MINIMAL TRIANGULATIONS ON ORIENTABLE SURFACES

BY

M. JUNGERMAN and G. RINGEL[1]

University of California, Santa Cruz, U.S.A.

Introduction

Let S be a compact 2-manifold. A polyhedron on S is called a *triangulation* if each face of the polyhedron is a triangle with 3 distinct vertices and the intersection of any two distinct triangles is either empty, a single vertex or a single edge (including the two vertices). A triangulation on S is called *minimal* if the number of triangles is minimal. For instance the tetrahedron is a minimal triangulation of the sphere and the well known embedding of the complete graph with 7 vertices in the torus is a minimal triangulation of the torus.

Let $\delta(S)$ be the number of triangles in a minimal triangulation of S. In 1950 at a seminar at the University of Bonn, E. Peschl mentioned the problem of determining $\delta(S)$ for each surface S. The question may well be older than this. In 1955 G. Ringel [9] gave a complete solution if S is non-orientable.

In this paper we present a complete solution of the orientable part of the problem. We prove a formula for $\delta(S_p)$ for the orientable surface S_p of genus p.

The proof of the formula is a problem similar in nature and at least equivalent in complexity to the problem of determining the genus of the complete graph K_n. In both problems one must exhibit triangular embeddings of graphs which are complete or nearly complete (where some edges are missing). In the genus problem for K_n one has to add handles in order to gain the missing edges. In the problem of determining $\delta(S_p)$ the situation is reversed: one must find ways to subtract handles in order to remove edges, while preserving the triangulation.

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1. Lower bound

Let T be a triangulation of S and α_0, α_1, α_2 the number of vertices, edges and faces of T respectively. Then

$$\alpha_0 - \alpha_1 + \alpha_2 = E(S) \tag{1.1}$$

is the Euler characteristic of S. Since T is a triangulation each pair of vertices is joined by at most one edge. So T has no more edges than pairs of vertices:

$$\alpha_1 \leqslant \binom{\alpha_0}{2}. \tag{1.2}$$

Since each face in T is a triangle and each edge is incident with two triangles

$$3\alpha_2 = 2\alpha_1. \tag{1.3}$$

Together with (1.1) we obtain

$$3\alpha_0 - \alpha_1 = 3\mathrm{E}(S). \tag{1.4}$$

From (1.2) it follows that

$$6\alpha_0 - \alpha_0(\alpha_0 - 1) \leqslant 6E(S). \tag{1.5}$$

This quadratic inequality has the solution:

$$\alpha_0 \geqslant \left\{\frac{7 + \sqrt{49 - 24E(S)}}{2}\right\}. \tag{1.6}$$

The symbol $\{x\}$ denotes the smallest integer $\geqslant x$.

So far we have obtained a lower bound (1.6) for the number of vertices in T. Now multiply (1.1) by 2 and use equation (1.3). We obtain

$$\alpha_2 = 2\alpha_0 - 2E(S). \tag{1.7}$$

From now on we assume that T is a minimal triangulation. Then $\alpha_2 = \delta(S)$ and from (1.6) and (1.7) we obtain

$$\delta(S) \geqslant 2\left\{\frac{7 + \sqrt{49 - 24E(S)}}{2}\right\} - 2E(S). \tag{1.8}$$

Ringel [9] has proved that equality holds in (1.8) for all non-orientable surfaces S with two exceptions: Klein's bottle, N_2, and the non-orientable surface N_3 of genus 3. In these two cases he showed that

$$\delta(N_2) = 16 \quad \text{and} \quad \delta(N_3) = 20.$$

We shall prove that equality holds in (1.8) for the orientable surface S_p of genus p for all p with the remarkable exception of the double-torus S_2. In other words, we shall prove the following theorem. Note that $E(S_p) = 2-2p$.

THEOREM 1.1. *Let $\delta(S_p)$ be the number of triangles in a minimal triangulation of S_p. Then*

$$\delta(S_p) = 2\left\{\frac{7+\sqrt{1+48p}}{2}\right\} + 4(p-1) \quad \textit{if } p \neq 2$$

and (1.9)

$$\delta(S_2) = 24.$$

For practical reasons it is convenient to first prove the following theorem.

THEOREM 1.2. *For each pair $(n, t) \neq (9, 3)$ of integers with the property*

$$\left.\begin{array}{l} 4 \leqslant n,\ 0 \leqslant t \leqslant n-6 \\ (n-3)(n-4) \equiv 2t \pmod{12} \end{array}\right\} \tag{1.10}$$

there exists a triangular embedding of a graph with n vertices and

$$\binom{n}{2} - t$$

edges into an orientable surface. Such an embedding also exists for the pair (10, 9) and does not exist for the pair (9, 3).

The two Theorems 1.1 and 1.2 are in fact equivalent. Here we only need to show that Theorem 1.1 follows from Theorem 1.2: We assume Theorem 1.2 and wish to prove Theorem 1.1. Given an integer $p \geqslant 0$, consider the two integers

$$n = \left\{\frac{7+\sqrt{1+48p}}{2}\right\} \quad \text{and} \quad t = \frac{(n-3)(n-4)-12p}{2}. \tag{1.11}$$

We intend to show that (n, t) satisfies (1.10). From (1.11) it follows that

$$\frac{7+\sqrt{1+48p}}{2} \leqslant n < \frac{7+\sqrt{1+48p}}{2} + 1,$$

and so

$$\sqrt{1+48p} \leqslant 2n-7 < \sqrt{1+48p}+2. \tag{1.12}$$

Squaring we obtain

$$12p \leqslant (n-3)(n-4) \tag{1.13}$$

from the first inequality of (1.12) and $(n-4)(n-5)<12p$ from the second one if $p\neq 0$. (If $p=0$ then $n=4$ and $t=0$, so (1.10) is obvious.) It follows that $2t=(n-3)(n-4)-12p<n^2-7n+12-(n^2-9n+20)=2n-8$ so that $t<n-4$. This is not yet what we want. We shall have to exclude the possibility that $t=n-5$. If $t=n-5$ we obtain from (1.11)

$$2n-10\equiv(n-3)(n-4)\,(\mathrm{mod}\ 12)$$

or

$$2\equiv n(n-9)\ (\mathrm{mod}\ 12). \tag{1.14}$$

It can easily be checked that there is no n satisfying (1.14). Therefore $t\neq n-5$ and together with (1.11) and (1.13) we obtain

$$0\leqslant t\leqslant n-6.$$

Therefore the integers n, t defined by (1.11) satisfy the requirement (1.10). Assume now $p\neq 2$. Then $(n, t)\neq(9, 3)$. By Theorem 1.2 there exists a triangulation T of an orientable surface S with n vertices and

$$\binom{n}{2}-t$$

edges. We determine the Euler characteristic of S using (1.4):

$$6E(S)=6n-n(n-1)+2t=12-(n-3)(n-4)+2t=12-12p.$$

So S is in fact the orientable surface S_p of genus p.

Moreover, from (1.3) we obtain $3\alpha_2=n(n-1)-2t=n^2-n-(n-3)(n-4)+12p$, so that $\alpha_2=2n+4(p-1)$. Therefore equation (1.9) gives the exact number of triangles in T. Together with (1.8) this proves Theorem 1.1 if $p\neq 2$.

Now assume $p=2$ and let T be a minimal triangulation of S_2. Since $E(S_2)=-2$, (1.8) gives

$$\delta(S_2)\geqslant 2\times 9+4=22.$$

If $\delta(S_2)=22$ then (1.8) is an equality and from (1.7) it follows that $\alpha_0=9$. From (1.3) we obtain $\alpha_1=33$. So the 1-skeleton of T is a graph with 9 vertices and

$$\binom{9}{2}-3=33$$

edges. But by Theorem 1.2 such a triangular embedding does not exist. Therefore $\delta(S_2)\neq 22$ and by (1.7), $\delta(S_2)\geqslant 24$. By Theorem 1.2 there exists an orientable triangular embedding of a graph with 10 vertices and

$$\binom{10}{2}-9=36$$

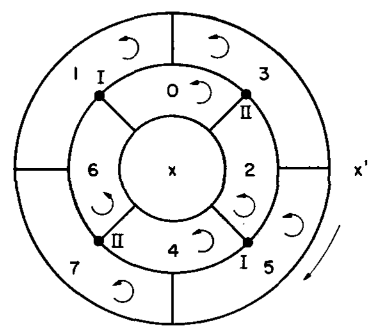

Figure 2.1

edges. From (1.4) we see that this embedding is on the surface S_2 and from (1.7) we see that the number of triangles is 24, so $\delta(S_2)=24$.

By a similar method one can also show that Theorem 1.1 implies Theorem 1.2.

Now we shall actually construct all the triangulations necessary to prove Theorem 1.2. The crucial observation for this is that the pairs (1.11) for all values of p actually exhaust the pairs (1.10).

2. Low order cases

We shall examine the situation for small values of the genus p and the corresponding pair (n, t) given by (1.11).

If $p=0$, then $n=4$, $t=0$, and the tetrahedron solves the problem. If $p=1$, then $n=7$, $t=0$ and we take the embedding of the complete graph K_7 in the torus first found by Heawood [3] (see also Ringel [11], page 5).

If $p=2$, then $n=9$, $t=3$. The authors became convinced that an orientable triangular embedding (abbreviated o.t.e.) of a graph with 9 vertices and 33 edges does not exist and confirmed this by computer. Independently, Huneke [5] gave an explicit proof. Figure 2.1 shows a map on the sphere with 10 countries. We *add* two handles in the following way. Two of the vertices are denoted by I. Around each of the two vertices labelled I draw a small circle. Excise the interior of both circles and identify the two boundaries as illustrated in figure 2.2. This gives 6 new adjacencies:

$$(1, 2),\ (1, 4),\ (4, 0),\ (0, 5),\ (5, 6),\ (5, 1).$$

A similar operation at the two vertices denoted II adds a second handle and gives 6 more adjacencies as illustrated in figure 2.3. The dual polyhedron of the resulting map on S_2 is a triangulation with $\alpha_0=10$ and $\alpha_1=36$. This shows Theorem 1.2 for the pair (10, 9).

