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Riemann surfaces in fibered polynomial hulls

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Abstract. Let Δ be the closed unit disk in $\mathbf{C}$, let Γ be the circle, let $\Pi: \Delta\times\mathbf{C}\to\Delta$ be projection, and let $A(\Delta)$ be the algebra of complex functions continuous on Δ and analytic in $\operatorname{int}\Delta$. Let K be a compact set in $\mathbf{C}^2$ such that $\Pi(K)=\Gamma$, and let $K_\lambda\equiv\{w\in\mathbf{C}\,|\,(\lambda,w)\in K\}$. Suppose further that (a) for every $\lambda\in\Gamma$, K_λ is the union of two nonempty disjoint connected compact sets with connected complement, (b) there exists a function $Q(\lambda,w)\equiv(w-R(\lambda))^2-S(\lambda)$ quadratic in w with $R,S\in A(\Delta)$ such that for all $\lambda\in\Gamma$, $\{w\in\mathbf{C}\,|\,Q(\lambda,w)=0\}\subset\operatorname{int}K_\lambda$, where S has only one zero in $\operatorname{int}\Delta$, counting multiplicity, and (c) for every $\lambda\in\Gamma$, the map $w\mapsto Q(\lambda,w)$ is injective on each component of K_λ. Then we prove that $\widehat{K}\setminus K$ is the union of analytic disks 2-sheeted over $\operatorname{int}\Delta$, where $\widehat{K}$ is the polynomial convex hull of K. Furthermore, we show that $\partial\widehat{K}\setminus K$ is the disjoint union of such disks.

Let Δ be the closed unit disk in $\mathbf{C}$, let Γ be the circle and let $\Pi: \Delta\times\mathbf{C}\longrightarrow\Delta$ be projection. Let K be a compact set such that $\Pi(K)=\Gamma$. Numerous authors (see [1], [5], [6], [8], [9], [12]) have studied features of the polynomial hull of K, denoted by $\widehat{K}$ or hull(K), frequently to investigate whether $\widehat{K}$ contains analytic structure in the form of graphs of analytic functions whose boundaries land in K. (Such functions are commonly called *analytic selectors* for K.) In this endeavour, it is natural to restrict oneself to the case where the *fiber* of K over $\lambda\in\Gamma$, $K_\lambda\equiv\{w\in\mathbf{C}\,|\,(\lambda,w)\in K\}$ is a connected compact set with connected complement (so also polynomially convex). (See [5], [6], [9].)

We now consider the case of a compact K where the fibers are not necessarily connected, but still have connected complements (and so are still polynomially convex). We shall specify circumstances where the part of the polynomial hull of K which projects through Π onto $\operatorname{int}\Delta$ is the union of analytic disks which are not graphs over $\operatorname{int}\Delta$ but are 2-sheeted over $\operatorname{int}\Delta$. Under the same circumstances, we shall show that $\partial\widehat{K}\setminus K$ is the disjoint union of such analytic disks. Let $A(\Delta)$ denote the disk algebra of functions continuous on Δ and analytic on $\operatorname{int}\Delta$, and let $H^\infty(\Delta)$ denote the algebra of bounded analytic functions on $\operatorname{int}\Delta$. We consider K with the following properties:

(1a) for every $\lambda\in\Gamma$, K_λ is the union of two nonempty disjoint connected compact sets with connected complement;

(1b) there exists a function $Q(\lambda,w)\equiv(w-R(\lambda))^2-S(\lambda)$ quadratic in w with $R,S\in A(\Delta)$ such that for all $\lambda\in\Gamma$, $\{w\in\mathbf{C}|Q(\lambda,w)=0\}\subset\operatorname{int}K_\lambda$, where S has only one zero in $\operatorname{int}\Delta$, counting multiplicity;

(1c) for every $\lambda\in\Gamma$, the map $w\mapsto Q(\lambda,w)$ is injective on each component of K_λ.

Note that (1c) implies that S has no zeroes on Γ and that the points in $\{w\in\mathbf{C}| Q(\lambda,w)=0\}$ lie in different components of K_λ, $\lambda\in\Gamma$. Property (1c) is easily obtained if, for example, the diameters of the components of K_λ are sufficiently small.

We shall prove the following result.

Theorem 1. *If K is a compact set satisfying* (1a–c) *then $\widehat{K}\setminus K$ is the union of the interiors of analytic disks of the form*

$$\begin{aligned} \operatorname{int}\Delta &\longrightarrow \widehat{K}, \\ \Gamma &\longrightarrow K \qquad\qquad \textit{for a.e. } \lambda\in\Gamma, \\ z &\longmapsto (B(z),f(z)), \end{aligned} \tag{2}$$

where B is a Blaschke product of order 2 *and $f\in H^\infty(\Delta)$* (*so the accumulation points on the boundary of the disk land in K*).

First we prove a theorem which allows more components in the fibers of K_λ but requires a relation among the components.

Theorem 2. *Let M and Y be compact sets fibered over the circle* (*i.e.*, $\Pi(M)=\Pi(Y)=\Gamma$) *such that $\widehat{M}\neq M$ and Y has fibers $Y_\lambda\subset\mathbf{C}$, $\lambda\in\Gamma$, which are connected with connected complement. Suppose that there exists a function*

$$Q(\lambda,w)=\sum_{n=0}^{d}a_n(\lambda)w^n,$$

with $a_n\in A(\Delta)$ for all n and $a_d\equiv 1$ such that for all $\lambda\in\Gamma$,

$$M_\lambda=\{w\in\mathbf{C}\mid Q(\lambda,w)\in Y_\lambda\}.$$

Then $\widehat{M}\setminus M$ is the union of analytic varieties d-sheeted over $\operatorname{int}\Delta$.

Proof. Let $(\lambda_0,w_0)\in\widehat{M}\setminus M$. Then we claim that $(\lambda_0,Q(\lambda_0,w_0))\in\widehat{Y}\setminus Y$. Given a polynomial P,

$$\begin{aligned} |P(\lambda_0,Q(\lambda_0,w_0))| &\le \sup_{(\lambda,w)\in M}|P(\lambda,Q(\lambda,w))| \\ &\le \sup_{\{(\lambda,w)|(\lambda,Q(\lambda,w))\in Y\}}|P(\lambda,Q(\lambda,w))| \le \sup_{(\lambda,w)\in Y}|P(\lambda,w)| \end{aligned}$$

as claimed.

