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Existence of the spectral gap for elliptic operators

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Abstract. Let M be a connected, noncompact, complete Riemannian manifold, consider the operator $L=\Delta+\nabla V$ for some $V\in C^2(M)$ with $\exp[V]$ integrable with respect to the Riemannian volume element. This paper studies the existence of the spectral gap of L. As a consequence of the main result, let ϱ be the distance function from a point o, then the spectral gap exists provided $\lim_{\varrho\to\infty}\sup L\varrho<0$ while the spectral gap does not exist if o is a pole and $\lim_{\varrho\to\infty}\inf L\varrho\geq 0$. Moreover, the elliptic operators on $\mathbf{R}^d$ are also studied.

1. Introduction

Let $\widetilde{M}$ be a d-dimensional, connected, noncompact, complete Riemannian manifold, and let M be either $\widetilde{M}$ or an unbounded regular closed domain in $\widetilde{M}$. Next, consider $L=\Delta+\nabla V$ for some $V\in C^\infty(M)$ with $Z:=\int_M \exp[V]\,dx<\infty$. Let $d\mu=Z^{-1}\exp[V]\,dx$ be defined on M. The spectral gap of the operator L (with Neumann boundary condition if $\partial M\neq\emptyset$) is characterized as

$$(1.1)\qquad \lambda_1=\inf\left\{\frac{\mu(|\nabla f|^2)}{\mu(f^2)-\mu(f)^2}: f\in C^1(M)\cap L^2(\mu),\ f\neq\text{constant}\right\}.$$

We say that the spectral gap of L exists if $\lambda_1>0$. From now on, we assume that L is regular in the sense that $C_0^\infty(M)$ is dense in $W^{1,2}(M,d\mu)$ with the Sobolev norm $\|\cdot\|_{L^2(\mu)}+\|\nabla\cdot\|_{L^2(\mu)}$.

According to Wang [12] and Chen–Wang [3], we have $\lambda_1>0$ provided the Ricci curvature is bounded below and Hess_V is uniformly negatively definite out of a compact domain. Actually, the recent work by the author [14] shows that this condition implies the logarithmic Sobolev inequality which is stronger than the existence of

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spectral gap. Moreover, [13] proved that the logarithmic Sobolev inequality is equivalent to an exponential integrability of the square of the distance function, which naturally refers to the negativity of Hess_V along the radial direction.

On the other hand, we know that the spectral gap may exist if the distance function itself is exponential integrable. For instance, let $M=[0,\infty)$ and $L=(d^2/dr^2)-c(d/dr)$, $c>0$, then (see [4, Example 2.8]) $\lambda_1=\frac{1}{4}c^2>0$. From this we may guess that the existence of spectral gap, unlike the logarithmic Sobolev inequality, essentially depends on the first order radial-direction derivative of V rather than the second order derivative. This observation is now supported by Corollary 1.4 in the paper.

Our study is based on the fact that $\lambda_1>0$ is equivalent to $\inf\sigma_{\mathrm{ess}}(-L)>0$, where $\sigma_{\mathrm{ess}}(-L)$ denotes the essential spectrum of $-L$ (with Neumann boundary condition if $\partial M\neq\emptyset$). To see this, one needs only to show that 0 is an eigenvalue with multiplicity 1, equivalently, for any $f\in L^2(\mu)$ with $Lf=0$ and $\nu f|_{\partial M}=0$ (where ν denotes the inward unit normal vector field of ∂M when it is nonempty), one has that f is constant. This is a consequence of a result in Sturm [10].

Next, for fixed $o\in\widetilde{M}$, let $\varrho(x)$ be the Riemannian distance function from o. For $r>0$, let $B_r=\{x\in M:\varrho(x)<r\}$ and $B_r^c=M\setminus B_r$. By Donnely–Li's decomposition theorem (see [5]), one has $\inf\sigma_{\mathrm{ess}}(-L)=\lim_{r\to\infty}\lambda^c(r)$, where

$$\lambda^c(r)=\inf\{\mu(|\nabla f|^2): f\in C^1(M),\ \mu(f^2)=1,\ f=0 \text{ on } B_r\}.$$

Thence, $\lambda_1>0$ is equivalent to $\lambda^c(r)>0$ for large r. More precisely, we have the following result.