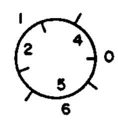

Figure 2.2

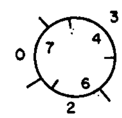

Figure 2.3

We remark here that the above map on S_2 may be described by a combinatorial scheme in the manner first introduced by Heffter [4] in 1891 as follows. Choose an orientation for each country as shown in figure 2.1. Note that these orientations carry over to the portions of the map shown in figures 2.2 and 2.3. The orientation of each country determines a cyclic sequence of the countries adjacent to it. For example the sequence for country x reads x. 6 4 2 0. The whole scheme consists of such a line for each country:

$$\begin{array}{lllllllll}
0. & 6 & x & 2 & 7 & 3 & 1 & 4 & 5 \\
2. & 0 & x & 4 & 1 & 5 & 3 & 6 & 7 \\
4. & 2 & x & 6 & 3 & 7 & 5 & 0 & 1 \\
6. & 4 & x & 0 & 5 & 1 & 7 & 2 & 3 \\
1. & 3 & x' & 7 & 6 & 5 & 2 & 4 & 0 \\
3. & 5 & x' & 1 & 0 & 7 & 4 & 6 & 2 \\
5. & 7 & x' & 3 & 2 & 1 & 6 & 0 & 4 \\
7. & 1 & x' & 5 & 4 & 3 & 0 & 2 & 6 \\
x. & 6 & 4 & 2 & 0 & & & & \\
x'. & 1 & 3 & 5 & 7. & & & &
\end{array} \tag{2.1}$$

Because every vertex in the map is of valence 3 the scheme satisfies

Rule R^*. If the ith line is of the form $i \ldots jkl \ldots$ then the kth line is of the form $\ldots lij \ldots$.

Conversely, every scheme satisfying this rule represents a map on an orientable surface with every vertex having valence 3 (providing the map is connected). Passing to the dual polyhedron, a scheme satisfying Rule R^* defines an o.t.e. of some graph. For further details see [11].

We now consider the situation where $p=3$. Then $(n, t)=(10, 3)$. In [11] page 23, an o.t.e. of the graph $K_{10}-K_3$ is constructed. Here we mean that from the complete graph K_{10} we have removed three edges forming a complete graph K_3.

If $p=4$ then $(n, t)=(11, 4)$. So we seek an o.t.e. of a graph of the type $K_{11}-4$ edges. The following scheme describes the desired triangulation.

1.	8	3	6	7	5	9	10	4	2	
2.	3	7	8	1	4	11	6	9	5	
3.	4	7	2	5	6	1	8	10	11	9
4.	5	7	3	9	6	8	11	2	1	10
5.	7	4	10	8	6	3	2	9	1	
6.	11	7	1	3	5	8	4	9	2	
7.	1	6	11	10	9	8	2	3	4	5
8.	5	10	3	1	2	7	9	11	4	6
9.	3	11	8	7	10	1	5	2	6	4
10.	11	3	8	5	4	1	9	7		
11.	10	7	6	2	4	8	9	3.		

Notice that the scheme satisfies Rule R^*.

In the first line number 11 is missing. So there is no edge (1, 11). We find that there are exactly four missing edges: (1, 11), (2, 10), (6, 10), (5, 11).

In some of the next cases we will often use the graph O_m with $2m$ vertices $1, 2, \ldots, m, 1', 2', \ldots, m'$, where each pair of vertices is joined by one edge except the pairs (i, i') for $i = 1, 2, \ldots, m$. For instance, O_3 is the octahedron and O_m can be considered as an m-dimensional analog of the octahedron. The authors have constructed an o.t.e. of O_m for each $m \not\equiv 2 \pmod 3$ in [8]. This means that Theorem 1.2 is proven for all pairs $(2m, m)$ where $m \not\equiv 2 \pmod 3$.

We continue with the next cases. To make it short, we list the genus p, then the corresponding pair (n, t), then the name of the graph with n vertices and

$$\binom{n}{2} - t$$

edges, then the place where an o.t.e. of it is published.

$p = 5$, (12, 6), O_6, Jungerman and Ringel [8].

$p = 6$, (12, 0), K_{12}, Heffter [4] in 1891 or Ringel [11].

$p = 7$, (13, 3), $K_{13} - K_3$, Jungerman [6] in 1974.

$p = 8$, (14, 7), O_7, Jungerman and Ringel [8].

$p = 9$, (14, 1), $K_{14} - K_2$, Ringel-Youngs [13] in 1969.

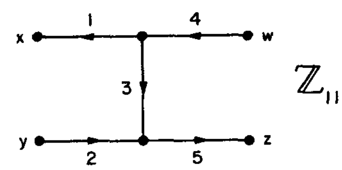

Figure 2.4

In most of the next cases we shall use the methods of current graphs invented by Gustin [1] and generalized by others. We assume the reader is familiar with about the first half of the book by Ringel [11]. It is impossible to repeat all the details of this method. However we do repeat the crucial construction principles for index one current graphs with abelian groups. We will use these throughout this paper. See the current graphs of figures 2.4 and 2.5 as examples.

(C1) *Each vertex has valence* 3, 2 *or* 1.

(C2) *The given rotation induces a single circuit.* Recall that ● indicates clockwise and ○ counterclockwise rotation.

(C3) *If* $a_1, a_2, \ldots, a_s$ *are all the currents used in the current graph then the set* $\{0, \pm a_1, \pm a_2, \ldots, \pm a_s\}$ *exhausts all elements of the group.*

We define the *excess* of a *vertex* P in a current graph as follows: the excess is the sum of the inward flowing currents minus the sum of the outward flowing currents.

(C4) *If a vertex of valence* 3 *is not identified by a letter then its excess is zero.* (Kirchhoff's Current Law.)

Note: A vertex which is identified by one or more letters is called a *vortex.*

(C5) *Each vertex identified by one letter*, such as x, *is of valence* 1 *and its excess is a generator of the group.*

(C6) *Each element of order* 2 *in the group must be a current of an end-arc.*

Here we skip (C7), which we will use and state later.

(C8) *If a vertex of valence* 1 *is not identified by a letter its excess is an element of order* 2 *or* 3 *in the group.*

The construction principle (C9) in [11] applies *only* to nonorientable triangulations, so we ignore it here. We add a new principle (C10), which was not used in [11].

(C10) *If there is a vertex of valence* 2 *then the group is of even order and the excess of the vertex generates a subgroup* U *of index* 2 (*the even elements*). *Neither of the currents of the two edges incident with the vertex are in* U, i.e., *they are both odd. Moreover, the vertex has to be identified by two letters such as* x *and* y.

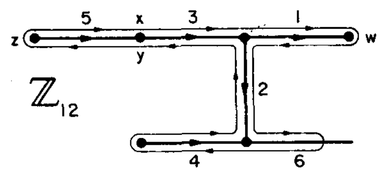

Figure 2.5

$p=10$, $(15, 6)$, $K_{15}-K_4$. Finding an o.t.e. of this graph is exactly the problem of exercise 2.3.5 on page 32 in [11]. The solution is given by the current graph of figure 2.4. Notice that the group is $\mathbf{Z}_{11}$ and the principles (C1) through (C5) apply.

$p=11$, $(15, 0)$, K_{15}. Ringel [10] in 1961 or [11], page 151.

$p=12$, $(16, 6)$, $K_{16}-K_4$. We denote the vertices of this graph by the elements of the cyclic group $\mathbf{Z}_{12}$ and the letters x, y, z, w. Then use the current graph of figure 2.5. The principles listed above hold. The induced circuit passes the vertex x, y of valence 2 twice. In writing the log of the circuit we write x the first time and y the second time. As usual, the log gives us the 0-line. The other lines are obtained as usual by the additive rule, however in the odd lines we interchange x and y. So the scheme is as follows.

0.	2	6	8	4	10	9	y	7	z	5	x	3	1	w	11
1.	3	7	9	5	11	10	x	8	z	6	y	4	2	w	0
2.	4	8	10	6	0	11	y	9	z	7	x	5	3	w	1
3.	5	9	11	7	1	0	x	10	z	8	y	6	4	w	2
4.	6	10	0	8	2	1	y	11	z	9	x	7	5	w	3
5.	7	11	1	9	3	2	x	0	z	10	y	8	6	w	4
6.	8	0	2	10	4	3	y	1	z	11	x	9	7	w	5
7.	9	1	3	11	5	4	x	2	z	0	y	10	8	w	6
8.	10	2	4	0	6	5	y	3	z	1	x	11	9	w	7
9.	11	3	5	1	7	6	x	4	z	2	y	0	10	w	8
10.	0	4	6	2	8	7	y	5	z	3	x	1	11	w	9
11.	1	5	7	3	9	8	x	6	z	4	y	2	0	w	10
x.	3	0	5	2	7	4	9	6	11	8	1	10			
y.	7	0	9	2	11	4	1	6	3	8	5	10			
z.	5	0	7	2	9	4	11	6	1	8	3	10			
w.	11	0	1	2	3	4	5	6	7	8	9	10.			

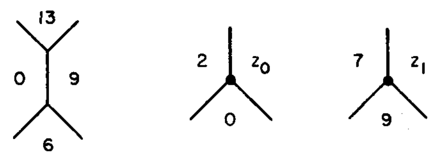

Figure 2.6

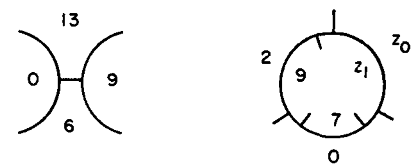

Figure 2.7

$p = 14$, (17, 7). We consider the scheme (2.2). Here we first have to reverse the order of all the odd numbered lines. Then Rule R^* is satisfied. The map defined by (2.2) contains the part shown in figure 2.6.

