ADDITIVE VOLUME INVARIANTS OF RIEMANNIAN MANIFOLDS

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Introduction

Let (M, g) be an analytic Riemannian manifold and $m \in M$ a point. It is known (see e.g. [2]) that the volume of a small geodesic ball with center m and radius r is given by a power series expansion

$$V_m(r) = V_0(r)(1 + B_2 r^2 + B_4 r^4 + \ldots + B_{2k} r^{2k} + \ldots)$$

where $V_0(r)$ is the volume of the Euclidean ball of the same dimension and radius. Here the "volume invariants" B_2, B_4, ... are analytic functions of $m \in M$, or, more specifically, they are scalar curvature invariants of orders 2, 4, ... respectively.

A. Gray and L. Vanhecke [4] have calculated the first three invariants B_2, B_4, B_6 in terms of the curvature tensor R, the Ricci tensor ϱ, the scalar curvature τ and their covariant derivatives. In the same work the following was proved:

Let (M, g) be an analytic Riemannian manifold such that $V_m(r) = V_0(r)(1 + O(r^6))$ for all $m \in M$, i.e. such that $B_2 = B_4 = 0$ identically. Then (M, g) is flat in each of the following cases: (a) $\dim M \leqslant 3$, (b) M has non-positive or non-negative Ricci curvature, (c) M is conformally flat, (d) M is a product of surfaces, (e) M is locally a product of classical symmetric spaces and symmetric spaces of rank 1, (f) under some other special conditions which we do not write down explicitly.

On the other hand, the following examples have been given:

(i) A 4-dimensional Riemannian manifold such that $R \neq 0$ and $V_m(r) = V_0(r)(1 + O(r^6))$ for all $m \in M$.

(ii) A 5-dimensional homogeneous Riemanian manifold such that $R \neq 0$ and $V_m(r) = V_0(r)(1 + O(r^6))$.

(iii) A direct product of non-flat homogeneous Riemannian manifolds of total dimension $n = 734$ and such that $V_m(r) = V_0(r)(1 + O(r^8))$.

In this paper we generalize the implicite idea involved in the construction of example (iii): besides the ordinary volume invariants $B_2, B_4, \ldots, B_{2k}, \ldots$ we shall define certain polynomial functions $A_2(B_2)$, $A_4(B_2, B_4)$, $\ldots$, $A_{2k}(B_2, \ldots, B_{2k}), \ldots$ which behave additively on direct products of Riemannian manifolds. Also, we have $B_2 = B_4 = \ldots = B_{2k} = 0$ if and only if $A_2 = A_4 = \ldots = A_{2k} = 0$. We call A_{2i} *additive volume invariants* of (M, g). (For homogeneous spaces, both sets of invariants are constants.)

Then we develop a method for the construction of homogeneous Riemannian spaces with the property $V_m(r) = V_0(r)(1 + O(r^{2k}))$, $k > 4$. After having calculated the first 6 invariants $A_2, \ldots, A_{12}$ on spheres, we construct a direct product of homogeneous spaces with the property $V_m(r) = V_0(r)(1 + O(r^{16}))$. Our main conjecture says that the same method may work for the construction of examples with arbitrary large k.

If the above conjecture proves to be true it may throw some light upon the difficult "volume conjecture" by A. Gray and L. Vanhecke: Assuming $V_m(r) = V_0(r)$ everywhere on (M, g), is (M, g) flat?

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1. Additive volumal invariants

Let $x_0^{-1}, x_1, \ldots, x_n, \ldots$ be independent variables and $\mathbf{Q}[x_0^{-1}, x_1, \ldots, x_n, \ldots]$ the corresponding ring of polynomials over rational numbers. We shall write briefly x_0^{-k} instead of $(x_0^{-1})^k$. Let us define a derivation D in $\mathbf{Q}[x_0^{-1}, x_1, \ldots, x_n, \ldots]$ as follows: $D(r) = 0$ for $r \in \mathbf{Q}$, $D(x_0^{-1}) = -x_1 x_0^{-2}$, $D(x_i) = x_{i+1}$ for $i \geq 1$. We also define formally

$$D(\ln x_0) = x_1 x_0^{-1}.$$

Then, for every $k \geq 1$, the kth iteration $D^{(k)}(\ln x_0) \in \mathbf{Q}[x_0^{-1}, x_1, \ldots]$ has the form

$$D^{(k)}(\ln x_0) = \sum_{\substack{i_1 + \ldots + i_l = k \\ i_1 \geq i_2 \geq \ldots \geq i_l > 0}} c_{i_1 \ldots i_l} x_{i_1} \ldots x_{i_l} x_0^{-l}, \qquad c_{i_1 \ldots i_l} \in \mathbf{Q}. \tag{1}$$

as we see easily by the induction. The coefficients $c_{i_1 \ldots i_l} \in \mathbf{Q}$ are uniquely determined.

We shall call the polynomial $D^{(k)}(\ln x_0)$ *the logarithmic operator form of order* k, and we denote it by L_k.

If X is a linear differential operator on a smooth manifold M, and if f is a smooth function on M, then we can consider a non-linear differential operator $L_k(X)$ on M defined by the following formula:

$$L_k(X)(f) = \sum_{\substack{i_1 + \ldots + i_l = k \\ i_1 \geq i_2 \geq \ldots \geq i_l > 0}} c_{i_1 \ldots i_l} (X^{(i_1)} f) \ldots (X^{(i_l)} f) f^{-l}. \tag{2}$$

An informal definition of $L_k(X)$ is the following: consider the arbitrary function $F(t)$ of one variable (of class C^∞) and calculate the expression $d^k/(dt)^k\,(\ln F(t))$. Then substituing $F \to f$, $F' \to Xf$, ..., $F^{(k)} \to X^{(k)}f$ everywhere, we obtain the value of $L_k(X)$ on f.

Let N be another smooth manifold, g a smooth function on N, and Y a linear differential operator on N. We shall consider the product manifold $M \times N$ with the projections $p_1\colon M \times N \to M$, $p_2\colon M \times N \to N$. The function $(f\circ p_1)(g\circ p_2)$ on $M \times N$ will be denoted briefly by fg, and the linear differential operator $p_1^* X + p_2^* Y$ on $M \times N$ will be denoted briefly by $X + Y$. Now, we have the basic

PROPOSITION 1.1. *If M, N, f, g, X, Y have the previous meaning, then*

$$L_k(X+Y)(fg) = L_k(X)(f) + L_k(Y)(g), \quad k = 1, 2, \ldots . \tag{3}$$

Proof. Let first $M = N = \mathbf{R}$, $X = d/dt$. Then for any smooth real function $\tilde{f}$ we get from (1) and (2)

$$L_k\left(\frac{d}{dt}\right)(\tilde{f}) = \frac{d^k(\ln \tilde{f})}{(dt)^k}$$

and hence

$$L_k\left(\frac{d}{dt}\right)(\tilde{f}\tilde{g}) = L_k\left(\frac{d}{dt}\right)(\tilde{f}) + L_k\left(\frac{d}{dt}\right)(\tilde{g}).$$

Now, it suffices to compare the Leibniz' rule

$$\frac{d^k}{(dt)^k}(\tilde{f}\tilde{g}) = \sum_{i=0}^{k} \binom{k}{i} \frac{d^i \tilde{f}}{(dt)^i} \cdot \frac{d^{k-i}\tilde{g}}{(dt)^{k-i}} \quad \text{on } \mathbf{R}$$

and the binomial formula

$$(X+Y)^{(k)}(fg) = \sum_{i=0}^{k} \binom{k}{i} (X^{(i)}f)(Y^{(k-i)}g) \quad \text{on } M \times N$$

to obtain the general formula (3).

