A CLASS OF SPECIAL $\mathcal{L}_\infty$ SPACES

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1. Introduction

We shall construct Banach spaces X and Y having some peculiar properties.

(a) X is a separable $\mathcal{L}_\infty$ space.

(b) X is a Radon-Nikodym space. Since a separable $\mathcal{L}_\infty$ space cannot be imbedded isomorphically into a separable dual space, this example solves negatively the following conjecture of Uhl: Is every separable Radon-Nikodym space isomorphic to a subspace of a separable dual space?

(c) X is a Schur space, i.e. weak and norm compactness coincide in X. This answers negatively a conjecture of Lindenstrauss who asked in [10] whether a space which has the weakly compact extension property is necessarily finite dimensional (see also Theorem 2.4). In [11] Pełczynski and Lindenstrauss and in [12] Lindenstrauss and Rosenthal asked whether every $\mathcal{L}_\infty$ space contains a subspace isomorphic to c_0. Our example disproves this conjecture.

(d) X is weakly sequentially complete. Since X^* is a $\mathcal{L}_1$ space, X^* is also weakly sequentially complete. For a long time it was conjectured that a Banach space is reflexive if and only if both X and X^* are weakly sequentially complete.

(a′) The Banach space Y is a separable $\mathcal{L}_\infty$ space.

(b′) The Banach space Y is a Radon-Nikodym space.

(c′) Y is somewhat reflexive, i.e. every infinite dimensional subspace of Y contains an infinite dimensional subspace which is reflexive.

Since Y does not contain a subspace isomorphic to l_1 it follows from results of Lewis and Stegall [9] that Y^* is isomorphic to l_1. It is strange that Y does not contain a copy of c_0. Since Y is a $\mathcal{L}_\infty$ space it also has the Dunford-Pettis property and hence there are

Dunford-Pettis spaces which are somewhat reflexive. For some time the Dunford-Pettis property was understood as a property opposite to reflexivity. Following these lines W. Davis asked whether a somewhat reflexive space could have the Dunford-Pettis property.

We think that the paper might be interesting for two reasons. First it solves some conjectures in Banach space theory. Secondly it gives a new way of constructing $\mathcal{L}_\infty$ spaces. We think that this paper is the first paper where $\mathcal{L}_\infty$ spaces are built using *isomorphic* copies of l_∞^n. The known examples of $\mathcal{L}_\infty$ spaces where constructed using isometric copies of l_∞^n and hence they all are isomorphic to preduals of L_1. We also show that there is a continuum number of mutually non-isomorphic $\mathcal{L}_\infty$ spaces.

The notations used in this paper are standard and coincide with the notations introduced in [14]. We also use this text as reference for any unexplained notion on Banach spaces. For more details on abstract vector measures we refer to [5]. Besides the introduction the paper contains four more chapters. Chapter two deals with a brief introduction in the theory of $\mathcal{L}_\infty$ spaces, for more details and for the origin of the problems the reader may consult [10], [11] and [12] (see also [13]). In [10] the Hahn-Banach problem is investigated and the $\mathcal{L}_\infty$ spaces are characterized as those Banach spaces which have the compact extension property. Chapter 3 contains some basic definitions of Radon-Nikodym spaces. For more details we refer to [5]. Chapters 4 and 5 contain the construction of the Banach spaces X and Y.

2. Hahn-Banach problems and $\mathcal{L}_\infty$ spaces

By a Hahn-Banach problem we mean the following: given Banach spaces Z_1, Z_2 and X, where Z_1 is a subspace of Z_2, given an operator $T_1: Z_1 \to X$, when does there exist an operator $T_2: Z_2 \to X$ extending T_1, i.e. $T_2|_{Z_1} = T_1$. We say that a Banach space X has the compact extension property (C.E.P.) if a compact extension T_2 exists whenever T_1 is compact. In the same way we say that X has the weakly compact extension property (W.C.E.P.) if a weakly compact extension T_2 exists for every weakly compact operator T_1. In [10], [11] and [12] (see [13] for the details) a characterization of Banach spaces with the C.E.P. is given. Since we use some of this theory we recall some definitions and theorems.

Definition 2.1. If X and X' are Banach spaces then the Banach-Mazur distance between X and X' is given by

$$d(X, X') = \inf \{\|T\| \cdot \|T^{-1}\| \mid T: X \to X' \text{ is an onto isomorphism}\}.$$

Definition 2.2. (i) A Banach space X is a $\mathcal{L}_{\infty,\lambda}$ space ($\lambda \geq 1$) if for all $E \subset X$, where E is a finite dimensional subspace of X, there is a finite dimensional subspace $F \subset X$, such that $E \subset F$ and $d(F, l_\infty^{\dim F}) \leq \lambda$. Here l_∞^n means the n-dimensional real vector space $\mathbf{R}^n$ endowed with the norm $\|(x_1, ..., x_n)\| = \max_{k \leq n} |x_k|$.

(ii) If X is a $\mathcal{L}_{\infty,\lambda}$ space for some λ then we simply say that X is a $\mathcal{L}_\infty$ space.

The characterization of Banach spaces having the C.E.P. is given in the following theorem, see [13] for proofs.

THEOREM 2.3. *The following are equivalent:*

(a) *X is a $\mathcal{L}_\infty$ space.*

(b) *X has the compact extension property, i.e. if Z_1, Z_2 and X are Banach spaces, Z_1 being a subspace of Z_2, if $T_1: Z_1 \to X$ is a compact operator, then there is a compact operator $T_2: Z_2 \subset X$ extending T_1.*

(c) *Same as* (b) *but the extension T_2 is not required to be compact, i.e. every compact operator has a continuous extension.*

(d) *X^* is isomorphic to a complemented subspace of an L_1 space.*

For X separable (a), (b), (c) *and* (d) *are equivalent with*

(e) *X^* the dual space of X is isomorphic to l_1 or to $M[0, 1]$, the Banach space of Radon measures on* $[0, 1]$. *Moreover X^* is isomorphic to l_1 if and only if X does not contain a subspace isomorphic to l_1.*

The following theorem was known for a long time but since there was no example it was useless.

THEOREM 2.4. *A Banach space X has the* W.C.E.P. *if and only if it is a $\mathcal{L}_\infty$ space having the Schur property.*

Proof. Suppose X is a $\mathcal{L}_\infty$ space having the Schur property. Since every weakly compact operator arriving in X is necessarily compact the W.C.E.P. and the C.E.P. coincide. Since X is a $\mathcal{L}_\infty$ space the preceding theorem shows that X has the W.C.E.P.

The converse is less obvious. Suppose that X has the W.C.E.P. and let $K \subset X$ be a weakly compact set in X. By the factorization theorem of Davis, Figiel, Johnson and Pełczynski [3] there is a reflexive space R as well as an operator $T: R \to X$ such that $K \subset T(B(R))$ ($B(R)$ is the closed unit ball of R). The Banach space R is canonically isometric to a subspace of $C(B(R^*))$, the space of real valued continuous functions on the unit ball of R^*, this unit ball being endowed with the weak* topology $\sigma(R^*, R)$. Since R is reflexive, the operator $T: R \to X$ is necessarily weakly compact and since X has the W.C.E.P. this operator has a weakly compact extension $T': C(B(R^*)) \to X$.

We now use the Dunford-Pettis property of $\mathcal{C}(B(R^*))$ (see [6], [5] or [14]). This property says that a weakly compact operator defined on the space of continuous functions maps weakly compact sets onto norm compact sets. It follows that $T'(B(R))$ is compact and hence also $T(B(R))$ and its subset K. This proves that X has the Schur property. Q.E.D.

The known theorem that follows is not difficult but it plays a central part in the solution of Uhl's conjecture. The theorem is a slight generalization of the Bessaga-Pełczynski theorem on the imbedding of c_0 in a dual space. See [1], [5] and [14].

THEOREM 2.5. *Let X be an infinite dimensional $\mathcal{L}_\infty$ space. If Y is a Banach space such that X is isomorphic to a subspace of Y^*, then Y contains a complemented subspace Z which is isomorphic to l_1. In particular Y^* is not separable.*

Proof. Let $i: X \to Y^*$ be the inclusion map. Transposition gives a map: $T: Y \to X^*$ defined as $T(y)(x) = i(x)(y)$. It follows that $T^*: X^{**} \to Y^*$ and that $T^*|_X = i$.