0.	12	z_0	2	10	1	y	11	5	4	6	9	13	x	3	8
1.	13	z_1	3	11	2	x	12	6	5	7	10	0	y	4	9
2.	0	z_0	4	12	3	y	13	7	6	8	11	1	x	5	10
3.	1	z_1	5	13	4	x	0	8	7	9	12	2	y	6	11
4.	2	z_0	6	0	5	y	1	9	8	10	13	3	x	7	12
5.	3	z_1	7	1	6	x	2	10	9	11	0	4	y	8	13
6.	4	z_0	8	2	7	y	3	11	10	12	1	5	x	9	0
7.	5	z_1	9	3	8	x	4	12	11	13	2	6	y	10	1
8.	6	z_0	10	4	9	y	5	13	12	0	3	7	x	11	2
9.	7	z_1	11	5	10	x	6	0	13	1	4	8	y	12	3
10.	8	z_0	12	6	11	y	7	1	0	2	5	9	x	13	4
11.	9	z_1	13	7	12	x	8	2	1	3	6	10	y	0	5
12.	10	z_0	0	8	13	y	9	3	2	4	7	11	x	1	6
13.	11	z_1	1	9	0	x	10	4	3	5	8	12	y	2	7
x.	11	8	7	4	3	0	13	10	9	6	5	2	1	12	
y.	0	1	4	5	8	9	12	13	2	3	6	7	10	11	
z_0.	12	10	8	6	4	2	0								
z_1.	1	3	5	7	9	11	13.								

(2.2)

Figure 2.7 shows how we switch the adjacency (0, 9) with the adjacency (6, 13) and how we add one handle. Now z_0 and z_1 are adjacent and are considered as one country z. In the resulting map there are exactly 7 adjacencies missing, namely

$$(1, 8),\ (3, 10),\ (4, 11),\ (5, 12),\ (x, y),\ (y, z),\ (x, z).$$

We remark that the original scheme (2.2) was obtained from the orientable cascade in figure 2.8. More details on orientable cascades may be found in [8], [11]. The embedding

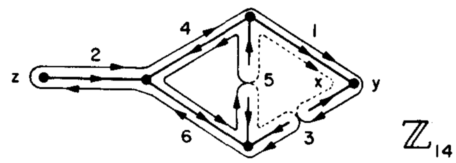

Figure 2.8

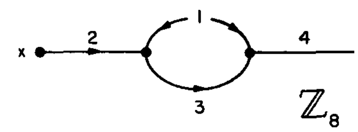

Figure 2.9

described by the scheme (2.1) may also be obtained from an orientable cascade, specifically the one given in figure 2.9. We will not use cascades in the rest of the paper.

$p = 15$, (17, 1), $K_{17} - K_2$. Ringel [10] in 1961 or [11].

$p = 16$, (18, 9), O_9, Jungerman and Ringel [8].

$p = 17$, (18, 3), $K_{18} - K_3$, Jungerman [6].

3. Handle subtraction

LEMMA 3.1. *Let a combinatorial scheme satisfying Rule R^* contain the portion*

$$\begin{array}{llllll} a. & \ldots\ b & c & d & e & \ldots \\ f. & \ldots\ c & b & e & d & \ldots \\ b. & \ldots\ e & f & c & a & \ldots \\ c. & \ldots\ d & a & b & f & \ldots \\ d. & \ldots\ f & e & a & c & \ldots \\ e. & \ldots\ a & d & f & b & \ldots. \end{array} \tag{3.1}$$

Then the scheme obtained by replacing the portion (3.1) *by*

$$\begin{array}{llll} a. & \ldots\ b & e & \ldots \\ f. & \ldots\ c & d & \ldots \\ b. & \ldots\ e & a & \ldots \\ c. & \ldots\ d & f & \ldots \\ d. & \ldots\ f & c & \ldots \\ e. & \ldots\ a & b & \ldots \end{array} \tag{3.2}$$

still satisfies Rule R^*.

This can be proven by checking Rule R^* directly. The change from (3.1) to (3.2) is shown geometrically in figures 3.1 and 3.2 and essentially consists of the *subtraction of a handle.*

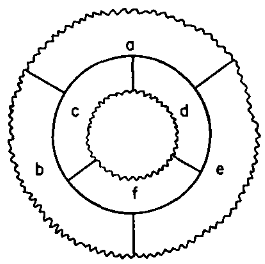

Figure 3.1

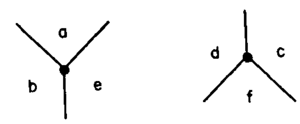

Figure 3.2

The operation has the following properties:

(1) *We lose 6 vertices and gain* 2.

(2) *We lose* 6 *edges and gain* 0.

The dual map interpretation of these properties is particularly relevant to Theorem 1.2: *If the scheme* (3.1) *represents an o.t.e. of a graph with n vertices and*

$$\binom{n}{2} - t$$

edges then there is an o.t.e. (3.2) *of a graph with n vertices and*

$$\binom{n}{2} - (t+6)$$

edges.

If we wish to find a portion of the form (3.1) in a given scheme it is not necessary to search for the whole portion (3.1). It suffices to look only for the part

$$\begin{array}{llllll} a. & \dots & b & c & d & e & \dots \\ f. & \dots & c & b & e & d & \dots . \end{array} \tag{3.3}$$

The rest of (3.1) then follows by applying Rule R^*.

Many current graphs used in [11] to determine the genus of K_m contain a portion called an arithmetic comb. More specifically they contain a portion as in figure 3.3.

It is important that the upper vertices have clockwise rotation, and the lower vertices have counterclockwise rotations. We shall prove that the map defined by a current graph having a part as in figure 3.3 contains portions of the form (3.1). Using Kirchhoff's Current Law (C4) we obtain from figure 3.3 the more complete figure 3.4. Then line 0 and line h (by the additive rule) of the constructed scheme read

$$\begin{array}{lllllllll} 0. & \dots, & r+h, & g+h, & -t, & g, & r, & g-h, & -t-h, & \dots \\ h. & & & & \dots, & g+h, & r+h, & g, & -t, & \dots . \end{array}$$

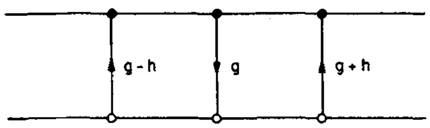

Figure 3.3

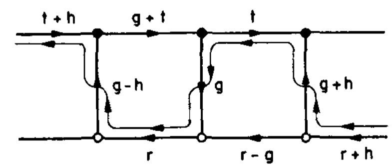

Figure 3.4

Therefore, if the current graph (index 1) has the group Z_m, we have

$$\begin{array}{llllll} i. & \ldots, & r+h+i, & g+h+i, & i-t, & g+i, \ \ldots \\ h+i. & \ldots, & g+h+i, & r+h+i, & g+i, & i-t, \ \ldots \end{array} \tag{3.4}$$

for $i=0, 1, 2, \ldots, m$. This is exactly the form (3.3), so we can actually subtract a handle *m-times*. One may easily check that the $6m$ vertices (6 for each handle) and $6m$ edges to be subtracted according to (3.4) are all distinct. Therefore the subtraction of the m handles can be done simultaneously or one after the other.

To summarize, whenever a current graph contains a portion of the form (3.3) we can subtract from one to m handles where m is the order of the group used.

Therefore if the current graph represents an o.t.e. of a graph of the type $K_n - t_0$ edges then we may obtain o.t.e.'s of graphs of the type $K_n-(t_0+6i)$ edges for $1 \leqslant i \leqslant m$, where m is the order of the group. In proving Theorem 1.2 we don't need to subtract this many handles. In fact, i need never be taken higher than $n/6$, since Theorem 1.2 only concerns pairs of the form (n, t) where $t \leqslant n-6$. In our applications we will always have $m > n/6$. Thus exhibiting a current graph representing an o.t.e. of $K_n - t_0$ edges which contains a portion of the form (3.3) will be sufficient to prove Theorem 1.2 for all pairs (n, t) with $t_0 \leqslant t \leqslant n-6$.

4. General cases

We now proceed with the proof of Theorem 1.2. The proof breaks down naturally into twelve cases depending on the residue class n (mod 12) in the given pair (n, t). We shall handle these case in the following order: 4.1. Case 3; 4.2. Case 0; 4.3. Case 4; 4.4. Case 1; 4.5 Case 6; 4.6. Case 2; 4.7. Case 10; 4.8. Case 7; 4.9 Case 5; 4.10. Case 9; 4.11. Case 11; 4.12. Case 8.

4.1. *Case* 3. Let $n=12s+3$. It is known that there exists an o.t.e. of K_n (see [11], page 151). So Theorem 1.2 holds for all pairs $(n, 0)$. For the pair $(15, 6)$ see chapter 2, $p=10$.

The current graph in figure 4.1.1. gives an o.t.e. of the graph $K_{27}-K_4$ and there are

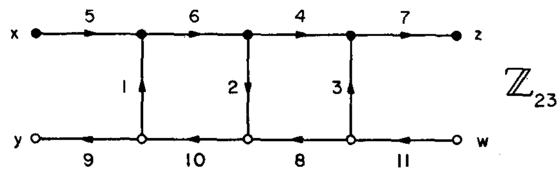

Figure 4.1.1

23 subtractable handles (figure 3.3 applies). This proves Theorem 1.2 for the pairs (27, 6), (27, 12) and (27, 18).

Remark. Figure 4.1.1 is identical to figure 6.5 of [11] except that the group $\mathbf{Z}_{23}$ is used instead of $\mathbf{Z}_{22}$. If we try the generalization for $n=12s+3$ as in figure 6.6 of [11] using the group $\mathbf{Z}_{12s-1}$, we conflict with property (C5). The current of the arc (directed edge) incident with vertex y would be $5s-1$ and $(5s-1, 12s-1)=(s-3, 7)$. So (C5) fails if $s\equiv 3 \pmod 7$.

However the current graph of figure 4.1.3 works for every $s\geqslant 3$ (figure 4.1.2 shows the example $s=3$) and defines an o.t.e. of $K_{12s+3}-K_4$. It also has parts as in figure 3.3 so handles can be subtracted. This gives the proof of Theorem 1.2 for the pairs $(12s+3, 6i)$ where $i=1, 2, \ldots, 2s-1$.

