SEPARATRICES AT SINGULAR POINTS OF PLANAR VECTOR FIELDS

BY

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§ 1. Introduction

Let (0, 0) be an isolated singular point of the real analytic vector field

$$\begin{aligned} \dot{x} &= X_d(x, y) + X_{d+1}(x, y) + \dots = X(x, y) \\ \dot{y} &= Y_d(x, y) + Y_{d+1}(x, y) + \dots = Y(x, y) \end{aligned} \tag{1.1}$$

where X_i, Y_i are homogeneous of degree i and $X_d^2 + Y_d^2 \not\equiv 0$. We will call the integer $d \geqslant 1$ the *degree* of the singularity at (0, 0). If (0, 0) is neither a center nor a focus, then a small enough neighborhood of (0, 0) can be decomposed into a finite number of elliptic, hyperbolic and parabolic sectors (for precise definitions and a proof see, for example, [1], [2] or [4]). Bendixson [2] noticed that each hyperbolic sector must contain a branch of $xX + yY = 0$ and each elliptic sector must contain a branch of $X = 0$. He concluded that there are at most $2d + 2$ hyperbolic and $2d$ elliptic sectors. By a *separatrix* at (0, 0) we mean a solution curve of (1.1) that is the boundary of a hyperbolic sector at (0, 0). Since each hyperbolic sector has two boundaries, there are at most $4d + 4$ separatrices at (0, 0). The main result of this paper, proved in Section 3, is that the number of separatrices at (0, 0) is actually bounded by four if $d = 1$, six if $d = 2$, and $4d - 4$ if $d \geqslant 3$. We give examples to show that these bounds are sharp.

Our result is proved by repeatedly blowing up the singularity (0, 0) of (1.1), a technique that goes back to Bendixson [2] and has been used by many authors (e.g. [1], [3], [5], [8], [9]). We review blowing up in Section 2. Most recent uses of blowing up focus on classifying degenerate singularities of low codimension (i.e., not *too* degenerate). By contrast, we use the technique to determine the most degenerate behavior that can occur at a singularity of given degree. To accomplish this we use counting arguments to keep track of what happens as we repeatedly blow up; these arguments seem to be new.

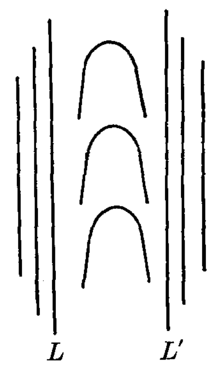


Figure 1. L and L' are inseparable leaves.

We became interested in counting the number of separatrices at a singular point by way of our interest in foliations of the plane given by the integral curves of a nowhere vanishing polynomial vector field. Recall that a leaf L of a foliation is said to be *inseparable* if there exists another leaf L' such that any two neighborhoods of L and L' in the leaf space intersect; see Figure 1. Inseparable leaves of a planar polynomial foliation correspond to separatrices of a "singular point at infinity". In [6] we report some results on the possible number of inseparable leaves of a planar foliation given by a polynomial vector field of fixed degree.

Bendixson knew his bound of $2d+2$ for the number of hyperbolic sectors at a singularity of degree d was sharp ([2], p. 32), and in fact it is easy to give examples (Example 1 of Section 2). On the other hand, Bendixson does not claim that his bound of $2d$ for the number of elliptic sectors is sharp ([2], p. 31). In fact, we have shown that the sharp bound for the number of elliptic sectors at a singularity of degree d is $2d-1$ for every $d \geq 1$, [7].

Finally we note that the assumption that $(0, 0)$ is an isolated singular point of an analytic vector field can be weakened to: $(0, 0)$ is a singular point of a C^∞ vector field that satisfies a Łojasiewicz condition at $(0, 0)$. This point is briefly discussed in Section 2. A vector field $\mathbf{X}(x, y)$ is said to satisfy a Łojasiewicz condition at $(0, 0)$ if there exist a positive integer k and positive constants c and δ such that $\|\mathbf{X}(x, y)\| \geq c\|(x, y)\|^k$ for all (x, y) with $\|(x, y)\| \leq \delta$.

§ 2. Blowing up

We will review blowing up in some detail and calculate a few relevant examples. Let $\mathbf{X}$ be a vector field on $\mathbf{R}^2$ of the form (1.1) with $d \geq 1$.

Consider the map η_1 from the $x\lambda$-plane to the xy-plane given by

$$\eta_1(x, \lambda) = (x, \lambda x).$$

The image of η_1 is the xy-plane with $\{(0, y): y \neq 0\}$ removed. η_1 collapses the λ-axis to $(0, 0)$ and takes each line $\lambda=\lambda_0$ to the line $y=\lambda_0 x$. The half plane $x>0$ (resp. $x<0$) of the $x\lambda$-plane is mapped diffeomorphically onto the half plane $x>0$ (resp. $x<0$) of the xy-plane.

Pull back $\mathbf{X}|\mathbf{R}^2-\{y\text{-axis}\}$ by η_1 to obtain a vector field $\eta_1^*\mathbf{X}$ on $\mathbf{R}^2-\{\lambda\text{-axis}\}$. $\eta_1^*\mathbf{X}$ is defined by

$$\begin{aligned} \dot{x} &= \sum_{i=0}^{\infty} x^{d+i} X_{d+i}(1,\lambda) = x^d X_d(1,\lambda) + x^{d+1} X_{d+1}(1,\lambda) + \dots \\ \dot{\lambda} &= \sum_{i=0}^{\infty} x^{d-1+i}[Y_{d+i}(1,\lambda) - \lambda X_{d+i}(1,\lambda)] \\ &= x^{d-1} Y_d(1,\lambda) + x^d Y_{d+1}(1,\lambda) + \dots - \lambda[x^{d-1} X_d(1,\lambda) + x^d X_{d+1}(1,\lambda) + \dots]. \end{aligned} \tag{2.1}$$

Since $d \geq 1$, $\eta_1^*\mathbf{X}$ extends to an analytic vector field on the $x\lambda$-plane which we still denote $\eta_1^*\mathbf{X}$. There are now two cases to consider.

1. $xY_d(x, y) - yX_d(x, y) \not\equiv 0$. We divide (2.1) by x^{d-1} to get a new analytic vector field $\bar{\mathbf{X}}^1$ defined by

$$\begin{aligned} \dot{x} &= \sum_{i=0}^{\infty} x^{1+i} X_{d+i}(1,\lambda) = x X_d(1,\lambda) + x^2 X_{d+1}(1,\lambda) + \dots \\ \dot{\lambda} &= \sum_{i=0}^{\infty} x^{i}[Y_{d+i}(1,\lambda) - \lambda X_{d+i}(1,\lambda)] \\ &= Y_d(1,\lambda) + x Y_{d+1}(1,\lambda) + \dots - \lambda[X_d(1,\lambda) + x X_{d+1}(1,\lambda) + \dots]. \end{aligned} \tag{2.2}$$

$\bar{\mathbf{X}}^1$ and $\eta_1^*\mathbf{X}$ have the same solution curves on $\mathbf{R}^2-\{\lambda\text{-axis}\}$, so η_1 sends solution curves of $\bar{\mathbf{X}}^1$ to solution curves of $\mathbf{X}$. The λ-axis is invariant under $\bar{\mathbf{X}}^1$ and there are at most $d+1$ singularities of $\bar{\mathbf{X}}^1$ on the λ-axis. The least degenerate case occurs when at each singularity of $\bar{\mathbf{X}}^1$ on the λ-axis, the linear part of $\bar{\mathbf{X}}^1$ has two nonzero eigenvalues. Figure 2 shows a typical example of the flow of $\bar{\mathbf{X}}^1$ near the λ-axis in this case.

2. $xY_d(x, y) - yX_d(x, y) \equiv 0$. In this case we call $(0, 0)$ a *special* singularity. We have $X_d = xQ_{d-1}(x, y)$ and $Y_d = yQ_{d-1}(x, y)$ where Q_{d-1} is homogeneous of degree $d-1$ and $Q_{d-1} \not\equiv 0$. (If $Q_{d-1} \equiv 0$ we would have $X_d \equiv Y_d \equiv 0$.) We divide by x^d to get an analytic vector field, again denoted $\bar{\mathbf{X}}^1$, defined by

$$\begin{aligned} \dot{x} &= \sum_{i=0}^{\infty} x^{i} X_{d+i}(1,\lambda) = Q_{d-1}(1,\lambda) + x X_{d+1}(1,\lambda) + \dots \\ \dot{\lambda} &= \sum_{i=1}^{\infty} x^{i-1}[Y_{d+i}(1,\lambda) - \lambda X_{d+i}(1,\lambda)] \\ &= Y_{d+1}(1,\lambda) + x Y_{d+2}(1,\lambda) + \dots - \lambda[X_{d+1}(1,\lambda) + x X_{d+2}(1,\lambda) + \dots]. \end{aligned}$$

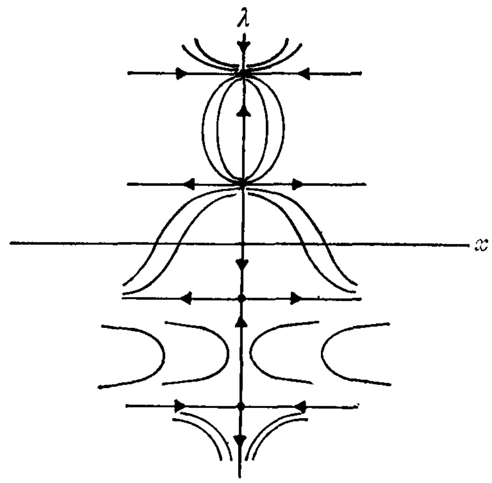


Figure 2

$\bar{\mathbf{X}}^1$ is transverse to the λ-axis except at at most $d-1$ points $(0, \lambda)$ with $Q_{d-1}(1, \lambda)=0$. The least degenerate case occurs when $Y_{d+1}(1, \lambda)-\lambda X_{d+1}(1, \lambda) \neq 0$ for each λ such that $Q_{d-1}(1, \lambda)=0$. Figure 3 shows a typical example of the flow of $\bar{\mathbf{X}}^1$ near the λ-axis when this condition is satisfied. In Figure 3, $Q_{d-1}(1, \lambda)$ changes sign at $\lambda=\lambda_1$ but not at $\lambda=\lambda_2$.

To see how the flow of $\bar{\mathbf{X}}^1$ near the λ-axis relates to the flow of $\mathbf{X}$ near $(0, 0)$ it is perhaps helpful to consider polar blowing up. Polar blowing up is conceptually simpler than algebraic blowing up, but is less tractable computationally. One can use the map φ: $S^1 \times \mathbf{R} \to \mathbf{R}^2$ defined by $\varphi(\theta, r)=(r \cos \theta, r \sin \theta)$ to replace $(0, 0)$ by a circle in the same way η_1 is used to replace $(0, 0)$ by a line. (In polar blowing up one divides by r^{d-1} or r^d.) The result, if $xY_d-yX_d \not\equiv 0$, is a vector field $\mathbf{X}^P$ on $S^1 \times \mathbf{R}$ that has $S^1 \times \{0\}$ as an invariant set on which there are at most $2d+2$ singularities of $\mathbf{X}^P$; if $xY_d-yX_d \equiv 0$, the result is a vector field $\bar{\mathbf{X}}^P$

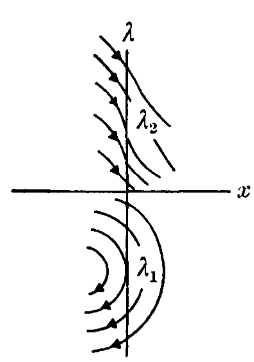


Figure 3

transverse to $S^1 \times \{0\}$ except at at most $2d-2$ points. When $xY_d - yX_d \not\equiv 0$, we have (after dividing by r^{d-1}) $\bar{\mathbf{X}}^P$ defined by

$$\dot{r} = \sum_{i=0}^{\infty} r^{i+1}[\cos\theta\, X_{d+i}(\cos\theta, \sin\theta) + \sin\theta\, Y_{d+i}(\cos\theta, \sin\theta)]$$

$$\dot{\theta} = \sum_{i=0}^{\infty} r^{i}[\cos\theta\, Y_{d+i}(\cos\theta, \sin\theta) - \sin\theta\, X_{d+i}(\cos\theta, \sin\theta)].$$

When $xY_d - yX_d \equiv 0$, we have (after dividing by r^d) $\bar{\mathbf{X}}^P$ defined by

$$\dot{r} = Q_{d-1}(\cos\theta, \sin\theta) + \sum_{i=1}^{\infty} r^{i}[\cos\theta\, X_{d+i}(\cos\theta, \sin\theta) + \sin\theta\, Y_{d+i}(\cos\theta, \sin\theta)]$$

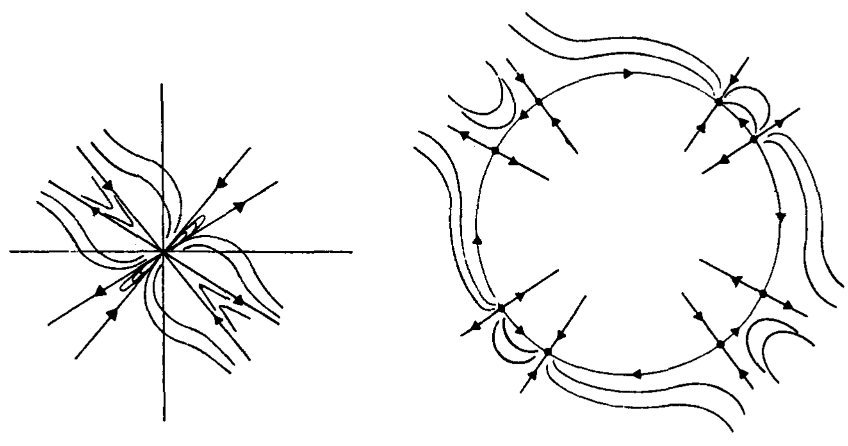
$$\dot{\theta} = \sum_{i=1}^{\infty} r^{i-1}[\cos\theta\, Y_{d+i}(\cos\theta, \sin\theta) - \sin\theta\, X_{d+i}(\cos\theta, \sin\theta)].$$

Figure 4 (a) shows the singular point of $\mathbf{X}$ that when blown up algebraically gave Figure 2; Figure 4 (b) shows the result of polar blowing up of this singularity. Figures 5 (a) and 5 (b) bear the same relation to Figure 3. The reader should notice that when one blows up algebraically, the phase portrait of $\bar{\mathbf{X}}^1$ in the half-plane $x<0$ is "upside down" by comparison to the phase portrait of $\mathbf{X}$ in the left half-plane, or by comparison to the phase portrait of $\bar{\mathbf{X}}^P$ in $\{(r, \theta): r>0, \pi/2<\theta<3\pi/2\}$.