If T is a weakly compact operator then also T^* is a weakly compact operator and hence the inclusion map i being a restriction of T is also a weakly compact operator. Since X is an infinite dimensional $\mathcal{L}_\infty$ space it is not a reflexive space and hence i is not weakly compact. It follows that T is not weakly compact. Hence the image $T(B(Y))$ is not a relatively weakly compact subset of X^*. From Theorem 2.3 we know that X^* is isomorphic to a complemented subspace of an L_1 space and hence the Kadec-Pełczynski theorem [8] applies. The set $T(B(Y))$ contains a sequence $(e_n)_{n\geqslant 1}$ such that $(e_n)_{n\geqslant 1}$ is equivalent to the usual basis of l_1 and such that $S = \overline{\text{span}}\,(e_n,\, n \geqslant 1)$ is complemented in X^*.

Let $P: X^* \to S$ be a continuous projection. Take for each $n \geqslant 1$ an element $y_n \in B(Y)$ such that $T(y_n) = e_n$. It is elementary to see that $(y_n)_{n\geqslant 1}$ is equivalent to the usual basis of l_1 and also to $(e_n)_{n\geqslant 1}$. Let now $Z = \overline{\text{span}}\,(y_n,\, n \geqslant 1)$ and let $V: S \to Z$ be the operator defined by the relation $V(y_n) = e_n$. If Q is defined as $Q = V \circ P \circ T$ then clearly Q is a projection $Y \to Z$. Q.E.D.

3. Uhl's conjecture on Radon-Nikodym spaces

From [5] we recall some definitions and notations. A probability space $(\Omega, \mathcal{A}, \mathbf{P})$ is a triple where Ω is a set, $\mathcal{A}$ is a σ-algebra on Ω and $\mathbf{P}$ is a σ-additive positive measure, defined on Ω and with total mass $\mathbf{P}(\Omega) = 1$. Measurability will be defined only for variables taking values in a separable Banach space. For more general definitions and problems the reader can consult an advanced textbook on measure theory. If X is a separable Banach space and $\xi: \Omega \to X$ is a mapping then ξ is measurable if $\xi^{-1}(0) \in \mathcal{A}$ for all open set

0 in X. If ξ is measurable then it is easily seen that $\|\xi(\omega)\|$ defines a real valued measurable function. If also $\int \|\xi(\omega)\| d\mathbf{P}$ is finite then ξ is called Bochner integrable and $\int \xi d\mathbf{P}$ denotes the integral of ξ with respect to $\mathbf{P}$. A function $\mu: \mathcal{A} \to X$ which is σ-additive is called a vector measure. The measure μ is $\mathbf{P}$-continuous if $\mathbf{P}(A) = 0$ implies $\mu(A) = 0$. For a vector measure μ we define the bounded variation norm $\|\mu\|$ as:

$$\|\mu\| = \sup\left\{\sum_{k=1}^{n} \|\mu(A_k)\| \mid A_1, \ldots, A_n \text{ a partition of } \Omega\right\}.$$

If ξ is Bochner integrable then $\mu(A) = \int_A \xi d\mathbf{P}$ defines a vector measure whose bounded variation norm is $\|\mu\| = \int \|\xi(\omega)\| d\mathbf{P} = \|\xi\|_1$. In general a measure with bounded variation is not of this form.

Definition 3.1. A Banach space X has the Radon-Nikodym property (RNP) if for each probability space $(\Omega, \mathcal{A}, \mathbf{P})$ and each $\mathbf{P}$-continuous vector measure μ of bounded variation there is a density, i.e. there is a $\xi: \Omega \to X$, Bochner integrable such that $\mu(A) = \int_A \xi d\mathbf{P}$ for all $A \in \mathcal{A}$.

As examples of RN spaces we can give separable dual spaces as well as subspaces of RN spaces. Since the only known examples were of this kind, Uhl conjectured that every separable RN space is isomorphic to a subspace of a separable dual space (see [5] page 82 and page 211–212). We know that if X is isomorphic to the dual space of a separable Banach space then X has RNP if and only if X is separable. We also know different geometric characterizations of separable RN spaces. If Uhl's conjecture were true then a beautiful geometric description of subspaces of separable dual spaces would be obtained. (Un)fortunately Uhl's conjecture is wrong.

For later use we mention the following, see [5] for a proof.

LEMMA 3.2. *If $\mu: \mathcal{A} \to X$ is a $\mathbf{P}$-continuous vector measure with bounded variation then μ has a density if and only if there is a sequence of vector measures μ_n such that μ_n has finite dimensional range and $\mu_n \to \mu$ in variation norm.*

4. The basic example

By means of a rather simple construction we will make a Banach space which is a $\mathcal{L}_\infty$ space, a Schur space and a RN space. Before passing to the details we will fix some notation. From [13] we recall that if $X = \overline{\bigcup_n F_n}$ where F_n is an increasing sequence of finite dimensional subspaces and if $d(F_n, l_\infty^{d_n}) \leq \lambda$ then X is a $\mathcal{L}_{\infty, \lambda+\varepsilon}$ space for all $\varepsilon > 0$. The sequence F_n will be constructed in the following way.

For each n the space E_n is the subspace of l_∞ spanned by the first d_n coordinates, i.e.

$E_n = \{x \in l_\infty \mid x_k = 0 \text{ for } k > d_n\}$. The space F_n is then an extension of E_n, more precisely there is an isomorphism $i_n\colon E_n \to l_\infty$ such that $\pi_n \circ i_n$ is the identity on E_n and such that $F_n = i_n(E_n)$. Here π_n means the natural projection of l_∞ in E_n, i.e. $\pi_n(x)$ is the restriction of x to the first d_n coordinates. To construct i_n we will use a weak* limit procedure on l_∞. For each $m < n$ we define an injection $i_{m,n}\colon E_m \to E_n$ in such a way that they satisfy

(α) $\pi_m \circ i_{m,n} = \mathrm{id}_{E_m}$ for $m < n$,

(β) $i_{m,n} \circ i_{l,m} = i_{l,n}$ if $l < m < n$.

The mapping i_n is then defined as $i_n(x) = \lim_{k\to\infty} i_{n,k}(x)$. We now pass to the technical description of $i_{m,n}$. First we give the general argument and afterwards we give details for small n.

Let $\lambda > 1$ be fixed and take $\delta > 0$ such that $1 + 2\delta\lambda \leqslant \lambda$. This δ is fixed for the complete construction. Let $d_1 = 1$, let d_m ($m \leqslant n$) be known, let $i_{m,l}$ ($m < l \leqslant n$) be constructed such that they satisfy (α) and (β). For $m < n$; $1 \leqslant i < d_m$; $1 \leqslant j \leqslant d_n$; $\varepsilon' = \pm 1$; $\varepsilon'' = \pm 1$, define the functional $f_{m,i,j,\varepsilon',\varepsilon''} \in E_n^*$ as follows:

$$f_{m,i,j,\varepsilon',\varepsilon''}(x) = \varepsilon' x_i + \delta\varepsilon''(x - i_{m,n}\pi_m(x))_j.$$

Consider the set of functionals

$$\mathfrak{F}_n = \{f_{m,i,j,\varepsilon',\varepsilon''} \mid m < n;\ 1 \leqslant i \leqslant d_m;\ 1 \leqslant j \leqslant d_n;\ \varepsilon' = \pm 1;\ \varepsilon'' = \pm 1\}.$$

Let $d_{n+1} = d_n + \operatorname{card}(\mathfrak{F}_n)$ and enumerate the elements of $\mathfrak{F}_n$ as $g_{d_n+1}, \ldots, g_{d_{n+1}}$. The mapping $i_{n,n+1}\colon E_n \to E_{n+1}$ is now defined as

$$i_{n,n+1}(x) = (x_1, x_2, \ldots, x_{d_n}, g_{d_n+1}(x), g_{d_n+2}(x), \ldots, g_{d_{n+1}}(x), 0, \ldots, 0, \ldots)$$

We also put $i_{m,n+1} = i_{n,n+1} \circ i_{m,n}$ for $m < n$. The properties (α) and (β) remain trivially verified. The heart of the example lies in the metric properties of the injection. Before studying these properties we give details for $n = 1, 2, 3$.

(i) $n = 1$. There is no possible value of m and hence $d_2 = d_1 = 1$ and $i_{1,2} = \mathrm{id}_{E_1}$.