Since for $\lambda\in\Gamma$ the Y_λ are connected with connected complement, there exists $f\in H^\infty(\Delta)$ such that

$$f(\lambda_0)=Q(\lambda_0,w_0)$$

and the accumulation points of the graph of f over Γ land in Y. Then we have that

$$\{(\lambda,w)\in \operatorname{int}\Delta\times\mathbf{C}\mid Q(\lambda,w)=f(\lambda),\ |\lambda|<1\}$$

is an analytic variety passing through (λ_0,w_0) whose accumulation points over Γ land in M. □

Corollary 1. *If M and Y are as in Theorem 2 then*

$$\{(\lambda,w)\in\widehat{M}\backslash M\}=\{(\lambda,w)\in\operatorname{int}\Delta\times\mathbf{C}\mid(\lambda,Q(\lambda,w))\in\widehat{Y}\backslash Y\}.$$

Proof. The inclusion $\subset$ was proven in the theorem. As for the opposite take (λ_0,w_0) with $(\lambda_0,Q(\lambda_0,w_0))\in\widehat{Y}\backslash Y$. Then there exists an $f\in H^\infty(\Delta)$ such that $f(\lambda_0)=Q(\lambda_0,w_0)$ and such that the set of accumulation points of the graph of f over Γ is contained in Y. Then

$$\{(\lambda,w)\in\operatorname{int}\Delta\times\mathbf{C}\mid(\lambda,Q(\lambda,w))\text{ belongs to the graph of } f \text{ over } \operatorname{int}\Delta\}$$

is an analytic variety over $\operatorname{int}\Delta$ passing through (λ_0,w_0) with accumulation points over Γ in M. Thus $(\lambda_0,w_0)\in\widehat{M}\backslash M$, as desired. □

Example. Suppose M is a compact set defined over Γ such that M_λ is the union of two disks of radius $\frac{1}{2}$ centered at $\pm\sqrt{\lambda}$. Let us take $Q(\lambda,w)=w^2$. We claim that M has the required properties described in Theorem 2. First, given a fixed $\lambda\in\Gamma$, choose a square root $\sqrt{\lambda}$. Then the image of $\{w\in\mathbf{C}\,\big|\,|w-\sqrt{\lambda}\,|\leq\frac{1}{2}\}$ under the map $w\mapsto w^2$ is the same as the image of $\{w\in\mathbf{C}\,\big|\,|w+\sqrt{\lambda}\,|\leq\frac{1}{2}\}$. We call the image Y_λ; since the squaring map is two-to-one, M_λ is the preimage of Y_λ under the squaring map. Letting Y be the set with fibers Y_λ, we see that Y is compact. Also Y has connected and simply connected fibers because the squaring map is one-to-one in a neighborhood of each of the components of M_λ so is a homeomorphism from each component to Y_λ. Hence $\widehat{M}\backslash M$ is the union of varieties of the form $w^2=f(\lambda)$, where $f\in H^\infty(\Delta)$ and $f(\lambda)\in Y_\lambda$ for a.e. $\lambda\in\Gamma$.

Next we require two lemmas.

Lemma 1. *If U and V are in $A(\Delta)$ and V has exactly one zero in Δ (not on Γ), then $\{(\lambda,w)\in\Delta\times\mathbf{C}\,|\,(w-U(\lambda))^2-V(\lambda)=0\}$ is a 2-sheeted analytic disk over Δ whose boundary is a continuous closed curve.*

Proof. We may write

$$V(\lambda)=\frac{\lambda-\alpha}{1-\bar{\alpha}\lambda}e^{\phi(\lambda)},$$

where $\phi\in A(\Delta)$, $|\alpha|<1$. Then our surface over Δ is

$$\left\{(\lambda,w)\in\Delta\times\mathbf{C}\;\middle|\;\left(\frac{w-U(\lambda)}{e^{\phi(\lambda)/2}}\right)^2-\frac{\lambda-\alpha}{1-\bar{\alpha}\lambda}=0\right\}$$

which, via the change of coordinates

$$(\lambda',w')=\left(\frac{\lambda-\alpha}{1-\bar{\alpha}\lambda},\frac{w-U(\lambda)}{e^{\phi(\lambda)/2}}\right),$$

biholomorphic in $\operatorname{int}\Delta\times\mathbf{C}$ and continuous in $\Delta\times\mathbf{C}$, is equivalent to

$$\{(\lambda',w')\in\Delta\times\mathbf{C}\,|\,(w')^2-\lambda'=0\},$$

a 2-sheeted disk. □

Lemma 2. *If $U,V\in A(\Delta)$ and for all $\lambda\in\Gamma$, the solutions of $(w-U(\lambda))^2-V(\lambda)=0$ lie in K_λ (one in each component) then V has exactly one zero in Δ, counting multiplicity, which is not on Γ.*

Proof. Choose ε small enough so that if $\lambda\in\Gamma$, the components of K_λ are at least 3ε apart in distance. From Lemma 1 and the remark following (1), we conclude that the analytic variety in (1) given by $\{(\lambda,w)\in\operatorname{int}\Delta\times\mathbf{C}\,|\,(w-R(\lambda))^2-S(\lambda)=0\}$ is an analytic disk 2-sheeted over $\operatorname{int}\Delta$. Suppose it is parametrized with $z\mapsto(B(z),g(z))$, $|z|\le 1$. Then B is analytic in $\operatorname{int}\Delta$ and maps the closed disk two-to-one onto itself. Clearly $B\in A(\Delta)$ (using the transformation from Lemma 1), and maps Γ to Γ. Thus B is a Blaschke product of order 2. Now the solutions of $(w-R(\lambda))^2-S(\lambda)=0$ over λ are $R(\lambda)\pm\sqrt{S(\lambda)}$, where $\sqrt{S(\lambda)}$ is not well defined over Γ. However, since $S\circ B$ has winding number 2 over Γ, $\sqrt{S\circ B}$ can be continuously well defined over Γ; we choose it so that $R(B(z))+\sqrt{S\circ B}\,(z)$ equals $g(z)$. Then we choose $\sqrt{V\circ B}$ so that $U(B(z))+\sqrt{V\circ B}\,(z)$ lies in the same component of $K_{B(z)}$ as $g(z)$. Construct a path $p(z,t)$ from $g(z)$ to $U(B(z))+\sqrt{V\circ B}\,(z)$ which varies continuously in (z,t) and always stays within ε of $K_{B(z)}$. Then we find through the homotopy p that

$$\begin{aligned}\operatorname{wind}\bigl(2\sqrt{V\circ B}\,\bigr)&=\operatorname{wind}\bigl(U\circ B+\sqrt{V\circ B}-\bigl(U\circ B-\sqrt{V\circ B}\,\bigr)\bigr)\\&=\operatorname{wind}\bigl(R\circ B+\sqrt{S\circ B}-\bigl(R\circ B-\sqrt{S\circ B}\,\bigr)\bigr)\\&=\operatorname{wind}\bigl(2\sqrt{S\circ B}\,\bigr)=1,\end{aligned}$$

so wind$(V\circ B)=2$ and hence the winding number of V is one over Γ. Thus V has exactly one zero on Δ, since it has none on Γ (the roots of $(w-U(\lambda))^2-V(\lambda)=0$ are distinct for $\lambda\in\Gamma$). $\square$

In order to distinguish between elements of the copy of Δ that we began with and elements of the domain of functions such as B and g above which parametrize the 2-sheeted disks, we generally use λ to refer to the elements of the former and z to refer to elements of the latter.