Theorem 1.1. (1) *If* $\mu(B_r)>0$, *then*

$$\lambda_1\leq\frac{\lambda^c(r)}{\mu(B_r)}. \tag{1.2}$$

(2) *Let* $\lambda(R)$ *be the smallest positive Neumann eigenvalue of* $-L$ *on* B_R. *If* $\lambda^c(r)>0$, *then*

$$\lambda_1\geq\sup_{R>r}\frac{\lambda^c(r)\lambda(R)\mu(B_R)(R-r)^2-2\lambda(R)(1-\mu(B_R))}{2\lambda(R)(R-r)^2+\lambda^c(r)(R-r)^2\mu(B_R)+2\mu(B_R)}>0. \tag{1.3}$$

We now go to estimate the quantity $\lambda^c(r)$. For $D\geq0$, define

$$\gamma(r)=\sup_{\substack{\varrho(x)=r\\ x\notin\mathrm{cut}(o)}}L\varrho(x),\quad C(r)=\int_{D+1}^r\gamma(s)\,ds,\quad r>D.$$

Here and in what follows, the point x runs over M.

Theorem 1.2. *Suppose that there exists $D>0$ such that either $\partial M\subset B_D$ or $\nu\varrho\le 0$ on $\partial M\cap(B_D^c\setminus\mathrm{cut}(o))$. For any $r_0>D$ and positive function $f\in C[r_0,\infty)$, we have*

$$\lambda^c(r_0)\ge\inf_{t\ge r_0} f(t)\bigg(\int_{r_0}^t \exp[-C(r)]\,dr\int_r^\infty \exp[C(s)]f(s)\,ds\bigg)^{-1}. \tag{1.4}$$

Consequently, we have $\lambda_1>0$ provided there exists a positive $f\in C[D+1,\infty)$ such that

$$\sup_{t\ge D+1}\frac{1}{f(t)}\int_{D+1}^t \exp[-C(r)]\,dr\int_r^\infty \exp[C(s)]f(s)\,ds<\infty. \tag{1.5}$$

We remark that the assumption in Theorem 1.2 holds if either ∂M is bounded or $o\in M$ and M is convex. Especially, for the case $M=[0,\infty)$ and $o=0$, we have $C(r)=V(r)$. By [4, Theorem 2.1], if $f'>0$ then

$$\lambda_1\ge\inf_{t\ge 0} f'(t)\exp[C(t)]\bigg(\int_t^\infty \exp[C(s)]f(s)\,ds\bigg)^{-1}. \tag{1.6}$$

Hence, the second assertion in Theorem 1.2 can be regarded as an extension of [4, Theorem 2.1] to Riemannian manifolds.

Corollary 1.3. *Under the assumption of Theorem 1.2 we have $\lambda_1>0$ if*

$$\sup_{t\ge D+1}\exp[-C(t)]\int_t^\infty \exp[C(s)]\,ds<\infty.$$

Consequently, $\lambda_1>0$ provided $\int_{D+1}^\infty(\gamma+\varepsilon)^+(r)\,dr<\infty$ for some $\varepsilon>0$.

Now, it is time to state the result mentioned in the abstract.

Corollary 1.4. (1) *Under the assumption of Theorem 1.2, we have $\lambda_1>0$ provided*

$$\lim_{r\to\infty}\sup_{\substack{\varrho(x)=r\\ x\notin\mathrm{cut}(o)}} L\varrho(x)<0.$$

(2) *Suppose that ∂M is bounded and o is a pole. If*

$$\lim_{r\to\infty}\inf_{\varrho(x)=r} L\varrho(x)\ge 0,$$

then $\lambda_1=0$.

Remark. (1) The first part of Corollary 1.4 follows from Corollary 1.3 directly. It was pointed out to the author by the referee that this can also be proved by using

Cheeger's inequality (cf. [1]), the argument goes as follows. Define the isoperimetric constant by

$$h_r = \inf \frac{A(\partial\Omega \cap \text{interior}(M))}{\mu(\Omega)},$$

where Ω runs over all bounded open subsets of B_r^c and A denotes the measure on $\partial\Omega$ induced by μ. Then, one has $\lambda^c(r) \geq \frac{1}{4}h_r^2$.