(ii) $n = 2$. Possible value of $m = 1$, possible value of $i = 1$, possible value of $j = 1$. It follows that card $(\mathfrak{F}_2) = 4$ and $d_3 = 5$.

$$f_{1,1,1,+1,+1}(x) = x_1 + \delta(x - \pi_1(x))_1 = x_1$$

$$f_{1,1,1,+1,-1}(x) = x_1$$

$$f_{1,1,1,-1,+1}(x) = -x_1 + \delta(x - \pi_1(x))_1 = -x_1$$

$$f_{1,1,1,+1,-1}(x) = -x_1.$$

(iii) $n=3$. Possible values of $m=1, 2$. In both cases there is one value of i and 5 values of j hence there are 40 possibilities. The number $d_4=d_3+40=45$. As a typical element we evaluate

$$\begin{aligned} f_{1,1,4,\varepsilon',\varepsilon''}(x) &= \varepsilon' x_1 + \delta\varepsilon''(x - i_{1,3}(x))_4 \\ &= \varepsilon' x_1 + \delta\varepsilon''(x_4 - (-x_1)) \\ &= (\varepsilon' + \delta\varepsilon'')x_1 + \delta\varepsilon'' x_4. \end{aligned}$$

The following lemma is crucial in the construction of the space X. In particular the estimate (2) will give that X is a $\mathcal{L}_\infty$ space. The estimate (5) will give that X is a RN space with the Schur property.

LEMMA 4.1. *$i_{m,n}$ $(m<n)$ satisfy:*

(1) $d(E_n, l_\infty^{d_n}) = 1$,

(2) $\|i_{m,n}\| \leqslant \lambda$,

(3) $\pi_m \circ i_{m,n} = \mathrm{id}_{E_m}$,

(4) $i_{m,n} \circ i_{l,m} = i_{l,n}$ $\quad (l<m<n)$,

(5) *for all $x \in E_n$ and all $m<n$ we have*

$$\|i_{n,n+1}(x)\| \geqslant \|\pi_m(x)\| + \delta\|x - i_{m,n}\pi_m(x)\|.$$

Proof. (1), (3) and (4) are obvious from the construction.

(5): For $m<n$, the elements $i, j, \varepsilon', \varepsilon''$ can be chosen such that $\varepsilon' x_i = \|\pi_m(x)\|$ and $\varepsilon''(x - i_{m,n}\pi_m(x))_j = \|x - i_{m,n}\pi_m(x)\|$. Hence

$$\|i_{n,n+1}(x)\| \geqslant \|\pi_m(x)\| + \delta\|x - i_{m,n}\pi_m(x)\|.$$

(2): Since $\|i_{1,2}\| = 1$ we can proceed by induction. Suppose we already have $\|i_{m,n}\| \leqslant \lambda$ for all $m<n$. We first prove that $\|i_{n,n+1}\| \leqslant \lambda$.

To see this take $f_{m,i,j,\varepsilon',\varepsilon''} \in \mathfrak{F}_n$ and observe that by induction we have

$$|f(x)| \leqslant \|\pi_m(x)\| + \delta\|x - i_{m,n}\pi_m(x)\| \leqslant \|x\| + \delta(\|x\| + \lambda\|x\|) \leqslant (1+2\delta\lambda)\|x\| \leqslant \lambda\|x\|.$$

By definition of $i_{n,n+1}$ we then have $\|i_{n,n+1}\| \leqslant \lambda$. Let now $m<n$ and calculate $i_{m,n+1}$. Since $i_{m,n+1} = i_{n,n+1} i_{m,n}$ we have that

$$\|i_{m,n+1}\| = \max\{\|i_{m,n}\|;\ \|f i_{m,n}\| \text{ where } f \in \mathfrak{F}_n\}.$$

Since also $\|i_{m,n}\| \leqslant \lambda$ by induction, it remains to prove $\|f i_{m,n}\| \leqslant \lambda$ for all $f \in \mathfrak{F}_n$. Let now $f = f_{m',i,j,\varepsilon',\varepsilon''} \in F_n$. Then

$$f i_{m,n}(x) = f_{m',i,j,\varepsilon',\varepsilon''}(i_{m,n}(x)) = \varepsilon'(\pi_{m'} i_{m,n}(x))_i + \varepsilon''\delta(i_{m,n}(x) - i_{m',n}\pi_{m'} i_{m,n}(x))_j.$$

Hence

$$\|fi_{m,n}\| \leqslant \|\pi_{m'} i_{m,n}\| + \delta \|i_{m,n} - i_{m',n} \pi_{m'} i_{m,n}\|.$$

We distinguish the two cases $m' \leqslant m$ and $m' > m$. If $m' \leqslant m$ we have $\pi_{m'} i_{m,n} = \pi_{m'}$ and hence

$$\begin{aligned} \|fi_{m,n}\| &\leqslant \|\pi_{m'}\| + \delta(\|i_{m,n}\| + \|i_{m',n} \circ \pi_{m'}\|) \\ &\leqslant 1 + \delta(\lambda + \lambda) \leqslant \lambda. \end{aligned}$$

If $m' > m$ we have $\pi_{m'} i_{m,n} = \pi_{m'} i_{m',n} i_{m,m'}$ and $i_{m',n} \pi_{m'} i_{m,n} = i_{m',n} i_{m,m'} = i_{m,n}$. Hence $\|fi_{m,n}\| \leqslant \|i_{m,n}\| + 0 \leqslant \lambda$. Q.E.D.

Fix now n and E_n. For each $k > n$ the function $i_{n,k}$: $E_n \to E_k$ is constructed as above and is injective. At each stage new coordinates are added to the existing ones and for each n and all $x \in E_n$ the elements $i_{n,k}(x)$, $k > n$ are uniformly bounded by $\lambda \|x\|$, hence the limit $\lim_{k\to\infty} i_{n,k}(x)$ exists in the weak* topology of l_∞ (i.e. for the topology $\sigma(l_\infty, l_1)$). Let us put $i_n(x) = \lim_{k\to\infty} i_{n,k}(x)$ and let F_n be the image of i_n, i.e. $F_n = i_n(E_n)$.

LEMMA 4.2. *The following properties hold:*

(6) $i_m = i_n i_{m,n}$ *for all* $m < n$,

(7) $F_m \subset F_n$ *for* $m < n$,

(8) $\|i_n\| \leqslant \lambda$,

(9) $d(E_n, F_n) = d(F_n, l_\infty^{d_n}) \leqslant \lambda$,

(10) *for* $x \in F_n$ *we have for all* $m < n$: $\|x\| \geqslant \|\pi_m(x)\| + \delta \|x - i_m \pi_m(x)\|$.

Proof. (6): For $x \in E_m$ we have $i_n i_{m,n}(x) = \lim_{k\to\infty} i_{n,k} i_{m,n}(x) = \lim_{k\to\infty} i_{m,k}(x) = i_m(x)$,

(7): $F_m = i_m(E_m) = i_m i_{m,n}(E) \subset i_n(E_n) = F_n$ for $m < n$,

(8): Since $\|i_{n,k}\| \leqslant \lambda$ for all k we have that $\|i_n\| \leqslant \lambda$,

(9): Since $\pi_n i_n = \mathrm{id}_{E_n}$ we obtain $d(E_n, F_n) \leqslant \|\pi_n\| \cdot \|i_n\| = \lambda$,

(10): For $x \in F_n$ we have $\|x\| = \lim_k \|i_{n,k}(x)\|$. In particular for $k \geqslant n$,

$$\begin{aligned} \|x\| &\geqslant \|i_{n,k+1} \pi_n(x)\| \\ &\geqslant \|\pi_m i_{n,k+1} \pi_n(x)\| + \delta \|i_{n,k+1} \pi_n(x) - i_{m,k+1} \pi_m i_{n,k+1} \pi_n(x)\| \\ &\geqslant \|\pi_m(x)\| + \delta \|\pi_{k+1}(x) - i_{m,k+1} \pi_m(x)\|. \end{aligned}$$

By passing to the limit:

$$\|x\| \geqslant \|\pi_m(x)\| + \delta \|x - i_m \pi_m(x)\|. \qquad \text{Q.E.D.}$$

Let us now put X equal to the closure of $\bigcup_{n\geq 1} F_n$. The space X can also be seen as the direct limit of the system

$$E_1 \xrightarrow{i_{1,2}} E_2 \xrightarrow{i_{2,3}} E_3 \longrightarrow \dots \longrightarrow E_n \xrightarrow{i_{n,n+1}} E_{n+1} \longrightarrow \dots .$$

The estimates $\|i_{n,n+1} \circ i_{n-1,n} \circ \dots \circ i_{2,1}\| \leq \lambda$ or the estimates (9) give that X is a $\mathcal{L}_{\infty,\lambda+\varepsilon}$ space for all $\varepsilon > 0$. The estimate (10) of Lemma 4.2 can now be restated as.