Combining Lemmas 1 and 2, we see that given any continuously bounded analytic variety $\{(\lambda,w)\in\Delta\times\mathbf{C}\,|\,(w-U(\lambda))^2-V(\lambda)=0\}$ with $U,V\in A(\Delta)$ over Δ, where the fiber of the variety over λ has one point in each component of K_λ, it must be a 2-sheeted analytic disk with boundary over Δ.

In order to prove Theorem 1, we shall first assume that K is a smoothly bounded solid torus, i.e., we shall assume that there exists a mapping

$$\mathcal{I}:\Gamma\times\Gamma\longrightarrow\Gamma\times\mathbf{C},$$
$$(z,w)\longmapsto(z^2,I(z,w))$$

such that the following hold, where K is the compact set whose fibers over $\lambda\in\Gamma$ are hull$(I(z,\Gamma))\cup$hull$(I(-z,\Gamma))$ for $z^2=\lambda$:

(3a) I is of class C^2;
(3b) $(\partial I/\partial w)(z,w)$ is never 0;
(3c) for any $z\in\Gamma$, $I(z,\cdot)$ is injective.

We shall need the fact that there exists a compact set M as in Theorem 2, also satisfying (1), such that $K_\lambda\subset M_\lambda$ for all $\lambda\in\Gamma$. To see this, let X denote the compact set whose fiber X_λ is $\{w\in\mathbf{C}\,|\,Q(\lambda,w)=Q(\lambda,w')$ for some $w'\in K_\lambda\}$. In other words, $X_\lambda=K_\lambda\cup(2R(\lambda)-K_\lambda)$, where $2R(\lambda)-K_\lambda=\{w\in\mathbf{C}\,|\,w=2R(\lambda)-w'$ for some $w'\in K_\lambda\}$. Then we claim that X_λ consists of two connected components. Let $K_{\lambda,1}$ and $K_{\lambda,2}$ denote the components of K_λ and let $K'_{\lambda,1}$ and $K'_{\lambda,2}$ denote their reflections $2R(\lambda)-K_{\lambda,1}$ and $2R(\lambda)-K_{\lambda,2}$ in $R(\lambda)$, respectively. Then $X_\lambda=K_{\lambda,1}\cup K_{\lambda,2}\cup K'_{\lambda,1}\cup K'_{\lambda,2}$. Clearly $K_{\lambda,1}\cap K'_{\lambda,2}\neq\emptyset$ and $K'_{\lambda,1}\cap K_{\lambda,2}\neq\emptyset$. Also $K_{\lambda,1}\cup K'_{\lambda,2}$ does not meet $K'_{\lambda,1}\cup K_{\lambda,2}$ because (i) $K_{\lambda,1}\cap K_{\lambda,2}=\emptyset$ and $K'_{\lambda,1}\cap K'_{\lambda,2}=\emptyset$ from (1a) and (ii) $K_{\lambda,1}\cap K'_{\lambda,1}=\emptyset$ and $K_{\lambda,2}\cap K'_{\lambda,2}=\emptyset$ from (1c). This establishes the claim. Since the components of X_λ are symmetric about $R(\lambda)$, the polynomial hulls of the components are as well, and are disjoint because the two components of X_λ are connected. Thus if we define X' over Γ to have fibers hull(X_λ) and M to be the closure of X' in $\Gamma\times\mathbf{C}$ then M satisfies (1) and the properties that M does in Theorem 2, and $M\supset K$.

We shall need (3) when invoking results from [5], [9] and [11].

Let w_1 be one of the elements of $\mathbf{C}$ such that $Q(1,w_1)=0$. Then in fact we will show that, with the additional conditions (3), $\widehat{K}\setminus K$ is the union of analytic disks of the form (2) where

$$\begin{gathered} B(z)=e^{i\theta}z\frac{z-\alpha}{1-\bar{\alpha}z}, \quad B(1)=1, \ |\alpha|\le 1-\varepsilon \text{ for some } \varepsilon>0, \\ f\in A(\Delta), \text{ and } f(1) \text{ is in the same component of } K_1 \text{ as } w_1. \end{gathered} \tag{4}$$

We shall also need the fact that K can be continuously expanded to a solid torus slightly larger than M. In other words, we construct mappings $\mathcal{I}_t(z,w)=(z^2,I_t(z,w))$, $0\le t\le 2$ having the same properties as $\mathcal{I}$ above in (3) and let K^t be the compact set whose fibers over $\lambda\in\Gamma$ are $\operatorname{hull}\big(I_t(\sqrt{\lambda},\Gamma)\big)\cup\operatorname{hull}\big(I_t(-\sqrt{\lambda},\Gamma)\big)$. We require that $K^{t_1}_\lambda\subset\operatorname{int}K^{t_2}_\lambda$ if $t_1<t_2$, $K^t=\bigcap_{s>t}K^s$, $K^0=K$, $M_\lambda\subset\operatorname{int}K^1_\lambda$ for all $\lambda\in\Gamma$ and for all t, $0\le t\le 2$, K^t satisfies the properties that K does in (1). To do this, we follow a method of Słodkowski [9, p. 371]. Suppose that we first construct a compact N satisfying the same properties K does in (1) and (3), and $M_\lambda\subset\operatorname{int}N_\lambda$ for all $\lambda\in\Gamma$. We may also construct N so that the associated map $\mathcal{I}_N$ extends to be a diffeomorphism of the interior of the solid torus by extending each $I_N(z,\cdot)$ from Γ to Δ. (We leave the verification of this intuitively obvious fact to the reader.) By composing the inverse of $\mathcal{I}_N$ with the diffeomorphism $(z,w)\mapsto(z,w/(1-|w|^2))$, we can map the sets K, M to sets K', M' in Słodkowski's setting in $\Gamma\times\mathbf{C}$, construct the associated $(K^t)'$ there, and pull them back through the above diffeomorphism to obtain the K^t. The only difference now is that Słodkowski only needed $(K^1)'$ large enough to contain the graph of a constant function. By using a compactness argument, we can extend this so that $(K^1)'$ contains any particular compact set in $\Gamma\times\mathbf{C}$, say M', so that K^1 contains M. The remaining properties are easily verified.