The formal correspondence between the local behavior at $(0, 0)$ of $\mathbf{X}$ and the local behavior near $S^1 \times \{0\}$ of $\bar{\mathbf{X}}^P$ is given by the following theorem of Bendixson ([1], [2] or [4]). By a *positive* (resp. *negative*) *semipath*, we mean a solution curve defined for all $t \geqslant t_0$ (resp. for all $t \leqslant t_0$).

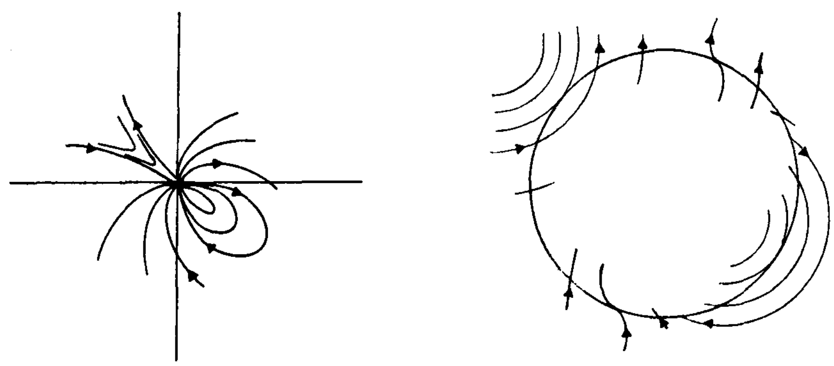
THEOREM 2.1. *Any semipath of* (1.1) *that tends to* $(0, 0)$ *is either a spiral or tends to* $(0, 0)$ *in a limiting direction* θ^*. *If at least one positive* (*resp. negative*) *semipath of* (1.1) *is a spiral tending to* $(0, 0)$ *as* $t \to \infty$ (*resp. as* $t \to -\infty$), *then all positive* (*resp. negative*) *semipaths passing through points of some neighborhood of* $(0, 0)$ *are also spirals. If* $xY_d - yX_d \not\equiv 0$, *all directions* θ^* *along which semipaths tend to* $(0, 0)$ *satisfy the equation* $\cos\theta\, Y_d(\cos\theta, \sin\theta) - \sin\theta\, X_d(\cos\theta, \sin\theta) = 0$. *If* $xY_d - yX_d \equiv 0$, *then* $Y_d = yQ_{d-1}$ *and* $X_d = xQ_{d-1}$. *In this case, for all* θ *not satisfying* $Q_{d-1}(\cos\theta, \sin\theta) = 0$, *we have exactly one semipath tending to* $(0, 0)$ *in that direction. If* θ^* *satisfies* $Q_{d-1}(\cos\theta, \sin\theta) = 0$, *there may be no semipaths tending to* $(0, 0)$ *in that direction, a finite number or infinitely many.*

We can formalize the relation between the flows of $\bar{\mathbf{X}}^P$ and $\bar{\mathbf{X}}^1$ as follows. Let U be a neighborhood of $S^1 \times \{0\}$ in $S^1 \times \mathbf{R}$ and let $U_+ = U \cap \{(\theta, r): r>0\}$. φ maps U_+ diffeomor-



(a) (b)

Figure 4



(a) (b)

Figure 5

phically onto a deleted neighborhood of $(0, 0)$ in $\mathbf{R}^2$ and takes solution curves of $\bar{\mathbf{X}}^P$ to those of $\mathbf{X}$. Let $U_{+R} = U_+ \cap \{(\theta, r)\colon -\pi/2 < \theta < \pi/2\}$, $U_{+L} = U_+ \cap \{(\theta, r)\colon \pi/2 < \theta < 3\pi/2\}$. Given any neighborhood V of the λ-axis we let $V_R = V \cap \{(x, \lambda)\colon x > 0\}$, $V_L = V \cap \{(x, \lambda)\colon x < 0\}$. Now let $V = \eta_1^{-1} \circ \varphi(U_+)$. Then $V_R = \eta_1^{-1} \circ \varphi(U_{+R})$, $V_L = \eta_1^{-1} \circ \varphi(U_{+L})$, and $\eta_1^{-1} \circ \varphi$ sends solution curves of $\bar{\mathbf{X}}^P$ to those of $\bar{\mathbf{X}}^1$.

Clearly $\bar{\mathbf{X}}^1$ gives us a complete picture of the solution curves of $\mathbf{X}$ in a neighborhood of $(0, 0)$ with the y-axis removed. Now suppose $xY_d - yX_d \not\equiv 0$ and x is not a factor of $xY_d - yX_d$. Then the polar vector field $\bar{\mathbf{X}}^P$ is nonsingular at $(\pi/2, 0)$ and $(-\pi/2, 0)$. It follows that $\bar{\mathbf{X}}^P$ is transverse to $\theta = \pi/2$ near $(\pi/2, 0)$ and to $\theta = -\pi/2$ near $(-\pi/2, 0)$. Then we can easily "fill in" the part of the phase portrait of $\mathbf{X}$ near 0 that blowing up via η_1 does not tell us. (Notice, however, that we will be unable to distinguish a center from a focus. This difficulty will not be important in the sequel.) Similarly, suppose $xY_d - yX_d \equiv 0$ but x is not a factor of Q_{d-1}. Then $\overline{\mathbf{X}}^P$ is transverse to $S^1 \times \{0\}$ near $(\pi/2, 0)$ and $(-\pi/2, 0)$, and again we can easily fill in the part of the phase portrait of $\mathbf{X}$ near 0 that blowing up via η_1 does not tell us.

There are two cases in which blowing up via η_1 is inadequate:

1 (a) $xY_d - yX_d \not\equiv 0$ and x divides $xY_d - yX_d$.

2 (a) $xY_d - yX_d \equiv 0$ and x divides Q_{d-1}.

To study these two cases, we will use the map η_2 from the μy-plane to the xy-plane given by

$$\eta_2(\mu, y) = (\mu y, y).$$

The image of η_2 is the xy-plane with $\{(x, 0): x \neq 0\}$ removed. η_2 collapses the μ-axis to $(0, 0)$ and takes each line $\mu = \mu_0$ to the line $x = \mu_0 y$. Pull back $\mathbf{X} \,|\, \mathbf{R}^2 - \{x\text{-axis}\}$ by η_2 to obtain a vector field $\eta_2^* \mathbf{X}$ on $\mathbf{R}^2 - \{\mu\text{-axis}\}$. In case 1 (a), define $\overline{\mathbf{X}}^2$ to be $(1/y^{d-1})\eta_2^* \mathbf{X}$, extended analytically to the μ-axis. We get

$$\begin{aligned} \dot{\mu} &= \sum_{i=0}^{\infty} y^i [X_{d+i}(\mu, 1) - \mu Y_{d+i}(\mu, 1)] \\ &= X_d(\mu, 1) + yX_{d+1}(\mu, 1) + \dots - \mu[Y_d(\mu, 1) + yY_{d+1}(\mu, 1) + \dots] \\ \dot{y} &= \sum_{i=0}^{\infty} y^{1+i} Y_{d+i}(\mu, 1) = yY_d(\mu, 1) + y^2 Y_{d+1}(\mu, 1) + \dots . \end{aligned}$$

In case 2 (a), define $\overline{\mathbf{X}}^2$ to be $(1/y^d)\eta_2^* \mathbf{X}$, extended analytically to the μ-axis. We get

$$\begin{aligned} \dot{\mu} &= \sum_{i=1}^{\infty} y^{i-1} [X_{d+i}(\mu, 1) - \mu Y_{d+i}(\mu, 1)] \\ &= X_{d+1}(\mu, 1) + yX_{d+2}(\mu, 1) + \dots - \mu[Y_{d+1}(\mu, 1) + yY_{d+2}(\mu, 1) + \dots] \\ \dot{y} &= \sum_{i=0}^{\infty} y^i Y_{d+i}(\mu, 1) = Q_{d-1}(\mu, 1) + yY_{d+1}(\mu, 1) + \dots . \end{aligned}$$

It is easy to see how the flow of $\overline{\mathbf{X}}^2$ near the μ-axis relates to the flow of $\overline{\mathbf{X}}^P$ near $S^1 \times \{0\}$ and to the flow of $\mathbf{X}$ near $(0, 0)$. We leave the details to the reader.

We will be interested in keeping track of how many new singularities are created when we blow up the singularity of $\mathbf{X}$ at $(0, 0)$. To avoid counting some new singularities

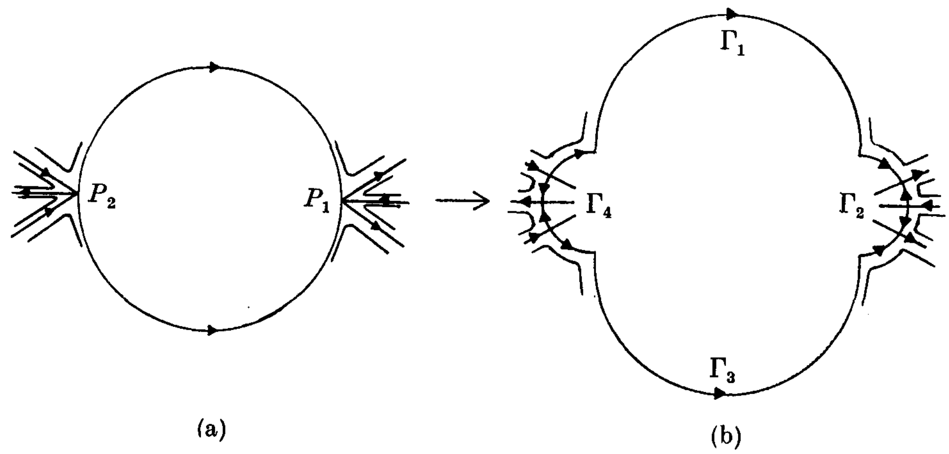


Figure 6

twice, and for the sake of consistency, we will always proceed as follows. If x does not divide $xY_d - yX_d$ (case 1) or Q_{d-1} (case 2) we will blow up using only η_1. Otherwise, choose a number $a>0$ such that every root of $Y_d(1,\lambda)-\lambda X_d(1,\lambda)=0$ (case 1) or of $Q_{d-1}(1,\lambda)=0$ (case 2) has absolute value less than a. Restrict $\bar{\mathbf{X}}^1$ to $\{(x,\lambda)\colon |\lambda|<a\}$ and restrict $\bar{\mathbf{X}}^2$ to $\{(\mu,y)\colon |\mu|<1/a\}$. Notice that in case 1 (a) the restricted $\bar{\mathbf{X}}^2$ has only one singularity on the μ-axis, at $(0,0)$; in case 2 (a) the restricted $\bar{\mathbf{X}}^2$ is transverse to the μ-axis except at the one point $(0,0)$. The restricted $\bar{\mathbf{X}}^1$ tells us the flow of $\mathbf{X}$ in $\{(x,y)\colon |y|<a|x|\}$ and the restricted $\bar{\mathbf{X}}^2$ tells us the flow of $\mathbf{X}$ in $\{(x,y)\colon |y|>a|x|\}$. Near $(0,0)$ it is easy to fill in the missing part of the flow of $\mathbf{X}$, subject to the caveat about distinguishing a center from a focus.

After blowing up we may wish to analyze a singularity of $\bar{\mathbf{X}}^1$ at $(0,\lambda_0)$ or a singularity of $\bar{\mathbf{X}}^2$ at $(0,0)$. In the first case, translate $(0,\lambda_0)$ to the origin by the map $(x,\lambda)\mapsto(x,\lambda-\lambda_0)$, relabel the second variable $\lambda=y$, and blow up using η_1 and, if necessary, η_2. In the second case, relabel $\mu=x$ and blow up. This procedure can be repeated as often as we wish.