We also define the *reduced logarithmic operator form $\hat{L}_k$ of order k* by substituing $x_0{}^1 = 1$ in (1). For the reduced operator forms we have the following:

COROLLARY 1.2. *Let M, N be smooth manifolds, $(a, b) \in M \times N$ a fixed point, f, g smooth functions on M, N respectively such that $f(a) = g(b) = 1$, and X, Y linear differential operators on M, N respectively. Then*

$$\hat{L}_k(X+Y)_{(a,b)}(fg) = \hat{L}_k(X)_a(f) + \hat{L}_k(Y)_b(g). \tag{4}$$

The proof is obvious. Let us remark that (4) has a local character: the linear differential operators X, Y and the functions f, g are to be defined only in some neighborhoods of the points a, b respectively.

We shall give an explicite form of the differential operators $\hat{L}_k(X)$ for $k \leqslant 6$ (here X means an arbitrary linear differential operator).

PROPOSITION 1.3. *We have*

$$\hat{L}_1(X) = X$$

$$\hat{L}_2(X) = X^{(2)} - (X^{(1)})^2$$

$$\hat{L}_3(X) = X^{(3)} - 3X^{(2)}X^{(1)} + 2(X^{(1)})^3$$

$$\hat{L}_4(X) = X^{(4)} - 4X^{(3)}X^{(1)} - 3(X^{(2)})^2 + 12X^{(2)}(X^{(1)})^2 - 6(X^{(1)})^4$$

$$\begin{aligned}\hat{L}_5(X) = {} & X^{(5)} - 6X^{(4)}X^{(1)} - 10X^{(3)}X^{(2)} + 20X^{(3)}(X^{(1)})^2 + 30(X^{(2)})^2X^{(1)} \\ & - 60X^{(2)}(X^{(1)})^3 + 24(X^{(1)})^5\end{aligned}$$

$$\begin{aligned}\hat{L}_6(X) = {} & X^{(6)} - 6X^{(5)}X^{(1)} - 15X^{(4)}X^{(2)} + 30X^{(4)}(X^{(1)})^2 - 10(X^{(3)})^2 + 120X^{(3)}X^{(2)}X^{(1)} \\ & - 120X^{(3)}(X^{(1)})^3 + 30(X^{(2)})^3 - 270(X^{(2)})^2(X^{(1)})^2 + 360X^{(2)}(X^{(1)})^4 - 120(X^{(1)})^6.\end{aligned}$$

We shall now recall some concepts and results from [5]. Let (M, g) be an analytic Riemannian manifold and $m \in M$. If $(x_1, \ldots, x_n)$ is any system of normal coordinates at m then the *Euclidean Laplacian* $\tilde{\Delta}_m$ is defined by the formula

$$\tilde{\Delta}_m = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}.$$

Further, the *normal volume function* θ *at* m is defined by $\theta_m = \omega(\partial/\partial x_1, \ldots, \partial/\partial x_n)$, where ω is a volume element of (M, g) near m (such that $\theta_m > 0$). The definitions of $\tilde{\Delta}_m$ and θ_m are independent of the choice of normal coordinates at m (here "independent" means in the sense of germs). $\tilde{\Delta}_m$ is a local linear differential operator on (M, g).

Let us recall the Pizzetti's formula [1, p. 287], expressing the mean value of an analytic function f on the sphere $S_m^{n-1}(r)$ of radius r and with center m in the Euclidean space $\mathbf{R}^n$:

$$M_m(r, f) = \Gamma\left(\frac{1}{2}n\right) \sum_{k=0}^{\infty} \left(\frac{r}{2}\right)^{2k} \frac{1}{k!\,\Gamma\left(\frac{n}{2}+k\right)} (\Delta^k f)_m.$$

Here Γ means the usual gamma-function and Δ^k is the kth iteration of the Laplacian in

$\mathbf{R}^n$. It is not difficult to get from here a formula for the volume $\mathfrak{S}_m(r)$ of a small geodesic sphere of an analytic Riemannian manifold (M, g):

$$\mathfrak{S}_m(r) = 2\pi^{n/2} r^{n-1} \sum_{k=0}^{\infty} \left(\frac{r}{2}\right)^{2k} \frac{1}{k!\,\Gamma\left(\frac{n}{2}+k\right)} \tilde{\Delta}_m^k(\theta_m)(m)$$

(cf. Corollary 3.3 of [5]).

Here $\tilde{\Delta}_m^k$ means the kth iteration of the Euclidean Laplacian $\tilde{\Delta}_m$. The last formula can be easily rewritten in the form

$$\mathfrak{S}_m(r) = \mathfrak{S}_0(r)\left(1 + \sum_{k=1}^{\infty} \frac{\tilde{\Delta}_m^k(\theta_m)(m)}{2^k \cdot k!\,(n+2k-2)\dots(n+2)\,n} r^{2k}\right) \tag{5}$$

where $\mathfrak{S}_0(r)$ is the volume of a Euclidean sphere of radius r in $\mathbf{R}^n$ $(n = \dim M)$.

Now, if we differentiate the power-series expansion

$$V_m(r) = V_0(r)\left(1 + \sum_{k=1}^{\infty} B_{2k}(m)\, r^{2k}\right) = V_0(1)\left(r^n + \sum_{k=1}^{\infty} B_{2k}(m)\, r^{2k+n}\right)$$

with respect to r, we get immediately

$$\mathfrak{S}_m(r) = \mathfrak{S}_0(r)\left(1 + \sum_{k=1}^{\infty} \frac{2k+n}{n} B_{2k}(m)\, r^{2k}\right). \tag{6}$$

Comparing (5) and (6) we get

$$\tilde{\Delta}_m^k(\theta_m)(m) = 2^k \cdot k!(n+2k)(n+2k-2)\dots(n+2)\,B_{2k}(m). \tag{7}$$

Let us define a *global* function $\tilde{\Delta}^k\theta$ on (M, g) putting $(\tilde{\Delta}^k\theta)(m) = \tilde{\Delta}_m^k(\theta_m)(m)$, $m \in M$. We obtain

$$\tilde{\Delta}^k\theta = 2^k \cdot k!(n+2k)(n+2k-2)\dots(n+2)\,B_{2k}. \tag{8}$$

In particular, the function $\tilde{\Delta}^k\theta$ is analytic.

If (M_i, g_i), $i = 1, 2$, are two analytic Riemannian manifolds and $(m_1, m_2) \in M_1 \times M_2$ we can consider an *adapted* normal coordinate system $(x_1, \dots, x_{n_1}, x_{n_1+1}, \dots, x_{n_1+n_2})$ defined in a "rectangular" normal neighborhood $U_{m_1} \times U'_{m_2}$ in $M_1 \times M_2$. With respect to these adapted normal coordinates, we can see easily that

$$\tilde{\Delta}_{(m_1, m_2)} = \tilde{\Delta}_{1, m_1} + \tilde{\Delta}_{2, m_2}, \quad \theta_{(m_1, m_2)} = \theta_{1, m_1}\theta_{2, m_2} \tag{9}$$

for the corresponding Euclidean Laplacians and normal volume functions (via the corresponding projections p_1, p_2). Moreover, $\theta_{1,m_1}(m_1) = \theta_{2,m_2}(m_2) = 1$.