Lemma 3. *There exists an $\varepsilon>0$ such that if $B(z)=e^{i\theta}z(z-\alpha)/(1-\bar{\alpha}z)$ and the mapping $z\mapsto(B(z),g(z))$ is an analytic disk continuous for $z\in\Delta$ with boundary in K^1 then $|\alpha|<1-\varepsilon$.*

Proof. Suppose that for a sequence of continuously bounded analytic disks with boundary in K^1, 2-sheeted over $\operatorname{int}\Delta$, we obtain $B_n, g_n\in A(\Delta)$ parametrizing them as above, with associated α_n tending to 1 in modulus, and $\theta_n\to\theta$. Then on compact subsets of $\operatorname{int}\Delta$, $B_n(z)$ converges to $e^{i\phi}z$ (for some real constant ϕ) and $g_n\to g$. Thus $\operatorname{hull}(K^1)\setminus K^1$ contains an analytic graph over $\operatorname{int}\Delta$. If we restrict the corresponding function to the region $|\lambda|<1-\delta$ where δ is chosen so small that for all λ on the circle of radius $1-\delta$, $\operatorname{hull}(K^1)_\lambda\subset K^2_\lambda$, (possible since $K^1\subset\operatorname{int}K^2$ in $\Gamma\times\mathbf{C}$) then we have a continuous selector for the set K^2 over $|\lambda|=1$. The topology of K^2 does not permit this. Thus the possible α must have modulus bounded above by $1-\varepsilon$ for some $\varepsilon>0$. $\square$

Letting ε be as found in Lemma 3, let $\widetilde{K}^t$ equal the union of K^t with the union over int Δ of all analytic disks possessing properties (2) and (4), replacing K by K^t.

Theorem 3. *Let K be a compact set fibered over Γ satisfying properties* (1) *and suppose there exist functions $\mathcal{I}(z,w)=(z^2, I(z,w))$ satisfying properties* (3). *Then $\widehat{K}\setminus K$ is the union of the interiors of analytic disks of the form*

$$\begin{aligned} \Delta &\longrightarrow \widehat{K}, \\ \Gamma &\longrightarrow K, \\ z &\longmapsto (B(z), f(z)), \end{aligned}$$

where B is a Blaschke product of order 2, *$f\in A(\Delta)$.*

Proof. If (B,f) is a pair satisfying (2) and (4), replacing K by K^t, let $\sqrt{S\circ B}$ denote the continuous square root of $S\circ B$ over Γ such that $R(1)+\sqrt{S\circ B}\,(1)$ is in the same component of $K^t_{B(1)}$ as w_1. (Note that the winding number of $S\circ B$ is 2 on Γ.) Let

$$L^t(B)=\big\{(z,w)\in\Gamma\times\mathbf{C}\,|\,(B(z),w)\in K^t,\ w\in \text{the same component of } K^t_{B(z)} \text{ as } R(B(z))+\sqrt{S\circ B}\,(z)\big\}$$

and let

$$\widetilde{K}^t(B)=K^t\cup\{(\lambda,w)\in\operatorname{int}\Delta\times\mathbf{C}\,|\,(\lambda,w)=(B(z),w) \text{ where } (z,w) \text{ lies on the graph of some element of } A(\Delta) \text{ which is an analytic selector for } L^t(B)\}.$$

Then $\widetilde{K}^t$ is the union of all sets $\widetilde{K}^t(B)$ ranging over all possible Blaschke products B satisfying (4). Now let s be the infimum of all t such that $\widetilde{K}^t\supset\widehat{K}$. We first show that $s\le 1$ and eventually $s=0$. We apply Theorem 2 to M and obtain the corresponding set Y with connected fibers specified in Theorem 2, that is $Y_\lambda\equiv\{w\in\mathbf{C}\,|\,w=Q(\lambda,w') \text{ for some } w'\in M_\lambda\}$. Following Słodkowski [9, p. 380] we write Y as the decreasing intersection of sets Y^n fibered over Γ whose boundaries are smooth tori; write $M^n_\lambda=\{w\in\mathbf{C}\,|\,Q(\lambda,w)\in Y^n_\lambda\}$. For some large n, $M^n\subset K^1$. Now Theorem 3 of [5], Theorem 1.1 of [9] and Theorem 4 of [11] show that $\operatorname{hull}(Y^n)\setminus Y^n$ is the union of graphs over int Δ of elements of $A(\Delta)$. Then Corollary 1 shows that $\operatorname{hull}(M^n)\setminus M^n$ is the union of varieties of the form $\{(\lambda,w)\in\operatorname{int}\Delta\times\mathbf{C}\,|\,Q(\lambda,w)=f(\lambda),\ f\in A(\Delta)\}$ with boundary in M^n. Lemmas 1 and 2 show that such a variety must be an analytic disk; in particular, suppose this disk is parametrized by $z\mapsto(B(z),g(z))$, $|z|\le 1$. Then as in Lemma 2, we find that B is a Blaschke product of order 2 and $g\in A(\Delta)$. By change of coordinates in z we may assume that $B(0)=0$, and $(B(1), g(1))=(1, v_1)$, where v_1 is in the same component of M_λ as w_1. Since $M^n \subset K^1$, this shows that indeed $\widetilde{K}^1 \supset \widetilde{M}^n = \text{hull}(M^n) \supset \widehat{M} \supset \widehat{K}$, as desired. Thus $s \leq 1$.

We want to prove that $s=0$, so we make the assumption

$$(5) \qquad s>0.$$

Claim. It is true that $\widetilde{K}^s \supset \widehat{K}$.

Take $(\lambda, w) \in \widehat{K}$. Then clearly $(\lambda, w) \in \widetilde{K}^s$ if $|\lambda|=1$. If $|\lambda|<1$, then for $n \geq 1$ take $\{B_n\}$ and $\{f_n\}$ possessing properties (2) and (4) (replacing K by $K^{s+1/n}$) such that $(\lambda, w) \in \widetilde{K}^{s+1/n}(B_n)$. Then there exist $z_n \in \text{int}\,\Delta$ and $f_n \in A(\Delta)$ which is an analytic selector for $L^{s+1/n}(B_n)$ with $(B_n(z_n), f_n(z_n))=(\lambda, w)$. If $B_n(z)= e^{i\theta_n} z(z-\alpha_n)/(1-\bar{\alpha}_n z)$ then without loss of generality we may assume that $\alpha_n \to \alpha_0$, $|\alpha_0| \leq 1-\varepsilon$, $\theta_n \to \theta_0$, $z_n \to z_0$, $|z_0|<1$, (if $|z_0|=1$ then since $B_n \to B_0$ uniformly, $|B_n(z_n)-B_0(z_0)| \leq |B_n(z_n)-B_0(z_n)|+|B_0(z_n)-B_0(z_0)|$ tends to zero, as $n \to \infty$, so $1>|\lambda|=|B_n(z_n)| \to |B_0(z_0)|=1$, which is impossible) and $f_n \to f_0$ uniformly on compact subsets of int Δ. Also note that we have chosen $\sqrt{S \circ B_n}$ and $\sqrt{S \circ B_0}$ such that $\sqrt{S \circ B_n}\,(z)$ converges to $\sqrt{S \circ B_0}\,(z)$ for all $z \in \Gamma$.