Dumortier [3] has described a useful way of thinking about repeated blowing up. We will describe Dumortier's idea pictorially; for a more computational description see [3]. Begin by using polar blowing up, and identify $\{(\theta,r)\colon r>-1\}$ with $\mathbf{R}^2-\{(0,0)\}$ via $(\theta,r)\mapsto((r+1)\cos\theta,(r+1)\sin\theta)$. We get a vector field on $\mathbf{R}^2-\{(0,0)\}$ which we denote $\mathbf{X}^P$. The circle $C\colon x^2+y^2=1$ corresponds to the original origin and its exterior corresponds to the original $\mathbf{R}^2-\{(0,0)\}$. Suppose there is a singularity $(0,\lambda_0)$ of $\bar{\mathbf{X}}^1$ that we wish to analyze further. It corresponds to singularities of $\mathbf{X}^P$ at P_1 and $P_2=-P_1$ on C. By using polar blowing up at P_1 and at P_2 we can obtain an analytic vector field $\mathbf{Y}$ on an open subset of $\mathbf{R}^2$ that includes $\Gamma\cup\operatorname{ext}\Gamma$ ($\operatorname{ext}\Gamma=$exterior of Γ), where Γ is a curve in $\mathbf{R}^2$ homeomorphic to S^1 (see Figure 6); $\mathbf{Y}$ and Γ have the following properties:

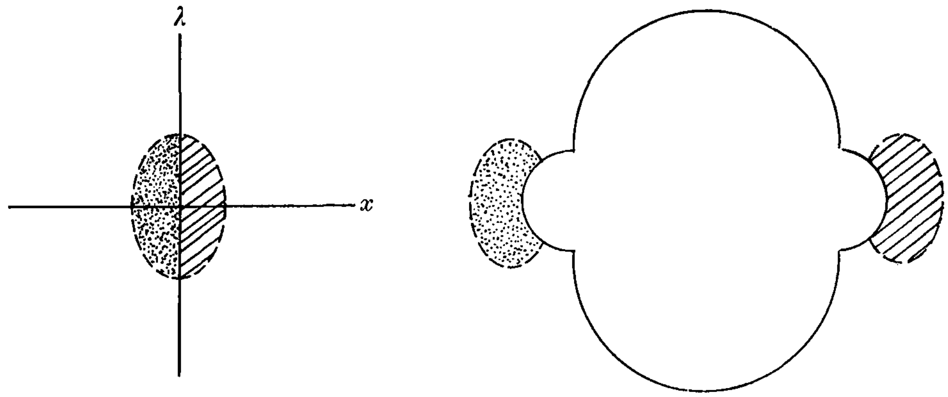


Figure 7

1. $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$, where each Γ_i is a closed analytic arc and successive Γ_i intersect transversally.

2. There is a continuous map from $\Gamma \cup \operatorname{ext} \Gamma$ to $\mathbf{R}^2$ that takes Γ to $(0, 0)$, takes $\operatorname{ext} \Gamma$ diffeomorphically onto $\mathbf{R}^2 - \{(0, 0)\}$, and takes solution curves of $\mathbf{Y}$ to solution curves of $\mathbf{X}$.

3. For $i = 1, 2, 3, 4$, $\mathbf{Y}|\Gamma_i$ is either tangent to Γ_i with a finite number of singularities or transverse to Γ_i except at a finite number of points.

4. Let $\mathring{\Gamma}_i$ denote Γ_i minus its endpoints. Then $\mathbf{Y}|\mathring{\Gamma}_1 \cup \mathring{\Gamma}_3$ is conjugate to $\mathbf{X}^P|C - \{P_1, P_2\}$.

In effect, we have replaced $\{P_1, P_2\}$ by $\Gamma_2 \cup \Gamma_4$.

To see the relationship of $\mathbf{Y}$ to the vector fields obtained by algebraic blowing up, assume for simplicity that $\lambda_0 = 0$. Then $\overline{\mathbf{X}}^1$ has a singularity at $(0, 0)$ that corresponds to $P_1 = (1, 0)$ and $P_2 = (-1, 0)$. Blowing up the singularity of $\overline{\mathbf{X}}^1$ at $(0, 0)$ via η_1, η_2 yields two new vector fields which we denote respectively $\overline{\mathbf{X}}^{11}$, $\overline{\mathbf{X}}^{12}$. The flows of $\overline{\mathbf{X}}^{11}$ and of $\mathbf{Y}$ in the slashed and dotted regions of Figure 7 correspond. The phase portrait of $\overline{\mathbf{X}}^{11}$ in the dotted region of the $x\lambda$-plane will be "right-side up" by comparison to the phase portrait of Y in the dotted region of the Dumortier picture, because two successive orientation reversals have cancelled each other. Notice that $\overline{\mathbf{X}}^{11}|\lambda$-axis corresponds to both $\mathbf{Y}|\Gamma_2$ and $\mathbf{Y}|\Gamma_4$. Similarly, in Figure 8 the flows of $\overline{\mathbf{X}}^{12}$ and of $\mathbf{Y}$ in the regions labeled 1, 2, 3, 4 correspond. The μ-axis corresponds to $\Gamma_2 \cup \Gamma_4$, and the y-axis to $\Gamma_1 \cup \Gamma_3$. The origin of the μy-plane corresponds to the four corners of Γ. Notice that $\overline{\mathbf{X}}^{11}$ alone would not allow us to study the behavior of $\mathbf{Y}$ at the corners of Γ.

Consider a singularity $(0, \lambda_0)$ of $\overline{\mathbf{X}}^{11}$ that we wish to analyze further. It corresponds to singularities of $\mathbf{Y}$ at $P_1 \in \Gamma_2$ and at $P_2 \in \Gamma_4$. In Dumortier's scheme blowing up the singu-

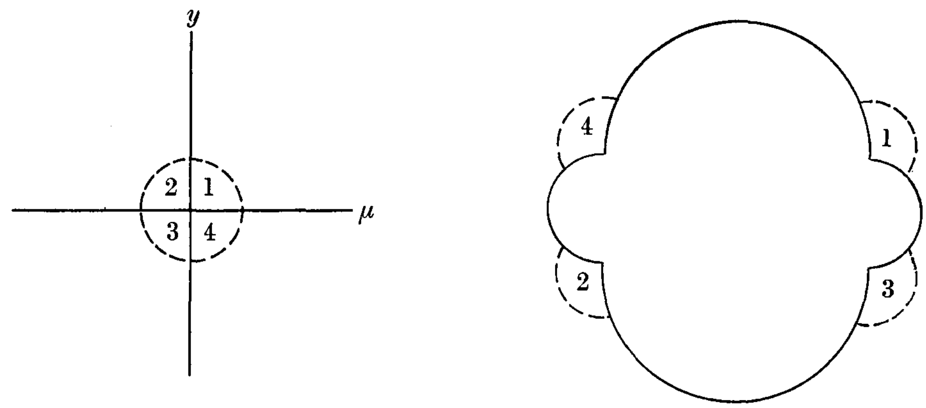


Figure 8

larity of $\bar{\mathbf{X}}^{11}$ at $(0, \lambda_0)$ corresponds to replacing each of P_1 and P_2 by a new arc of Γ. See Figure 9.

On the other hand, blowing up a singularity of $\bar{\mathbf{X}}^{12}$ at the origin of the μy-plane corresponds to replacing the four corners of Γ by arcs. See Figure 10.

In general, each time we blow up a singularity algebraically (whether we use η_1 or η_1 and η_2) we replace two or four points of Γ by arcs—two points if the singularity we blow up does not correspond to a corner of Γ, four points if it does. At each stage, in Dumortier's scheme, we have a piecewise analytic curve Γ in $\mathbf{R}^2$ homeomorphic to S^1, a vector field $\mathbf{Y}$ defined on an open subset of $\mathbf{R}^2$ that includes $\Gamma \cup \operatorname{ext} \Gamma$, and a continuous map $\Gamma \cup \operatorname{ext} \Gamma \to \mathbf{R}^2$ that takes Γ to $(0, 0)$, takes $\operatorname{ext} \Gamma$ diffeomorphically onto $\mathbf{R}^2 - \{(0, 0\}$, and takes solution curves of $\mathbf{Y}$ to those of $\mathbf{X}$.

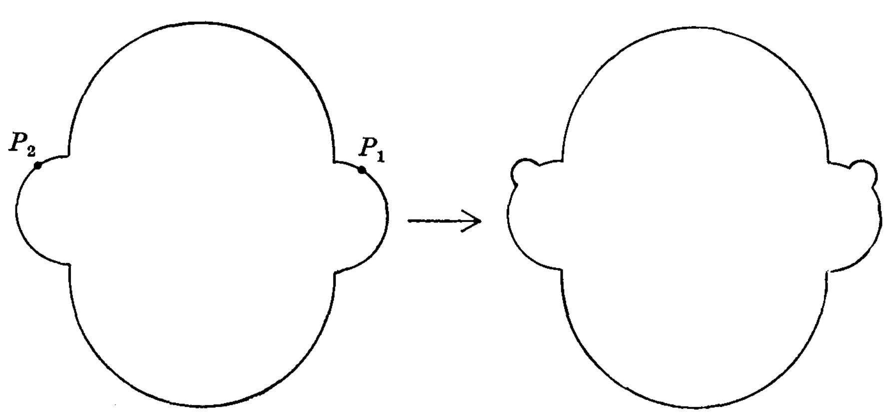


Figure 9

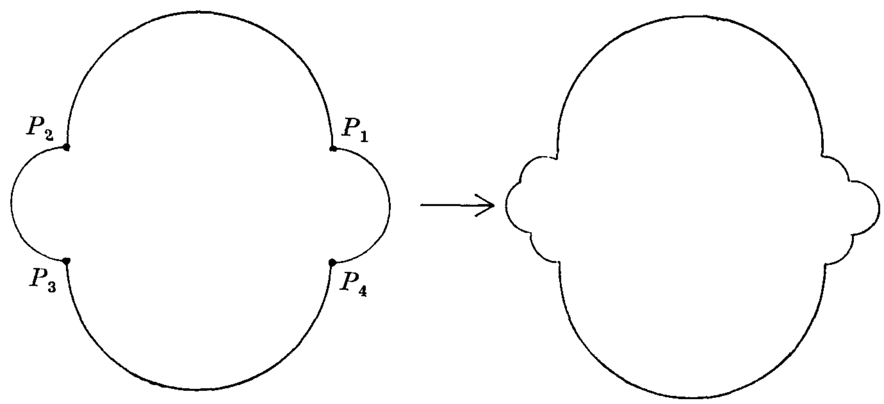


Figure 10

Suppose after some number of algebraic blow-ups we have a vector field $\bar{\mathbf{X}}$ in the $x\lambda$-plane with a singularity at $(0, \lambda_0)$. Let $\mathbf{Y}$ denote the Dumortier vector field arising at the same stage, and let Γ denote the homeomorph of S^1 that corresponds to the origin of the original xy-plane. Suppose $(0, \lambda_0)$ corresponds to exactly two points P_1, P_2 of Γ. Then there is a one-to-one correspondence between semipaths of $\bar{\mathbf{X}}$ in $\mathbf{R}^2 - \{\lambda\text{-axis}\}$ that converge to $(0, \lambda_0)$ and semipaths of $\mathbf{Y}$ in ext Γ that converge to P_1 or P_2. The λ-axis corresponds to two arcs of Γ. On the other hand suppose $(0, 0) \in x\lambda$-plane corresponds to four corners P_1, P_2, P_3, P_4 of Γ. (We have arranged that only the origin of the $x\lambda$-plane or μy-plane can correspond to corners of Γ.) Then there is a one-to-one correspondence between semipaths of $\bar{\mathbf{X}}$ in $\mathbf{R}^2 - \{x\text{-axis} \cup \lambda\text{-axis}\}$ that converge to $(0, 0)$ and semipaths of $\mathbf{Y}$ in ext Γ that converge to P_1, P_2, P_3, P_4. The x-axis and λ-axis both correspond to arcs of Γ. Similar remarks apply to $(0, 0) \in \mu y$-plane.

We now describe the *tree* $\mathcal{J}$ of a singularity, which is a certain directed graph. The vertices of $\mathcal{J}$ represent certain germs of vector fields at $(0, 0)$, which will always be written in the form (1.1). The initial vertex of $\mathcal{J}$, from which the tree will grow, represents the singularity we wish to analyze. Edges of $\mathcal{J}$ originate at this initial vertex and terminate at vertices representing the singularities (or tangencies if $xY_d - yX_d \equiv 0$) that appear when we blow up the original singularity. If one of the new singularities requires further blowing up, edges originate at the vertex that represents it and terminate at vertices that represent the resulting singularities (or tangencies). More precisely:

A vertex V of a directed graph G is called *terminal* (in G) if no directed edge of G originates at V. Nonterminal vertices of $\mathcal{J}$ all represent singularities. Given a nonterminal vertex V of $\mathcal{J}$, we now describe the edges of $\mathcal{J}$ that originate at V and the vertices at which they terminate. Let V represent a singular point $(0, 0)$ of type (1.1).

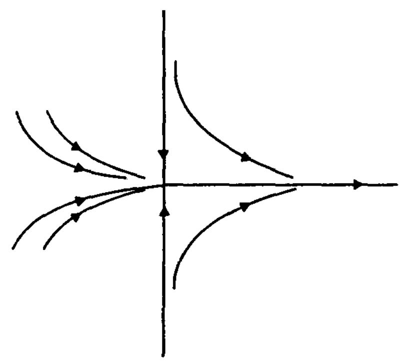


Figure 11. A saddle-node.

1. If $xY_d - yX_d \not\equiv 0$ compute $\bar{X}^1$ and, if necessary, $\bar{X}^2$. There is one edge originating at V for each singularity of $\bar{X}^1$ on the λ-axis. Each such edge terminates at a vertex representing the corresponding singularity of $\bar{X}^1$. In order that this singularity be of the form (1.1), we translate to the origin and relabel $\lambda = y$. If $\bar{X}^2$ is needed, there is one additional edge originating at V. It terminates at a vertex that represents the singularity of $\bar{X}^2$ at (0, 0), with the relabeling $\mu = x$.

2. If $xY_d - yX_d \equiv 0$, compute $\bar{X}^1$ and, if necessary, $\bar{X}^2$. There is one edge originating at V for each point on the λ-axis where $\bar{X}^1$ is not transverse to the λ-axis. Each of these edges terminates at a vertex representing the corresponding germ of $\bar{X}^1$, translated to the origin and relabeled as above. If $\bar{X}^2$ is needed, there is an additional edge originating at V; it terminates at a vertex that represents the germ at (0, 0) of $\bar{X}^2$, with the relabeling $\mu = x$. Note that these new vertices may not represent singularities.