Next, let ν be the inward normal vector field of $\partial\Omega$. Noting that $\nu\varrho \leq 0$ on $\partial M \cap \partial\Omega$, we obtain

$$A(\partial\Omega \cap \text{interior}(M)) \geq \int_{\partial\Omega} \nu\varrho \, dA - \int_{\partial\Omega\cap\partial M} \nu\varrho \, dA \geq \int_{\partial\Omega} \nu\varrho \, dA = -\int_{\Omega} L\varrho \, d\mu,$$

where $L\varrho$ is understood in distribution sense in the case that $\text{cut}(o) \neq \emptyset$. Then, under the condition we have $\lim_{r\to\infty} \lambda^c(r) > 0$.

(2) The proof of Corollary 1.4(2) is based on the following upper bound estimate (cf. [9, Proposition 2.13]):

$$\lambda_1 \leq \tfrac{1}{4} \sup\{\varepsilon^2 : \mu(\exp[\varepsilon\varrho]) < \infty\}. \tag{1.7}$$

This estimate can be proved by taking the test function $f_n = \exp\left[\varepsilon\frac{1}{2}(\varrho \wedge n)\right]$ and then letting $n \to \infty$, referring to the proof of Theorem 3.2 below.

The proofs of the above results are given in the next section, and along the same line the spectral gap of elliptic operators on $\mathbf{R}^d$ is studied in Section 3.

2. Proofs

Proof of Theorem 1.1. We prove (1.2) and (1.3) respectively.

(a) The proof of (1.2) is modified from Thomas [10] which studies the upper bound of the spectral gap for discrete systems.

For any $\varepsilon > 0$, choose $f_\varepsilon \in C^1(M)$ with $f_\varepsilon|_{B_r} = 0$ and such that $\mu(f_\varepsilon^2) = 1$ and $\mu(|\nabla f_\varepsilon|^2) \leq \varepsilon + \lambda^c(r)$. Noting that $\mu(f_\varepsilon) = \mu(f_\varepsilon 1_{B_r^c}) \leq \sqrt{\mu(B_r^c)}$, we have that $\mu(f_\varepsilon^2) - \mu(f_\varepsilon)^2 \geq \mu(B_r)$, then

$$\lambda_1 \leq \frac{\mu(|\nabla f_\varepsilon|^2)}{\mu(f_\varepsilon^2) - \mu(f_\varepsilon)^2} \leq \frac{\varepsilon + \lambda^c(r)}{\mu(B_r)}.$$

This proves (1.2) by letting $\varepsilon \downarrow 0$.

(b) Next, we go to prove (1.3). It suffices to show that for any $f \in C^1(M)$ with $\mu(f^2) = 1$, $\mu(f) = 0$ and any $R > r$,

$$\mu(|\nabla f|^2) \geq \frac{\lambda^c(r)\lambda(R)\mu(B_R)(R-r)^2 - 2\lambda(R)(1-\mu(B_R))}{2\lambda(R)(R-r)^2 + \lambda^c(r)(R-r)^2\mu(B_R) + 2\mu(B_R)}. \tag{2.1}$$

Let $a=\mu(f^2 1_{B_R})$. Noting that $\mu(f)=0$, we obtain

$$\frac{1}{\lambda(R)}\mu_{B_R}(|\nabla f|^2)\geq \mu_{B_R}(f^2 1_{B_R})-\mu_{B_R}(f 1_{B_R})^2=\frac{a}{\mu(B_R)}-\frac{\mu(f 1_{B_R^c})^2}{\mu(B_R)^2}$$
$$\geq \frac{a}{\mu(B_R)}-\frac{(1-a)(1-\mu(B_R))}{\mu(B_R)^2}=\frac{a+\mu(B_R)-1}{\mu(B_R)^2},$$

where $\mu_{B_R}=\mu/\mu(B_R)$. This implies

$$\mu(|\nabla f|^2)\geq \frac{\lambda(R)}{\mu(B_R)}(a+\mu(B_R)-1)=:g_1(a). \tag{2.2}$$