Subclaim. It is true that $(z_0, f_0(z_0)) \in \text{hull}(L^s(B_0))$, where the function $B_0(z)= e^{i\theta_0} z(z-\alpha_0)/(1-\bar{\alpha}_0 z)$.

We have $(z_n, f_n(z_n)) \in \text{hull}(L^{s+1/n}(B_n))$ for $n \geq 1$. Fix polynomial $P(z, w)$, fix $\varepsilon>0$, let

$$C = \sup_{(z,w) \in L^s(B_0)} |P(z, w)|$$

and take N_1 so large that

$$L^{s+1/N_1}(B_0) \subset \{(z, w) \in \Delta \times \mathbf{C} \mid |P(z, w)| < C+\varepsilon\},$$

(using the fact that $K^s = \bigcap_{t>s} K^t$, so $L^s(B_0) = \bigcap_{t>s} L^t(B_0)$) and choose $N_2 \geq N_1$ so large that for $n \geq N_2$

$$L^{s+1/N_1}(B_n) \subset \{(z, w) \in \Delta \times \mathbf{C} \mid |P(z, w)| < C+\varepsilon\}.$$

Then for $n>N_2$,

$$L^{s+1/n}(B_n) \subset \{(z, w) \in \Delta \times \mathbf{C} \mid |P(z, w)| < C+\varepsilon\},$$

and choose N_2 even larger so that for $n>N_2$,

$$|P(z_n, f_n(z_n)) - P(z_0, f_0(z_0))| \leq \varepsilon,$$

possible since $z_n \to z_0$, $|z_0|<1$ and $f_n \to f_0$ uniformly on compact subsets of $\operatorname{int}\Delta$. Then $|P(z_0, f_0(z_0))| \le \sup_{L^s(B_0)} |P| + 2\varepsilon$, and this holds for any $\varepsilon>0$, so

$$|P(z_0, f_0(z_0))| \le \sup_{L^s(B_0)} |P|$$

and hence $(z_0, f_0(z_0)) \in \operatorname{hull}(L^s(B_0))$. This proves the subclaim.

Take $f \in A(\Delta)$ which is an analytic selector for $L^s(B_0)$ (see Theorem 3 of [5], Theorem 1.1 of [9] and Theorem 4 of [11]) and whose graph passes through the point $(z_0, f_0(z_0))$. This shows that $(B_0(z_0), f(z_0)) = (\lambda, w) \in \widetilde{K}^s$, as desired. Hence $\widetilde{K}^s \supset \widehat{K}$, which was our claim.

We now claim that $\widehat{K} \setminus \bigcup_{t<s} \widetilde{K}^t$ is nonempty. Relative to the topology of $\Delta \times \mathbf{C}$, $\widetilde{K}^{t_1}$ contains a neighborhood of $\widetilde{K}^{t_2}$ if $t_1 > t_2$ since $L^{t_1}(B)$ contains a neighborhood of $L^{t_2}(B)$ in $\Gamma \times \mathbf{C}$ for any B. Thus for any r, $\bigcup_{t<r} \widetilde{K}^t$ is relatively open in $\Delta \times \mathbf{C}$. Thus if $\bigcup_{t<s} \widetilde{K}^t$ contains $\widehat{K}$, then for some $r<s$, $\widetilde{K}^r$ contains a neighborhood of $\widehat{K}$ in $\Delta \times \mathbf{C}$. (This holds because the interiors of the $\widetilde{K}^t$ in $\Delta \times \mathbf{C}$ form an open cover of $\widehat{K}$, and $\widehat{K}$ is compact.) This contradicts the minimality of s.

Thus there exists some $p = (B(z_0), f(z_0)) \in \widehat{K} \setminus \bigcup_{t<s} \widetilde{K}^t$. Clearly $|z_0|<1$. Then $(z_0, f(z_0)) \in \operatorname{hull}(L^s(B)) \setminus \bigcup_{t<s} \operatorname{hull}(L^t(B))$.

We claim that this means that $f(z) \in \partial L^s_z(B)$ for all $z \in \Gamma$. To see this, suppose that at some point $\zeta \in \Gamma$, $f(\zeta) \in \operatorname{int} L^s_\zeta(B)$. By continuity of f, this holds in a neighborhood of ζ in Γ. Now let $N(z)$ be the inward pointing unit normal to $\partial L^s_z(B)$ at $f(z)$, if $f(z) \in \partial L^s_z(B)$. Choose a polynomial $G(z)$ such that $\arg G$ is within $\frac{1}{10}\pi$ of $\arg N(z)/((z-z_0)/(1-\bar{z}_0 z))$, where $N(z)$ is defined, and arbitrary elsewhere on Γ except that $G(z) \ne 0$ on Γ and $\operatorname{wind} G$ equals 0. If we let $F(z) = G(z)(z-z_0)/(1-\bar{z}_0 z)$ then $F \in A(\Delta)$, $\arg F$ is within $\frac{1}{10}\pi$ (modulo 2π) of $\arg N(z)$ where $N(z)$ is defined, F is never zero on Γ and $F(z_0)=0$. Hence for sufficiently small positive τ, $f(z) + \tau F(z) \in \operatorname{int} L^s_z(B)$ for all z in Γ. (This is obvious pointwise for $z \in \Gamma$ and can be extended to the entire circle uniformly in τ by a compactness argument.) Furthermore, the graph of $f + \tau F$ passes through $(z_0, f(z_0))$. This contradicts the minimality of s and we conclude that $f(z) \in \partial L^s_z(B)$ for all $z \in \Gamma$.

We consider the various possibilities for the value of the winding number of $\left(f - R\circ B - \sqrt{S\circ B}\right)$ over Γ. We may show through an argument like the above that if the winding number were positive, s would not be minimal. We next show that this winding number is either 0 or -1.