Our formula (4) now implies

$$\hat{L}_k(\tilde{\Delta}_{(m_1,m_2)})(\theta_{(m_1,m_2)})(m_1,m_2) = \hat{L}_k(\tilde{\Delta}_{1,m_1})_{m_1}(\theta_{1,m_1}) + \hat{L}_k(\tilde{\Delta}_{2,m_2})_{m_2}(\theta_{2,m_2}). \tag{10}$$

We are ready to introduce our basic concept:

Definition 1.4. Let (M, g) be an analytic Riemannian manifold. *The additive volume invariant of order* $2k$ is a function $A_{2k}: M \to R$ defined by the rule

$$A_{2k}(m) = \hat{L}_k(\tilde{\Delta}_m)_m(\theta_m), \quad m \in M. \tag{11}$$

We make the following *conventions*:

(a) The right-hand side of (11) will also be denoted by the symbol $\hat{L}_k(\tilde{\Delta})(\theta)(m)$.

(b) For a given Riemannian manifold M, the corresponding invariant A_{2k} will also be written as A_{2k}^M, and similarly for B_{2k}.

(c) For a homogeneous Riemanian manifold M the invariants A_{2k}^M are constant functions. The corresponding constants will be denoted by $A_{2k}(M)$ $(k = 1, 2, \ldots)$.

THEOREM 1.5. *For each* $k = 1, 2, \ldots$ *there exists a countable set* $\{P_{n,k}\}$ *of polynomials,* $P_{n,k} \in \mathbf{Q}[t_1, \ldots, t_k]$ *for* $n = 1, 2, \ldots$, *with the following property: for each analytic Riemannian manifold* (M, g) *of dimension* n *we have*

$$A_{2k}^M = P_{n,k}(B_2^M, B_4^M, \ldots, B_{2k}^M)$$

where $B_2^M, \ldots, B_{2k}^M$ *are the ordinary volume invariants of* (M, g). *In particular, the functions* A_{2k}^M *are analytic.*

Further, we have $A_2^M = \ldots = A_{2k}^M = 0$ *iff* $B_2^M = B_4^M = \ldots = B_{2k}^M = 0$.

Finally, if (M_i, g_i), $i = 1, 2$, *are two analytic Riemannian manifolds, then*

$$A_{2k}^{M_1 \times M_2} = A_{2k}^{M_1} \circ p_1 + A_{2k}^{M_2} \circ p_2, \quad k = 1, 2, \ldots, \tag{12}$$

where $p_i: M_1 \times M_2 \to M_i$ *are projections.*

Proof. From Definition 1.4 and (2) we get

$$A_{2k} = \hat{L}_k(\tilde{\Delta})(\theta) = \sum_{\substack{i_1+\ldots+i_l=k \\ i_1 \geq i_2 \geq \ldots \geq i_l > 0}} c_{i_1\ldots i_l}(\tilde{\Delta}^{i_1}\theta)\ldots(\tilde{\Delta}^{i_l}\theta) \tag{13}$$

where $c_{i_1\ldots i_l}$ are uniquely determined rational constants. The first assertion now follows from (8).

The second assertion follows from the fact that in (1) we always have the coefficient $c_k=1$ at the term $x_k x_0^{-1}$. It means that we can rewrite (13) in the form

$$A_{2k}=\tilde{\Delta}^k\theta-q_{k-1}(\tilde{\Delta}\theta,\tilde{\Delta}^2\theta,\ldots,\tilde{\Delta}^{k-1}\theta) \tag{14}$$

where $q_{k-1}(t_1,\ldots,t_{k-1})$ is a polynomial over $\mathbf{Q}$.

Finally, (12) is nothing but the global form of (10).

COROLLARY 1.6. *If (M_i,g_i) are homogeneous, then*

$$A_{2k}(M_1\times M_2)=A_{2k}(M_1)+A_{2k}(M_2),\quad k=1,2,\ldots.$$

COROLLARY 1.7. *There exist polynomials $Q_k(t_1,\ldots,t_k)$ such that*

$$A_{2k}=\tilde{\Delta}^k\theta-Q_{k-1}(A_2,\ldots,A_{2k-2}),\quad k=1,2,\ldots. \tag{15}$$

Proof. It follows immediately by the induction from (14).

PROPOSITION 1.8. *The first 6 polynomials Q_{k-1} are given by the following formulas:*

$$\begin{aligned}
&Q_0 &&=0\\
&Q_1(A_2) &&=(A_2)^2\\
&Q_2(A_2,A_4) &&=3A_4A_2+(A_2)^3\\
&Q_3(A_2,A_4,A_6) &&=4A_6A_2+3(A_4)^2+6A_4(A_2)^2+(A_2)^4\\
&Q_4(A_2,\ldots,A_8) &&=5A_8A_2+10A_6(A_2)^2+10A_6A_4+15(A_4)^2A_2+10A_4(A_2)^3+(A_2)^5\\
&Q_5(A_2,\ldots,A_{10}) &&=6A_{10}A_2+15A_8A_4+15A_8(A_2)^2+10(A_6)^2+60A_6A_4A_2+20A_6(A_2)^3\\
& &&\quad+15(A_4)^3+45(A_4)^2(A_2)^2+15A_4(A_2)^4+(A_2)^6.
\end{aligned}$$

Proof. Using the formulas of Proposition 1.3 we get the expression for $A_{2k}=\hat{L}_k(\tilde{\Delta})(\theta)$ $(k=1,\ldots,6)$ in the form (14). Then we proceed step by step to rewrite (14) in the form (15).

According to Theorem 1.5, the invariant A_{2k} is a polynomial in $B_2,\ldots,B_{2k}$, for each k. On the other hand, the formulas (8) and (15) show that B_{2k} is a polynomial in $A_2,\ldots,A_{2k}$. Because A_{2k} are additive on direct products in the sense of (12), each invariant $B_{2k}^{M_1\times M_2}$ is a polynomial in $B_{2i}^{M_1}$, $B_{2i}^{M_2}$, $i=1,\ldots,k$. It is easy to get an explicit formula: at a fixed point $(m_1,m_2)\in M_1\times M_2$ we can use (9) and hence

$$\tilde{\Delta}^p\theta=\sum_{q=0}^{p}\binom{p}{q}(\tilde{\Delta}_1^q\theta_1)(\tilde{\Delta}_2^{p-q}\theta_2).$$

Substituing from (8) we get

$$(n+2)(n+4)\dots(n+2p)B_{2p}^{M_1\times M_2}$$
$$=\sum_{q=0}^{p}(m+2)\dots(m+2q)(n-m+2)\dots(n-m+2p-2q)(B_{2q}^{M_1}\circ p_1)(B_{2p-2q}^{M_2}\circ p_2)$$

where $m=\dim M_1$, $n=\dim(M_1\times M_2)$.

If we define a *formal power series*

$$\hat{B}_s^M(m)=\sum_{p=0}^{\infty}2^p(n+2)\,\dots\,(n+2p)\,B_{2p}^M(m)\cdot s^p,\quad n=\dim M,\ m\in M,$$

then we obtain easily the (slightly modified) *product formula* by A. Gray:

$$\hat{B}_s^{M_1\times M_2}=(\hat{B}_s^{M_1}\circ p_1)(\hat{B}_s^{M_2}\circ p_2).$$

This result was obtained in [3] by a different method (using the Laplace transformation).

Using the power series $\hat{B}_s^M$ we can get a new formula for the additive volume invariants:

PROPOSITION 1.9. *Let $\hat{B}_s^M$ have a non-zero convergence radius η at a point $m\in M$. Then, in a neighborhood of $s=0$ we have*

$$\ln\hat{B}_s^M(m)=\sum_{k=0}^{\infty}\frac{A_{2k}^M(m)}{k!}s^k.$$

Proof. Put $h(s)=\hat{B}_s^M(m)$ for $s\in(-\eta,\eta)$. We can see from (8) that $h(s)=\sum_{k=0}^{\infty}((\tilde{\Delta}^k\theta)_m/k!)s^k$ and hence $\tilde{\Delta}^k(\theta)_m=h^{(k)}(0)$ for $k=0,1,\dots$.