Next, define

$$h(x)=\begin{cases} 0, & \text{if } \varrho(x)\leq r,\\ 1, & \text{if } \varrho(x)\geq R,\\ \dfrac{\varrho(x)-r}{R-r}, & \text{otherwise.}\end{cases}$$

Then $fh=0$ on $\{x\in M:\varrho(x)=r\}$. By the definition of $\lambda^c(r)$,

$$1-a\leq \mu(f^2h^2)\leq \frac{\mu(|\nabla(fh)|^2)}{\lambda^c(r)}\leq \frac{2}{\lambda^c(r)}\left(\mu(|\nabla f|^2)+\frac{a}{(R-r)^2}\right).$$

Therefore,

$$\mu(|\nabla f|^2)\geq \frac{\lambda^c(r)}{2}-\left(\frac{\lambda^c(r)}{2}+\frac{1}{(R-r)^2}\right)a=:g_2(a). \tag{2.3}$$

By combining (2.2) with (2.3) we obtain

$$\mu(|\nabla f|^2)\geq \inf_{\varepsilon\in[0,1]}\max\{g_1(\varepsilon),g_2(\varepsilon)\}. \tag{2.4}$$

Since $g_1(\varepsilon)$ is increasing in ε while $g_2(\varepsilon)$ is decreasing in ε, the above infimum is attained at

$$\varepsilon_0=\frac{\frac{1}{2}\lambda^c(r)+\lambda(R)(1-\mu(B_R))/\mu(B_R)}{\lambda(R)/\mu(B_R)+\frac{1}{2}\lambda^c(r)+1/(R-r)^2}$$

which solves $g_1(\varepsilon)=g_2(\varepsilon)$. Then $\mu(|\nabla f|)\geq g_1(\varepsilon_0)=g_2(\varepsilon_0)$, which is equal to the right-hand side of (2.1). □

Proof of Theorem 1.2. For any $m>r_0$, let $\Omega_m=B_m\setminus\overline{B}_{r_0}$. Since L is regular,

$$\lambda^c(r_0)=\lim_{m\to\infty}\lambda_0(\Omega_m), \tag{2.5}$$

where $\lambda_0(\Omega_m)$ denotes the smallest eigenvalue of $-L$ on Ω_m with a Neumann condition on interior$(\partial M \cap \partial\Omega_m)$ and a Dirichlet condition on the remainder of $\partial\Omega_m$. Let $u(>0)$ be the corresponding eigenfunction. Define

$$F(t) = \int_{r_0}^{t} \exp[-C(r)]\, dr \int_{r}^{m} \exp[C(s)] f(s)\, ds, \quad t \in [r_0, m].$$

We claim that there exists $c(m)>0$ such that $u(x) \le c(m) F(\varrho(x))$ on Ω_m. Actually, since $|\nabla u|$ is bounded on Ω_m, it suffices to show that $u=0$ on $S := \{x \in \partial\Omega_m : \varrho(x) = r_0\}$. If there exists $x_0 \in S$ such that $u(x_0)>0$, then $x_0 \in$ interior$(\partial M \cap S)$ by the boundary conditions. This means that $\nu(x_0) = \nabla\varrho(x_0)$ which contradicts the assumption that $\nu\varrho(x_0) \le 0$.

Next, let $c := \inf_{t \ge r_0} f(t)/F(t)$, and let x_t be the L-diffusion process with reflecting boundary on ∂B_m. By the assumption and Itô's formula for $\varrho(x_t)$ (see [7]), we have, before the time $\tau := \{t \ge 0 : \varrho(x_t) = r_0\}$,

$$d\varrho(x_t) = \sqrt{2}\, db_t + L\varrho(x_t)\, dt - dL_t, \quad x_0 \in \Omega_m, \tag{2.6}$$

where b_t is a one-dimensional Brownian motion, $L\varrho$ is taken to be zero on cut(o) and L_t is an increasing process with support contained in $\{t \ge 0 : x_t \in \text{cut}(o) \cup \partial B_m\}$. Noting that $L\varrho(x_t) \le \gamma(\varrho(x_t))$ for $x_t \notin \text{cut}(o)$, by (2.6) and Itô's formula we obtain

$$dF\circ\varrho(x_t) \le \sqrt{2}\, F'\circ\varrho(x_t)\, db_t - f(x_t)\, dt \le \sqrt{2}\, F'\circ\varrho(x_t)\, db_t - cF\circ\varrho(x_t)\, dt.$$