If $\operatorname{wind}\left(f(z) - R(B(z)) - \sqrt{S\circ B}\,(z)\right) = d < 0$ then

$$\begin{aligned}\operatorname{wind}\left(f(z) - R(B(z)) - \sqrt{S\circ B}\,(z)\right)\left(f - R(B(z)) + \sqrt{S\circ B}\,(z)\right) \\ = 1 + d = \operatorname{wind}((f(z) - R(B(z)))^2 - S(B(z)))\end{aligned}$$

which is ≥ 0 since $(f(z)-R(B(z)))^2-S(B(z))$ is analytic, and nonzero on Γ since $s>0$. Hence $d=-1$.

Case 1. Assume that $\operatorname{wind}(f-R\circ B-\sqrt{S\circ B})=0$.

Let $Q(\lambda,w)=(w-U(\lambda))^2-V(\lambda)$ be analytic in $\operatorname{int}\Delta\times\mathbf{C}$, continuous on $\Delta\times\mathbf{C}$, and zero on points $(B(z),f(z))$, $z\in\Delta$.

In the proof Lemma 1 we found a change of coordinates in $\Delta\times\mathbf{C}$ which is analytic in $\operatorname{int}\Delta\times\mathbf{C}$ and which carries $Q(\lambda,w)$ to $w^2-\lambda$. Let us switch to these coordinates, obtaining sets J^t as the image of the sets K^t. Under this transformation we observe that $\widetilde{K}^t$ maps to $\tilde{J}^t$ and $\widehat{K}$ to $\hat{J}$. Then s satisfies the same extremal property with respect to the J^t as the K^t. Now in the new coordinates the J^t are not as smooth as the K^t but needed properties will be preserved. In particular, (i) the winding number above is the same since the function $e^{\phi(\lambda)/2}$ in Lemma 1 has no zeroes in Δ, and (ii) for fixed t the fibers of J^t are still smoothly bounded. Now with our change of coordinates we find that B is transformed into the squaring map and f into the identity. We write $p=(w_0^2,w_0)$.

Let $n(w)$ be the inward unit normal to $J^s_{w^2}$ at w and let $N(w)=2wn(w)$ be the image of $n(w)$ under the differential of $w\mapsto w^2-\lambda$. (Note that $n(w)$ is still continuous under the change of coordinates.) Then $\operatorname{wind}(N(w))=1$. Choose a polynomial g such that $|\arg g(w)-\arg N(w)|<\frac{1}{10}\pi$ (modulo 2π) and $g(w_0)=0$. Now consider the set where

$$w^2-\lambda=\tau g(w)$$

for some fixed small positive constant τ. We need a lemma. Let D be a closed disk in $\mathbf{C}$ centered at 0 such that $J^2\subset\Gamma\times\operatorname{int}D$.

Lemma 4. *For τ sufficiently small,*

$$w^2-\lambda=\tau g(w) \tag{6}$$

has exactly two solutions for w in $\operatorname{int}D$ *for all $\lambda\in\Delta$, the solutions actually lie in* $\operatorname{hull}(J^2)_\lambda$ *as well, and for $\lambda\in\Gamma$, the solutions lie in different components of* $\operatorname{hull}(J^2)_\lambda$.

Proof. Suppose the assertion for $\lambda\in\Delta$ does not hold. Then take $\tau_n\downarrow 0$, $\lambda_n\to\lambda\in\Delta$ such that (6) does not have exactly two solutions for $w\in\operatorname{int}D$, where we replace λ, τ in (6) by λ_n, τ_n. Suppose this number of solutions is equal to k_n.

Since $w^2-\lambda_n-\tau_n g(w)\to w^2-\lambda$ uniformly for w in a compact set in $\mathbf{C}$, as $n\to\infty$, $w^2-\lambda_n-\tau_n g(w)$ has the same number of zeroes in D as $w^2-\lambda$. So for large n, $k_n=2$, a contradiction. The argument regarding $\operatorname{hull}(J^2)_\lambda$ is similar.

To prove the assertion regarding $\lambda\in\Gamma$, we may proceed by contradiction again and use a similar argument to come to the conclusion that for some λ, $w^2-\lambda$ does not vanish at one of $\pm\sqrt{\lambda}$, an obvious contradiction. □

We claim that for $\lambda\in\Gamma$, these zeroes are in fact in int J^s_λ for small τ. Let $h(z,\tau)$ denote the location of the zero for $w^2-z^2-\tau g(w)$ which is in the same component of $\operatorname{hull}(J^2)_{z^2}$ as z. (Note $\lambda=z^2$.) We claim that h is a C^∞ function in (z,τ) for sufficiently small τ. We know that h satisfies the equation

$$F(h,z,\tau)\equiv h^2-z^2-\tau g(h)=0.$$

Fix v, $|v|=1$. Then since $\partial F/\partial h=2h-\tau g'(h)=2v\neq 0$ when $(h,z,\tau)=(v,v,0)$, the implicit function theorem shows that h is a C^∞ function of (z,τ) in a neighborhood of $(v,0)$. Choosing finitely many such neighborhoods covering all $v\in\Gamma$ we find that indeed h has the required smoothness.

We check that the set $\{(\lambda,w)\in\Delta\times\operatorname{int}D\,|\,w^2-\lambda=\tau g(w)\}$ is in fact (for the above small τ) given by $\{(\lambda,w)\in\Delta\times\mathbf{C}\,|\,w^2+a_1(\lambda)w+a_0(\lambda)=0\}$ for some $a_1(\lambda),a_0(\lambda)\in A(\Delta)$. Let $r^1_\tau(\lambda)$, $r^2_\tau(\lambda)$ be the two solutions, not well-defined, of (6) for $w\in\operatorname{int}D$. Then we just have to show that $r^1_\tau+r^2_\tau$ and $r^1_\tau r^2_\tau$ are both elements of $A(\Delta)$. Consider the well-defined continuous function $(r^1_\tau(\lambda)-r^2_\tau(\lambda))^2$ on Δ; near where r^1_τ is different from r^2_τ, $r^1_\tau(\lambda)$ and $r^2_\tau(\lambda)$ can be well-defined and are analytic; thus $(r^1_\tau(\lambda)-r^2_\tau(\lambda))^2$ is continuous and analytic on Δ where it is nonzero. By Radó's theorem, $(r^1_\tau(\lambda)-r^2_\tau(\lambda))^2$ is in $A(\Delta)$. Thus its zeroes are isolated in $\operatorname{int}\Delta$. We conclude that $r^1_\tau+r^2_\tau$ and $r^1_\tau r^2_\tau$ are analytic except at isolated points where $r^1_\tau(\lambda)=r^2_\tau(\lambda)$. But both functions are clearly bounded on Δ so such singularities are removable. Hence $r^1_\tau+r^2_\tau$ and $r^1_\tau r^2_\tau$ are both elements of $A(\Delta)$, as desired.