According to (13) and the proof of Proposition 1.1

$$A_{2k}^M(m)=\hat{L}_k(\tilde{\Delta})(\theta)(m)=\sum c_{i_1\dots i_l}(\tilde{\Delta}^{i_1}\theta)_m\dots(\tilde{\Delta}^{i_l}\theta)_m$$
$$=\sum c_{i_1\dots i_l}h^{(i_1)}(0)\dots h^{(i_l)}(0)=\hat{L}_k\left(\frac{d}{ds}\right)_0(h)=\left.\frac{d^k(\ln h)}{(ds)^k}\right|_{s=0}.$$

Now, because $h(s)$ is analytic in a neighborhood of $s=0$, and $h(0)=1$, then $\ln h(s)$ is also analytic in a neighborhood of $s=0$. Q.E.D.

2. The invariants A_{2k} for some homogeneous spaces

Our basic problem now is to construct Riemannian manifolds satisfying $A_2=A_4=\dots=A_{2k}=0$ for possibly large k. Here is the main idea: we find out a finite collection $\{M_1, ..., M_l\}$ of homogeneous spaces such that

(i) all values $A_{2i}(M_j)$, $i=1, ..., k$; $j=1, ..., l$ are rational numbers,

(ii) for each $i=1, ..., k$, the invariant A_{2i} is negative for some M_α and it is positive for some M_β.

A family $\{M_1, ..., M_l\}$ of spaces with the properties (i), (ii) will be said to be *k-splitting* (or simply *splitting*).

Consider a direct product $M = M_1^{n_1} \times M_2^{n_2} \times ... \times M_l^{n_l}$. We have

$$A_{2i}(M) = \sum_{j=1}^{l} n_j A_{2i}(M_j), \quad i=1, \ldots, k.$$

Thus, in order to satisfy the identities $A_2(M) = ... = A_{2k}(M) = 0$ we have to solve a system of equations with rational coefficients

$$\sum_{j=1}^{l} x_j A_{2i}(M_j) = 0, \quad i=1, \ldots, k, \tag{16}$$

in *positive integers* $x_1, ..., x_l$.

Now, what does it mean "to solve" our system? Whereas the examples given in [4] were very concrete, we shall prefer rather theoretical constructions, i.e., pure existence theorems. The reason is obvious: the numerical calculation of the coefficients $A_{2i}(M_j)$ gets always very tiresome for a large k, and the complete list of the invariants A_{2k} is not known even for the simplest non-flat homogeneous spaces (e.g. for the sphere S^2). On the other hand, to get the information contained in our conditions (i), (ii) is much easier. The problem how to solve the system (16) with such a minimum information will be discussed in the next section. Here we shall add one more remark:

Even if $l > k$, our system (16) with only l unknowns may turn out "too rigid". To make it more flexible, we have to consider together with each space M_j the class of all homothetic spaces. In this way, we bild into our system of equations new parameters which can be arbitrary positive rational numbers. In fact, we have

PROPOSITION 2.1. *Let (M, g) be an analytic Riemannian manifold and $\lambda > 0$ a real number. Let $M(\lambda)$ denote the manifold (M, g^λ), where $g^\lambda = \lambda^{-1} g$. Then $A_{2k}^{M(\lambda)} = \lambda^k A_{2k}^M$ for each $k = 1, 2, \ldots$.*

Proof. It is obvious that the geodesic ball of radius r in the space $M(\lambda)$ coincides with the geodesic ball of radius $r\sqrt{\lambda}$ in (M, g). Also, we have the relation $\omega^\lambda = \lambda^{-n/2}\omega$ between the corresponding volume elements ($n = \dim M$). Hence we obtain for the corresponding volumes: $V_m(r) = \lambda^{-n/2} V_m(r\sqrt{\lambda})$. Using the ordinary volume expansions, we get

$$V_0(1) r^n (1 + \sum B_{2k}^{\lambda} r^{2k}) = \lambda^{-n/2} V_0(1) (\sqrt{\lambda} r)^n (1 + \sum B_{2k} r^{2k} \lambda^k)$$

and hence $B_{2j}^{\lambda} = B_{2j} \cdot \lambda^j$ for each j. Now, due to (8) and (13), we can write

$$A_{2k} = P_{n,k}(B_2, \ldots, B_{2k}) = \sum_{\substack{i_1 + \ldots + i_l = k \\ i_1 \geqslant i_2 \geqslant \ldots \geqslant i_l > 0}} \alpha_{i_1 \ldots i_l} B_{2i_1} \ldots B_{2i_l}$$

where the coefficients $\alpha_{i_1 \ldots i_l}$ depend only on $n = \dim M$. Hence the proposition follows.

Let R, ϱ, τ denote the Riemann curvature tensor, the Ricci tensor and the scalar curvature respectively. $\| \, \|$ will denote the lengtht of a tensor in the corresponding tangent space of (M, g).

PROPOSITION 2.2. *For any analytic Riemannian manifold* (M, g) *we have*

$$A_2 = -\frac{\tau}{3}, \quad A_4 = \frac{1}{45}\left(-3\|R\|^2 + 8\|\varrho\|^2 - 18\Delta\tau\right).$$

In particular, if M is homogeneous, then $A_4 = (1/45)(-3\|R\|^2 + 8\|\varrho\|^2)$.

Proof. Let us consider the ordinary volume expansion $V_m(r) = V_0(r)(1 + B_2 r^2 + B_4 r^4 + \ldots)$. Following [2] or [4], we have

$$B_2 = -\frac{\tau}{6(n+2)}, \quad B_4 = \frac{1}{360(n+2)(n+4)}(-3\|R\|^2 + 8\|\varrho\|^2 + 5\tau^2 - 18\Delta\tau),$$

where $n = \dim M$. Now, using (8), (15) and Proposition 1.3 we obtain easily

$$A_2 = \tilde{\Delta}\theta = 2(n+2) B_2 = -\frac{\tau}{3},$$

$$A_4 = \tilde{\Delta}^2 \theta - (A_2)^2 = 8(n+4)(n+2) B_4 - \frac{\tau^2}{9} = \frac{1}{45}(-3\|R\|^2 + 8\|\varrho\|^2 - 18\Delta\tau),$$

For M homogeneous we have $\tau = \text{const.}$, and $\Delta\tau = 0$. Q.E.D.

It is an interesting observation to see that A_4 is essentially obtained from B_4 by the removal of the term $5\tau^2$ which does not behave additively on direct products. Exactly in the same way we can calculate an additive invariant of order 6 from the known coefficient B_6. For homogeneous spaces this calculation has been done in [4, § 7]. (In fact, the additive invariant of order 6 presented there is equal to $3 \cdot 5 \cdot 7 \cdot 9 \cdot A_6$.)

The explicit formula for the next invariant B_8 is not known in general (with the exception of dimension $n=2$) and we have no formula for A_8, as well. It seems that such general formulas for A_8, A_{10}, ... etc. would be very complicated (even for the homogeneous spaces) and of little use for our purposes. Thus, our next question is *for what special kind* of homogeneous spaces we can calculate the higher order invariants with a minimum effort. Our first choice will be *the symmetric spaces of rank* 1. In this case, there are explicit formulas for the volume of a geodesic ball (or equivalently, of a geodesic sphere) containing only elementary functions. Thus the ordinary volume invariants B_{2i} are known, and each A_{2k} is given as a polynomial in the variables $B_2, \ldots, B_{2k}$. The only obstacle is the growing complexity of these polynomials for the higher orders.