4.2. *Case* 0. Let n be of the form $n=12s$. There exists an o.t.e. of K_n (see Terry, Welch, Youngs [14], or [11]). Here we construct an o.t.e. of the graph K_n-K_4 using the current graphs in figure 4.2.1 for $s\geqslant 4$, figure 4.2.2 if $s=3$, and figure 4.2.3 if $s=2$. The first two figures have big enough portions of the arithmetic comb (as figure 3.3), so that a handle can be subtracted i-times ($i=1, 2, \ldots, (n-6)/6$). This leaves the case $s=2$ to consider. Here we have another way to find subtractable handles. Assume figure 4.2.4 is part of a current graph as for instance in figures 4.2.3 and 4.3.1. The line 0 and line 2t read

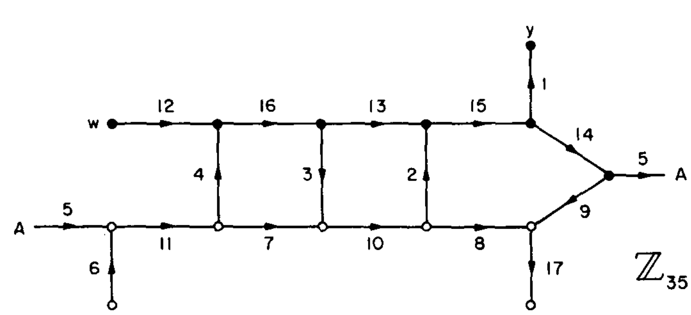

Figure 4.1.2

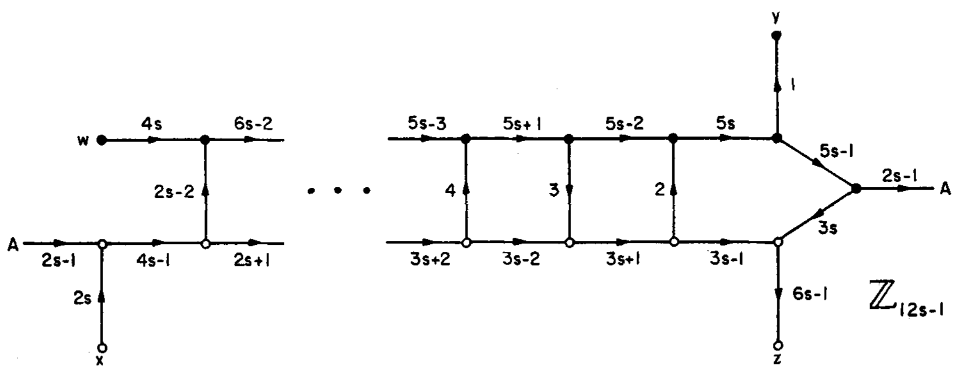

Figure 4.1.3

$$
\begin{array}{lllllll}
0. & \ldots, & t, & w, & -t, & x, & -3t, \ \ldots \\
2t. & \ldots, & & w, & t, & x, & -t, \ \ldots .
\end{array}
\tag{4.2.1}
$$

This contains a part as in (3.3) which guarantees that a handle is subtractable.

We can get more subtractable handles by using the additive rule [11], page 25. However, one has to be very careful because some of these handles may interfere with one another (cannot be subtracted simultaneously). For instance, in the map defined by the current graph in figure 4.2.3, the following four handles are subtractable:

$$
\begin{array}{llllll}
0. & 3 & w & 17 & x & \\
6. & & w & 3 & x & 17 \\
\\
2. & 5 & w & 19 & x & \\
8. & & w & 5 & x & 19 \\
\\
4. & 7 & w & 1 & x & \\
10. & & w & 7 & x & 1 \\
\\
6. & 9 & w & 3 & x & \\
12. & & w & 9 & x & 3.
\end{array}
$$

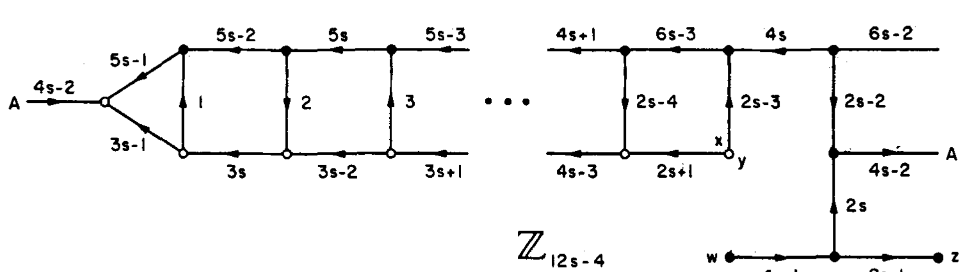

Figure 4.2.1

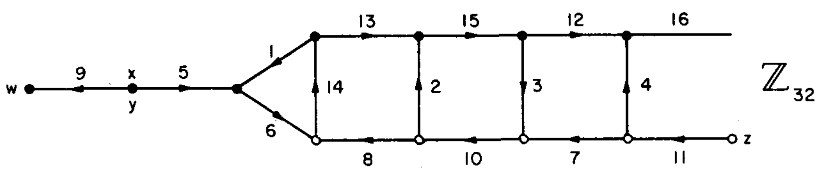

Figure 4.2.2

But the first and the fourth are not independent because one of the six triangles involved in the first handle, namely $(6, w, 3)$ is also involved in the fourth.

But this doesn't matter. We just take the first two handles. This proves Theorem 1.2 for the pairs (24, 6), (24, 12), (24, 18). For the pair (12, 6) see chapter 2, $p=5$.

4.3. *Case* 4. In [11] page 90, an o.t.e. of K_n is constructed for each n of the form $n=12s+4$. For $s\geqslant 3$, the current graph contains portions as in figure 3.3, so handles can be subtracted $(2s-1)$ times.

If $s=2$, we consider the current graph in figure 4.3.1 which gives us an o.t.e. of $K_{28}-K_4$. Since figure 4.2.4 applies with $t=1$, the portion (4.2.1) of the scheme becomes

0.	.	1	w	23	x	.	.
2.	.	.	w	1	x	23	.

This gives one subtactable handle. We need two more which we get by the additive rule:

4.	.	5	w	3	x	.	.
6.	.	.	w	5	x	3	.
8.	.	9	w	7	x	.	.
10.	.		w	9	x	7	.

Note that these three handles do not interfere with each other. For $s=1$ see chapter 2, $p=12$.

4.4. *Case* 1. It is known that there exists an o.t.e. of $K_{12s+1}-K_3$. For $s=1$ see chapter 2, $p=7$. For $s=2$ use figure 6.5 of [11] and for $s>2$ use figures 6.7 and 6.8 in [11] and their

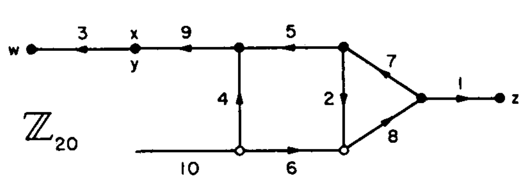

Figure 4.2.3

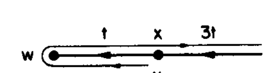

Figure 4.2.4

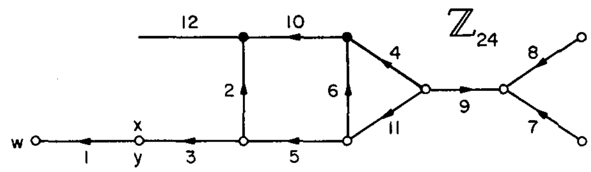

Figure 4.3.1

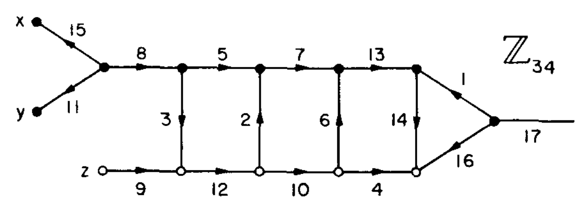

Figure 4.4.1

generalizations. All these current graphs have the standard arithmetic comb with at least 3 verticals, which allows handle subtractions, except for $s=3$.

For $s=3$ we use the current graph in figure 4.4.1. Here for the three verticals on the right, figure 3.3 applies with $g=-6$, $h=-8$. So we can subtract handles.

4.5. *Case* 6. In [11] page 155, an o.t.e. of K_n-K_3 is constructed for each n of the form $n=12s+6$. ($s\geqslant 2$.) If $s=1$ see chapter 2, $p=17$. This proves Theorem 1.2 for the pairs $(n, 3)$ where $n\equiv 6 \pmod{12}$ and $n\geqslant 18$.

We are going to construct an o.t.e. of a graph of the form $K_{..}-9$ edges. If s is even, consider the current graph in figure 4.5.1. All the construction principles (C1), (C2), (C3), (C4), (C5), (C6) and (C8) are valid except that the vortex w violates the rule (C5). In fact the current $2s$ does not generate the group $\mathbf{Z}_{12s+2}$. It only generates the subgroup of all even elements. This causes (see Case 2 in [11]) line w in the scheme to consist of two cycles instead of one; one with all the even elements, one with all the odd ones.

Therefore the map defined by the current graph has two countries, w_0 and w_1. w_0 is adjacent to all the even numbered countries. w_1 is adjacent to all the odd ones.

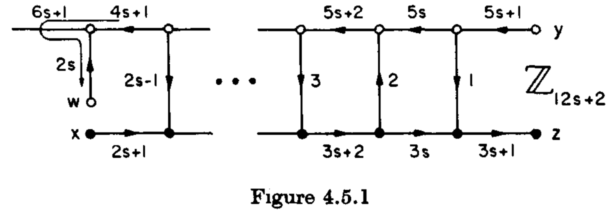

Figure 4.5.1

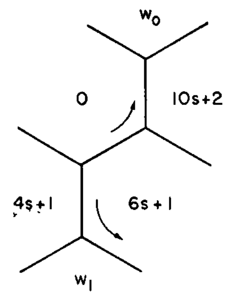

Figure 4.5.2

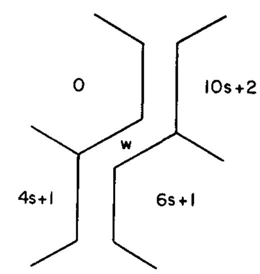

Figure 4.5.3

Parts of line 0 and line $6s+1$ of the scheme are:

$$\begin{array}{rlllll} 0. & \dots\ 4s+1, & 6s+1, & 10s+2, & w_0 & \dots \\ 6s+1. & \dots\ 10s+2, & 0, & 4s+1, & w_1 & \dots . \end{array}$$

Therefore the map has the local portion in figure 4.5.2. This can be modified as in figure 4.5.3. The result is that w_0 and w_1 have been merged into a single country w and adjacencies

$$(10s+2, 0) \quad (0, 6s+1), \quad (6s+1, 4s+1)$$

have been lost. There are 6 more missing adjacencies:

$$(x, y), \quad (x, z), \quad (x, w), \quad (y, z), \quad (y, w), \quad (z, w).$$

The dual of this map is an o.t.e. of a graph of the type $K_n - 9$ edges. This proves Theorem 1.2 for the pair $(n, 9)$. There are subtractable handles (figure 3.3 applies). So in fact we have proved Theorem 1.2 for the pairs $(n, 9)$, $(n, 15)$, ..., $(n, n-9)$.