A vertex V is terminal in $\mathcal{J}$ iff V represents one of the following types of germs:

1. A nonsingular germ. These can only occur when the vertex immediately preceding V represents a special singularity.

2. A germ of form (1.1) such that $xY_d - yX_d$ has no nonconstant linear factors, or such that $xY_d - yX_d \equiv 0$ and Q_{d-1} has no nonconstant linear factors. Such a germ is a node or a focus.

3. A germ of form (1.1) with $d = 1$ and at least one nonzero eigenvalue of its linear part. Such a germ is a node, focus, saddle, or saddle-node ([1], [2] or [4]; see Figure 11).

The following result about blowing up is fundamental:

THEOREM 2.2. *Let* (0, 0) *be a singularity of* (1.1) *with tree* $\mathcal{J}$. *Then* $\mathcal{J}$ *is a finite tree.*

Theorem 2.2 says that the blowing up process eventually reduces any singularity to easily understood singularities. Bendixson [2] proved Theorem 2.2 for analytic singularities

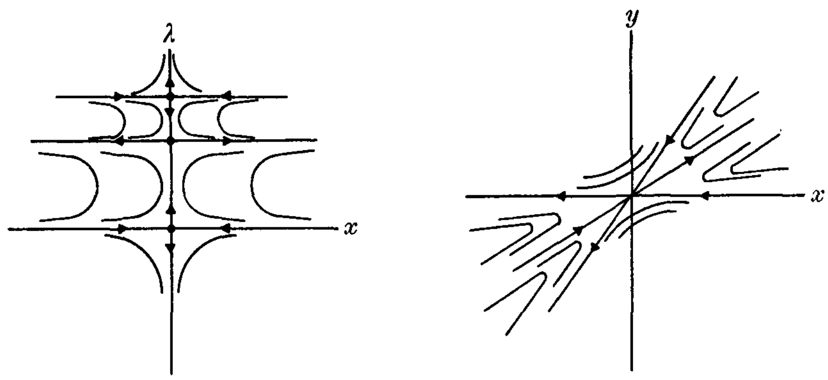


Figure 12. Example 1 with $d=2$. At left: $\overline{\mathbf{X}}^1$. At right: $\mathbf{X}$.

(1.1) with $\dot{x}$ and $\dot{y}$ relatively prime in the ring of real analytic functions. His proof also works for C^∞ singularities such that the Taylor series of $\dot{x}$ and $\dot{y}$ about the singularity are relatively prime in the ring of real power series. Dumortier [3] extended Theorem 2.2 to C^∞ singularities that satisfy a Łojasiewicz condition. It is Dumortier's version of Theorem 2.2 that allows us to consider any isolated analytic singularity. Other versions of Theorem 2.2 occur in [5] and [8].

We shall now give several examples of vector fields that illustrate phenomena regarding hyperbolic and elliptic sectors and separatrices.

Example 1. A singularity of degree d with $2d+2$ hyperbolic sectors and $2d+2$ separatrices. This example is a singular point with $xY_d-yX_d\not\equiv 0$, such that $\overline{\mathbf{X}}^1$ has $d+1$ singularities on the λ-axis, each a saddle. See Figure 12.

Let

$$F(x,y)=\begin{cases}\displaystyle\prod_{j=1}^{d}(y-(d+j+1)x) & \text{if } d \text{ is even}\\[2ex] \displaystyle-\prod_{j=1}^{d}(y-(d+j+1)x) & \text{if } d \text{ is odd.}\end{cases}$$

Note that $F(1,0)>0$, $(\partial F/\partial y)(1,0)<0$ and $F(1,0)+(\partial F/\partial y)(1,0)>0$. Let $X_d=F(x,y)+x(\partial F/\partial y)(x,y)$ and $Y_d=y(\partial F/\partial y)$ and consider the vector field

$$\dot{x}=X_d$$
$$\dot{y}=Y_d.$$

In this case, $xY_d-yX_d=-yF(x,y)$. Since x does not divide $-yF(x,y)$, we need only blow up using η_1. $\overline{\mathbf{X}}^1$ is then given by

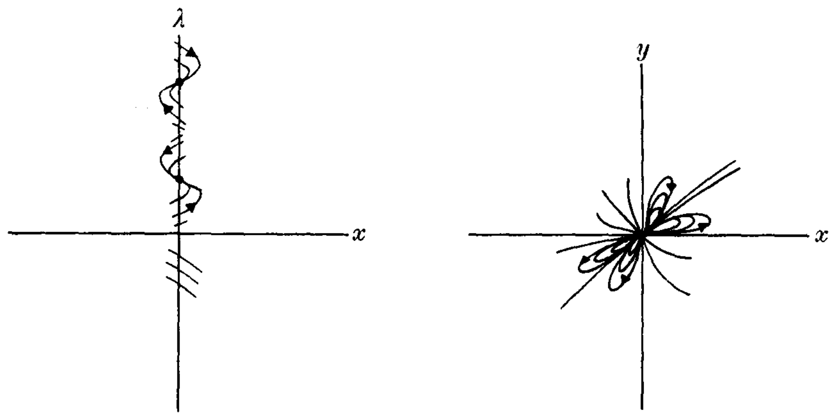


Figure 13. Example 2 with $d=3$. At left: $\bar{\mathbf{X}}^1$. At right: $\mathbf{X}$.

$$\dot{x} = x(F(1,\lambda) + (\partial F/\partial y)(1,\lambda))$$
$$\dot{\lambda} = -\lambda F(1,\lambda).$$

The points $(0, d+j+1)$, $j=1, ..., d$, and $(0, 0)$ are the only singularities of $\bar{\mathbf{X}}^1$ on the λ-axis. Each is of degree 1. Calculating $D\bar{\mathbf{X}}^1(0,\lambda)$ we get:

$$D\bar{\mathbf{X}}^1(0,\lambda) = \begin{pmatrix} F(1,\lambda) + (\partial F/\partial y)(1,\lambda) & 0 \\ 0 & -F(1,\lambda) - \lambda(\partial F/\partial y)(1,\lambda) \end{pmatrix}.$$

If $\lambda = d+j+1$, $j=1, ..., d$, we get $\det D\bar{\mathbf{X}}^1(0,\lambda) = -\lambda[(\partial F/\partial y)(1,\lambda)]^2 < 0$. If $\lambda = 0$ we get $\det D\bar{\mathbf{X}}^1(0,0) = -F(1,0)(F(1,0) + (\partial F/\partial y)(1,0)) < 0$. Thus all singularities are saddles.

Example 2. A singularity of degree d with $2d-2$ elliptic sectors. We will construct a special singularity with $d-1$ nodes in the blown up picture. See Figure 13.

Let

$$Q_{d-1} = \prod_{j=1}^{d-1} (y - jx),$$
$$X_d = xQ_{d-1}, \qquad Y_d = yQ_{d-1},$$
$$X_{d+1} = -xyQ_{d-1}, \quad Y_{d+1} = x^2 Q_{d-1},$$
$$X_{d+2} = 0, \qquad Y_{d+2} = -x^4 \frac{\partial Q_{d-1}}{\partial y};$$
$$X_{d+i} = Y_{d+i} = 0 \qquad \text{for} \quad i \geqslant 3.$$

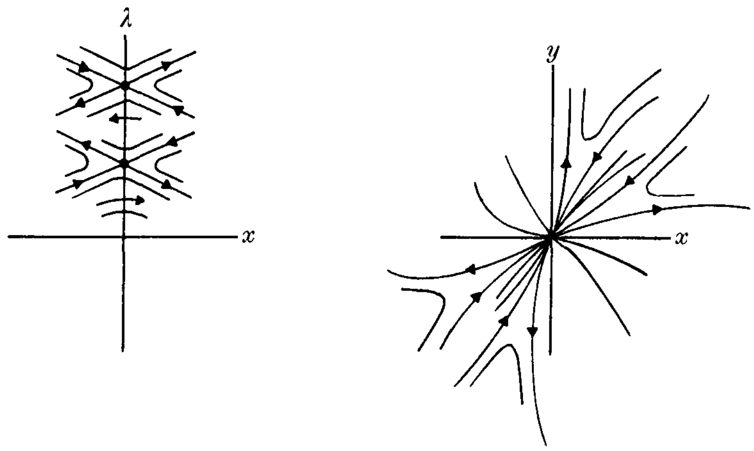


Figure 14. Example 3 with $d=3$. At left: $\bar{\mathbf{X}}^1$. At right: $\mathbf{X}$.

Notice $xY_d - yX_d \equiv 0$. Since x does not divide Q_{d-1}, we only have to blow up via η_1. For $\bar{\mathbf{X}}^1$ we get

$$\dot{x} = Q_{d-1}(1,\lambda) + xX_{d+1}(1,\lambda)$$

$$\dot{\lambda} = -\lambda^2 Q_{d-1}(1,\lambda) + \lambda Q_{d-1}(1,\lambda) - x\frac{\partial Q_{d-1}}{\partial y}(1,\lambda).$$

$\bar{\mathbf{X}}^1$ is transverse to the λ-axis except at $(0, i)$, $i=1, \ldots, d-1$. At each of these points $\bar{\mathbf{X}}^1$ has a singularity. We calculate for $i=1, \ldots, d-1$:

$$D\bar{\mathbf{X}}^1(0,i) = \begin{pmatrix} 0 & (\partial Q_{d-1}/\partial y)(1,i) \\ -(\partial Q_{d-1}/\partial y)(1,i) & \cdots \end{pmatrix}$$

The determinant is $[(\partial Q_{d-1}/\partial y)(1, i)]^2 > 0$, so each singularity of $\bar{\mathbf{X}}^1$ on the λ-axis is a node.

Example 3. A singularity of degree d with $4d-4$ separatrices. This example is a special singularity that when blown up yields $d-1$ saddles on the λ-axis. See Figure 14. Consider Example 2 and change Y_{d+2} to $x^4(\partial Q_{d-1}/\partial y)$. The calculation is similar to that for Example 2.

Example 4. Another singularity of degree d with $4d-4$ separatrices. This example appears at first glance to have $4d-2$ separatrices, but in fact there are only $4d-4$. We will return to this example when we prove Theorem 3.13.

The first blow-up yields two singularities: a saddle of degree 1 and a special singularity of degree d. The special singularity then blows up to yield $d-1$ degree 1 saddles. See Figure 15. Notice that the separatrices of the first saddle do not correspond to separa-

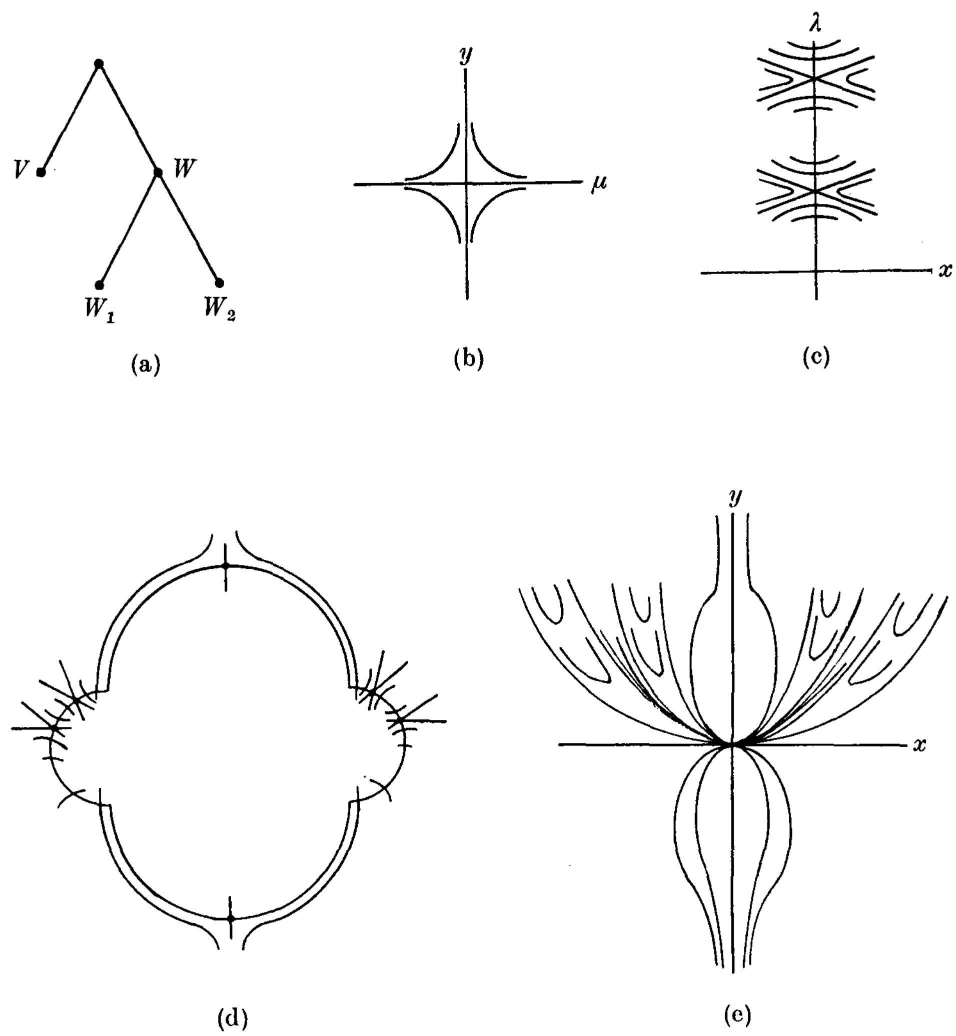


Figure 15. Example 4 with $d=3$. 15(a) shows the tree of the singularity: V represents a degree 1 saddle, W represents a special singularity of degree 3, and W_1 and W_2 represent degree 1 saddles. 15(b) shows the singularity represented by V. 15(c) shows the blow-up of the singularity represented by W. 15(d) shows the Dumortier picture. 15(e) shows the original singularity.

trices of the original singularity. Instead, they correspond to curves that lie in the middle of parabolic sectors of the original singularity.