We shall now summarize the explicit formulas for the geodesic *spheres* in the symmetric spaces of rank 1. The corresponding formulas for the geodesic balls are obtained by the integration. With respect to Proposition 2.1, it suffices to choose one representant in each homothety class.

THEOREM A ([2]). *The volume of a small geodesic sphere in a symmetric space of rank* 1 *is given by the following formulas:*

(1) *For the sphere* S^n *with constant curvature* 1*:*

$$\mathfrak{S}_m(r) = \mathfrak{S}_0(1) \sin^{n-1}(r).$$

(2) *For the complex projective space* $\mathbf{CP}^n$ *with constant holomorphic sectional curvature* 4 ($\dim M = 2n$)*:*

$$\mathfrak{S}_m(r) = \mathfrak{S}_0(1) \sin^{2n-1}(r) \cdot \cos r.$$

(3) *For the quaternionic projective space* $\mathbf{QP}^n$ *with maximum sectional curvature* 4 ($\dim M = 4n$)*:*

$$\mathfrak{S}_m(r) = \mathfrak{S}_0(1) \tfrac{1}{8} \sin^3(2r) \sin^{4n-4}(r).$$

(4) *For the Cayley plane* $\mathrm{Cay}\, P^2$ *with maximum sectional curvature* 4 ($\dim M = 16$)*:*

$$\mathfrak{S}_m(r) = \mathfrak{S}_0(1) 2^{-7} \sin^7(2r) \sin^8 r.$$

(5) *For the non-compact duals of the spaces* (1)–(4), *the corresponding formulas are obtained by substituting* sinh *for* sin *and* cosh *for* cos *everywhere.*

In each formula, $\mathfrak{S}_0(1)$ *denotes the volume of the unit sphere in the Euclidean space of the corresponding dimension.*

PROPOSITION 2.3. *Let* $S^2(\lambda)$ *denote the sphere with the constant curvature* $\lambda > 0$ *and* $H^2(-\lambda)$ *the hyperbolic plane with the constant curvature* $-\lambda < 0$. *Put* $a_k = A_{2k}(S^2(1))$ *for*

$k=1, 2, \ldots$. *Then we have*

$$A_{2k}(S^2(\lambda)) = \lambda^k a_k, \quad A_{2k}(H^2(-\lambda)) = (-\lambda)^k a_k, \quad k=1, 2, \ldots .$$

In particular, the invariants A_2, A_6, $A_{10}, \ldots$ *acquire opposite values on the spaces* $S^2(1)$, $H^2(-1)$.

Proof. Using (1) and (5) of Theorem A and the integration, we obtain

$$V_m(r) = 2V_0(1)(1-\cos r), \quad V_m(r) = 2V_0(1)(\cosh r - 1)$$

for $S^2(1)$ and for $H^2(-1)$ respectively. Hence we get

$$B_{2k}(S^2(1)) = 2(-1)^k/(2k+2)!, \quad B_{2k}(H^2(-1)) = 2/(2k+2)!.$$

Consequently, we have $B_{2i}(H^2(-1)) = (-1)^i B_{2i}(S^2(1))$ for $i=1, 2, \ldots$. Now, we can use the method of the proof of Proposition 2.1.

PROPOSITION 2.4. *All the invariants* A_{2k} *are rational numbers for the spaces* (1)–(4) *from Theorem A and for their non-compact duals.*

Proof. It is obvious that the ordinary volume invariants B_{2i} are always rational numbers. The rest follows from Theorem 1.5.

From the Propositions 2.3 and 2.4 we see the following: if we take $S^2(1)$ and $H^2(-1)$ (or, more precisely, their rational homothety classes) for the first members of our splitting family $\{M_1, \ldots, M_l\}$, we have "splitted" all the invariants of the form A_{2+4k}. It would be very comfortable if we could find out, for each k, a k-splitting family $\{M_1, \ldots, M_l\}$ consisting only of the symmetric spaces of rank 1. Unfortunately, such a scheme cannot be carried out due to the following result:

THEOREM B ([4]). *For all (non-flat) symmetric spaces of rank* 1 *and symmetric spaces of classical type, the expression* $-3\|R\|^2 + 8\|\varrho\|^2$ *is positive. Consequently, the invariant* A_4 *is positive for all these symmetric spaces.*

Remark. The problem remains open for the exceptional symmetric spaces of rank greater than one.

Thus, we have to look for a non-symmetric homogeneous space for which $A_4 < 0$. The most simple example is given in [4]: consider the 3-dimensional matrix group

$$G_3 = \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$$

naturally diffeomorphic to $\mathbf{R}^3(x, y, z)$, with the invariant Riemannian metric $dx^2+dz^2+(dy-x\,dz)^2$. The vector fields $X_1=(\partial/\partial x)$, $X_2=(\partial/\partial y)$, $X_3=x(\partial/\partial y)+(\partial/\partial z)$ form an orthogonal basis of the corresponding Lie algebra $(G_3)_e$. We can determine the Riemann connection ∇ in the standard way:

$$\nabla_{X_1}X_2=\nabla_{X_2}X_1=-\tfrac{1}{2}X_3,\quad \nabla_{X_1}X_3=-\nabla_{X_3}X_1=\tfrac{1}{2}X_2,\quad \nabla_{X_2}X_3=\nabla_{X_3}X_2=\tfrac{1}{2}X_1.$$

The tensor fields R, ϱ, τ are obtained easily from here, and using Proposition 2.2 we get $A_2=\frac{1}{6}$, $A_4=-\frac{1}{20}<0$. According to [4], we find easily $A_6=26/(3\cdot5\cdot7\cdot9)\in\mathbf{Q}$. Thus, *if we join the group manifold G_3 to our previous family $\{S^2(1), H^2(-1)\}$, we obtain a 3-splitting family.*

The next invariant to be splitted is A_8. If we want to use our group space G_3 in *any* splitting family without restriction, we have to prove the rationality of all invariants A_{2k} first.

PROPOSITION 2.5. *All invariants $A_{2k}(G_3)$ are rational numbers. Moreover, we have $A_8(G_3)\approx-0.021455<0$.*

Proof. We shall first construct explicitly the exponential map at the origin 0. The equations of the geodesics emanating from 0 can be written in the form

$$\begin{cases}\dot u(t)+w(t)g(t)=0\\ \dot w(t)-u(t)g(t)=0\\ \dot g(t)=0\end{cases}\quad\text{where}\quad\begin{cases}u(t)=\dot x(t)\\ w(t)=\dot z(t)\\ g(t)=\dot y(t)-x(t)\dot z(t).\end{cases}$$

The initial conditions are

$$x(0)=y(0)=z(0)=0,$$

$$u(0)=u,\quad g(0)=v,\quad w(0)=w.$$

Now, the exponential map Exp: $(u, v, w)\mapsto(x, y, z)$, defining normal coordinates at the origin, is given by the formulas

$$\begin{cases}x=\dfrac{u\cdot\sin v+w\cdot\cos v-w}{v}\\[2ex] y=\left(\dfrac{2v^2+u^2+w^2}{2v}-\dfrac{uw}{2v^2}\right)+\dfrac{1}{4v^2}[(w^2-u^2)\sin(2v)-2uw\cos(2v)+4uw\cos v-4w^2\sin v]\\[2ex] z=\dfrac{-u\cdot\cos v+w\cdot\sin v+u}{v}\end{cases}$$

(we take the limits to define our map at the origin (0, 0, 0)). The normal volume function θ is equal to the Jacobian $D(x, y, z)/D(u, v, w)$. After some algebraic transformations we obtain

$$\theta = \frac{(u^2 + w^2(2-2\cos v - v\sin v)}{v^4} + \frac{2(1-\cos v)}{v^2}.$$

It is easy to write down the power series for θ, which has rational coefficients. Now, the values $(\tilde{\Delta}^k\theta)_0$, $k=1, 2, \ldots$, at the origin are obviously rational numbers. Hence, the invariants $A_{2k}(G_3)$ are also rational.