Suppose $s=3$. Consider the current graph in figure 4.5.4 and apply the construction using figures 4.5.2 and 4.5.3. Since there are subtractable handles we have proved Theorem 1.2 for the pairs (42, 9), (42, 15), ..., (42, 33).

If $s \geqslant 5$ and odd consider the current graph in figure 4.5.5. Replace $10s+2$ by $3s+1$ and $4s+1$ by $9s+2$ in figures 4.5.2 and 4.5.3. Then apply the same construction as before. □

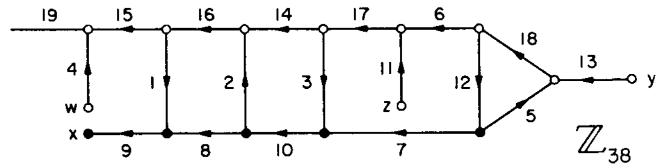

Figure 4.5.4

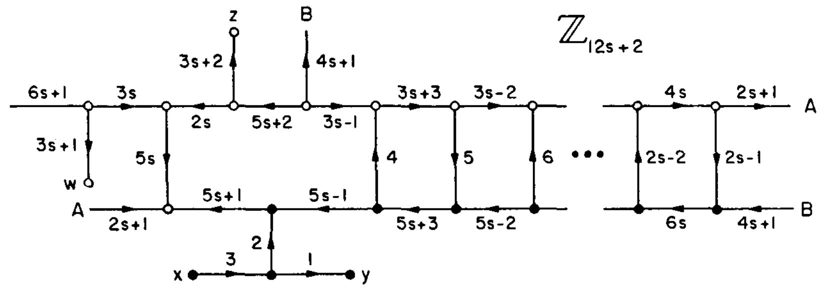

Figure 4.5.5

4.6. *Case* 2. This time let n be given in the form $n = 12s + 14$. It is known that there exists an o.t.e. of $K_n - K_2$. See Ringel–Youngs [13] if s is even and Jungerman [7] if s is arbitrary. This proves Theorem 1.2 for the pairs $(n, 1)$.

Now consider the index one current graph ($s \geq 3$) in figure 4.6.1 using the cyclic group $\mathbf{Z}_{12s+6}$ and the extra symbols x, y, u, v, a, b, c, and w. If $s = 1$, we use the current graph in figure 4.6.2. If $s = 2$, we use the current graph in figure 4.6.3. There are four vortices in figure 4.6.1:

the excess of vortex	$(x, y) = 2$	and	$(2, 12s+6) = 2,$
the excess of vortex	$(u, v) = 6s+8$	and	$(6s+8, 12s+6) = 2,$
the excess of vortex	$(a, b, c) = -3$	and	$(-3, 12s+6) = 3,$
the excess of vortex	$w = 2$	and	$(2, 12s+6) = 2.$

The first two agree with the construction principle (C10). The third agrees with the previously omitted following principle.

(C7) *The currents leading into a vortex identified by three letters, such as a, b, c, are congruent to one another* (mod 3) *but not congruent to* 0 (mod 3). *The excess of the vortex generates a subgroup of index* 3.

The current graphs in figures 4.6.2 and 4.6.3 also satisfy these principles.

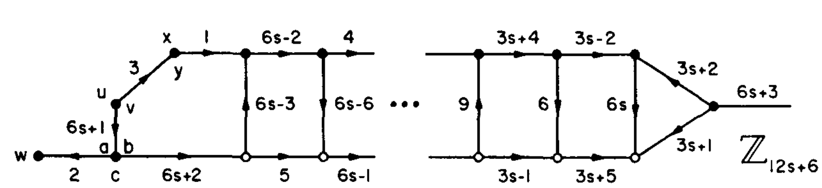

Figure 4.6.1

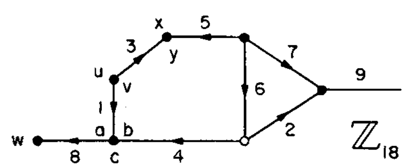

Figure 4.6.2

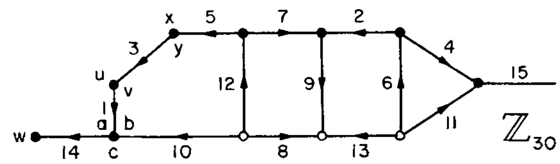

Figure 4.6.3

As in Case 6 in section 4.5 the fourth vortex w generates two countries w_0, w_1. $w_0(w_1)$ is adjacent to all even (odd) numbered countries.

We note that the currents of the verticals in figure 4.6.1 form the arithmetic sequence:

$$6s-3, \quad \ldots, \quad 12, 9, 6. \tag{4.6.1}$$

Since $s \geqslant 3$ this means we have subtractable handles. We also have subtractable handles in figure 4.6.3. In figure 4.6.2 we don't need them.

We consider the lines 0 and $6s+1$ of the scheme generated by the current graphs of figure 4.6.1. The lines have the following parts

$$\begin{array}{rl} 0. & \ldots c,\ 2,\ w_0,\ 12s+4,\ a,\ 6s+5,\ u,\ 3,\ x,\ \ldots,\ y,\ 12s+3,\ v,\ 6s+1,\ b \ldots \\ 6s+1. & \ldots,\quad w_1,\ 6s-1,\ b,\ 0,\ \ldots. \end{array}$$

This gives a partial picture of the map around the country 0 as illustrated in figure 4.6.4. We modify the map as illustrated in figure 4.6.5. We now proceed to add handles in order to unify w_0 and w_1 and gain additional adjacencies.

First handle. Consider the map on a torus shown in figure 4.6.6. Excise country 0 from figure 4.6.5 and the unnamed country in the middle of figure 4.6.6. Then identify the two resulting boundaries in the obvious way. After this there is a new country, named 0, that is adjacent to the same countries as the old country 0 was in figure 4.6.5.

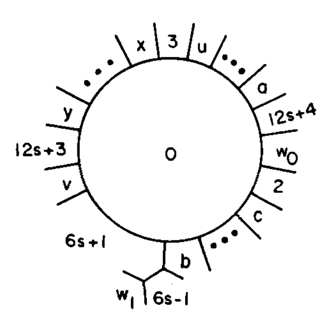

Figure 4.6.4

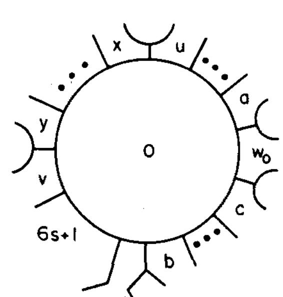

Figure 4.6.5

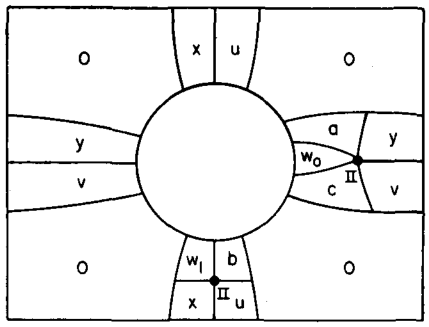

Figure 4.6.6

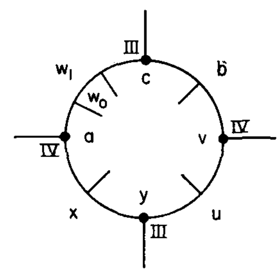

Figure 4.6.7

Second handle. Consider the two vertices both named II in figure 4.6.6. Excise a little disc around each of them. Identify the two resulting boundaries as shown in figure 4.6.7. That way several adjacencies are achieved and w_0 and w_1 also become adjacent. We remove the boundary between w_0 and w_1 and call the new country w.

Third handle. Excise a small neighborhood around each of the two vertices III in figure 4.6.7 and identify the boundaries as in figure 4.6.8.

Fourth handle. Do the same with vertices IV according to figure 4.6.9.

After all this we have constructed a map with the countries $0, 1, \ldots, 12s+5$, a, b, c, x, y, u, v, w. Exactly seven pairs of countries are not adjacent:

$$\begin{array}{llll} (0, 12s+3), & (0, 3), & (0, 12s+4), & (0, 2), \\ (6s+1, 6s-1), & (6s+1, b), & (a, c). & \end{array}$$

The first five of them were lost in the transformation from figure 6.4.4 to figure 4.6.5.

The construction is the same in the cases $s=1$ (from figure 4.6.2) and $s=2$ (from figure 4.6.3). Only the numbers on certain countries in figure 4.6.4 are changed. Since the lettered countries are unchanged, the rest of the construction is identical to the case $s \geqslant 3$. The dual of the map obtained by the construction is an o.t.e. of a graph of the form $K_n - 7$ edges. This proves Theorem 1.2 for the pair $(n, 7)$. If we stop the process after the

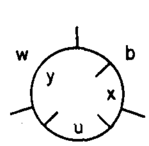

Figure 4.6.8

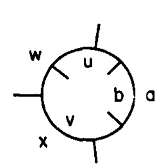

Figure 4.6.9

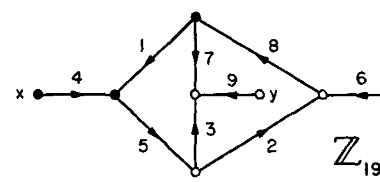

Figure 4.7.1

third (second) handle we get the proof of Theorem 1.2 for the pair $(n, 13)$ (the pair $(n, 19)$). This already proves Theorem 1.2 if $n=26$. If $n \geqslant 38$ then we need to subtract handles. Fortunately, as we said before, we can apply figure 3.3 to do so.

4.7. *Case* 10. If $n=10 \pmod{12}$, there exists an o.t.e. of the graph $K_n - K_3$. The standard proof [11] page 29 for $n \geqslant 34$ uses a nice index one current graph which has big enough portions of an airithmetic comb to subtract handles.