COROLLARY 4.3. *For all $x \in X$ and all m we have*

$$\|x\| \geq \|\pi_m(x)\| + \delta\|x - i_m \pi_m(x)\|.$$

Proof. First let $x \in \bigcup_n F_n$, i.e. suppose $x \in F_n$ for some n. If $m < n$ then the inequality is precisely estimate (10). If $m \geq n$ then $x = i_m \pi_m(x)$ hence the inequality is trivial. For $x \in X$ we proceed by a limit argument. Q.E.D.

THEOREM 4.4. *X is a $\mathcal{L}_\infty$ space having the Schur property.*

Proof. The fact that X is a $\mathcal{L}_\infty$ space is already observed above. Let us prove the Schur property. We will prove that if $x_k \in X$, $\|x_k\| = 1$, $x_k \to 0$ coordinatewise then there is a subsequence $(y_n)_{n\geq 1}$ of $(x_k)_{k\geq 1}$ such that y_k is equivalent to the usual basis of l_1. This will prove the Schur property as well as the statement that every infinite dimensional subspace of X contains a subspace isomorphic to l_1 (both properties are related by [15] but in this case it is easy to prove them explicitly). To see that the above property is sufficient we proceed as follows. Let z_k be a sequence in X such that $z_k \to 0$ weakly, we have to prove that $\|z_k\| \to 0$. If this is not the case we may suppose that $\|z_k\| > \varepsilon > 0$ for some and all k (eventually we take a subsequence). Since $\|z_k\| > \varepsilon > 0$ we have that $\|z_k\|^{-1}$ is bounded above and hence $z_k \|z_k\|^{-1} = x_k \to 0$ weakly. Clearly $x_k \to 0$ coordinatewise and $\|x_k\| = 1$. By the statement above x_k contains a subsequence y_n equivalent to the l_1 basis. Since $x_k \to 0$ weakly we also have $y_n = 0$ weakly but this is a contradiction to the fact that y_n is equivalent to the usual basis of l_1. Let now Z be an infinite dimensional subspace of X. Since Z is infinite dimensional there is a bounded sequence z_k in Z and $\varepsilon > 0$ such that $\|z_k - z_1\| > \varepsilon$ for $k \neq 1$. Since z_k is a bounded sequence we may suppose (eventually take a subsequence) that for each coordinate i we have $z_{k,i}$ is convergent. Put now $x_k = (z_{2k} - z_{2k+1})\|z_{2k} - z_{2k+1}\|^{-1}$. By construction $x_k \in Z$, $\|x_k\| = 1$ and $x_k \to 0$ coordinatewise.

By our statement z_k contains a subsequence $y_n \in Z$, equivalent to the usual l_1 basis. Hence Z contains an l_1 copy.

We now construct the subsequence y_k. For each k take $\varepsilon_k>0$ such that

$$\prod_{j=1}^{\infty}(1-\varepsilon_j)>\tfrac{1}{2}$$

and

$$\sum_{p=1}^{\infty}\varepsilon_p\leqslant\frac{\delta}{4(1+\delta)}.$$

Inductively we construct $1=n_1<n_2<\ldots$ and $s_1<s_2<\ldots$ so that

$$\|\pi_{s_k}(x)\|\geqslant(1-\varepsilon_{k+1})\|x\| \quad \text{for } x\in\operatorname{span}(x_{n_1},\ldots,x_{n_k})$$

$$\|x_{n_p}-i_{s_k}\pi_{s_k}(x_{n_p})\|\leqslant\varepsilon_k \quad \text{for } p\leqslant k$$

$$\|i_{s_p}\pi_{s_p}(x_{n_k})\|\leqslant\varepsilon_k \quad \text{for } p<k.$$

Put now $y_k=x_{n_k}$ and let $a_1,\ldots,a_k$ be real numbres. From Corollary 4.3 it follows that

$$\begin{aligned}
\|a_1y_1+\ldots+a_ky_k\| &\geqslant \|\pi_{s_k}(a_1y_1+\ldots+a_ky_k)\|\\
&\geqslant \|\pi_{s_{k-1}}(a_1y_1+\ldots+a_ky_k)\|\\
&\qquad +\delta\|(a_1y_1+\ldots+a_ky_k)-i_{s_{k-1}}\pi_{s_{k-1}}(a_1y_1+\ldots+a_ky_k)\|\\
&\geqslant \|\pi_{s_{k-1}}(a_1y_1+\ldots+a_{k-1}y_{k-1})\|-|a_k|\varepsilon_k+\delta(1-\varepsilon_k)|a_k|-\sum_{p\leqslant k}\delta|a_p|\varepsilon_{k-1}\\
&\geqslant (1-\varepsilon_k)\|a_1y_1+\ldots+a_{k-1}y_{k-1}\|+\delta(1-\varepsilon_k)|a_k|-(1+\delta)\varepsilon_{k-1}\sum_{p\leqslant k}|a_p|.
\end{aligned}$$

Inductively

$$\begin{aligned}
\|a_1y_1+\ldots+a_ky_k\| &\geqslant \delta\left(\sum_{j=1}^{k}|a_j|\prod_{i=j}^{k}(1-\varepsilon_i)\right)-(1+\delta)\left(\sum_{p\leqslant k}\varepsilon_p\right)\left(\sum_{p\leqslant k}|a_p|\right)\\
&\geqslant \frac{\delta}{4}\sum_{j=1}^{k}|a_j|.
\end{aligned}$$

This proves that (y_k) is equivalent to the usual l_1 basis. Q.E.D.

Remarks. If Z is a $\mathcal{L}_{\infty,\mu}$ space for all $\mu>1$ then it turns out that Z^* is isometric to an L_1 space. In this case Z is called predual of L_1. In [17] Zippin has shown that such a space contains a subspace isometric to c_0. In [10], [11], [12] and [13] it is conjectured that

(i) If a Banach space V has the W.C.E.P. then V is finite dimensional,

(ii) every $\mathcal{L}_\infty$ space contains a subspace isomorphic to c_0,

(iii) every $\mathcal{L}_\infty$ space is isomorphic to a predual of L_1,

(iv) every $\mathcal{L}_\infty$ space is isomorphic to a quotient of a $C(K)$ space.

From Theorem 2.4 it follows that (i) is the weakest of all four conjectures. The Banach space X constructed above satisfies the hypothesis of Theorem 2.4 and hence has the WCEP. This proves that all four conjectures are wrong. From the remarks made in chapter 2 it also follows that X^* is weakly sequentially complete (X^* is isomorphic to an L_1 space). Since X is a Schur space it is weakly sequentially complete. As far as we know this space is the first non reflexive space such that both X and X^* are weakly sequentially complete.

THEOREM 4.5. *The Banach space X has the Radon-Nikodym property.*

Proof. Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space and let μ: $\mathcal{A}\to X$ be a $\mathbf{P}$-continuous measure. We will show that μ can be approximated by "finite dimensional" measures.

For every set $A\in\mathcal{A}$ and every n we have

$$\|\mu(A)\| \geqslant \|\pi_n\mu(A)\| + \delta\|\mu(A)-i_n\pi_n\mu(A)\|.$$

Taking the supremum over all partitions we obtain

$$|\mu| \geqslant |\pi_n\mu| + \delta|\mu - i_n\pi_n\mu|.$$

Let now $A_1, ..., A_k$ be partition of Ω such that

$$|\mu| \leqslant \sum_{p=1}^{k} \|\mu(A_p)\| + \varepsilon/2.$$

Let n_0 be large enough so that

$$\sum_{p=1}^{k} \|\pi_n\mu(A_p)\| \geqslant \sum_{p=1}^{k} \|\mu(A_p)\| - \varepsilon/2 \geqslant |\mu| - \varepsilon$$

for all $n\geqslant n_0$. Since $|\pi_n\mu| \geqslant \sum_{p=1}^{k}\|\pi_n\mu(A_p)\|$ we obtain $|\pi_n\mu| \geqslant |\mu| - \varepsilon$ and hence $|\mu - i_n\pi_n\mu| \leqslant \varepsilon\delta^{-1}$ for all $n\geqslant n_0$. This shows that $\lim i_n\pi_n\mu = \mu$ in variation norm. Q.E.D.

Remark 1. Combining Theorem 4.5 and Theorem 2.5 gives that X is a separable RN space which does not imbed in a separable dual space. Also X is not complemented in its second dual space.

Remark 2. The proof that X has the Schur property is related to the following concept.