Using our power series up to degree 8, we can find after routine calculations $\tilde{\Delta}^1\theta = \frac{1}{6}$, $\tilde{\Delta}^2\theta = -\frac{1}{45}$, $\tilde{\Delta}^3\theta = \frac{1}{140}$, $\tilde{\Delta}^4\theta = -1/(7\cdot 45)$, and hence $A_2 = \frac{1}{6}$, $A_4 = -\frac{1}{20}$, $A_6 = 26/(3\cdot 5\cdot 7\cdot 9) \approx 0.0275132$ $A_8 \approx -0.021455 < 0$. Q.E.D.

Using (15), Proposition 1.3 and the proof of Proposition 2.3, we can calculate

$$A_8(S^2(1)) = A_8(H^2(-1)) = -32/(3^3\cdot 5^2\cdot 7) < 0.$$

Thus, the family $\{S^2(1), H^2(-1), G_3\}$ *is not* 4*-splitting.*

Now, what else should we join to our family to split the invariant A_8? If we change the metric on G_3 to another invariant metric, we can show easily that the new space will be homothetic to G_3. Thus the change of the metric does not help. If we change the group, e.g., if we take the matrix group

$$\begin{pmatrix} e^z & 0 & x \\ 0 & e^{-z} & y \\ 0 & 0 & 1 \end{pmatrix}$$

with an invariant metric (the second most simple case) we can see easily that $A_4 < 0$ again, but the calculation of the exponential map leads to elliptic functions. Thus, the calculation of A_8 would lead to tedious numerical calculations. The situation with the other group spaces seems to be even worse.

Fortunately (and perhaps surprisingly) this difficulty can be taken care of by using the simplest symmetric spaces of rank 1 – the spheres.

PROPOSITION 2.6. *The first* 6 *additive volumal invariants on the unit spheres are given by the following formulas:*

$$A_2(S^{n+2}) = -\frac{1}{3}(n+1)(n+2)$$

$$A_4(S^{n+2}) = \frac{2}{3^2\cdot 5}(n+1)(n+2)(4n+1)$$

$$A_6(S^{n+2}) = \frac{8(n+1)(n+2)}{3^3\cdot 5\cdot 7}(-16n^2+15+1)$$

$$A_8(S^{n+2}) = \frac{16(n+1)(n+2)}{3^3\cdot 5^2\cdot 7}(16n^3-209n^2+89n-1)$$

$$A_{10}(S^{n+2}) = \frac{128(n+1)(n+2)}{3^4\cdot 5\cdot 7\cdot 11}(64n^4+461n^3-1008n^2+484n-1)$$

$$A_{12}(S^{n+2}) = \frac{256(n+1)(n+2)}{3^5\cdot 5^2\cdot 7^2\cdot 11\cdot 13}(-207744n^5-95125n^4+4349166n^3-6862618n^2 +3266748n+23).$$

The proof is given by the following procedure: first we compare the Pizzetti-like formula (5) with the formula $\mathfrak{S}_m(r)=\mathfrak{S}_0(1)\sin^{n+1}(r)$ for the $(n+2)$-dimensional unit sphere. Hence we calculate the numbers $\tilde{\Delta}^k\theta$, and using formula (15) together with Proposition 1.8, we can calculate our invariants step by step. The calculation of A_{10} and A_{12} is already pretty laborious. The calculation of A_{14} exceeds the capacity of a pocket calculator. [1]

From our formulas we can see immediately:

PROPOSITION 2.7. *$A_8(S^{n+2})$ is negative for $n\leqslant 12$ and positive for $n>12$. $A_{12}(S^{n+2})$ is positive for $n\leqslant 3$ and negative for $n>3$. Hence the family $\{S^2(1), H^2(-1), G_3, S^6(1), S^{15}(1)\}$ is 7-splitting.*

Remark. In fact, we need not write S^6 in our family to be 7-splitting. But the construction (see the next section) would not be simplified in this way and therefore we prefer write down the "well-deserved" spheres explicitly (which may be more instructive).

Now, the following two observations are remarkable:

(a) Every invariant $A_{2k}(S^{n+2})$ $(k=1, ..., 6)$ is a polynomial in one variable n which is *formally* of degree $2k$ but *actually* of degree $k+1$. (To make this observation, one has to look at the calculations involved.)

(b) Look at the polynomials of degree $k-1$ at the very right-hand sides of our formulas. For $k\geqslant 3$, the leading coefficient of our polynomial has the opposite sign to the last coefficient.

We *conjecture* that this may be true for the higher order invariants, too, with possible modifications. In other words, we make

Conjecture 1. Each of the invariants A_{2k} $(k>6)$ splits on a family of spheres.

For the calculation of the additive invariants in the special case S^2 we can also use a differential equation, as the next proposition shows. This direct method does not work

(1) I have used Sharp El-500S

for S^n, $n>2$, but it is still possible that some generalization or modification of this method exists.

PROPOSITION 2.8. *There exists a unique real analytic function $\psi(r)$ defined in a neighborhood of $r=0$ and satisfying the differential equation*

$$r(\psi'(r)+\psi^2(r))+(r+\tfrac{3}{2})\psi(r)+1=0. \tag{17}$$

For this function we have $\psi^{(k)}(0)=A_{2k+2}(S^2(1))$ for $k=0, 1, 2, \ldots$.

Remark. By the successive differentiations of (17) we can calculate the additive invariants of S^2 much easier than by the standard method using (15).

Proof of the proposition. For the unit sphere S^2 we have $B_{2k}=2(-1)^k/(2k+2)!$ (see Proposition 2.3). We obtain from (8)

$$\tilde{\Delta}^k\theta=\frac{2^{2k}\cdot(k!)^2(-1)^k}{(2k+1)!}.$$

Define a real analytic function on $\mathbf{R}$ by the formula

$$h(r)=\sum_{k=0}^{\infty}\frac{(\tilde{\Delta}^k\theta)}{k!}r^k=\sum_{k=0}^{\infty}\frac{2^{2k}\cdot k!(-1)^k}{(2k+1)!}r^k.$$

We can check directly that $h(r)$ is a (singular) solution of the differential equation $rh''(r)+(r+3/2)h'(r)+h(r)=0$ satisfying the initial condition $h(0)=1$. Now, $\varphi(r)=\ln h(r)$ is defined and real analytic in a neighborhood of $r=0$ and it satisfies the equation $r(\varphi''+(\varphi')^2)+(r+3/2)\varphi'+1=0$. Then $\psi(r)=\varphi'(r)$ satisfies (17) and we can see easily that $\psi(r)$ is a unique analytic solution of (17) defined in a neighborhood of the origin.

Finally, according to the proof of Proposition 1.9 we have $A_{2k}(S^2)=\varphi^{(k)}(0)=\psi^{(k-1)}(0)$.
Q.E.D.

Example 2.9. Using Propositions 2.3, 2.5 and 2.6 we can check easily the 734-dimensional example by A. Gray and L. Vanhecke: put

$$M=[G_3]^{169}\times H^2\left(-\frac{13}{2}\right)\times\left[S^3\left(\frac{13}{60}\right)\right]^{75}.$$

Then $A_2(M)=A_4(M)=A_6(M)=0$. Moreover, we see that $A_8(M)<0$.