Let

$$\dot{x} = xy^{d-1} + \sum_{i=2}^{d} a_i x^{2i-1} y^{d-i}$$

$$\dot{y} = 2y^d + 2\sum_{i=2}^{d} a_i x^{2i-2} y^{d+1-i} + \sum_{i=2}^{d} b_i x^{2i+2} y^{d-i}$$

where the a_i's and b_i's are chosen so that

$$\lambda^{d-1}+\sum_{i=2}^{d} a_i\lambda^{d-i}=\prod_{j=1}^{d-1}(\lambda-j)$$

$$\sum_{i=2}^{d} b_i\lambda^{d-i}=\frac{d}{d\lambda}\prod_{j=1}^{d-1}(\lambda-j).$$

Notice that $xY_d-yX_d=xy^d$, so we will have to blow up using both η_1 and η_2.

Blowing up via η_2, we get for $\bar{\mathbf{X}}^2$:

$$\dot{\mu}=-\mu+\mu^2(\ldots)$$
$$\dot{y}=2y+y^2(\ldots).$$

Therefore $(0, 0)$ is a saddle of $\bar{\mathbf{X}}^2$.

Blowing up via η_1, we get for $\bar{\mathbf{X}}^1$:

$$\begin{aligned}\dot{x}&=x\lambda^{d-1}+\sum_{i=2}^{d} a_i x^i\lambda^{d-i}=\bar{X}_d(x,\lambda)\\ \dot{\lambda}&=\lambda^d+\sum_{i=2}^{d} a_i x^{i-1}\lambda^{d+1-i}+\sum_{i=2}^{d} b_i x^{i+2}\lambda^{d-i}=\bar{\Lambda}_d(x,\lambda)+\bar{\Lambda}_{d+2}(x,\lambda).\end{aligned} \tag{2.3}$$

The origin is the only singularity of $\bar{\mathbf{X}}^1$ on the λ-axis (which, of course, we knew to expect). Since $x\bar{\Lambda}_d-\lambda\bar{X}_d\equiv 0$, the origin is a special singularity of $\bar{\mathbf{X}}^1$. We can write $\bar{X}_d(x,\lambda)=xQ_{d-1}(x,\lambda)$ and $\bar{\Lambda}_d(x,\lambda)=\lambda Q_{d-1}(x,\lambda)$ where $Q_{d-1}(x,\lambda)=\prod_{j=1}^{d-1}(\lambda-jx)$.

Now we change λ to y in (2.3) and blow up again via η_1. We get a vector field $\bar{\mathbf{X}}^{11}$ given by:

$$\dot{x}=Q_{d-1}(1,\lambda)$$
$$\dot{\lambda}=x\bar{\Lambda}_{d+2}(1,\lambda)=x(\partial Q_{d-1}/\partial\lambda)(1,\lambda).$$

$\bar{\mathbf{X}}^{11}$ has zeros on the λ-axis at $(0, j)$, $j=1, \ldots, d-1$. Calculating $D\bar{\mathbf{X}}^{11}$ at points on the λ-axis, we get

$$D\bar{\mathbf{X}}^{11}(0,\lambda)=\begin{pmatrix}0 & (\partial Q_{d-1}/\partial\lambda)(1,\lambda)\\ (\partial Q_{d-1}/\partial\lambda)(1,\lambda) & 0\end{pmatrix}.$$

The determinant is negative at the points $(0, j)$, $j=1, \ldots, d-1$, so these points are all saddles.

§ 3. Counting separatrices

The idea in counting the number of separatrices at a singular point is to go down the associated tree, obtaining at each stage a more accurate estimate.

We begin with some terminology. There is an obvious partial ordering on the vertex set of a directed tree. If V and W are vertices of a directed tree, the expressions "*V precedes W*" and "*W follows V*" refer to this ordering. We say *V immediately precedes W* or *W immediately follows V* if there is a directed edge with origin V and terminus W. The *predecessor of W* is the vertex that immediately precedes W (there is at most one in the tree of a singularity); the *successors of V* are the vertices that immediately follow V.

Let $\mathcal{T}$ be the tree of a singularity and V a vertex of $\mathcal{T}$. We say V is *special* (resp. *nonspecial*) if V represents a special singularity (resp. any other germ). We say V is a *corner* if V represents a singularity that corresponds to a corner of Γ in the associated Dumortier picture. The *degree of V, $d(V)$*, is defined iff V represents a singularity; $d(V)$ is the degree of the singularity that V represents.

We define a function S from the vertex set of $\mathcal{T}$ into the nonnegative integers as follows, If V represents a nonsingular germ, set $S(V)=1$. Now suppose V represents a singularity of degree $m \geqslant 1$. If V is the initial vertex of $\mathcal{T}$, or if V immediately follows a special vertex, set $S(V)=m+1$. If V immediately follows a nonspecial vertex, set $S(V)=m-1$ if V is a corner, and set $S(V)=m$ otherwise. Note that if V is a nonterminal vertex of $\mathcal{T}$ with $d(V)=m$, then the number of corners that immediately follow V is 0, 1, or 2 according as $S(V)=m+1$, m, or $m-1$. Given a subtree $\mathcal{T}'$ of $\mathcal{T}$, we define $S(\mathcal{T}')=\sum S(V)$, this sum taken over all vertices V of $\mathcal{T}'$ that are terminal in $\mathcal{T}'$ (i.e., no edge of $\mathcal{T}'$ originates at V).

The number $2S(V)$ can be thought of as an estimate of the maximum number of separatrices that the vertex V can "contribute" to the singularity we are studying. We will see that there are problems with this estimate, but first we show that it is quite good for terminal vertices of $\mathcal{T}$.

PROPOSITION 3.1. *Let* $(0, 0)$ *be a singularity of* (1.1) *with tree* $\mathcal{T}$. *Then the number of separatrices at* $(0, 0)$ *is* $\leqslant 2S(\mathcal{T})$.

Proof. Let $\mathbf{Y}$ be the Dumortier vector field arising at the last stage of the construction of $\mathcal{T}$ and let Γ denote the homeomorph of S^1 in $\mathbf{R}^2$ that corresponds to the origin of our original vector field. It is easy to see that if c is a separatrix of $\mathbf{X}$ at $(0, 0)$, then c corresponds to a semipath of $\mathbf{Y}$ in $\operatorname{ext}\Gamma$ that limits either at a singularity of $\mathbf{Y}$ on Γ, or at an isolated tangency of $\mathbf{Y}$ with Γ. For each terminal vertex V of $\mathcal{T}$, let $\sigma(V)$ denote the number of semipaths of $\mathbf{Y}$ in $\operatorname{ext}\Gamma$ that correspond to separatrices of $\mathbf{X}$ at $(0, 0)$ and that limit at one of the two or four points of Γ corresponding to V. We will show that $\sigma(V) \leqslant 2S(V)$, from which the result follows.

If V represents a nonsingular germ, this germ is topologically equivalent to the germ of the vector field pictured in Figure 3 at $(0, \lambda_1)$ or $(0, \lambda_2)$. It follows that $\sigma(V) \leqslant 2 = 2S(V)$.

If V represents a singularity of degree $m \geq 2$ which has the property that $xY_m - yX_m$ (or Q_{m-1}) has no nonconstant linear factor, then $\sigma(V)=0$.

Now suppose V represents a singularity of degree 1 whose linear part has at least one nonzero eigenvalue. If V represents a node, focus or center, than $\sigma(V)=0$. If V represents a saddle or saddle-node, we must consider three cases. We note first that if V represents a saddle, then each pair of opposite separatrices of the saddle share a common tangent line at the origin; if V represents a saddle-node, the two separatrices that each bound one hyperbolic and one parabolic sector share a common tangent line at the origin.

1. $S(V)=0$. Then V is a corner. The germ $\mathbf{Z}$ at $(0, 0)$ represented by V has the positive and negative x- and y-axes as four of its solution curves. From the above description of saddles and saddle-nodes, it follows that each separatrix of $\mathbf{Z}$ at $(0, 0)$ must be a half-axis. Thus all separatrices of $\mathbf{Z}$ at $(0, 0)$ correspond to arcs of Γ in the associated Dumortier picture. It follows that $\sigma(V)=0$.

2. $S(V)=1$. V represents a germ $\mathbf{Z}$ at $(0, 0)$ that has two arcs of Γ as solution curves in the associated Dumortier picture. For definiteness, let us say that the positive and negative y-axis correspond to these arcs. If $\mathbf{Z}$ has four separatrices at $(0, 0)$, two must be the positive and negative y-axis; if $\mathbf{Z}$ has three separatrices at $(0, 0)$, at least one must be the positive or negative y-axis. Thus, $\sigma(V) \leq 2$.

3. $S(V)=2$. No separatrix of $\mathbf{Z}$ at $(0, 0)$ corresponds to an arc of Γ, so $\sigma(V) \leq 4$. ∎

In view of Proposition 3.1, we ask what is the maximum $S(\mathcal{J})$ can be for a singularity of given degree. The next proposition and its corollary are first steps toward answering this question.

LEMMA 3.2. *Let V be a nonterminal vertex of the tree $\mathcal{J}$ of some singularity. Let $d(V) = m \geq 1$.*

(1) *Suppose V is nonspecial. Let $V_1, \ldots, V_r$ be the successors of V. Then each V_i represents a singularity of degree $d_i \geq 1$, and $\sum_{i=1}^{k} d_i \leq m+1$.*

(2) *Suppose V is special. Let $V_1, \ldots, V_k$ be the successors of V that represent singularities; let $V_{k+1}, \ldots, V_r$ be the successors of V that represent nonsingular germs. Let $d(V_i)=d_i$, $i=1, \ldots, k$. Then $\sum_{i=1}^{k} d_i + (r-k) \leq m-1$.*

Proof. 1. Let $\mathbf{X}$ be the germ represented by V. If V_i represents a singularity $(0, \lambda_0)$ of $\overline{\mathbf{X}}^1$, then d_i is less than or equal to the multiplicity of λ_0 as a root of $Y_m(1, \lambda) - \lambda X_m(1, \lambda) = 0$; hence d_i is less than or equal to the multiplicity of $y - \lambda_0 x$ as a factor of $xY_m - yX_m$. If V_i represents a singularity $(0, 0)$ of $\overline{\mathbf{X}}^2$, then d_i is less than or equal to the multiplicity of 0 as a root of $X_m(\mu, 1) - \mu Y_m(\mu, 1) = 0$, hence less than or equal to the multiplicity of x as a factor of $xY_m - yX_m$. Since $xY_m - yX_m$ is homogeneous of degree $m+1$, the first statement follows.

2. The second statement is proved by an analogous argument using $Q_{m-1}(x, y)$ in place of $xY_m(x, y) - yX_m(x, y)$. ■

PROPOSITION 3.3. *Let V be a nonterminal, nonspecial vertex of the tree $\mathcal{T}$ of some singularity. Let $V_1, \ldots, V_r$ be the successors of V in $\mathcal{T}$. Then $\sum_{i=1}^{r} S(V_i) \leqslant S(V)$.*

COROLLARY 3.4. *Let* $(0, 0)$ *be a singularity of* (1.1) *of degree d with tree $\mathcal{T}$. If all special vertices of $\mathcal{T}$ are terminal, then $S(\mathcal{T}) \leqslant d+1$ and there are at most $2d+2$ separatrices of* (1.1) *at* $(0, 0)$.

Proof of Corollary 3.4. Since S (initial vertex of $\mathcal{T}) = d+1$, $S(\mathcal{T}) \leqslant d+1$ by repeated application of Proposition 3.3. The rest follows from Proposition 3.1. ■

Proof of Proposition 3.3. There are three cases:

1. V is the initial vertex of $\mathcal{T}$. Then $S(V_i) = d_i$ for all i since V is not special, so by Lemma 3.2, $\sum_{i=1}^{r} S(V_i) \leqslant d+1 = S(V)$.

2. $S(V) = d(V) = m$. One of the V_i, say V_1, must be a corner. Then $S(V_1) = d_1 - 1$, $S(V_i) = d_i$ for $i \geqslant 2$, so $\sum_{i=1}^{r} S(V_i) \leqslant m = S(V)$.

3. V is a corner of degree m. Then $S(V) = m-1$. Two of the V_i are corners, so $\sum_{i=1}^{r} S(V_i) \leqslant m-1 = S(V)$. ■

If we drop the assumption of Corollary 3.4, then $S(\mathcal{T})$ can be greater than $d+1$ and the number of separatrices can be greater than $2d+2$. Example 3 of Section 2 shows how S can increase when we blow up a special singularity. To get an upper bound for $S(\mathcal{T})$, we will use the function P defined on the vertex set of $\mathcal{T}$ as follows:

Let $\mathcal{T}$ be the tree of a singularity and let V be a vertex of $\mathcal{T}$. If V represents a nonsingular germ, let $P(V) = 0$. If V represents a singularity of degree m, let $P(V) = m-1$. If $\mathcal{T}'$ is a subtree of $\mathcal{T}$, let $P(\mathcal{T}') = \sum P(V)$, this sum taken over all vertices of $\mathcal{T}'$ that are terminal in $\mathcal{T}'$.

It turns out that when we blow up a special singularity, any increase in S is offset by a decrease in P. This is the idea behind the next result.