Definition (see [2]). A sequence $(\mathcal{G}_i)$ of finite dimensional subspaces of a Banach space Z is an l_1-skipped blocking decomposition of Z provided following conditions are fulfilled:

(a) $Z=[\mathcal{G}_i]_{i=1}^{\infty}$ = closed linear span generated by $(\mathcal{G}_i)$, $i=1,2,\ldots$,

(b) $\mathcal{G}_i \cap [\mathcal{G}_j]_{j\neq i}=\{0\}$ for all i,

(c) If m_k and n_k are sequences of positive integers so that $m_k<n_k+1<m_{k+1}$, then the sequence of spaces $F_k=[\mathcal{G}_i]_{i=m_k}^{n_k}$ is an l_1 decomposition of $[F_k]_{k=1}^{\infty}$.

In [2] it is proved that if Z has an l_1-skipped decomposition then Z has the strong Schur property and Z is an RN space. The proof of Theorem 4.4 shows in fact that X has an l_1-skipped decomposition. We prefer to include a proof of the RN property since the results of [2] are much more technical than the given proof.

Remark 3. The idea of the proof of Theorem 4.5 is due to Uhl. The original proof involved martingale theory and was a little more complicated, although it used the same principle.

5. Another example

A look at the example of the preceding paragraph might suggest that a $\mathcal{L}_\infty$ space either contains a copy of c_0 space or a copy of l_1. The Banach space Y constructed in this chapter does not contain subspaces isomorphic to c_0 or to l_1. It turns out that the space Y is somewhat reflexive, i.e. every infinite dimensional subspace Z of Y contains an infinite dimensional subspace which is reflexive.

The construction follows the same line as in the preceding chapter. Again the same numbers $(d_n)_{n\geq 1}$ are introduced. The spaces E_m are the same as in chapter 4. The functionals $f\in\mathfrak{F}_n$ are defined in a similar way, for $m<n$; $1\leqslant i\leqslant d_m$; $1\leqslant j\leqslant d_n$; $\varepsilon'=\pm 1$, $\varepsilon''=\pm 1$, we define

$$f_{m,i,j,\varepsilon',\varepsilon''}(x)=\varepsilon' a x_i+\varepsilon'' b(x-i_{mn}\pi_m(x))_j \quad \text{for all } x\in E_n.$$

The numbers a and b are chosen so that

(1) $0<b<a<1$,

(2) $a+2b\lambda\leqslant\lambda$; hence $a+b(1+\lambda)\leqslant\lambda$,

(3) $a+b>1$.

The existence of such numbers follows from the inequality $\lambda>1$. We define $i_{n,n+1}$: $E_n\to E_{n+1}$ as in chapter 4 and $i_{m,n+1}$ is defined as $i_{m,n+1}=i_{n,n+1}i_{m,n}$ for $m<n$.

LEMMA 5.1. *The following properties hold:*

(1) $d(E_n, l_\infty^{d_n})=1$,

(2) $\|i_{m,n}\|\leqslant\lambda$ *for all* $m<n$,

(3) $\pi_m i_{m,n} = \mathrm{id}_{E_m}$ *for all* $m<n$,

(4) $i_{m,n} i_{l,m} = i_{l,n}$ *for* $l<m<n$,

(5) *for* $x \in E_n$:

$$\|i_{n,n+1}(x)\| = \max \begin{cases} \|x\| \\ a\|\pi_m(x)\| + b\|x - i_{m,n}\pi_m(x)\|;\ m<n. \end{cases}$$

Proof. (1), (3) and (4) are obvious. (2) follows from $\|i_{n,n+1}(x)\| = \max_{f\in\mathfrak{F}_n}(\|x\|, |f(x)|)$ and $|f(x)| \leqslant a\|x\| + b\|x - i_{m,n}\pi_m(x)\| \leqslant a + b(1+\lambda) \leqslant \lambda$.

The proof in the general case ($m<n$) as well as the proof of (5) are now done in the same way as in Lemma 4.1. Q.E.D.

As in chapter 4 we now define i_m: $E_n \to l_\infty$ and we put $F_n = i_n(E_n)$.

LEMMA 5.2. *The following properties hold:*

(6) $i_m = i_n i_{m,n}$ for $m<n$,

(7) $F_m \subset F_n$ *for* $m<n$,

(8) $\|i_n\| \leqslant \lambda$,

(9) $d(F_n, l_\infty^{d_n}) \leqslant \lambda$,

(10) *for* $x\in F_n$ *we have*

$$\|x\| = \max \begin{cases} \|\pi_n(x)\| \\ a\|\pi_m(x)\| + b\|x - i_m\pi_m(x)\|;\ m<n. \end{cases}$$

Proof. Only the estimate need to be proved. For $x \in F_n$ it is clear that

$$\|x\| = \lim_{k\to\infty} \|i_{n,k}(x)\|.$$

Let us fix $k>n$ and let us calculate $\|\pi_{k+1}(x)\|$. Since

$$\pi_{k+1}(x) = i_{n,k+1}\pi_n(x) = i_{k,k+1} i_{n,k}\pi_n(x) = i_{k,k+1}\pi_k(x)$$

it follows that

$$\|\pi_{k+1}(x)\| = \max \begin{cases} \|\pi_k(x)\| \\ a\|\pi_m(x)\| + b\|\pi_k(x) - i_{m,k}\pi_m(x)\|, \quad m\leqslant k. \end{cases}$$

But for $k>m\geqslant n$ we have $\pi_k(x) = i_{m,k}\pi_m(x)$ and hence we have

$$\|\pi_{k+1}(x)\| = \max \begin{cases} \|\pi_k(x)\| \\ a\|\pi_m(x)\| + b\|\pi_k(x) - i_{m,k}\pi_m(x)\|, \quad m<n. \end{cases}$$

An easy induction on k then shows

$$\|\pi_{k+1}(x)\| = \max \begin{cases} \|\pi_n(x)\| \\ a\|\pi_m(x)\| + b\|\pi_k(x) - i_{m,k}\pi_m(x)\|, \quad m < n. \end{cases}$$

Letting $k \to \infty$ we obtain the desired estimate. Q.E.D.

Let now Y be the direct limit of the system

$$E_1 \xrightarrow{i_{1,2}} E_2 \xrightarrow{i_{2,3}} E_3 \longrightarrow \dots,$$

i.e. Y = closure of $\bigcup_{n\geq 1} F_n$ in l_∞. Again Y is a $\mathcal{L}_\infty$ space.

LEMMA 5.3. *Y does not contain a copy of l_1.*

Proof. Let $e_n \in Y$ be a sequence equivalent to the usual basis of l_1. First suppose (eventually take a subsequence) that e_n is weak* convergent. Take now $y_n = e_{2n} - e_{2n-1}$. This sequence is still an l_1 basis but $\pi_m(y_n) \to 0$ as $n \to \infty$. By density of $\bigcup_{n\geq 1} F_n$ we can replace y_n by a sequence u_n such that $\pi_m(u_n) = 0$ for $m < n$ and $u_n \in F_{m'_n}$, where $m'_1 < m'_2 < \dots$ is a good subsequence. (We have to pass to a subsequence and we have to apply Theorem 10.1 p. 93 of [16].) By a result of James [7] there is a block basis z_n of (y_k) such that

(1) $\|z_n\| = 1$,
(2) $\pi_m(z_n) = 0$ for $m < n$,
(3) $z_k \in F_{m''_k}$ where $m''_1 < m''_2 < \dots$,
(4) $\left\| \sum_{k=1}^{N} a_k z_k \right\| \geq (1-\varepsilon) \sum_{n=1}^{N} |a_n|$.

Here $\varepsilon > 0$ is chosen such that $4\varepsilon < 1 - a$. Let now $x_1 = z_1$ and let $m_1 > m''_1$ be such that $\|\pi_{m_1}(z_1)\| > 1 - \varepsilon$. Let $x_2 = z_{k_2}$ where k_2 is such that $\pi_{m_1}(z_k) = 0$ for $k \geq k_2$. Take now m_2 such that $m_2 > m''_{k_2}$, $m_2 > m_1$ and $\|\pi_{m_2}(x_2)\| > 1 - \varepsilon$.