This then implies

$$E^x F\circ\varrho(x_{t\wedge\tau}) \le F\circ\varrho(x) \exp[-ct].$$

Let $\tau' = \inf\{t \ge 0 : x_t \in \overline{\partial\Omega_m \setminus \partial M}\}$, we have $\tau' \le \tau$ and $u(x_{t\wedge\tau'}) \le u(x_t \wedge \tau)$. Noting that $E^x u(x_{t\wedge\tau'}) = u(x) \exp[-\lambda_0(\Omega_m) t]$, we obtain

$$u(x) \exp[-\lambda_0(\Omega_m) t] \le c(m) E^x F\circ\varrho(x_t \wedge \tau) \le c(m) F\circ\varrho(x) \exp[-ct].$$

This implies $\lambda_0(m) \ge c$ for any $m > r_0$. Therefore, $\lambda^c(r_0) \ge c$. □

It was pointed out by the referee that there is an equivalent analysis proof of Theorem 1.2 (refer to [6, Lemma 1.1]). Let F and u be as above with $\int_{\Omega_m} u^2 = 1$, then $LF\circ\varrho \le -cF\circ\varrho$ on Ω_m in the distribution sense. Let $f = u/F\circ\varrho$, then f is bounded as was shown in the proof of Theorem 1.2. We have $uf\nu F\circ\varrho \le 0$ on $\partial\Omega_m$ since $\nu\varrho \le 0$ on ∂M and $u=0$ on $\partial\Omega_m \setminus \partial M$. Therefore, by Green's formula, we obtain

$$\begin{aligned}\lambda_0(\Omega_m) &= -\int_{\Omega_m} uLu\,d\mu = -\int_{\Omega_m} uL(fF\circ\varrho)\,d\mu\\ &= -\int_{\Omega_m} [ufLF\circ\varrho + uF\circ\varrho Lf + 2u\langle\nabla f, \nabla F\circ\varrho\rangle]\,d\mu\\ &\geq \int_{\Omega_m} [cufF\circ\varrho + \langle\nabla(fF^2\circ\varrho), \nabla f\rangle - 2u\langle\nabla f, \nabla F\circ\varrho\rangle]\,d\mu + \int_{\partial\Omega_m} [uF\circ\varrho\nu f]\,dA\\ &= c + \int_{\Omega_m} |\nabla f|^2 F\circ\varrho\,d\mu + \int_{\partial\Omega_m} [u\nu u - uf\nu F\circ\varrho]\,dA \geq c.\end{aligned}$$

Here, we have used the fact that $u\nu u=0$ by the mixed boundary condition.

Proof of Corollary 1.3. The proof of the first assertion is essentially due to [4]. Under the condition we have

$$\int_t^\infty \exp[C(s)]\,ds \leq c\exp[C(t)], \quad t\geq D+1,$$

for some constant $c>0$. This implies (see [4, Lemma 6.1])

$$\int_t^\infty \exp[\varepsilon s + C(s)]\,ds \leq \frac{c}{1-c\varepsilon}\exp[C(t)+\varepsilon t], \quad \varepsilon\in(0,c^{-1}).$$

By taking $f(r)=\exp[r/2c]$ in (1.4), we prove the first assertion.

Next, if there exists $\varepsilon>0$ such that $c_1 := \int_{D+1}^\infty (\gamma+\varepsilon)^+ ds < \infty$. Let $C_\varepsilon(r) = C(r) - \int_{D+1}^r (\gamma+\varepsilon)^+(s)\,ds$. Then $C_\varepsilon'(r) = \gamma(r) - (\gamma+\varepsilon)^+(r) \leq -\varepsilon$. Therefore,

$$\begin{aligned}\exp[-C(t)]\int_t^\infty \exp[C(s)]\,ds &\leq \exp[-C_\varepsilon(t)]\int_t^\infty \exp[C_\varepsilon(r)+c_1]\,dr\\ &\leq \exp[-C_\varepsilon(t)]\int_t^\infty \exp[C_\varepsilon(t) - \varepsilon(r-t) + c_1]\,dr\\ &= \frac{\exp[c_1]}{\varepsilon} < \infty.\end{aligned}$$

