let

$$y_1, y_2, \ldots, y_N$$

be the N linear functions that arise from y_1 by the substitutions of the group. When the x's undergo the substitutions of the group, the N y's undergo the permutations of the regular permutation-group in N symbols which is always one form of representation of G. The y's may or may not be linearly independent; in particular cases a number of them, when regarded as functions of the x's, may be actually identical.

Form now from the N y's, the n_i sets of n_i symbols each $(i = 1, 2, \ldots, r)$ in terms of which the regular form of G is completely reduced. Each set of n_i are transformed irreducibly among themselves by every operation of G. Hence when these variables are expressed in terms of the original x's, those given by any one set must either be linearly independent, or must all identically vanish. Further for any two sets which do not identically vanish, the variables of each set must either be independent of those of the other, or those of each set must be expressible linearly in terms of those of the other. The latter alternative is possible only when the irreducible components corresponding to the two sets are not distinct.

In this way a certain number of linearly independent sets of linear functions of the original x's are formed such that each set are transformed linearly and irreducibly among themselves, the corresponding group being one that arises in the complete reduction of the regular form of G. If the original x's can be expressed in terms of the N y's, the complete reduction is thus arrived at. If not, let z_1 be a linear function of the x's which is linearly independent of the y's. Then if

$$z_1, z_2, \ldots, z_N$$

are the linear functions of the x's that arise from z_1 by the operations of the group, the z's may be dealt with as the y's have been; and a further number of sets of linear functions of the x's obtained each of which are transformed irreducibly among themselves, the corresponding groups being again those that arise from the reduction of the regular form of G. This process may be continued till the x's have been replaced by an equal number of reduced variables linearly independent of each other. Hence:

Theorem. The only distinct irreducible groups of linear substitutions with which an abstract group G, of finite order, is simply or multiply isomorphic are the m distinct irreducible components that arise from the complete reduction of the regular form of G.

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