For $n=22$ we shall use the index one current graph of figure 4.7.1. It generates an embedding of $K_{22}-K_3$. The 0-line of the scheme is:

0. z 6 8 7 16 14 15 x 4 18 11 17 3 10 y 9 12 1 5 2 13.

Therefore the scheme contains the portions

0.	...	8	7	.	.	4	18	.	.	9	12	1	...
1.	...	0	12	18	4	.	.	.	.				
4.	...	1	18	0	.	.	.	.	.				
7.	...	0	8	12	9	.	.	.	.				
8.	...	12	7	0	.	.	.	.	.				
9.	...	7	12	0	.	.	.	.	.				
12.	...	18	1	0	9	7	8	.	.				
18.	...	0	4	1	12	.	.	.	.				

(4.7.1)

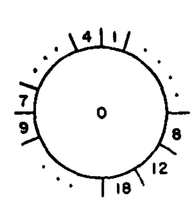

Figure 4.7.2

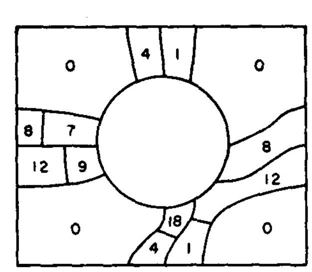

Figure 4.7.3

Now we replace the portions (4.7.1) by

$$\begin{array}{lccccccccccccc}
0. & \ldots & 8 & 12 & 18 & . & . & 9 & 7 & . & . & 4 & 1 & \ldots \\
1. & \ldots & 0 & 4 & . & . & . & . \\
4. & \ldots & 1 & 0 & . & . & . & . \\
7. & \ldots & 0 & 9 & . & . & . & . \\
8. & \ldots & 12 & 0 & . & . & . & . \\
9. & \ldots & 7 & 0 & . & . & . & . \\
12. & \ldots & 18 & 0 & 8 & . & . & . \\
18. & \ldots & 0 & 12 & . & . & . & .
\end{array} \tag{4.7.2}$$

Notice that we only changed the order in line 0. In the six other lines we dropped one or two or three numbers and did not change the order. We can easily check that after this operation Rule R^* still holds throughout the whole scheme. We lost the six adjacencies (1, 18), (4, 18), (7, 8), (7, 12), (9, 12), (1, 12).

So we really have constructed an o.t.e. of a graph of the type $K_{22}-9$ edges which proves Theorem 1.2 for the pairs (22, 3) and (22, 9). The transformation from (4.7.1) to (4.7.2) can be interpreted as a subtraction of a handle. But it is not easy to illustrate geometrically. It is more instructive to explain the reverse operation which goes from the map described by (4.7.2) back to the map described by (4.7.1). For (4.7.2) the surroundings of country 0 are pictured in figure 4.7.2. Consider figure 4.7.3 as a map on a torus (identify opposite sides of the rectangle) and excise the unnamed country from figure 4.7.3 and the country 0 from figure 4.7.2. Then identify the two boundaries in the obvious way. The result is a new map with six more adjacencies. It is in fact the map described by (4.7.1).

In order to prove Theorem 1.2 for the remaining pair (22, 15) we need to subtract one more handle. This can easily be done in the following way: add $+1$ to all the elements occurring in the schemes (4.7.1) and (4.7.2). □

4.8. *Case* 7. We shall prove Theorem 1.2 in case $n\equiv 7 \pmod{12}$: In [11] page 26 an o.t.e. of K_n is constructed based on the current graph in figure 2.15 of [11] with the group Z_n. If $n \geqslant 31$ it has a portion exactly as in figure 3.3 so we can subtract handles. This proves Theorem 1.2 for the pairs (n, t) with $t=0, 6, 12, \ldots, n-7$. If $n=19$ the current graph in figure 2.13 of [11] does not have enough vertical arcs to use the above method.

So we use the index 3 current graph in figure 4.8.1, which is a slight modification of figure 10.1 in [11]. It defines an o.t.e. of K_{19}.

The theory of index 2 and index 3 current graphs appears in [11] chapter 9. In the

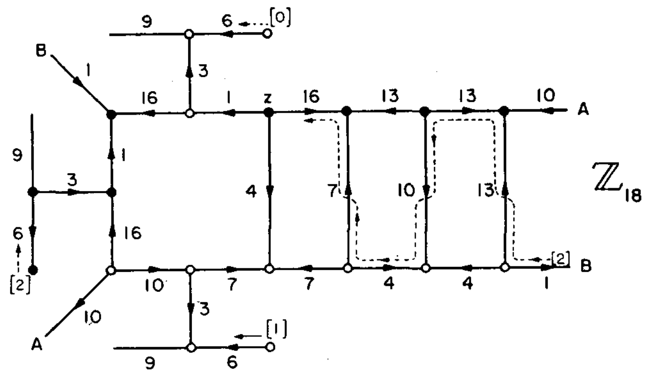

Figure 4.8.1

interest of brevity we omit the details here. Note that the current graph of figure 4.8.1 has a portion of the type shown in figure 3.3. We wish to show that this means that there are subtractable handles just as in the index one case, providing h is a multiple of 3. We note that one of the circuits $[i]$ with $i=0$, 1, or 2 has the form

$$[i]. \quad \ldots \; r+h, \; g+h, \; -t, \; g, \; r, \; g-h, \; -t-h, \; \ldots .$$

Since h is a multiple of 3 it follows that for any $j \equiv i \pmod 3$ the scheme contains the portion

$$\begin{array}{ll} j. & \ldots, \; r+h+j, \; g+h+j, \; j-t, \; g+j, \; \ldots \\ j+h. & \ldots, \; g+h+j, \; r+h+j, \; g+j, \; j-t, \; \ldots . \end{array}$$

This is exactly of the form (3.3), which implies that there are subtractable handles for $j=i$, $i+3$, $i+6$, ..., $i+m-3$, where m is the order of the group. In every application this will be more than enough handle subtractions.

In the present case $[i]=[2]$, and we can subtract 2 handles. This proves Theorem 1.2 for the pairs (19, 6) and (19, 12).

Remark. For later application (see section 4.10) we note that by subtracting one more handle we obtain an o.t.e. of a graph of the type $K_{19}-18$ edges.

4.9. *Case* 5. For each $n \equiv 5 \pmod{12}$ there exists an o.t.e. of the graph $K_n - K_2$. The proof is given by an easy index 3 current graph in [11]. It does not exactly contain an arithmetic comb. In fact every second rung in the ladder like graph is a "globular" rung. However, this can easily be changed so that the left half of the ladder has ordinary rungs and the rungs on the right hand half are all globular. See figure 4.9.1 as an example for $n=29$. For $n \geqslant 29$ the left part of the ladder is big enough that figure 3.3 applies and therefore handles can be subtracted. For $n=17$ see chapter 2, $p=14$ and 15.

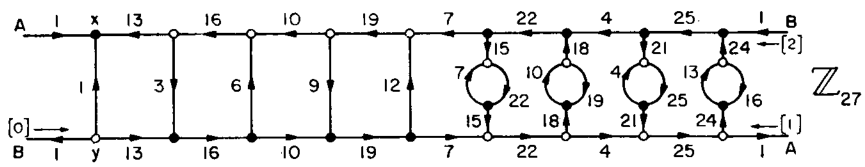

Figure 4.9.1

4.10. *Case* 9. Guy and Ringel [2] have constructed an o.t.e. for $K_{12s+9}-K_6$ for $s \geqslant 4$ using an index 3 current graph with the group $\mathbf{Z}_{12s+3}$. Fortunately it has the standard arithmetic comb of at least length 3 if $s \geqslant 5$. So we can subtract handles and this proves Theorem 1.2 for the pairs $(12s+9, t)$ with $t=15, 21, 27, \ldots, 12s+3$ and $s \geqslant 5$.

In the same paper Theorem 1.2 is proven for $(12s+9, 9)$ and $(12s+9, 3)$ when $s \geqslant 4$. Just take the dual of the constructed map after adding the first and the second handle.

Several cases remain to be treated. One of these, namely (33, 9) requires its own method, which we now give. Consider the index two graph of figure 4.10.1. Each of the vertices w_0, w_1, u_0, u_1 has a current excess generating the even subgroup. Thus each of w_0 and u_0, which lie in the [0] circuit, becomes a country adjacent to all the even-numbered countries, while w_1 and u_1 are adjacent to the odd-numbered countries. The logs of the portions of the circuits depicted near the vertices w_0 and w_1 are

$$\begin{array}{ll} [0] & \ldots, w_0, \; 8, \; 7, \ldots \\ [1] & \ldots, w_1, \; 16, \; 23, \ldots. \end{array}$$

So the scheme has the portion

$$\begin{array}{ll} 0. & \ldots, w_0, \; 8, \; 7, \ldots \\ 7. & \ldots, w_1, \; 23, \; 0, \ldots \end{array}$$

leading to figure 4.10.2. The alternative figure 4.10.3 merges w_0 and w_1 into a single country w and removes the three adjacencies (8, 0), (0, 7) and (7, 23). The same method creates a country u from u_0 and u_1 and removes three more adjacencies. The end result is an o.t.e. of a graph $K_{33}-9$ edges. The 9 missing adjacencies are the six above plus (x, u), (u, w) and (w, x).

In order to handle the rest of the cases we use the following induction type theorem.

THEOREM 4.10.1. *If there exists o.t.e.'s of graphs of the type $K_{2t+1}-h_1$ edges, $K_{2t+1}-h_2$ edges and $K_{2t+1}-h_3$ edges, $t \geqslant 2$, and each of these three graphs has a vertex of valence $2t$, then there exists an o.t.e. of a graph of the type $K_{6t+3}-(h_1+h_2+h_3+3)$ edges.*

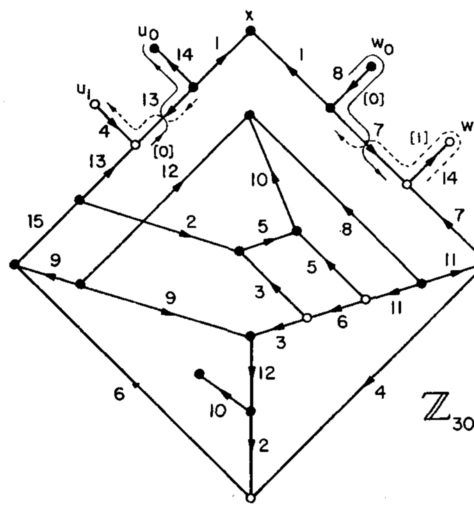

Figure 4.10.1

This theorem is proven for $h_1 = h_2 = h_3 = 0$ on page 162 of [11]. The generalization is obvious enough that we do not repeat the proof here.