We have $h\colon\Gamma\times(-\delta,\delta)\to\mathbf{C}$ for some small δ and

$$h(w,\tau)^2-\lambda-\tau g(h(w,\tau))=0.$$

Differentiating implicitly with respect to τ,

$$2h\frac{\partial h}{\partial\tau}-g(h(w,\tau))-\tau g'(h(w,\tau))\frac{\partial h}{\partial\tau}=0$$

for small $|\tau|$. When $\tau=0$,

$$2w\frac{\partial h}{\partial\tau}-g(w)=0,$$

so

$$\frac{\partial h}{\partial\tau}=\frac{g(w)}{2w}=\frac{\dfrac{g(w)}{N(w)}2wn(w)}{2w}=r(w)n(w),$$

where r is a continuous nonzero function in $w\in\Gamma$ with argument within $\frac{1}{10}\pi$ of 0. This means that for small positive τ, the zeroes of $w^2-\lambda-\tau g(w)$ in $\operatorname{int}D$ over λ lie in the interior of J^s_λ. Deferring the verification of this for a moment, we see that from

Lemmas 1 and 2 this means we have constructed a continuously bounded 2-sheeted analytic disk in J^t for some $t<s$. This disk passes through (w_0^2, w_0) since $g(w_0)=0$. This is a contradiction of the minimality of s and hence Case 1 is impossible.

To check the above assertion, first choose ε so small that a vector pointing with argument within $\frac{1}{5}\pi$ of the inward pointing normal to J_λ^s at w (where $w^2=\lambda$) lies entirely in int J_λ^s (except for w) if its length is less than ε. (First choose such small vectors on K^s since it is smooth. Then map these vectors to J^s under the affine coordinate change. Some ε will work for all λ because the dilation constant $e^{\phi(\lambda)/2}$ is bounded away from 0 and ∞ uniformly in λ.) Then choose δ so small that for $|\tau|<\delta$ and $z\in\Gamma$, (i) $|h(z,\tau)-h(z,0)|<\varepsilon$ and (ii) $|\arg(\partial h/\partial\tau)(z,\tau)-\arg(\partial h/\partial\tau)(z,0)|<\frac{1}{10}\pi$. Then for $0<|\tau|<\delta$, $\arg((h(z,\tau)-h(z,0))/\tau)=\arg((\partial h/\partial\tau)(z,\tau_z))$ for some τ_z between 0 and τ, by the mean value theorem. Hence for $0<\tau<\delta$, $h(z,\tau)-h(z,0)$ has length less than ε and has argument within $\frac{1}{5}\pi$ of the inward pointing normal to $J_{z^2}^s$ at z so $h(z,\tau)$ lies in int $J_{z^2}^s$ for all $0<\tau<\delta$ and $z\in\Gamma$.

Case 2. Assume that $\operatorname{wind}(f-R\circ B-\sqrt{S\circ B})=-1$.

Let us apply the same coordinate transformation as in Case 1. Let $n(w)$ and $N(w)$ be as before; then $\operatorname{wind}(N(w))=0$. Choose g analytic in a neighborhood of Δ such that $\arg(g(w))$ is within $\frac{1}{10}\pi$ of $\arg(-N(w))$ for $|w|=1$ and consider the set

$$T=\{(\lambda,w)\in\Delta\times\mathbf{C}\mid w^2-\lambda=\tau g(w)\}\cap\operatorname{hull}(J^2) \tag{7}$$

for small τ; using an argument similar to that in Case 1, this is an analytic 2-sheeted disk whose fiber over $\lambda\in\Gamma$ consists of 2 points outside of J_λ^s. As $\tau\downarrow 0$, T approaches the point (w_0^2, w_0). We also claim that T does not meet $\hat{J}$. We show this by proving that T does not meet any 2-sheeted analytic disk in $\tilde{J}^s\setminus J^s$ for any sufficiently small τ.

Parametrize T by $\lambda\mapsto(B_\tau(\lambda), f_\tau(\lambda))$ which possesses property (4). Let $\sqrt{B_\tau}$ be the continuous square root of B_τ on Γ such that $\sqrt{B_\tau}\,(1)=1$. Using reasoning similar to that at the end of Case 1, choose δ so small that T does not meet J^s for $0<\tau<\delta$ and $f_\tau(w)-\sqrt{B_\tau}\,(w)$ has argument within $\frac{1}{5}\pi$ of $\arg(-n(w))$, modulo 2π. Consider a disk in $\tilde{J}^s$ given by $\{(\lambda,w)\in\Delta\times\mathbf{C}|U(\lambda,w)=0\}$, U monic quadratic in w. Now there are two well-defined continuous functions $R_\tau^1(z)$, $R_\tau^2(z)$ such that the zeroes of $U(\lambda,w)$ over $\lambda=B_\tau(z)$ are $R_\tau^1(z), R_\tau^2(z)$; just let $R_\tau^1(z)$ be the zero of $U(B_\tau(z),w)$ which lies in the same component of $J_{B_\tau(z)}^1$ as $\sqrt{B_\tau}\,(z)$ and let $R_\tau^2(z)$ be the other zero.

Then $U(B_\tau(w), f_\tau(w))=(f_\tau(w)-R_\tau^1(w))(f_\tau(w)-R_\tau^2(w))$. Now over $|w|=1$,

$$\operatorname{wind}(f_\tau(w)-R_\tau^1(w))=\operatorname{wind}\big(f_\tau(w)-\sqrt{B_\tau}\,(w)\big)=\operatorname{wind} n(w)=-1,$$

since T does not meet J^s for $0<\tau<\delta$ and $f_\tau(w)-\sqrt{B_\tau}\,(w)$ has argument within $\frac{1}{5}\pi$ of $\arg(-n(w))$. Also for $|w|=1$,

$$\begin{aligned}\operatorname{wind}(f_\tau(w)-R_\tau^2(w)) &= \operatorname{wind}\big(\sqrt{B_\tau}\,(w)-R_\tau^2(w)\big)\\ &= \operatorname{wind}\big(\sqrt{B_\tau}\,(w)-(-\sqrt{B_\tau}\,(w))\big)=1\end{aligned}$$

so $\operatorname{wind}(U(B_\tau(w),f_\tau(w)))=0$; this means that U is never 0 on T for such τ. This holds for all U defining a 2-sheeted disk in $\tilde{J}^s$, so for $0<\tau<\delta$, T does not meet $\widetilde{J}^s$, so does not meet $\hat{J}$.