3. An existence theorem and conjectures

We shall start with an algebraic result.

LEMMA 3.1. *Consider the system of algebraic equations with* 14 *unknowns* $x_1, \ldots, x_7$, $u_1, \ldots, u_7$:

$$\begin{aligned}&\sum_{i=1}^{7} x_i u_i = \alpha_1, \quad \sum_{i=1}^{7} x_i u_i^2 = \beta_1, \quad \sum_{i=1}^{7} x_i u_i^3 = \alpha_2, \quad \sum_{i=1}^{7} x_i u_i^4 = \beta_2,\\ &\sum_{i=1}^{7} x_i u_i^5 = \alpha_3, \quad \sum_{i=1}^{7} x_i u_i^6 = \beta_3, \quad \sum_{i=1}^{7} x_i u_i^7 = \alpha_4\end{aligned} \tag{18}$$

where the constants on the right-hand sides satisfy the following conditions:

(a) $\beta_1 > 0, \quad \alpha_2 > 0, \quad \beta_2 > 0, \quad \beta_3 > 0$

(b) $\beta_1 \beta_2 - (\alpha_2)^2 > 0$

(c) $\beta_3 > \dfrac{\beta_1 \beta_2 - (\alpha_2)^2}{\beta_1 + \alpha_2} \left(1 + \left|\dfrac{\alpha_3}{\alpha_2}\right| + \left|\dfrac{\alpha_3(\beta_1 + \alpha_2) - \beta_2(\beta_2 + \alpha_2)}{\beta_1 \beta_2 - (\alpha_2)^2}\right| + \dfrac{\beta_2 + \alpha_2}{\beta_1 + \alpha_1}\right)^2$
$+ (\beta_2 + \alpha_2)|\alpha_3| / (\beta_1 + \alpha_2)$.

Then there exists a solution (x_i, u_i) *such that* $x_i > 0$ *for* $i = 1, \ldots, 7$, $u_j > 0$ *for* $j = 1, \ldots, 4$, $u_k < 0$ *for* $k = 5, 6, 7$. *If* α_j, β_j *are rational numbers, then we can find* x_i, y_i *to be rational numbers, too.*

Proof. Let us consider $u_1, \ldots, u_7$ as parameters which are subordinated to the conditions $u_1 > u_2 > u_3 > u_4 > 0 > u_5 > u_6 > u_7$. We shall solve the system (18) with respect to $x_1, \ldots, x_7$ using the Cramer's rule. Then we obtain $x_k = D_k / D$ $(k = 1, \ldots, 7)$ where

$$D = -\prod_{i=1}^{7} u_i \left(\prod_{1 \leqslant i < j \leqslant 7} (u_i - u_j) \right) > 0.$$

(The second factor is equal to the well-known Vandermonde's determinant — up to a sign.) Further, we can find without difficulties

$$D_k = (-1)^k \prod_{\substack{1 \leqslant i < j \leqslant 7 \\ i, j \neq k}} (u_i - u_j)\, s_6^{(k)}(u) \cdot F_k(u, \alpha, \beta), \quad k = 1, \ldots, 7,$$

where $F_k = \alpha_1 s_6^{(k)}(u) - \beta_1 s_5^{(k)}(u) + \alpha_2 s_4^{(k)}(u) - \beta_2 s_3^{(k)}(u) + \alpha_3 s_2^{(k)}(u) - \beta_3 s_1^{(k)}(u) + \alpha_4$ and $s_l^{(k)}(u)$ $(l = 1, \ldots, 6)$ denotes the lth symmetric function of the unknowns $u_1, \ldots, u_{k-1}, u_{k+1}, \ldots, u_7$.

Then, in order to satisfy the inequalities $x_k>0$ for $k=1, \ldots, 7$, we must have $D_k>0$ for $k=1, \ldots, 7$, and hence

$$F_1>0, \quad F_2<0, \quad F_3>0, \quad F_4<0, \quad F_5<0, \quad F_6>0, \quad F_7<0. \tag{19}$$

Here we have used the fact that $s_6^{(k)}=u_1 \ldots u_{k-1}u_{k+1} \ldots u_7$ is negative for $k\leqslant 4$ and positive for $k>4$.

Now, we put on the parameters u_i the following additional conditions:

$$u_1+u_7=0, \quad u_4+u_5=0, \quad u_4=(u_1)^{-3}.$$

We can rearrange each F_k in the form

$$F_k(u, \alpha, \beta)=\sum_{-6\leqslant l\leqslant 2} a_l^{(k)}(u_2, u_3, u_6)\cdot(u_1)^l$$

where $a_l^{(k)}$ are polynomials, and we always have either $a_2^{(k)}\neq 0$ or $a_1^{(k)}\neq 0$. For u_1 *sufficiently large with respect to* u_2, u_3, u_6 the expression F_k has the same sign as the leading coefficient $a_2^{(k)}$ or $a_1^{(k)}$. We can show easily by the direct calculation that the system of inequalities (19) is *then* equivalent to the following:

$$\left.\begin{aligned}
&-\alpha_2u_2u_3u_6+\beta_2(u_2u_3+u_2u_6+u_3u_6)-\alpha_3(u_2+u_3+u_6)+\beta_3>0\\
&\quad\alpha_2u_3u_6-\beta_2(u_3+u_6)+\alpha_3>0\\
&-\alpha_2u_2u_6+\beta_2(u_2+u_6)-\alpha_3>0\\
&-\beta_1u_2u_3u_6+\alpha_2(u_2u_3+u_2u_6+u_3u_6)-\beta_2(u_2+u_3+u_6)+\alpha_3>0\\
&-\alpha_2u_2u_3+\beta_2(u_2+u_3)-\alpha_3>0.
\end{aligned}\right\} \tag{20}$$

Here the first inequality comes from $F_1>0$, $F_7<0$, the second from $F_2<0$, the third from $F_3>0$, the fourth from $F_4<0$, $F_5<0$, and the last from $F_6>0$.

Now, we shall require the additional condition $u_2+u_6=0$, and we put, for the sake of brevity, $u=u_2$, $v=u_3$. Then (20) yields

$$\begin{aligned}
&(\alpha_2v-\beta_2)u^2+\beta_3-\alpha_3v>0\\
&(\beta_1v-\alpha_2)u^2+\alpha_3-\beta_2v>0\\
&(\beta_2-\alpha_2v)u+\alpha_3-\beta_2v>0\\
&\alpha_2u^2-\alpha_3>0\\
&(\beta_2-\alpha_2v)u+\beta_2v-\alpha_3>0.
\end{aligned}$$

Putting $v=(\beta_2+\alpha_2)/(\beta_1+\alpha_2)>0$ we get

$$\beta_2-\alpha_2v=\beta_1v-\alpha_2=\frac{\beta_1\beta_2-(\alpha_2)^2}{\beta_1+\alpha_2}>0.$$

The only free parameter is now $u>0$. If we denote

$$\gamma_1=\frac{\beta_2+\alpha_2}{\beta_1+\alpha_2},\quad \gamma_2=\frac{\beta_1\beta_2-(\alpha_2)^2}{\beta_1+\alpha_2},\quad \gamma_3=\alpha_3-\gamma_1\beta_2,$$

we have to satisfy the inequalities

$$\begin{gathered} u^2<\frac{\beta_3-\gamma_1\alpha_3}{\gamma_2} \\ u^2>-\frac{\gamma_3}{\gamma_2},\quad u>-\frac{\gamma_3}{\gamma_2} \\ u^2>\frac{\alpha_3}{\alpha_2},\quad u>\frac{\gamma_3}{\gamma_2}, \end{gathered} \tag{21}$$

where $\gamma_1>0$, $\gamma_2>0$, $\alpha_2>0$.