We will apply Theorem 4.10.1 using o.t.e.'s of the following types of graphs:

K_{19},	$K_{19} - 6$ edges,	$K_{19} - 12$ edges,	$K_{19} - 18$ edges,
K_{15},	$K_{15} - 6$ edges,	$K_{15} - 12$ edges,	
$K_{11} - 4$ edges,	$K_{11} - 10$ edges,		
K_7,	$K_7 - 6$ edges.		

Except for the three which will be described below these o.t.e.'s have been constructed earlier (see section 4.8 and chapter 2). We note that each time a handle is subtracted the valence of only six vertices decreases. Therefore the graphs used above satisfy the valence hypothesis of Theorem 4.10.1.

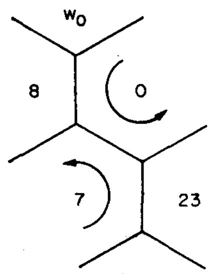

Figure 4.10.2

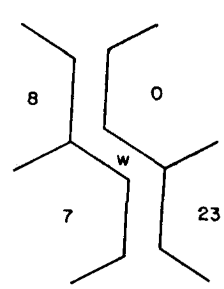

Figure 4.10.3

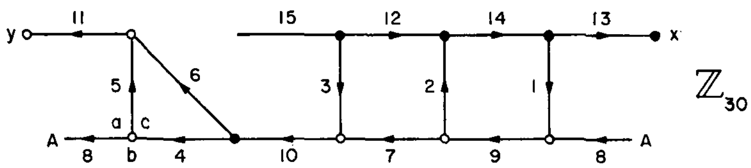

Figure 4.11.1

The three o.t.e.'s which remain to be constructed are the following.

(a) In order to find an o.t.e. of a graph of the type $K_{15}-12$ edges, take an o.t.e. of $K_{14}-K_2$ (see chapter 2, $p=9$). In the interior of any triangle place a new vertex v connected to the vertices of the triangle using three new edges. The result is an o.t.e. of a graph $K_{15}-12$ edges. At least one of the vertices of the original triangle has valence 14.

(b) Using the same methods as in (a) we get an o.t.e. of a graph of the form $K_{11}-10$ edges from an o.t.e. of $K_{10}-K_3$ (see chapter 2, $p=3$).

(c) An o.t.e. of a graph of the type K_7-6 edges is given by the following scheme. Note that vertex x has valence 6.

x.	1	0	3	2	5	4
o.	2	3	x	1	4	
2.	4	5	x	3	0	
4.	0	1	x	5	2	
1.	0	x	4			
3.	2	x	0			
5.	4	x	2.			

(K_7-6 edges)

Using various combinations of these o.t.e.'s in Theorem 4.10.1 and the o.t.e. of $K_{33}-K_3$ in [15] we obtain all the remaining embeddings necessary to complete the proof of Theorem 1.2. For instance let $t=9$, $h_1=h_2=18$, $h_2=12$. Then we obtain an o.t.e. of $K_{57}-51$ edges.

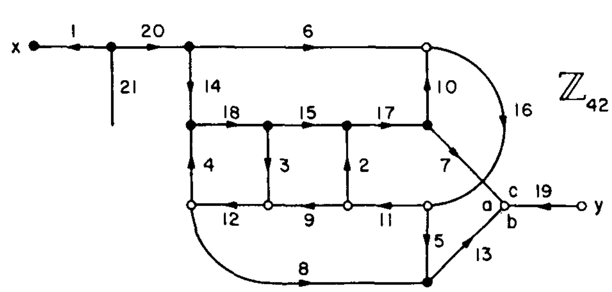

Figure 4.11.2

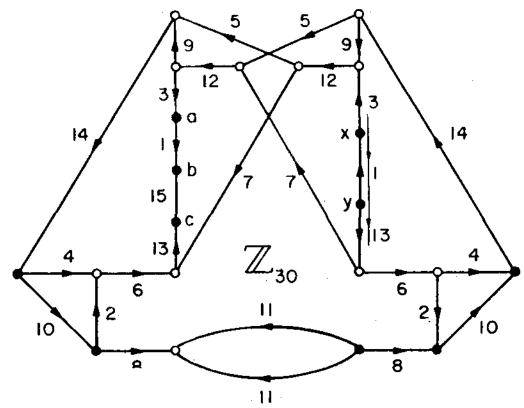

Figure 4.11.3

4.11. *Case* 11. Let n be of the form $n=12s+11$. An o.t.e. of K_n-K_5 is constructed in [11] or in Ringel–Youngs [12]. It has subtractable handles for $s\geqslant 4$.

For $s=2$ see figure 4.11.1 and for $s=3$ see figure 4.11.2. These current graphs give o.t.e.'s of K_n-K_5 with subtractable handles. This proves Theorem 1.2 for the pairs $(n, 10)$, $(n, 16)$, ..., $(n, n-7)$ if $n\geqslant 35$.

We now construct an o.t.e. for a graph of the form K_n-4 edges for $n\geqslant 35$. Consider first the case $n=35$. The index 2 current graph of figure 4.11.3 gives an o.t.e. of $K_{35}-K_5$. The 0 line reads

$$0. \quad \ldots \ a \ 1 \ b \ 15 \ c \ \ldots \ x \ 29 \ y \ \ldots .$$

So the (dual) map contains the portion shown in figure 4.11.4. We modify this as shown in figure 4.11.5. We add one handle at the vertices marked I according to figure 4.11.6, then the only missing adjacencies are