Let $P(\hat{J})$ be the set of continuous complex functions on $\hat{J}$ which are uniform limits of polynomials. Let $Q_\tau(\lambda,w)$ be monic quadratic in w, $0<\tau<\delta$, such that $Q_\tau(B_\tau(w),f_\tau(w))=0$ for all w. Then by the Oka–Weil theorem, $Q_\tau(\lambda,w)^{-1}$ is an element of $P(\hat{J})$, since T does not meet $\hat{J}$. Also $Q_\tau(\lambda,w)^{-1}$ is bounded on $J=J^0$ uniformly in τ, $0<\tau<\delta$, since $s>0$ but as $\tau\to 0$, $Q_\tau(w_0^2,w_0)^{-1}\to\infty$, a contradiction. Thus the original assumption (5) that $s>0$ must be false; $s=0$ and $\widetilde{K}\supset\widehat{K}$. (This concludes Case 2.)

We already know $\widetilde{K}\subset\widehat{K}$, so $\widetilde{K}=\widehat{K}$, as desired. □

Proof of Theorem 1. Following Słodkowski [9] choose compact sets $K(n)$ satisfying (1) such that $K=\bigcap_{n=1}^\infty K(n)$ and the $K(n)$ are solid tori whose boundaries arise from mappings $\mathcal{I}(n)$ which are restricted by (3). Also choose the $K(n)$ such that for all (λ,n), $K_\lambda\Subset K(n+1)_\lambda\Subset K(n)_\lambda$.

We now invoke Theorem 3, replacing K by $K(n)$ and conclude that $\widetilde{K(n)}=\operatorname{hull}(K(n))$, so $\widetilde{K(n)}\supset\widehat{K}$. Thus $\widehat{K}\subset\widetilde{K(n)}$ for all n. Thus every point p in $\widehat{K}\setminus K$ lies on a sequence of analytic disks parametrized by $z\mapsto(B^n(z),f^n(z))$, where B^n, f^n possess properties (4) with respect to $K(n)$. If $B^n\to B$ uniformly and $f^n\to f$ uniformly on compact sets then using an argument similar to that in the claim of the proof of Theorem 3, we can conclude that every point of the form $(B(\lambda),f(\lambda))$ for $|\lambda|<1$ lies in $\widehat{K}\setminus K$. The associated disk contains the point p and its boundary accumulation points lie in K, as desired. (Note that this shows $\widehat{K}=\bigcap_{n=1}^\infty\operatorname{hull}(K(n))$.) □

Theorem 4. *If K is as in Theorem 1, then $\partial\widehat{K}\setminus K$ is the disjoint union of 2-sheeted analytic disks.*

Proof. First suppose K has the special form in Theorem 3. Choose a point $(\lambda,w)\in\partial\widehat{K}$, where $|\lambda|<1$, and suppose (λ,w) lies on a disk parametrized by $z\mapsto(B(z),f(z))$. The analysis in the proof of Theorem 3 shows that we can choose f to be continuous on Δ, $f(z)\in\partial L_z(B)$ for all $z\in\Gamma$ and Case 2 of the proof of Theorem 3 holds. (Otherwise we can construct $g\in A(\Delta)$ such that $g(z)\in\operatorname{int}L_z(B)$

for all $z\in\Gamma$ and the disk parametrized by $z\mapsto(B(z),g(z))$ passes through (λ,w). Small perturbations of g then show that $(\lambda,w)\notin\partial\widehat{K}$.) Then in Case 2 we showed that every point on the disk $z\mapsto(B(z),f(z))$ is the limit of points on 2-sheeted disks external to $\widehat{K}$. (Actually we proved this for $\hat{J}$ but the coordinate transformation allows us to pull it back to $\widehat{K}$.) Hence all of the disk $z\mapsto(B(z),f(z))$ lies in $\partial(\widehat{K})\setminus K$. Thus $\partial(\widehat{K})\setminus K$ is the union of 2-sheeted analytic disks over $\operatorname{int}\Delta$.

To see that these disks are disjoint, suppose that two of them given by $z\mapsto(B^1(z),f^1(z))$ and $z\mapsto(B^2(z),f^2(z))$ meet in $\operatorname{int}\Delta\times\mathbf{C}$. Assume without loss of generality that $(B^1(z_1),f^1(z_1))=(B^2(z_2),f^2(z_2))$ for some $z_1,z_2\in\operatorname{int}\Delta$. Now construct the sequence of disks $z\mapsto(B^1_\tau(z),f^1_\tau(z))$ external to $\widehat{K}$, as in Case 2 of the proof of Theorem 3. By changing coordinates in z, assume that $(B^1_\tau(z_1),f^1_\tau(z_1))\to(B^1(z_1),f^1(z_1))$. Let $P^i(\lambda,w)$ be monic quadratic in w such that $P^i(B^i(z),f^i(z))=0$ for all $z\in\Gamma$, $i=1$, 2. Then the functions $P^i(B^1_\tau(z),f^1_\tau(z))$ are nonzero analytic functions in z which tend to 0 at $z=z_1$, as $\tau\to0$, for $i=1$, 2. Pass to a subsequence of (B^1_τ,f^1_τ) which converges locally uniformly (and nontrivially, without loss of generality) to $((B^1)',(f^1)')$. By Hurwitz' theorem, $\{P^i(B^1_\tau(z),f^1_\tau(z))\}_\tau$ tends to zero uniformly for z in compact subsets of $\operatorname{int}\Delta$ as $\tau\to0$, and we conclude that the two disks $z\mapsto(B^1(z),f^1(z))$ and $z\mapsto(B^2(z),f^2(z))$ parametrize the same analytic disk because for every $\lambda\in\operatorname{int}\Delta$, $P^1(\lambda,w)$ and $P^2(\lambda,w)$ vanish for the same two values of w.

For general K, write K as a decreasing intersection of $K(n)$ as before; then $\widehat{K}=\bigcap_{n=1}^\infty \operatorname{hull}(K(n))$, as noted at the end of the proof of Theorem 1. Choose $(\lambda,w)\in\partial\widehat{K}\setminus K$. Then, passing to a subsequence of the $K(n)$, there exist points $(\lambda_n,w_n)\in\partial\operatorname{hull}(K(n))\setminus K(n)$ converging to (λ,w). With them are associated 2-sheeted disks $z\mapsto(B^n(z),f^n(z))$ in $\partial\operatorname{hull}(K(n))$ which pass through (λ_n,w_n). A local uniform limit can be chosen as before so that $z\mapsto(B(z),f(z))$ passes through (λ,w) and lies in $\partial\operatorname{hull}(K)\setminus K$. To show that no two 2-sheeted disks in $\partial\operatorname{hull}(K(n))\setminus K(n)$ meet, we can employ an argument similar to that in the previous paragraph, using the (B^n,f^n) instead of the (B^1_τ,f^1_τ). □

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