Take then k_3 such that $\pi_{m_2}(z_k) = 0$ for $k \geq k_3$. Let $x_3 = z_{k_3}$ and take n such that $\|\pi_n(x_3)\| > 1 - \varepsilon$ and $x_3 \in F_n$. We now calculate the norm of $x = x_1 + x_2 + x_3$. Of course by James' construction $\|x\| \geq 3 - 3\varepsilon$. Successive applications of Lemma 5.2 will give upper bounds on $\|x\|$.

$$\|x\| = \max \begin{cases} \|\pi_n(x)\| \\ a\|\pi_m(x)\| + b\|x - i_m\pi_m(x)\|, \quad m < n. \end{cases}$$

(α) $m \leqslant m_1$:

$$\begin{aligned} &a\|\pi_m(x)\| + b\|x - i_m\pi_m(x)\| \\ &\quad = a\|\pi_m(x_1+x_2+x_3)\| + b\|x_1+x_2+x_3 - i_m\pi_m(x_1+x_2+x_3)\| \\ &\quad \leqslant a\|\pi_m(x_1)\| + b\|x_1 - i_m\pi_m(x_1)\| + b\|x_2+x_3\| \\ &\quad \leqslant \|x_1\| + b\|x_2+x_3\| \\ &\quad \leqslant 1+2b. \end{aligned}$$

(β) $m_1 < m \leqslant m_2$:

$$\begin{aligned} &a\|\pi_m(x)\| + b\|x - i_m\pi_m(x)\| \\ &\quad = a\|\pi_m(x_1+x_2+x_3)\| + b\|x_1+x_2+x_3 - i_m\pi_m(x_1+x_2+x_3)\| \\ &\quad = a\|\pi_m(x_1+x_2)\| + b\|x_1+x_2 - i_m\pi_m(x_1+x_2)\| + b\|x_3\| \\ &\quad \leqslant \|x_1+x_2\| + b\|x_3\| \\ &\quad \leqslant 2+b. \end{aligned}$$

(γ) $m_2 < m < n$:

$$\begin{aligned} &a\|\pi_m(x)\| + b\|x - i_m\pi_m(x)\| \\ &\quad = a\|\pi_m(x_1+x_2+x_3)\| + b\|x_1+x_2+x_3 - i_m\pi_m(x_1+x_2+x_3)\| \\ &\quad \leqslant a\|\pi_m(x_1+x_2)\| + a\|\pi_m(x_3)\| + b\|x_1+x_2 - i_m\pi_m(x_1+x_2)\| + b\|x_3 - i_m\pi_m(x_3)\| \\ &\quad \leqslant a\|x_1+x_2\| + \|x_3\| \quad (\text{since } x_1+x_2 = i_m\pi_m(x_1+x_2)) \\ &\quad \leqslant 2a+1. \end{aligned}$$

(δ) If the norm of x is at least $3-3\varepsilon$ then this norm cannot be attained in the first d_{m_2} coordinates.

Indeed for these coordinates x_3 is zero. Hence we have to give bounds for the coordinates of x, situated between $d_{m_2}+1$ and d_n. These coordinates are bounded by $\|\pi_n(x_3)\|$ + coordinates of (x_1+x_2). From Lemma 5.2 we know that these coordinates are bounded by quantities of the form

$$a\|\pi_m(x_1+x_2)\| + b\|\pi_n(x_1+x_2) - i_{mn}\pi_m(x_1+x_2)\|,$$

where $m < m_2$. We distinguish two cases:

($\delta 1$) $m \leqslant m_1$:

$$\begin{aligned} &a\|\pi_m(x_1+x_2)\| + b\|\pi_n(x_1+x_2) - i_{m,n}\pi_m(x_1+x_2)\| \\ &\quad \leqslant a\|\pi_m(x_1)\| + b\|\pi_n(x_1) - i_{m,n}\pi_m(x_1)\| + b\|\pi_n(x_2)\| \\ &\quad \leqslant \|x_1\| + b\|\pi_n(x_2)\| \\ &\quad \leqslant 1+b. \end{aligned}$$

($\delta 2$) $m_1 < m \leqslant m_2$:

$$\begin{aligned} a\|\pi_m(x_1+x_2)\| + b\|\pi_n(x_1+x_2) - i_{m,n}\pi_m(x_1+x_2)\| \\ \leqslant a\|\pi_m(x_1)\| + a\|\pi_m(x_2)\| + b\|\pi_n(x_2) - i_{m,n}\pi_m(x_2)\| \\ \leqslant a\|x_1\| + \|x_2\|. \end{aligned}$$

Summarizing (α), (β), (γ), ($\delta 1$) and ($\delta 2$) gives:

$$\|x\| \leqslant \max(1+2b,\ 2+b,\ 1+2a,\ 2+b,\ 2+a) \leqslant 2+a < 3-4\varepsilon \text{ by the choice of } \varepsilon.$$

This inequality contradicts $\|x\| \geqslant 3-3\varepsilon$ and hence proves Lemma 5.5. Q.E.D.

LEMMA 5.4. *Y is a Radon-Nikodym space.*

Proof. We again use Uhl's argument. Let μ: $\mathcal{A} \to Y$ be a vector measure with bounded variation. Put $\nu = \mu - i_n\pi_n\mu$. As in Theorem 4.5 we have $|\nu| \geqslant a|\pi_m\nu| + b|\nu - i_m\pi_m\nu|$. Let $A_1, \ldots, A_k$ be a partition of Ω such that $|\nu| \leqslant \sum_{p=1}^{k}\|\nu(A_p)\| + \varepsilon/2$ and let m_0 be large enough so that for $m \geqslant m_0$: $\sum_p \|\pi_m\nu(A_p)\| \geqslant \sum_{p=1}^{k}\|\nu(A_k)\| - \varepsilon/2$. It follows that $|\pi_m\nu| \geqslant \sum_{p=1}^{k}\|\pi_m\nu(A_p)\| \geqslant |\nu| - \varepsilon$, and hence $|\nu - i_m\pi_m\nu| \leqslant ((1-a)/b)|\nu| + a\varepsilon/b$ for all $m \geqslant m_0$. But for $m \geqslant n$ we have $\nu - i_m\pi_m\nu = \mu - i_n\pi_n\mu - i_m\pi_m\mu + i_m\pi_m i_n\pi_n\mu = \mu - i_m\pi_m\mu$ and hence for $m \geqslant \max(m_0, n)$:

$$|\mu - i_m\pi_m\mu| \leqslant ((1-a)/b)|\mu - i_n\pi_n\mu| + a\varepsilon/b.$$

Hence

$$\overline{\lim_{m\to\infty}}\ |\mu - i_m\pi_m\mu| \leqslant ((1-a)/b)|\mu - i_n\pi_n\mu| + a\varepsilon/b$$

and

$$\overline{\lim_{m\to\infty}}\ |\mu - i_m\pi_m\mu| \leqslant ((1-a)/b)\ \overline{\lim_{n\to\infty}}\ |\mu - i_n\pi_n\mu| + a\varepsilon/b.$$

Since this inequality holds for all $\varepsilon > 0$ and since $a+b > 1$ we obtain $\lim_{m\to\infty}|\mu - i_m\pi_m\mu| = 0$. Q.E.D.

LEMMA 5.5. *Y is somewhat reflexive.*

Proof. Let Z be an infinite dimensional subspace of Y. We will construct a basic sequence in Z which is shrinking and boundedly complete. From the results in [16] it follows that such a sequence spans a reflexive space. Since Y does not contain a copy of l_1 and since Z is infinite dimensional there is a basic sequence (z_n) in Z such that $\|z_n\| = 1$ and $z_n \to 0$ weakly. We use the perturbation theorem on basis to produce a sequence $(y_n)_{n\geqslant 1}$ equivalent to a subsequence $(z_{n'})$ of z_n and with following properties:

(1) $\|y_n\| = 1$,

(2) there is a sequence $m_1 < m_2 < \ldots$ such that

$$\pi_{m_k}\left(\sum_{s=1}^{t} a_s y_s\right) = \pi_{m_k}\left(\sum_{s=1}^{k} a_s y_s\right) \quad \text{for all } k < t,$$

$$(3)\quad \left\|\pi_{m_k}\left(\sum_{s=1}^{k} a_s y_s - i_m \pi_m\left(\sum_{s=1}^{k} a_s y_s\right)\right)\right\| \geqslant (1-\varepsilon)\left\|\sum_{s=1}^{k} a_s y_s - i_m \pi_m\left(\sum_{s=1}^{k} a_s y_s\right)\right\|$$

for all $m<m_{k-1}$ and all $a_1, \ldots, a_k$. Here $\varepsilon>0$ is chosen so that $\gamma=(a+b)(1-\varepsilon)^2>1$.