$$(0, 29), \quad (0, 1), \quad (0, 15), \quad (a, y).$$

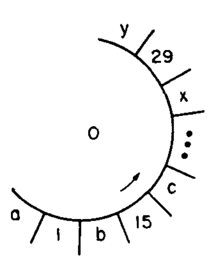

Figure 4.11.4

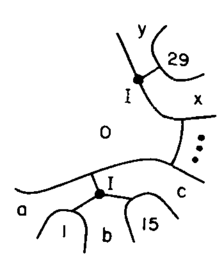

Figure 4.11.5

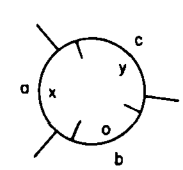

Figure 4.11.6

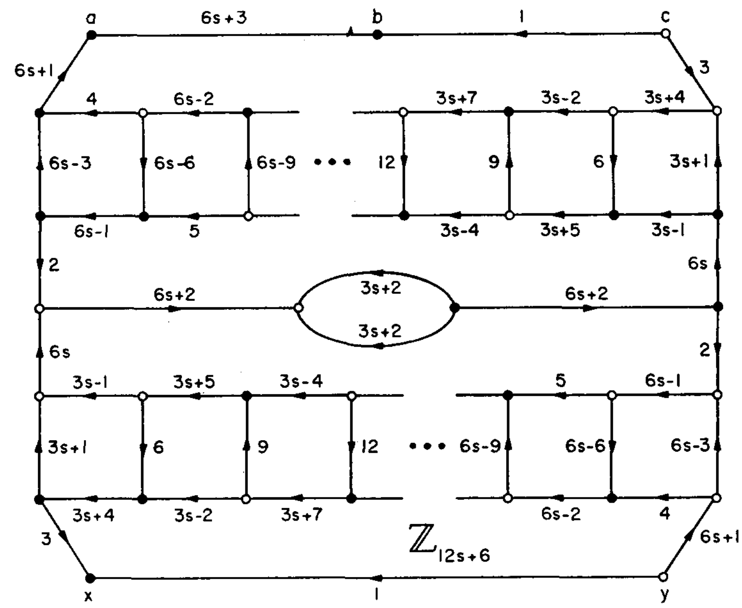

Figure 4.11.7

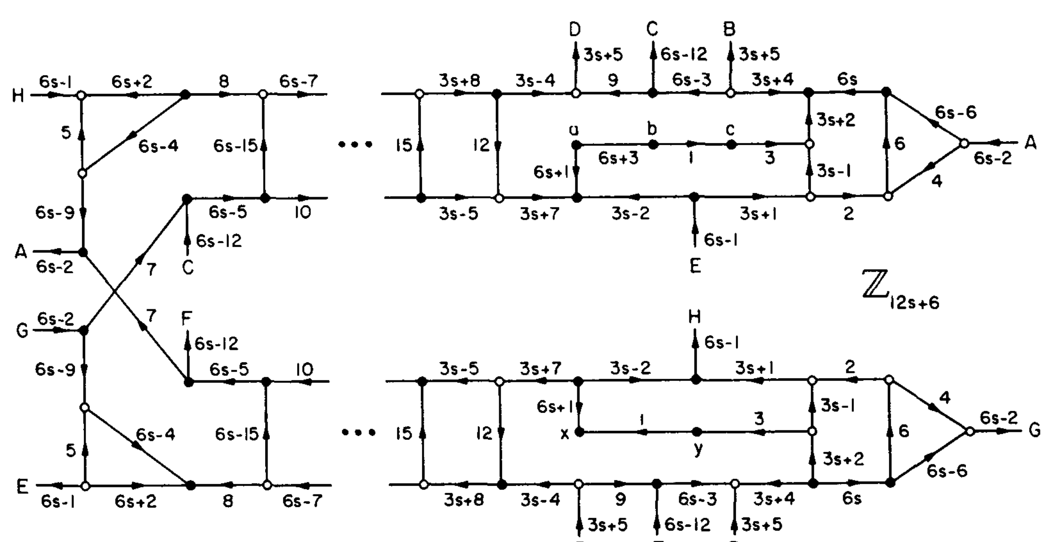

Figure 4.11.8

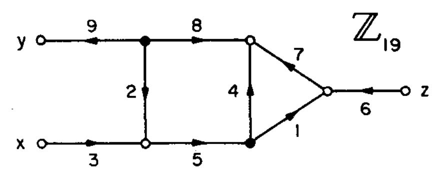

Figure 4.11.9

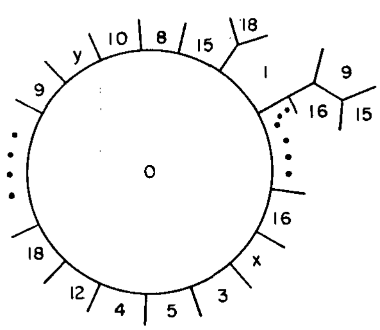

Figure 4.11.10

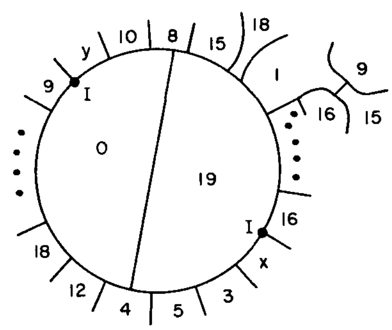

Figure 4.11.11

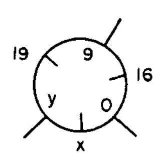

Figure 4.11.12

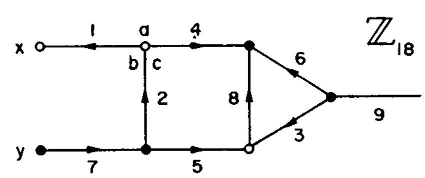

Figure 4.11.13

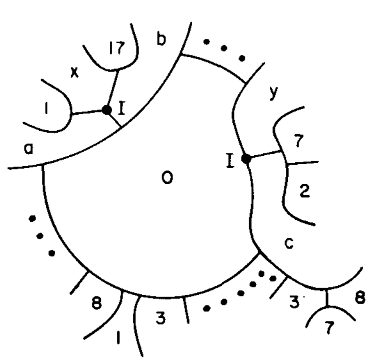

Figure 4.11.14

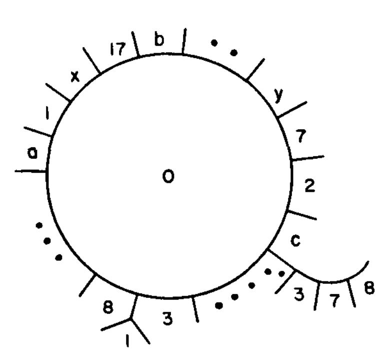

Figure 4.11.15

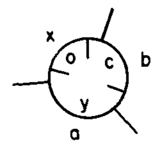

Figure 4.11.16

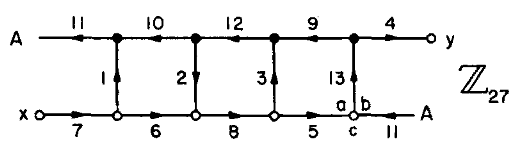

Figure 4.12.1

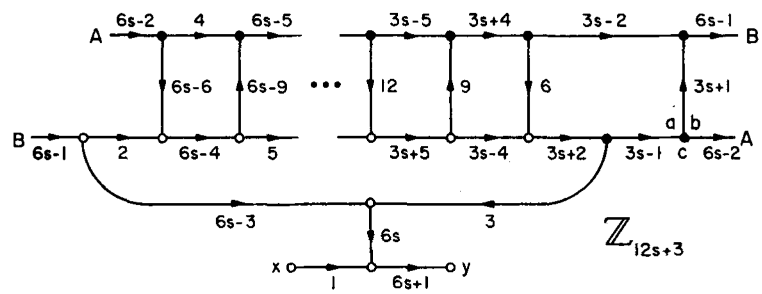

Figure 4.12.2

We use figure 4.11.7 for s odd and $s \geqslant 3$ and figure 4.11.8 for s even and $s \geqslant 4$ to obtain o.t.e.'s of $K_n - K_5$.

In each of these cases the vertices a, b, c, x and y are arranged as in the above case $n = 35$. We can add a handle in exactly the same manner. This proves Theorem 1.2 for the pairs $(n, 4)$, $n \geqslant 35$. For $n = 11$ see chapter 2, $p = 4$.

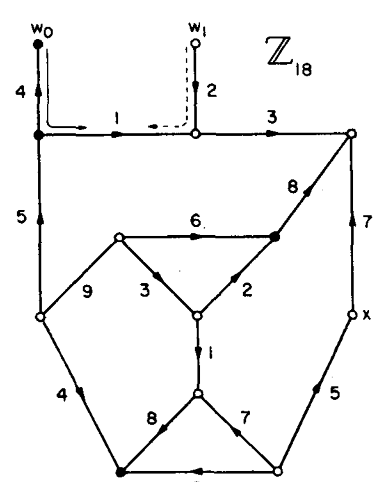

Figure 4.12.3

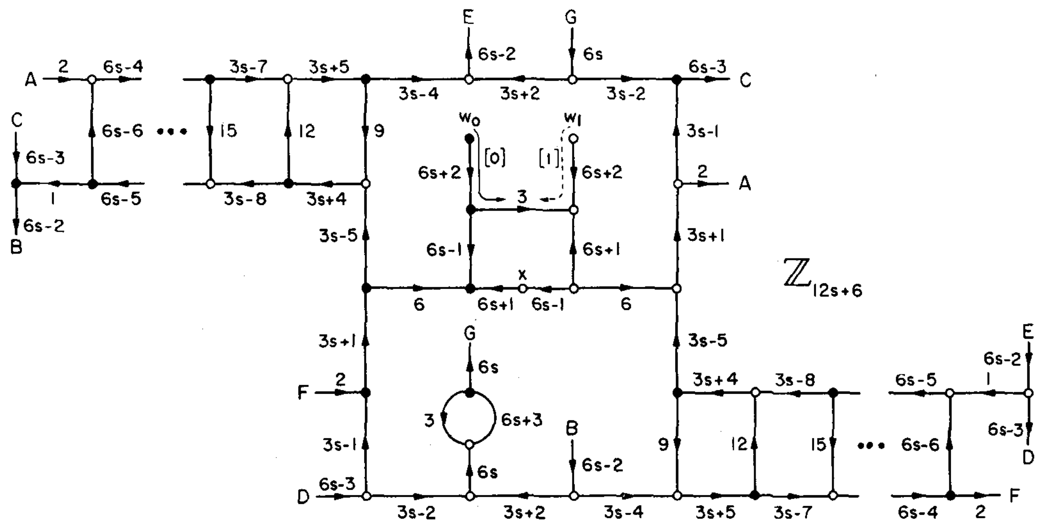

Figure 4.12.4

It remains to consider case $n=23$, which unfortunately has to be handled by an ad hoc method.

The current graph in figure 4.11.9 gives an o.t.e. of the graph $K_{22}-K_3$. The dual map contains the portion shown in figure 4.11.10. We make the modification shown in figure 4.11.11. In particular we divide country 0 into two parts named 0 and 19. Finally we add

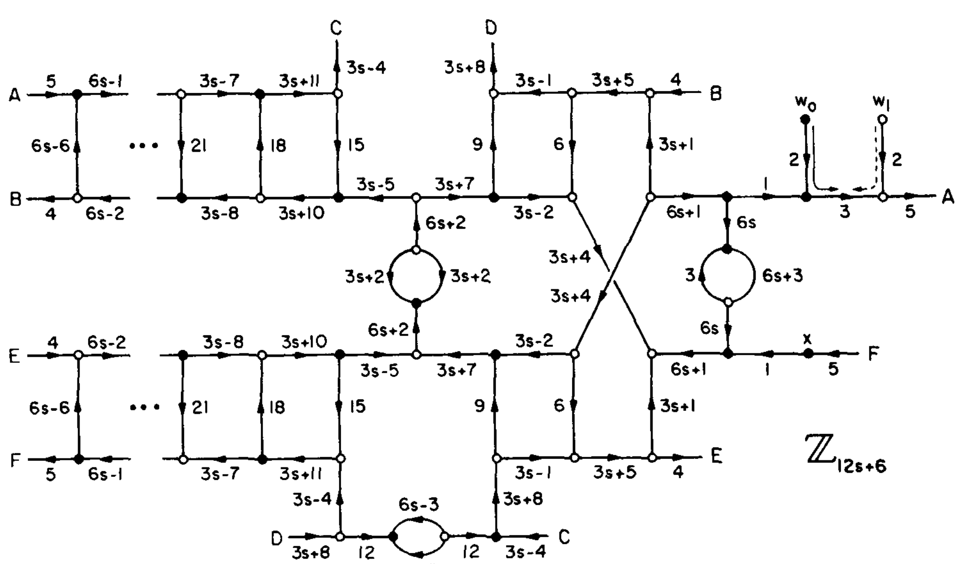

Figure 4.12.5

a handle at the vertices marked I to obtain 6 adjacencies shown in figure 4.11.12. This gives an o.t.e. of a graph of the type $K_{23}-16$ edges.

The current graph in figure 4.11.13 gives an o.t.e. of $K_{23}-K_5$. The dual contains the portion shown in figure 4.11.14, which we modify as in figure 4.11.15. Then we add a handle replacing the vertices marked I by the portion shown in figure 4.11.16. This results in an o.t.e. of K_{23} minus the four edges (0, 7), (0, 2), (0, 17), and (a, c).

4.12. *Case* 8. Let n be of the form $n=12s+8$. First we consider the pairs $(n, 10)$, $(n, 16)$, ..., $(n, n-10)$. If $n=20$ we use the o.t.e. of the octahedron graph O_{10} which is constructed in [8]. We use figure 4.12.1 for $n=32$ and figure 4.12.2 for $n \geqslant 44$, to obtain o.t.e.'s of $K_n - K_5$. All these have subtractable handles (figure 3.3 applies). Now only the pairs $(n, 4)$ remain. If $n=20$ consider the index two current graph of figure 4.12.3. Note that the two vortices w_0, w_1 are arranged just as in figure 4.10.1. Using the same construction as in figures 4.10.2 and 4.10.3 we merge w_0 and w_1 into one country w. We obtain a map with 20 countries missing only four adjacencies.

For $n=12s+8 \geqslant 32$ and s even we use the same method with figure 4.12.4. For s odd and $s \geqslant 3$ use figure 4.12.5. We note in both cases that the vortices w_0 and w_1 are arranged in the manner which allows the modification as shown in figures 4.10.2 and 4.10.3.

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