Let $\mathcal{T}$ be the tree of a singularity. A connected subtree $\mathcal{T}'$ of $\mathcal{T}$ is called *full* if for every vertex V of $\mathcal{T}'$ that is not the initial vertex of $\mathcal{T}$, the predecessor W of V in $\mathcal{T}$ is a vertex of $\mathcal{T}'$, and all successors of W in $\mathcal{T}$ are vertices of $\mathcal{T}'$. It follows that a full subtree of $\mathcal{T}$ must contain the initial vertex of $\mathcal{T}$.

PROPOSITION 3.5. *Let* $(0, 0)$ *be a singularity of* (1.1) *of degree* d *with tree* $\mathcal{J}$ *and let* $\mathcal{J}'$ *be a full subtree of* $\mathcal{J}$. *Then* $S(\mathcal{J}) \leqslant S(\mathcal{J}') + P(\mathcal{J}')$.

Letting $\mathcal{J}'$ be the initial vertex of $\mathcal{J}$, we get $S(\mathcal{J}) \leqslant 2d$. Then by Proposition 3.1 the number of separatrices at any singularity of degree d is $\leqslant 4d$. This is already an improvement over the bound $4d+4$ mentioned in the Introduction.

Proof. We would like to say that if V is any nonterminal vertex of $\mathcal{J}$ and $V_1, ..., V_r$ are its successors, then $\sum_{i=1}^r S(V_i) + P(V_i) \leqslant S(V) + P(V)$. We could then easily conclude that $S(\mathcal{J}) \leqslant S(\mathcal{J}) + P(\mathcal{J}) \leqslant S(\mathcal{J}') + P(\mathcal{J}')$. In fact, we will see that while at most stages in the construction of $\mathcal{J}$, $S+P$ does not increase, $S+P$ can increase at certain stages. Fortunately, it turns out that in the latter case $S+P$ goes back down before we finish.

Suppose first that V is a nonterminal special vertex of $\mathcal{J}$ with $d(V) = m$. Let $V_1, ..., V_k$ be the successors of V that represent singularities; let $V_{k+1}, ..., V_r$ be the successors of V that represent nonsingular germs. We have for $i = 1, ..., k$, $S(V_i) + P(V_i) = (d_i + 1) + (d_i - 1) = 2d_i$; for $i = k+1, ..., r$, $S(V_i) = 1$ and $P(V_i) = 0$. Therefore, using Lemma 3.2, $\sum_{i=1}^r (S(V_i) + P(V_i)) = 2\sum_{i=1}^k d_i + (r-k) \leqslant 2(m-1) \leqslant S(V) + P(V)$.

Now suppose V represents a nonspecial singularity of degree m and $V_1, ..., V_r$ are the successors of V. Each V_i represents a singularity of degree $d_i \geqslant 1$. We consider three cases.

1. $S(V) = m-1$. Here V is a corner, so two of the V_i are corners; therefore, $r \geqslant 2$. By Proposition 3.3, $\sum_{i=1}^r S(V_i) \leqslant m-1$; moreover, since $r \geqslant 2$, using Lemma 3.2 we have $\sum_{i=1}^r (d_i - 1) \leqslant (m+1) - 2 = m-1$. Thus $\sum_{i=1}^r (S(V_i) + P(V_i)) \leqslant 2(m-1) = S(V) + P(V)$.

2. $S(V) = m$. Then V represents the germ of a singularity at $(0, 0) \in xy$-plane for which the y-axis, say, is invariant. This singularity is written in the form

$$\dot{x} = X_m(x, y) + X_{m+1}(x, y) + \dots$$
$$\dot{y} = Y_m(x, y) + Y_{m+1}(x, y) + \dots.$$

Notice x is a factor of $xY_m - yX_m$. If V has more than one successor, then $\sum_{i=1}^r (S(V_i) + P(V_i)) \leqslant 2m - 1 = S(V) + P(V)$. Therefore, assume V has exactly one successor V_1, which must be a corner. If V_1 has degree $\leqslant m$, then $S(V_1) + P(V_1) \leqslant 2m - 2 < S(V) + P(V)$. Thus, we can assume V_1 is a corner of degree $m+1$. Then $S(V_1) + P(V_1) = 2m = S(V) + P(V) + 1$. The singularity represented by V_1 is obtained by blowing up the singularity represented by V via η_2. We get

$$\begin{aligned} \dot{\mu} &= X_m(\mu, 1) + yX_{m+1}(\mu, 1) + \dots - \mu[Y_m(\mu, 1) + yY_{m+1}(\mu, 1) + \dots] \\ \dot{y} &= yY_m(\mu, 1) + y^2 Y_{m+1}(\mu, 1) + \dots. \end{aligned} \tag{3.1}$$

In order that $(0, 0)$ be a singularity of (3.1) of degree $m+1$, we must have $Y_m(x, y) = ax^m$, $a \neq 0$, and $X_m(x, y) \equiv 0$. We can rewrite (3.1) as

$$\begin{aligned} \dot{\mu} &= \bar{M}_{m+1}(\mu, y) + \bar{M}_{m+2}(\mu, y) + \dots \quad \text{where} \quad \bar{M}_{m+1}(\mu, y) = -a\mu^{m+1} + y(\dots) \\ \dot{y} &= \bar{Y}_{m+1}(\mu, y) + \bar{Y}_{m+2}(\mu, y) + \dots \quad \text{where} \quad \bar{Y}_{m+1}(\mu, y) = a\mu^m y + y^2(\dots). \end{aligned} \tag{3.2}$$

Therefore,

$$y\bar{M}_{m+1}(\mu, y) - \mu\bar{Y}_{m+1}(\mu, y) = -2a\mu^{m+1}y + y^2(\dots) \not\equiv 0. \tag{3.3}$$

Hence V_1 is a nonspecial corner of degree $m+1$. V_1 is not terminal since y divides (3.3). It follows that among the successors of V_1 are two corners.

If V_1 has more than two successors, then summing $S+P$ over these successors, we would get a number $\leq 2m-1 = S(V) + P(V)$. We can therefore assume that V_1 has exactly two successors, each a corner. It follows that the only linear factors of $y\bar{M}_{m+1} - \mu\bar{Y}_{m+1}$ are μ and y. Let V_1' (resp. V_2') be the successor of V_1 corresponding to the factor μ (resp. y). By (3.3) y has multiplicity 1 as a factor of $y\bar{M}_{m+1} - \mu\bar{Y}_{m+1}$. It follows easily that V_2' is a terminal corner of degree 1, so $S(V_2') = P(V_2') = 0$. If the multiplicity of μ as a factor of $y\bar{M}_{m+1} - \mu\bar{Y}_{m+1}$ were $\leq m$, we would have $\sum_{i=1}^{2} (S(V_i') + P(V_i')) = S(V_1') + P(V_1') \leq (m-1) + (m-1) < S(V) + P(V)$. Therefore, we need only consider the case $y\bar{M}_{m+1} - \mu\bar{Y}_{m+1} = -2a\mu^{m+1}y$. To get the singularity represented by V_1' we replace μ by x in (3.2) and blow up via η_2. Then V_1' represents the singularity at $(0, 0) \in \mu y$-plane. If the degree of this singularity were $\leq m$, we would again have $S(V_1') + P(V_1') < S(V) + P(V)$. Therefore, we can assume V_1' represents a singularity of degree $m+1$. Notice $\sum_{i=1}^{2} (S(V_i') + P(V_i')) = 2m = S(V) + P(V) + 1$, so $S+P$ has not increased further. We compute:

$$\begin{aligned} \dot{\mu} &= \bar{M}'_{m+1}(\mu, y) + \bar{M}'_{m+2}(\mu, y) + \dots \quad \text{where} \quad \bar{M}'_{m+1}(\mu, y) = -2a\mu^{m+1} + y(\dots) \\ \dot{y} &= \bar{Y}'_{m+1}(\mu, y) + \bar{Y}'_{m+2}(\mu, y) + \dots \quad \text{where} \quad \bar{Y}'_{m+1}(\mu, y) = a\mu^m y + y^2(\dots). \end{aligned} \tag{3.4}$$

Notice the similarity to (3.2). The similarity of (3.4) to (3.2) is the key to the argument.

Continuing, we blow up the singularity represented by V_1', which is not terminal. Unless the result is exactly one corner V_1'' of degree $m+1$ and one terminal corner V_2'' of degree 1, $S+P$ must return to $\leq 2m-1$. In the remaining case, $S+P$ stays at $2m$ and for the singularity represented by V_1'' we compute

$$\begin{aligned} \dot{\mu} &= \bar{M}''_{m+1}(\mu, y) + \bar{M}''_{m+2}(\mu, y) + \dots \quad \text{where} \quad \bar{M}''_{m+1}(\mu, y) = -3a\mu^{m+1} + y(\dots) \\ \dot{y} &= \bar{Y}''_{m+1}(\mu, y) + \bar{Y}''_{m+2}(\mu, y) + \dots \quad \text{where} \quad \bar{Y}''_{m+1}(\mu, y) = a\mu^m y + y^2(\dots). \end{aligned}$$

Therefore, V_1'' is a nonterminal nonspecial corner of degree $m+1$.

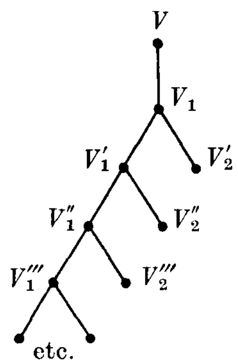


Figure 16. V is a vertex of degree m with $S(V)=m$. $V_1, V_1', V_1'', \ldots$ are corners of degree $m+1$. $V_2', V_2'', \ldots$ are terminal corners of degree 1.

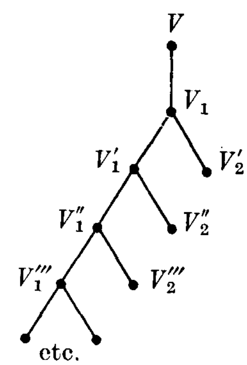


Figure 17. V is a vertex of degree m with $S(V)=m+1$. $V_1, V_1', V_1'', \ldots$ are vertices of degree $m+1$ with $S=m+1$. $V_2', V_2'', \ldots$ are terminal corners of degree 1.

We now see that if $S+P$ never returns to $\leqslant 2m-1$, we must get an infinite subtree of $\mathcal{T}$ that looks like Figure 16. This is impossible by Theorem 2.2.

3. $S(V)=m+1$. $S+P$ goes up when we blow up the singularity represented by V iff V is immediately followed by exactly one vertex V_1 of degree $m+1$. Then $S(V_1)+P(V_1)=2m+1=S(V)+P(V)+1$. In particular, we must have $xY_m-yX_m=(\text{linear factor})^{m+1}$; we assume for simplicity that $xY_m-yX_m=ax^{m+1}$, $a\neq 0$. Then V_1 represents the singularity at $(0,0)\in\mu y$-plane obtained by blowing up the singularity represented by V via η_2. We compute

$$\dot{\mu}=\bar{M}_{m+1}(\mu,y)+\bar{M}_{m+2}(\mu,y)+\ldots \quad \text{where} \quad \bar{M}_{m+1}(\mu,y)=-a\mu^{m+1}+y(\ldots)$$

$$\dot{y}=\bar{Y}_{m+1}(\mu,y)+\bar{Y}_{m+2}(\mu,y)+\ldots \quad \text{where} \quad \bar{Y}_{m+1}(\mu,y)=a\mu^{m}y+y^2(\ldots).$$

It follows that V_1 represents a nonterminal nonspecial singularity. We see by arguments like those for case 2 that if $S+P$ never returns to $\leqslant 2m$, $\mathcal{T}$ must contain the infinite subtree of Figure 17. This is impossible by Theorem 2.2. ∎

If $V_1, \ldots, V_k$ are vertices of $\mathcal{T}$, let $\mathcal{T}(V_1, \ldots, V_k)$ denote the smallest full subtree of $\mathcal{T}$ that contains $V_1, \ldots, V_k$. For example, if V is any vertex of $\mathcal{T}$, the vertex set of $\mathcal{T}(V)$ is just the predecessor W of V and all its successors (including V); the predecessor W' of W and all its successors; ...; the initial vertex of $\mathcal{T}$ and all its successors.

For later convenience, we gather three technical facts in the following lemma. Each can be proved by examining the proof of Proposition 3.5.

LEMMA 3.6. *Let* $(0, 0)$ *be a singularity of* (1.1) *of degree* d *with tree* $\mathcal{T}$. *Let* V *be a vertex of* $\mathcal{T}$. *Then*

(1) $S(\mathcal{T}(V))+P(\mathcal{T}(V))\leqslant 2d+1$.

(2) *If* V *is special*, $S(\mathcal{T}(V))+P(\mathcal{T}(V))\leqslant 2d$.

(3) *If* V *is special*, $d(V)\leqslant d$.

The following improvement on Proposition 3.5 will be proved by examining the function $S+P$ more closely.

PROPOSITION 3.7. *If* $(0, 0)$ *is a singularity of* (1.1) *of degree* $d\geqslant 2$ *with tree* $\mathcal{T}$, *then* $S(\mathcal{T})\leqslant 2d-1$.

It follows that if $d\geqslant 2$, the number of separatrices at $(0, 0)$ is at most $4d-2$. Before proving Proposition 3.7 we will prove some lemmas.

Let V be a vertex whose predecessor W represents a singularity of degree l, i.e.,

$$\begin{aligned} \dot{x} &= X_l(x, y)+X_{l+1}(x, y)+ \dots \\ \dot{y} &= Y_l(x, y)+ Y_{l+1}(x, y)+ \dots \end{aligned} \tag{3.5}$$

If W is nonspecial (resp. special), the singularity represented by V corresponds to a factor of xY_l-yX_l (resp. Q_{l-1}), and $d(V)$ is less than or equal to the multiplicity of this factor. V is called *irregular* if one of the following is true:

1. W is nonspecial and $d(V)$ is less than the multiplicity of the corresponding factor of xY_l-yX_l.

2. W is special and $d(V)$ is less than the multiplicity of the corresponding factor of Q_{l-1}.

Other vertices of $\mathcal{T}$ (including the initial vertex) are called *regular*.