We now show that y_n is boundedly complete. Let (a_k) be a sequence of real numbers such that $v_N=\sum_{s=1}^{N} a_s y_s$ is bounded. We have to prove that v_N is convergent or what is the same: v_N is a precompact sequence. Suppose now that $\|v_N\|\leqslant 1$ for all N and that there is $\beta>0$ with

$$\overline{\lim_{N\to\infty}}\, \|v_N - i_m \pi_m v_N\| > \beta. \tag{*}$$

For all $p<s<t$ and all $m<m_{p-1}$ the following estimates hold (Lemma 5.2):

$$\begin{aligned}
&\left\|\sum_{k=1}^{t} a_k y_k - i_m \pi_m\left(\sum_{k=1}^{t} a_k y_k\right)\right\| \\
&\quad \geqslant a\left\|\pi_{m_p}\left(\sum_{k=1}^{t} a_k y_k - i_m \pi_m\left(\sum_{k=1}^{t} a_k y_k\right)\right)\right\| \\
&\qquad + b\left\|\pi_{m_t}\left(\sum_{k=1}^{t} a_k y_k - i_m \pi_m\left(\sum_{k=1}^{t} a_k y_k\right)\right) - i_{m_p, m_t}\pi_{m_p}\left(\sum_{k=1}^{t} a_k y_k - i_m \pi_m\left(\sum_{k=1}^{t} a_k y_k\right)\right)\right\| \\
&\quad = a\left\|\pi_{m_p}\left(\sum_{k=1}^{t} a_k y_k - i_m \pi_m\left(\sum_{k=1}^{t} a_k y_k\right)\right)\right\| \\
&\qquad + b\left\|\pi_{m_t}\left(\sum_{k=1}^{t} a_k y_k\right) - i_{m, m_t}\pi_m\left(\sum_{k=1}^{t} a_k y_k\right) - i_{m_p, m_t}\pi_{n_p}\left(\sum_{k=1}^{t} a_k y_k\right) + i_{m, m_t}\pi_m\left(\sum_{k=1}^{t} a_k y_k\right)\right\| \\
&\quad = a\left\|\pi_{m_p}\left(\sum_{k=1}^{t} a_k y_k - i_m \pi_m\left(\sum_{k=1}^{t} a_k y_k\right)\right)\right\| + b\left\|\pi_{m_t}\left(\sum_{k=1}^{t} a_k y_k - i_{m_p}\pi_{m_p}\left(\sum_{k=1}^{t} a_k y_k\right)\right)\right\| \\
&\quad = a\left\|\pi_{m_p}\left(\sum_{k=1}^{p} a_k y_k - i_m \pi_m\left(\sum_{k=1}^{p} a_k y_k\right)\right)\right\| + b\left\|\pi_{m_t}\left(\sum_{k=1}^{t} a_k y_k - i_{m_p}\pi_{m_p}\left(\sum_{k=1}^{t} a_k y_k\right)\right)\right\|.
\end{aligned}$$

Given m take now p such that

$$\left\|\sum_{k=1}^{p} a_k y_k - i_m \pi_m\left(\sum_{k=1}^{p} a_k y_k\right)\right\| \geqslant \beta(1-\varepsilon),$$

hence also

$$\left\|\pi_{m_p}\left(\sum_{k=1}^{p} a_k y_k - i_m \pi_m\left(\sum_{k=1}^{p} a_k y_k\right)\right)\right\| \geqslant \beta(1-\varepsilon)^2.$$

For this p take now t such that

$$\left\|\sum_{k=1}^{t} a_k y_k - i_{m_p}\pi_{m_p}\left(\sum_{k=1}^{t} a_k y_k\right)\right\| \geqslant \beta(1-\varepsilon),$$

hence also

$$\left\|\pi_{m_t}\left(\sum_{k=1}^{t} a_k y_k - i_m \pi_m\left(\sum_{k=1}^{t} a_k y_k\right)\right)\right\| \geq \beta(1-\varepsilon)^2.$$

From these choices it follows that

$$\left\|\sum_{k=1}^{t} a_k y_k - i_m \pi_m\left(\sum_{k=1}^{t} a_k y_k\right)\right\| \geq (a+b)\,\beta(1-\varepsilon)^2 = \gamma\beta.$$

This inequality proves that (*) holds for $\gamma\beta$ instead of β.

Repeating the argument M times where M is such that $\gamma^M\beta > 1+\lambda$ gives a contradiction with $\|v_N - i_m\pi_m(v_N)\| \leq 1+\lambda$ for all N.

It follows that (y_n) and hence $(z_{n'})$ is a boundedly complete basic sequence.

To construct a shrinking basic sequence we proceed as follows. Since Y is a separable $\mathcal{L}_\infty$ space without a copy of l_1 it follows from [9] that Y^* is isomorphic to l_1 and hence is separable. Since $U = \overline{\text{span}\,(z_{n'})}$ is a subspace of Y, its dual as a quotient of l_1 is also separable. From results of Dean, Singer and Sternbach [4] it follows that U contains a sequence x_n which is a block basic sequence of $(z_{n'})_{n'}$ and which is shrinking. Since $(x_n)_n$ is equivalent to a block basis of $(y_n)_n$ it is also boundedly complete. Q.E.D.

Remark. The Dunford-Pettis property was understood as a property opposite to reflexivity. From this reasoning Davis asked following questions:

(1) Suppose Z is a Banach space without an infinite dimensional reflexive subspace. Is Z a Dunford-Pettis space?

(2) Suppose Z is a somewhat reflexive Banach space. Need Z to fail the Dunford-Pettis property?

Both questions have a negative answer. Question 2 has a negative answer because the Banach space Y is somewhat reflexive and as a $\mathcal{L}_\infty$ space it has the D–P property. (We remark that if Z is a Banach space which has the D–P property and does not contain an l_1 subspace then Z^* is a Schur space.)

Question 1 also has a negative answer. Take for instance the Orlicz function $\varphi(x) = e^{-1/x^2}$ $(x>0)$ and let Z be the Orlicz sequence space construted with φ. It can be seen that Z is not a D–P space but it does not have an infinite dimensional reflexive subspace.

For each a and b we constructed a Banach space $Y_{a,b}$. The aim of the following reasoning is to show that for fixed a the Banach spaces $(Y_{a,b})_b$ are mutually non-isomorphic. This gives a continuum number of separable mutually non isomorphic $\mathcal{L}_\infty$ spaces. (We remark that there are only uncountably many separable mutually non-isomorphic $C(K)$ spaces.) To prove that the spaces $Y_{a,b}$ are not isomorphic we will construct a basic sequence $(e_n)_n$ in $Y_{a,b}$ such that for all n we have the estimate $\|e_1+\dots+e_n\| \leqslant n^\alpha$. Here α is the unique number such that $(a^{1/(1-\alpha)}+b^{1/(1-\alpha)})^{1-\alpha}=1$. On the other hand we show that if $\|x_n\|=1$ and $x_n \to 0$ weakly then there is a subsequence (z_n) and a constant $C>0$ such that $\|z_1+\dots+z_n\| \geqslant Cn^\alpha$. Since for fixed a the parameter α is in one–one correspondence with b, this shows that the spaces $Y_{a,b}$ are not isomorphic. More precisely $Y_{a,b'}$ is not isomorphic to a subspace of $Y_{a,b}$ if $b<b'$. We now pass to the details.

LEMMA 5.6. *There is a sequence of natural numbers $(n_k)_{k\geqslant 1}$ with following properties:*

(1) $d_3<n_1 \leqslant d_4<n_2 \leqslant d_5< \dots \leqslant d_{k+2}<n_k \leqslant d_{k+3} \dots$.

(2) *If $x \in E_m$ and $x_1=x_2=0$ then $i_m(x)_{n_k}=0$ for all $k>m$.*

LEMMA 5.7. *If $x \in E_m$ and $\pi_{m-1}(x)=0$ then $\|i_m(x)\|=\|x\|$.*

Both statements are obvious. Let now e_k' be the element of E_{k+3} defined as $(e_k')_i=1$ if $i=n_k$ and $(e_k')_i=0$ if $i \neq n_k$. Put then $e_k=i_{k+3}(e_k')$. By Lemma 5.7 we have $\|e_k\|=1$.

From the estimates of Lemma 5.2 and from Lemma 5.6 we deduce that the sequence e_k is an unconditional basis (with unconditional constant $=1$) and that the norm of $e_1+\dots+e_N$ is attained in the extension beyond the coordinate d_{N+3}. Hence

$$\begin{aligned}\|e_1+\dots+e_N\| &= \max_{m<N+3} (a\|\pi_m(e_1+\dots+e_N)\|+b\|e_1+\dots+e_N-i_m\pi_m(e_1+\dots+e_N)\|)\\ &\leqslant \max_{m<N+3} (a\|e_1+\dots+e_{m-3}\|+b\|e_{m-2}+\dots+e_N\|)\\ &\leqslant \max_{k<N} (a\|e_1+\dots+e_k\|+b\|e_{k+1}+\dots+e_N\|).\end{aligned}$$

Put now $\gamma_N=\sup_{k\geqslant 0}\|e_{k+1}+\dots+e_{N+k}\|$. The above estimate then gives

(1) $\gamma_N \leqslant \max_{k<N}(a\gamma_k+b\gamma_{N-k})$,

(2) $\gamma_1=1$.