Put $u=1+|\alpha_3/\alpha_2|+|\gamma_3/\gamma_2|+\gamma_1$; then the last four inequalities (21) are fulfilled. The first inequality (21) is also satisfied because $(1+|\alpha_3/\alpha_2|+|\gamma_3/\gamma_2|+\gamma_1)^2<(\beta_3-\gamma_1|\alpha_3|)/\gamma_2$ is nothing but the condition (c). Also, we have $u>\gamma_1$; it means that $u_2>u_3$. (The other inequalities among the parameters u_i are obvious from our substitutions and limit procedures.)

The "rationality part" also follows from the construction. Q.E.D.

Now, we can prove our basic existence theorem.

THEOREM 3.2. *Let M_1, M_2, M_3 denote homogeneous Riemannian manifolds satisfying $A_4(M_1)<0$, $A_8(M_2)>0$, $A_{12}(M_3)<0$ and $A_{2k}(M_i)\in\mathbf{Q}$ for all k, and $i=1, 2, 3$ (e.g., $M_1=G_3$, $M_2=S^{15}(1)$, $M_3=S^6(1)$). Then there exist positive integers $n_1, \dots, n_{11}$ and positive rational numbers $r_1, \dots, r_4$, $s_1, \dots, s_4$, c, d such that the Riemannian space*

$$\begin{aligned} M=[S^2(r_1)]^{n_1}\times\dots\times[S^2(r_4)]^{n_4}\times[H^2(-s_1)]^{n_5}\times\dots\times[H^2(-s_4)]^{n_8}\times(M_1)^{n_9}\times(M_2(c))^{n_{10}} \\ \times(M_3(d))^{n_{11}} \end{aligned}$$

satisfies $A_2(M)=A_4(M)=\dots=A_{14}(M)=0$, i.e., the condition $V_m(r)=V_0(r)(1+O(r^{16}))$ at each point $m\in M$.

Proof. In accordance with Proposition 2.3 we put $A_{2k}(S^2(1))=a_k$ for $k=1, 2, \dots$; hence $A_{2k}(S^2(\lambda))=a_k\lambda^k$, $A_{2k}(H^2(-\lambda))=a_k(-\lambda)^k$. Further, we denote $A_{2k}(M_1)=b_k$, $A_{2k}(M_2)=c_k$, $A_{2k}(M_3)=d_k$. Recall that $a_1<0$, $a_2>0$, $a_3>0$, $a_4<0$, $a_5<0$, $a_6>0$ (see Proposition 2.6) and $b_2<0$, $c_4>0$, $d_6<0$.

Put $\delta_k^{(1)} = -b_k/a_k$ for $k=1, \ldots, 7$; then $\delta_2^{(1)} > 0$.

Put $\delta_k^{(2)} = \delta_k^{(1)} - (-1)^k \lambda^{2k-5}$. For a large rational $\lambda > 0$ we have $\delta_2^{(2)} > 0$, $\delta_3^{(2)} > 0$.

Put $\delta_k^{(3)} = \delta_k^{(2)} - (c_k/a_k)\mu^{2k-7}$. For a large rational $\mu > 0$ we have $\delta_2^{(3)} > 0$, $\delta_3^{(3)} > 0$, $\delta_4^{(3)} > (\delta_3^{(3)})^2/\delta_2^{(3)}$.

Finally, put $\delta_k^{(4)} = \delta_k^{(3)} - (d_k/a_k)\nu^{2k-11}$. For a large rational $\nu > 0$ we have $\delta_2^{(4)} > 0$, $\delta_3^{(4)} > 0$, $\delta_4^{(4)}\delta_2^{(4)} - (\delta_3^{(4)})^2 > 0$ and the condition (c) from our lemma is satisfied for

$$\alpha_1 = \delta_1^{(4)}, \quad \beta_1 = \delta_2^{(4)}, \quad \alpha_2 = \delta_3^{(4)}, \quad \beta_2 = \delta_4^{(4)}, \quad \alpha_3 = \delta_5^{(4)}, \quad \beta_3 = \delta_6^{(4)}, \quad \alpha_4 = \delta_7^{(4)}. \tag{22}$$

We see that the conditions (a), (b) of Lemma 3.1 are also fulfilled.

Let (x_i, u_i) be a (rational) solution of the system (18) in which the right-hand sides are given by (22). According to our lemma, we can suppose that $x_i > 0$, $u_1 > 0, \ldots, u_4 > 0$, $u_5 < 0, \ldots, u_7 < 0$. Then we have for $k = 1, \ldots, 7$:

$$a_k \sum_{i=1}^{7} x_i u_i^k + b_k + a_k \cdot \lambda^{-5}(-\lambda^2)^k + c_k \mu^{-7}(\mu^2)^k + d_k \nu^{-11}(\nu^2)^k = 0,$$

i.e.,

$$\sum_{i=1}^{4} x_i A_{2k}(S^2(u_i)) + \sum_{j=5}^{7} x_j A_{2k}(H^2(u_j)) + A_{2k}(M_1) + \lambda^{-5} A_{2k}(H^2(-\lambda^2))$$
$$+ \mu^{-7} A_{2k}(M_2(\mu^2)) + \nu^{-11} A_{2k}(M_3(\nu^2)) = 0.$$

Now, we only have to multiply this relation by the least common multiple of all denominators of our coefficients. Q.E.D.

At the end, we shall discuss the possible generalizations of our construction. Suppose for a moment that we can construct a k-splitting family of spaces for each k – it will be true if our Conjecture 1 holds, for instance. Then it seems prospective to try to generalize our algebraic procedure from Lemma 3.1, and from the proof of Theorem 3.2. To see it closer, let us look at out lemma first. The condition (c) seems to be rather special and complicated. But in fact this condition is only one of the possible forms how to specify the following requirement: the coefficient β_3 should be sufficiently large in comparison with the "influence" of the previous coefficients. Also, the condition (b) means only that β_2 has an analogous property. Looking at the procedure used in the proof of Theorem 3.2 (in fact, we have used the limits of auxiliary parameters) the possible directions of the generalizations are almost clear. Hence we make:

Conjecture 2. For every $k > 0$ there is a homogeneous space (M, g) satisfying $A_2(M) = \ldots = A_{2k}(M) = 0$.

Consider the following "volume condition" in an analytic Riemannian manifold M: the volume of a small geodesic ball is always equal to the volume of a Euclidean ball of the same dimension and radius. We know that this volume condition is equivalent with the vanishing of all scalar curvature invariants B_{2k}^{M}.

Conjecture 3. The volume condition cannot be reduced to a finite number of conditions $B_2 = B_4 = \dots = B_{2k} = 0$, in general. In other words, the volume condition is equivalent to an infinite system of partial differential equations (with the orders growing to the infinity) which cannot be reduced to its finite subsystem. (Cf. Conjecture (*) in [4].)

In accordance with the Conjecture 1 we can put the following, more specific conjectures:

Conjecture 4. For each $k > 0$ there is a symmetric space M_k' and a positive integer n_k such that the homogeneous space $M = M_k' \times (G_3)^{n_k}$ satisfies $A_2 = A_4 = \dots = A_{2k} = 0$.

Conjecture 5. For each $k > 0$ there is a symmetric space M satisfying the condition $V_m(r) = V_0(r)(1 + \alpha_k r_4 + O(r^{2k}))$, where α_k is a positive constant.

References

The references are reduced to a minimum which is absolutely necessary. A numerous bibliography can be found in [4] and [5], for instance.

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