LEMMA 3.8. *Let* $\mathcal{T}$ *be the tree of a singularity. Let* V *be a nonterminal, nonspecial, regular vertex of* $\mathcal{T}$ *with* $S(V)\leqslant d(V)$. *Then among the successors of* V *is a terminal corner of degree* 1.

Proof. Assume for simplicity that V corresponds to the factor y of xY_l-yX_l, which has multiplicity k. Blowing up (3.5) via η_1, we get

$$\begin{aligned} \dot{x} &= xX_l(1, \lambda)+x^2X_{l+1}(1, \lambda)+ \dots \\ \dot{\lambda} &= Y_l(1, \lambda)+xY_{l+1}(1, \lambda)+ \dots -\lambda[X_l(1, \lambda)+xX_{l+1}(1, \lambda)+ \dots]. \end{aligned} \tag{3.6}$$

V represents the singularity of (3.6) at $(0, 0)$, and $Y_l(1, \lambda)-\lambda X_l(1, \lambda)=a\lambda^k+$higher order terms, $a\neq 0$. After changing λ to y in (3.6), by the assumption that V is regular we can rewrite (3.6) as

$$\begin{aligned} \dot{x} &= \bar{X}_k(x, y) + \bar{X}_{k+1}(x, y) + \dots \\ \dot{y} &= \bar{Y}_k(x, y) + \bar{Y}_{k+1}(x, y) + \dots, \quad \text{where} \quad \bar{Y}_k(x, y) = ay^k + x(\dots) \end{aligned} \tag{3.7}$$

Now blow up (3.7) via η_2. One of the successors of V is a corner vertex that represents the singularity at (0, 0) of the resulting vector field. But for the $\dot{y}$ component of the resulting vector field we compute

$$\dot{y} = y\bar{Y}_k(\mu, 1) + y^2(\dots) = ay + \mu y(\dots) + y^2(\dots)$$

Therefore this corner is terminal of degree 1. ∎

The reader should notice that the terminal corner of Lemma 3.8 need not be regular.

COROLLARY 3.9. *Let $\mathfrak{T}$ be the tree of a singularity, and suppose some vertex V of $\mathfrak{T}$ is a nonterminal corner. Let $\mathcal{S}$ denote the totally ordered set of vertices consisting of V and the vertices that precede V. Write $\mathcal{S} = \{V_1, \dots, V_k\}$, where V_1 = initial vertex of $\mathfrak{T}$, $V_k = V$, and V_i precedes V_j if and only if $i < j$. Then there is some i, $1 < i < k$, such that V_i is irregular and V_j is a nonterminal corner for all j such that $i < j \leq k$.*

Proof. Let i be the greatest integer in $1, \dots, k$ such that V_i is not a corner. Since a successor of V_i is a corner, we must have V_i nonspecial and $S(V_i) = d(V_i)$. If V_i were regular, by Lemma 3.8 the unique corner that follows V_i would be terminal. We conclude that V_i is irregular. ∎

LEMMA 3.10. *Let $\mathfrak{T}$ be the tree of a singularity of degree d, and suppose $\mathfrak{T}$ has an irregular vertex W_1. Then $S(\mathfrak{T}) \leq 2d - 1$.*

Proof. Let W be the predecessor of W_1. By Lemma 3.6(1), $S(\mathfrak{T}(W)) + P(\mathfrak{T}(W)) \leq 2d + 1$. Let $W_1, \dots, W_r$ denote the successors of W, with $r \geq 1$. Since W_1 is irregular we can calculate that $\sum_{i=1}^{r} (S(W_i) + P(W_i)) \leq S(W) + P(W) - 1$. Therefore, if $S(\mathfrak{T}(W)) + P(\mathfrak{T}(W)) \leq 2d$ (the usual case), then $S(\mathfrak{T}(W_1)) + P(\mathfrak{T}(W_1)) \leq 2d - 1$, and the result follows from Proposition 3.5.

In the exceptional case $S(\mathfrak{T}(W)) + P(\mathfrak{T}(W)) = 2d + 1$, we will use the information about such trees obtained in the course of proving Proposition 3.5 to show that $S + P$ must decrease by at least *two* when we blow up W, so that again $S(\mathfrak{T}(W_1)) + P(\mathfrak{T}(W_1)) \leq 2d - 1$.

Suppose $S(\mathfrak{T}(W)) + P(\mathfrak{T}(W)) = 2d + 1$. Then W corresponds to V_1 or V_1' or V_1'' or ... in Figure 16 or Figure 17. Therefore $S(W) \leq d(W)$; W is nonspecial; and $d(W) \geq 2$, so $S(W) + P(W) \geq 2$. Also, W must be regular in order that $S(\mathfrak{T}(W)) + P(\mathfrak{T}(W)) = 2d + 1$. By Lemma

3.8, among the successors of W is a terminal corner of degree 1. If this corner is the only successor of W (so that it equals W_1), we have $S(W_1)+P(W_1)=0$ and it follows easily that $S(\mathcal{T}(W_1))+P(\mathcal{T}(W_1))\leqslant 2d-1$. The result then follows from Proposition 3.5. If W has more than one successor, let $W_2, \ldots, W_r$ denote the successors of W other than W_1, with $r\geqslant 2$. Since $r\geqslant 2$ and W_1 is irregular, we can calculate that $\sum_{i=1}^{r}(S(W_i)+P(W_i))\leqslant S(W)+P(W)-2$. Then $S(\mathcal{T}(W_1))+P(\mathcal{T}(W_1))\leqslant 2d-1$, and the result again follows from Proposition 3.5. ■

Proof of Proposition 3.7. If $\mathcal{T}$ has no nonterminal special vertices, then by Corollary 3.4, $S(\mathcal{T})\leqslant d+1\leqslant 2d-1$ for $d\geqslant 2$. We therefore assume $\mathcal{T}$ has a nonterminal special vertex V of degree m.

If V is a corner, then by Corollary 3.9, some vertex of $\mathcal{T}$ is irregular, so Lemma 3.10 implies that $S(\mathcal{T})\leqslant 2d-1$.

If V is not a corner, then $S(V)=m$ or $m+1$. By Lemma 3.6(2), $S(\mathcal{T}(V))+P(\mathcal{T}(V))\leqslant 2d$. Let $V_1, \ldots, V_r$ be the successors of V. We have $S(V)+P(V)\geqslant 2m-1$, and as in the proof of Proposition 3.5, $\sum_{i=1}^{r}(S(V_i)+P(V_i))\leqslant 2m-2$. Therefore, $S(\mathcal{T}(V_1))+P(\mathcal{T}(V_1))\leqslant 2d-1$. The result follows from Proposition 3.5. ■

To make further progress, we require two more technical preliminaries. The first is a strengthening of Lemma 3.10.

LEMMA 3.11. *Let $\mathcal{T}$ be the tree of a singularity of degree d. Suppose $\mathcal{T}$ has k irregular vertices $W_1, \ldots, W_k$; $k\geqslant 1$. Then $S(\mathcal{T}(W_1, \ldots, W_k))+P(\mathcal{T}(W_1, \ldots, W_k))\leqslant 2d-k$.*

Proof. By induction on k. If $k=1$, the proof of Lemma 3.10 gives the result. If $k>1$, let the k irregular vertices be numbered so that none of $W_1, \ldots, W_{k-1}$ follows W_k. Then either W_k is terminal in $\mathcal{T}(W_1, \ldots, W_{k-1})$ or W_k is not a vertex of $\mathcal{T}(W_1, \ldots, W_{k-1})$. By the induction hypothesis, $S(\mathcal{T}(W_1, \ldots, W_{k-1}))+P(\mathcal{T}(W_1, \ldots, W_{k-1}))\leqslant 2d-k+1$. If W_k is a vertex of $\mathcal{T}(W_1, \ldots, W_{k-1})$, it is not hard to see that in fact $S(\mathcal{T}(W_1, \ldots, W_k))+P(\mathcal{T}(W_1, \ldots, W_k))\leqslant 2d-k-1$. Otherwise, consider $\mathcal{T}(W_1, \ldots, W_k)$. Let W be the predecessor of W_k. If $S+P$ has not increased in the course of constructing $\mathcal{T}(W_1, \ldots, W_{k-1}, W)$ from $\mathcal{T}(W_1, \ldots, W_{k-1})$, we will have $S(\mathcal{T}(W_1, \ldots, W_{k-1}, W))+P(\mathcal{T}(W_1, \ldots, W_{k-1}, W))\leqslant 2d-k+1$. Since W_k is irregular, $S+P$ must decrease by at least one when we blow up W, so $S(\mathcal{T}(W_1, \ldots, W_k))+P(\mathcal{T}(W_1, \ldots, W_k))\leqslant 2d-k$. On the other hand, if $S+P$ has increased in the course of constructing $\mathcal{T}(W_1, \ldots, W_{k-1}, W)$ from $\mathcal{T}(W_1, \ldots, W_{k-1})$ it can have increased by at most one. (This fact is an obvious extension of Lemma 3.6(1).) In this case, $S(\mathcal{T}(W_1, \ldots, W_{k-1}, W))+P(\mathcal{T}(W_1, \ldots, W_{k-1}, W))=2d-k+2$, and we see as in the proof of Lemma 3.10 that $S+P$ must decrease by at least two when we blow up W. Again we conclude that $S(\mathcal{T}(W_1, \ldots, W_k))+P(\mathcal{T}(W_1, \ldots, W_k))\leqslant 2d-k$. ■

Lemma 3.12. *Let $\mathcal{T}$ be the tree of a singularity, and suppose $\mathcal{T}$ has exactly one irregular vertex W. Then $\mathcal{T}$ has at most one nonterminal special corner.*

Proof. We consider three cases.

1. $S(W)=d(W)-1$. Then W is a corner. Corollary 3.9 implies that W is terminal. Since any nonterminal corner of $\mathcal{T}$ must follow W (again Corollary 3.9), we conclude that in this case $\mathcal{T}$ has no nonterminal corners.

2. $S(W)=d(W)+1$. Then no successor of W is a corner. Corollary 3.9 shows that in this case also $\mathcal{T}$ has no nonterminal corners.

3. $S(W)=d(W)$. One successor of W, say W_1, is a corner; the other successors are not corners. Since W is the only irregular vertex of $\mathcal{T}$, by Corollary 3.9 any nonterminal corner of $\mathcal{T}$ must be W_1 or a vertex that follows W_1. If W_1 is a nonterminal *special* corner then no successor of W_1 is a corner. Therefore Corollary 3.9 implies that W_1 is the only nonterminal special corner in $\mathcal{T}$. If W_1 is not special, let W_1', W_2' be the corners that follow W_1. Since W_1 is regular, one of W_1', W_2', say W_2', must be terminal. Again any nonterminal corner of $\mathcal{T}$ must be W_1' or a vertex that follows W_1', and again if W_1' is special then it is the only nonterminal special corner in $\mathcal{T}$. Proceeding inductively, we conclude that $\mathcal{T}$ has at most one nonterminal special corner. ∎

Theorem 3.13. *Let $(0, 0)$ be a singularity of (1.1) of degree d with tree $\mathcal{T}$. If $d=1$, there are at most four separatrices at $(0, 0)$. If $d=2$, there are at most six separatrices at $(0, 0)$. If $d\geqslant 3$, there are at most $4d-4$ separatrices at $(0, 0)$. These bounds are sharp.*

Proof. By the remark following the statement of Proposition 3.5, if $d=1$, there are at most four separatrices at $(0, 0)$, and any saddle has exactly four. By the remark following the statement of Proposition 3.7, if $d=2$, there are at most six separatrices, and Example 1 of Section 2 shows there can be exactly six. Example 3 of Section 2 shows that for any $d\geqslant 3$ there exist singularities of degree d with $4d-4$ separatrices. It remains to show that if $d\geqslant 3$, there can be at most $4d-4$ separatrices.

Assume $d\geqslant 3$ and let $\mathcal{T}$ be the tree of $(0, 0)$. The natural attack would be to try to show that $S(\mathcal{T})\leqslant 2d-2$ and invoke Proposition 3.1. However, Example 4 of Section 2 has $S(\mathcal{T})=2d-1$. We will show that even when $S(\mathcal{T})=2d-1$, there are at most $4d-4$ separatrices.

If all special vertices of $\mathcal{T}$ are terminal, by Corollary 3.4, $S(\mathcal{T})\leqslant d+1\leqslant 2d-2$ (since $d\geqslant 3$), so we can ignore this case. We therefore assume $\mathcal{T}$ has at least one nonterminal special vertex.

First suppose $\mathcal{T}$ has a nonterminal special vertex V with $d(V)=m$ and $S(V)=m+1$.

By Lemma 3.6(2), $S(\mathcal{T}(V))+P(\mathcal{T}(V))\leqslant 2d$. Also, $S(V)+P(V)=2m$. Let $V_1, \dots, V_r$ be the successors of V. Then $\sum_{i=1}^{r}(S(V_i)+P(V_i))\leqslant 2m-2$. Therefore $S(\mathcal{T}(V_1))+P(\mathcal{T}(V_1))\leqslant 2d-2$, so $S(\mathcal{T})\leqslant 2d-2$ by Proposition 3.5, so our singularity has at most $4d-4$ separatrices. We therefore assume all nonterminal special vertices V of $\mathcal{T}$ have $S(V)\leqslant d(V)$.