LEMMA 5.8. *If (γ_N) is a sequence of real numbers satisfying (1) and (2) then*

$$\gamma_N \leqslant N^\alpha,$$

where α is the unique number between 0 and 1 satisfying

$$a^{1/(1-\alpha)}+b^{1/(1-\alpha)}=1.$$

Proof. By induction on N. Suppose the estimate holds up to $N-1$. Then

$$\gamma_N \leqslant \max \{ax^\alpha + b(N-x)^\alpha \mid x \text{ real and } 0<x<N\}.$$

Elementary calculus gives that the maximum is attained at $x=N\cdot a^{1/(1-\alpha)}$ and hence $\gamma_N \leqslant N^\alpha$. Q.E.D.

Suppose now that x_n is a sequence in $Y_{a,b}$ such that $\|x_n\|=1$ and $x_n \to 0$ weakly. By taking subsequences and using the perturbation theorem on basic sequences we may suppose that

(1) there is a sequence of natural numbers $s_1 < s_2 \ldots$,
(2) $x_k \in F_{s_k}$ for all k and $\|x_k\|=1$,
(3) $\|\pi_{s_k}(x_l)\|=0$ for $l>k$,
(4) $\|\pi_{s_k}(x)\| \geqslant (1-\varepsilon_k)\|x\|$ for all $x \in \operatorname{span}(x_1, \ldots, x_k)$,
(5) $\prod_{k=1}^\infty (1-\varepsilon_k)>0$ and ε_k is decreasing.

Put now $\delta_N = \inf_k \|x_{k+1}+\ldots+x_{k+N}\|$. Clearly $\delta_1=1$ and by Lemma 5.2,

$$\begin{aligned}\|x_{k+1}+\ldots+x_{k+N}\| &\geqslant a\|\pi_{s_{k+p}}(x_{k+1}+\ldots+x_{k+N})\| \\ &\qquad + b\|x_{k+1}+\ldots+x_{k+N} - i_{s_{k+p}}\pi_{s_{k+p}}(x_{k+1}+\ldots+x_{k+N})\| \\ &\geqslant a\|\pi_{s_{k+p}}(x_{k+1}+\ldots+x_{k+p})\| + b\|x_{k+p+1}+\ldots+x_{k+N}\| \\ &\geqslant a(1-\varepsilon_p)\cdot\delta_p + b\cdot\delta_{N-p}.\end{aligned}$$

Hence

(1) $\delta_N \geqslant \max\limits_{p<N} (a(1-\varepsilon_p)\delta_p + b\delta_{N-p})$,
(2) $\delta_1 = 1$.

LEMMA 5.9. *Under the above conditions there is a constant $C>0$ such that*

$$\delta_N \geqslant CN^\alpha$$

where α is given by $a^{1/(1-\alpha)}+b^{1/(1-\alpha)}=1$.

Proof. To improve clarity we put $\eta = a^{1/(1-\alpha)}<1$ and we put $[x]$ equal to the greatest natural number $\leqslant x$. We also put $\lambda_0(N)=N$ and $\lambda_k(N)=[\lambda_{k-1}(N)\eta]$. Since λ_k is decreasing there is a unique number $k_0(N)$ such that $\lambda_{k_0-1}(N)>1$ and $\lambda_{k_0}(N)=1$. In this case we have $\lambda_{k_0-1}(N)\eta \geqslant 1$ and hence $\lambda_{k_0-1}(N)\geqslant \eta^{-1}$. Inductively we obtain $\lambda_{k_0-j}(N)\geqslant \eta^{-j}$ and $N=\lambda_0(N)\geqslant \eta^{-k_0}$. Since $a>b$ we have that $\eta>\frac{1}{2}$ and hence there is N_0 such that $[N\eta]\geqslant$

$N-[N\eta]$ for all $N \geqslant N_0$. Since $\delta_N>0$ there also is $C'>0$ such that $\delta_N \geqslant C'N^\alpha$ for all $N \leqslant N_0$. Let us put

$$C_N=(1-\varepsilon_{\lambda_1(N)})\dots(1-\varepsilon_{\lambda_{k_0}(N)})\left(1-\frac{1}{\lambda_1(N)}\right)^\alpha\dots\left(1-\frac{1}{\lambda_{k_0-1}(N)}\right)^\alpha C'.$$

By induction we will prove that $\delta_N \geqslant C_N N^\alpha$.

For $N \leqslant N_0$ this is clear by definition of C'. Now suppose that $N>N_0$.

$$\begin{aligned}\delta_N &\geqslant (1-\varepsilon_p)\,a\delta_p+b\delta_{N-p} \quad \text{for all } p<N\\ &\geqslant (1-\varepsilon_{\lambda_1(N)})\,(a\delta_{\lambda_1(N)}+b\delta_{N-\lambda_1(N)})\\ &\geqslant (1-\varepsilon_{\lambda_1(N)})\,(aC_{\lambda_1(N)}\lambda_1(N)^\alpha+bC_{N-\lambda_1(N)}(N-\lambda_1(N))^\alpha).\end{aligned}$$

Since $N \geqslant N_0$ we have $\lambda_1(N)>N-\lambda_1(N)$ and hence $C_{\lambda_1(N)} \leqslant C_{N-\lambda_1(N)}$. We obtain

$$\begin{aligned}\delta_N &\geqslant (1-\varepsilon_{\lambda_1(N)})\,C_{\lambda_1(N)}(a\lambda_1(N)^\alpha+b(N-\lambda_1(N))^\alpha)\\ &\geqslant (1-\varepsilon_{\lambda_1(N)})\,(a(N\eta)^\alpha+b(N-N\eta)^\alpha)\,\frac{\lambda_1(N)^\alpha}{(N\eta)^\alpha}\,C_{\lambda_1(N)}\\ &\geqslant (1-\varepsilon_{\lambda_1(N)})\left(1-\frac{1}{\lambda_1(N)}\right)^\alpha C_{\lambda_1(N)}N^\alpha\\ &\geqslant C_N N^\alpha.\end{aligned}$$

To complete the proof we only need to show that $\inf_N C_N>0$:

$$\begin{aligned}C_N &= (1-\varepsilon_{\lambda_1})\dots(1-\varepsilon_1)\left(1-\frac{1}{\lambda_1}\right)^\alpha\dots\left(1-\frac{1}{\lambda_{k_0-1}}\right)^\alpha C'\\ &\geqslant (1-\varepsilon_{\lambda_1})\dots(1-\varepsilon_1)\,(1-\eta)^\alpha\dots(1-\eta^{k_0-1})^\alpha C'\\ &\geqslant C'\prod_{j=1}^{\infty}(1-\varepsilon_j)\,(1-\eta^j)^\alpha>0.\end{aligned}$$

Q.E.D.

THEOREM 5.10. *If $b'>b$ then $Y_{a,b}$ is not isomorphic to a subspace of $Y_{a,b'}$.*

Proof. If $b<b'$ then $\alpha'>\alpha$ where $a^{1/(1-\alpha)}+b^{1/(1-\alpha)}=1$ and $a^{1/(1-\alpha')}+b'^{1/(1-\alpha')}=1$. Let e_n be the sequence constructed above. Since $\overline{\text{span}}\,(e_n, n\geqslant 1)$ does not contain a copy of l_1 and since e_n is unconditional we obtain $e_n\to 0$ weakly. Suppose now that $Y_{a,b}$ is isomorphic to a subspace Z of $Y_{a,b'}$. Let x_n be the element of Z corresponding to e_n. Since $x_n\to 0$ we also have $z_n=x_n/\|x_n\|\to 0$ weakly. By passing to a subsequence and by applying Lemma 5.9 above we obtain

$$\exists C>0 \text{ such that } \|z_1+\dots+z_N\|\geqslant CN^{\alpha'}.$$

Hence $\exists C''>0$ such that $\|x_1+\dots+x_N\|\geqslant C''N^{\alpha'}$ (note that x_n is unconditional).

On the other hand the estimation of Lemma 5.8 remains true for subsequences of e_n. We obtain,

$$\exists C''' \text{ with } \|x_1 + \ldots + x_N\| \leqslant C''' N^{\alpha},$$

which is a contradiction. Q.E.D.

COROLLARY 5.11. *For all $\lambda > 1$ there is a continuum number of mutually non-isomorphic $\mathcal{L}_{\infty,\lambda}$ spaces.*

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Received February 5, 1979