If $\mathcal{T}$ has more than one irregular vertex, then it follows from Lemma 3.11 and Proposition 3.5 that $S(\mathcal{T})\leqslant 2d-2$, so our singularity has at most $4d-4$ separatrices. We will separately consider the two cases, $\mathcal{T}$ has no irregular vertex and $\mathcal{T}$ has one irregular vertex.

1. $\mathcal{T}$ has no irregular vertex. By Corollary 3.9, the nonterminal special vertices of $\mathcal{T}$ cannot be corners. Let $\mathcal{T}'$ be the subtree of $\mathcal{T}$ obtained by deleting from $\mathcal{T}$ every vertex that follows a nonterminal special vertex of $\mathcal{T}$. Then the terminal vertices of $\mathcal{T}'$ are: (1) possibly certain vertices W that are also terminal in $\mathcal{T}$; and (2) a nonempty set of special vertices $U_1, \dots, U_k$, each of which is nonterminal in $\mathcal{T}$. Let $d(U_i)=m_i$, $i=1, \dots, k$. Then $S(U_i)=m_i$ for each i. Let $a=\sum S(W)$, this sum taken over all terminal vertices W of $\mathcal{T}$ that are in $\mathcal{T}'$. Using Proposition 3.3, we have

$$S(\mathcal{T}') = a+\sum_{i=1}^{k} m_i \leqslant d+1. \tag{3.8}$$

Let $\mathcal{T}''$ be the connected subtree of $\mathcal{T}$ whose vertex set is the vertices of $\mathcal{T}'$ plus the successors of $U_1, \dots, U_k$. Let U_{ij}, $j=1, \dots, r_i$ denote the successors of U_i. Then

$$\sum_{j=1}^{r_i}(S(U_{ij})+P(U_{ij}))\leqslant 2m_i-2,$$

so

$$S(\mathcal{T}'')+P(\mathcal{T}'')\leqslant a+\sum_{i=1}^{k}(2m_i-2). \tag{3.9}$$

From (3.8) and (3.9) we get

$$S(\mathcal{T}'')+P(\mathcal{T}'') \leqslant a+2(d+1-a)-2k = 2d-a-2(k-1).$$

If $k\geqslant 2$, then $S(\mathcal{T}'')+P(\mathcal{T}'')\leqslant 2d-2$, so there are at most $4d-4$ separatrices. Therefore we assume $k=1$. If $a\geqslant 2$, again $S(\mathcal{T}'')+P(\mathcal{T}'')\leqslant 2d-2$, so we assume $a=0$ or $a=1$. Writing $m_1=m$, (3.9) becomes

$$S(\mathcal{T}'')+P(\mathcal{T}'')\leqslant a+2m-2.$$

Now $m\leqslant d$ by Lemma 3.6(3). Hence if $a=0$ or $m<d$, we have $S(\mathcal{T})\leqslant 2d-2$. Thus we can assume $a=1$ and $m=d$.

We now let V denote the unique nonterminal special vertex of $\mathcal{T}''$ (so $V=U_1$), and we let $V_1, \dots, V_r$ denote the successors of V.

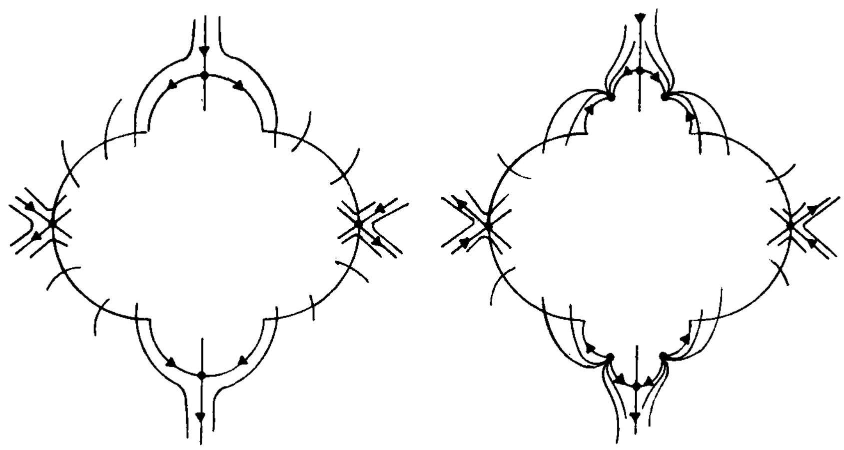


Figure 18. Two possible Dumortier pictures for Y.

In order that $S(\mathfrak{T})=2d-1$, we must have $d-1$ vertices immediately following V, each of degree 1. To see this, notice that Lemma 3.2 implies $\sum_{i=1}^{r} S(V_i)\leqslant d-1+r$. Then $S(\mathfrak{T}'')\leqslant d+r$, so $S(\mathfrak{T})\leqslant d+r$ from Proposition 3.3. If $r\leqslant d-2$, we would have $S(\mathfrak{T})\leqslant 2d-2$. Therefore, $r=d-1$.

It follows that each V_i represents either a nonsingular germ or a singularity of degree 1. If some V_i represents a nonsingular germ, we again have $S(\mathfrak{T})\leqslant 2d-2$, so each V_i represents a singularity of degree 1. Thus the terminal vertices of $\mathfrak{T}''$ are: (1) one vertex W_1 that is also terminal in $\mathfrak{T}$ and has $S(W_1)=1$; (2) possibly other vertices W that are also terminal in $\mathfrak{T}$ and have $S(W)=0$; (3) $V_1, \ldots, V_{d-1}$, each of which represents a singularity of degree 1 and hence has $S(V_i)=2$. The V_i need not be terminal in $\mathfrak{T}$.

Let $\mathbf{Y}$ denote the Dumortier vector field associated with $\mathfrak{T}''$ and let Γ denote the homeomorph of S^1 in $\mathbf{R}^2$ that corresponds to the origin of our original vector field. As in the proof of Proposition 3.1, for each terminal vertex U of $\mathfrak{T}''$ we let $\sigma(U)$ denote the number of semipaths of $\mathbf{Y}$ in ext Γ that correspond to separatrices of our original vector field at $(0, 0)$ and that limit at one of the points of Γ corresponding to U. We saw in the proof of Proposition 3.1 that $\sigma(W_1)\leqslant 2$ and $\sigma(W)=0$ for those terminal vertices of $\mathfrak{T}''$ with $S(W)=0$. Moreover, since each V_i represents a singularity of degree 1, we have seen that Proposition 3.5 implies that each V_i represents a singularity with at most four separatrices. It follows easily that $\sigma(V_i)\leqslant 4$ for each i.

If no V_i is a corner, one sees easily that in fact $\sigma(W_1)=0$. The reason is that all solution curves of $\mathbf{Y}$ that start near the points of Γ corresponding to W_1 must cross Γ or limit at points of Γ. See Figure 18. In this case our original vector field has at most $4d-4$ separatrices at $(0, 0)$.

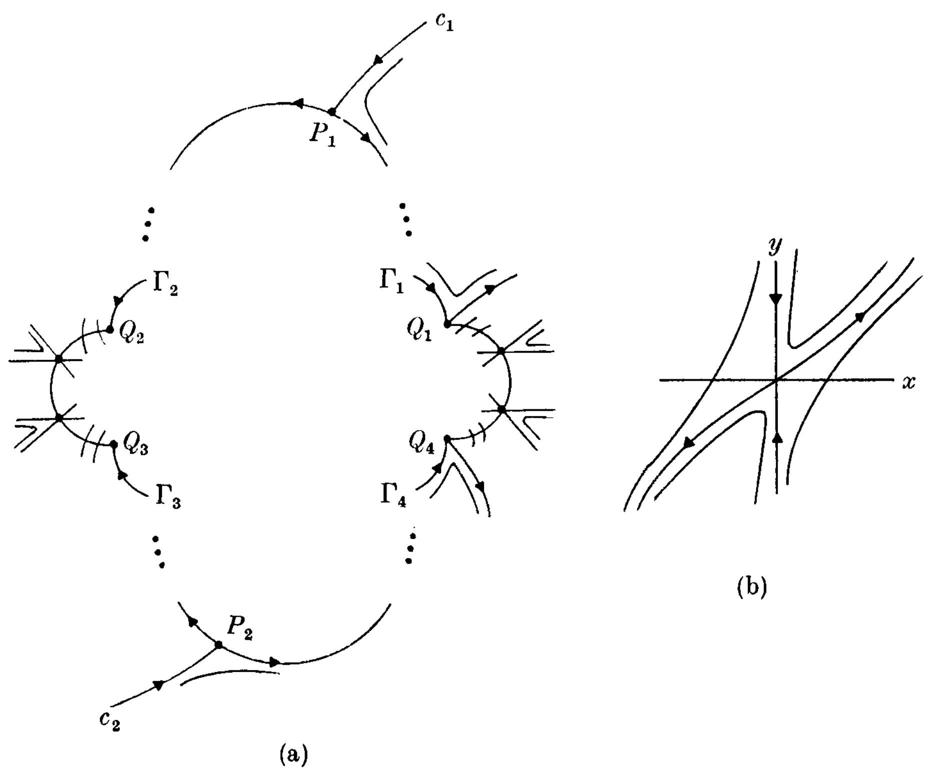


Figure 19

Now suppose some V_i, say V_1, is a corner. Consider Figure 19. In Figure 19(a), P_1 and P_2 are the points of Γ corresponding to W_1; $Q_1, \ldots, Q_4$ are the corners of Γ corresponding to V_1. For $i=1, 2,$ in ext Γ there is at most one semipath c_i converging to P_i that might correspond to a separatrix of our original singularity.

First suppose V_1 represents a singularity with four separatrices. Then this singularity is a saddle ([1], pp. 340, 357, 362). The curves $\Gamma_1, \ldots, \Gamma_4$ in Figure 19(a) are invariant because V is the unique nonterminal special vertex of $\mathcal{J}''$. These curves must correspond to separatrices of the saddle repretented by V_1 (the positive and negative y-axes in Figure 19(b)). Therefore $\sigma(V_1) \leqslant 2$. so our original singularity has at most $4d-4$ separatrices.

Next suppose V_1 represents a singularity with at most three separatrices. If $\sigma(W_1) \neq 0$ we see from Figure 19(a) that some Γ_i must bound a hyperbolic sector at the corresponding Q_i. This Γ_i corresponds to a separatrix of the singularity represented by V_1, so $\sigma(V_1) \leqslant 2$. Thus whether or not $\sigma(W_1)=0$ we get $\sigma(V_1)+\sigma(W_1) \leqslant 4$. Again our original singularity has at most $4d-4$ separatrices.

2. $\mathcal{J}$ has one irregular vertex. X. If $\mathcal{J}$ has no nonterminal special corner, the proof

proceeds as in the first case. Thus we assume $\mathcal{J}$ has a nonterminal special corner V. It is unique by Lemma 3.12. As in the first case, let $\mathcal{J}'$ be the subtree of $\mathcal{J}$ obtained by deleting from $\mathcal{J}$ every vertex that follows a nonterminal special vertex of $\mathcal{J}$. If V is not a vertex of $\mathcal{J}'$, the proof again proceeds as in the first case. If V is a vertex of $\mathcal{J}'$, then the terminal vertices of $\mathcal{J}'$ are: (1) possibly certain vertices W that are also terminal in $\mathcal{J}$; (2) V; and (3) possibly a set of special vertices $U_1, \ldots, U_k$, each of which is nonterminal in $\mathcal{J}$ and none of which is a corner. Since V follows X (by Corollary 3.9), X is a vertex of $\mathcal{J}'$. It is easy to see that $S(\mathcal{J}(X)) \leqslant d$. Then repeated application of Proposition 3.3 shows that $S(\mathcal{J}') \leqslant d$. Let $a = \sum S(W)$, this sum taken over all terminal vertices W of $\mathcal{J}$ that are in $\mathcal{J}'$. Let $m = d(V)$ and let $m_i = d(U_i)$, $i = 1, \ldots, k$. Using Proposition 3.3,

$$S(\mathcal{J}') = a + (m-1) + \sum_{i=1}^{k} m_i \leqslant d. \tag{3.10}$$

Let $\mathcal{J}''$ be the connected subtree of $\mathcal{J}$ whose vertex set is the vertices of $\mathcal{J}'$ plus the successors of $U_1, \ldots, U_k$. Then

$$S(\mathcal{J}'') + P(\mathcal{J}'') \leqslant a + (2m-2) + \sum_{i=1}^{k} (2m_i - 2). \tag{3.11}$$

From (3.10) and (3.11) we get

$$S(\mathcal{J}'') + P(\mathcal{J}'') \leqslant a + 2(d-a) - 2k = 2d - a - 2k.$$

If $k \geqslant 1$ then $S(\mathcal{J}'') + P(\mathcal{J}'') \leqslant 2d-2$, so we can assume that V is the only nonterminal special vertex of $\mathcal{J}$. Similarly, we can assume $a = 0$ or $a = 1$. Rewriting (3.11) as

$$S(\mathcal{J}'') + P(\mathcal{J}'') \leqslant a + 2m - 2$$

and recalling that $m \leqslant d$, we see that we can assume $a = 1$ and $m = d$. We now argue just as in the first case that the number of separatrices of our original vector field at $(0, 0)$ is $\leqslant 4d - 4$. ∎

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Added March 1980. We recently learned of four Russian papers, listed below, on separatrices and elliptic sectors. In [S1] Sagalovich shows that the number of separatrices of (1.1) at (0, 0) is $\leqslant 4d-2$ if $d \geqslant 2$. In [S2] Sagalovich claims to establish by examples that this bound is the best possible, but the examples are wrong. In [B1] Berlinskii proves that the number of elliptic sectors of (1.1) at (0, 0) is $\leqslant 2d-1$ and gives examples to show that this bound is the best possible. All four papers contain further discussion of the possible topological types of degree d singularities.

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