Hence, $\lambda_1>0$ by the first assertion. □

Remark. From (1.3) we may derive explicit lower bounds of λ_1. For instance, assume that B_R is convex for any R, let $K\geq 0$ be such that $\mathrm{Ric}-\mathrm{Hess}_V \geq -K$. We have [3]

$$\lambda(R) \geq \tfrac{1}{8}\pi^2 K\bigl(\exp\bigl[\tfrac{1}{2}KR^2\bigr]-1\bigr)^{-1}. \tag{2.7}$$

Next, if $\lim_{r\to\infty}\sup\gamma(r)<0$, let $\beta(r)=\inf_{s\ge r}(-\gamma(s))^+$, by taking $f(t)=\exp\left[\frac{1}{2}\beta(r)t\right]$ in (1.4), we obtain

$$
\begin{aligned}
\lambda^c(r) &\ge \inf_{t\ge r}\exp\left[\frac{\beta(r)t}{2}\right]\left(\int_r^t ds\int_s^\infty \exp\left[\frac{\beta(r)u}{2}+\int_s^u\gamma(v)\,dv\right]du\right)^{-1}\\
&\ge \inf_{t\ge r}\exp\left[\frac{\beta(r)t}{2}\right]\left(\int_r^t ds\int_s^\infty \exp\left[\frac{\beta(r)u}{2}-\beta(r)(u-s)\right]du\right)^{-1}=\frac{\beta(r)^2}{4}.
\end{aligned}
\tag{2.8}
$$

Here, in the second step, we have assumed that $\beta(r)>0$ so that $\gamma(v)\le-\beta(r)$ for $v\ge r$. Then the estimate $\lambda^c(r)\ge\frac{1}{4}\beta(r)^2$ is true for any r since $\beta(r)$ is nonnegative. The explicit lower bound of λ_1 then follows from (1.3), (2.7) and (2.8).

Proof of Corollary 1.4(2). Suppose that $\partial M\subset B_D$. Under the polar coordinate at o, we have $x=(r,\xi)$ for $r=\varrho(x)$ and $\xi\in\mathbf{S}^{d-1}$, the $(d-1)$-dimensional unit sphere which is considered as the bundle of unit tangent vectors at o. Under this coordinate, the Riemannian volume element can be written as $dx=g(r,\xi)\,dr\,d\xi$ and $\Delta\varrho=(\partial/\partial r)(\log g(r,\xi))|_{r=\varrho}$. Suppose that $\lim_{r\to\infty}\inf_{\varrho(x)=r}L\varrho\ge0$. Then, for any $\varepsilon>0$ there exists $r_1>D$ such that

$$\frac{\partial}{\partial r}\log g(r,\xi)\ge-\frac{\varepsilon}{2}-\frac{\partial}{\partial r}V(r,\xi),\quad r\ge r_1.$$

This implies

$$g(r,\xi)\ge g(r_1,\xi)\exp\left[-\tfrac{1}{2}\varepsilon(r-r_1)-V(r,\xi)+V(r_1,\xi)\right]\ge c\exp\left[-\tfrac{1}{2}\varepsilon\varrho-V\right],\quad r\ge r_1,$$

for some constant $c>0$. Therefore

$$
\begin{aligned}
\mu(\exp[\varepsilon\varrho])&\ge\int_{[D,\infty)\times\mathbf{S}^{d-1}}\exp[\varepsilon r+V(r,\xi)]g(r,\xi)\,dr\,d\xi\\
&\ge c\int_{[r_1,\infty)\times\mathbf{S}^{d-1}}\exp\left[\tfrac{1}{2}\varepsilon r\right]dr\,d\xi=\infty.
\end{aligned}
$$

By (1.7), we have $\lambda_1=0$. □

Remark. (1) According to the above proof, the function ϱ in Corollary 1.4(2) can be replaced by the distance from any bounded regular domain such that the outward-pointing normal exponential map on the boundary induces a diffeomorphism. See e.g. Kumura [8] for some discussions on such manifolds.

(2) In general, for any $r>D>0$, let

$$\Xi_r^D=\{\xi\in\mathbf{S}^{d-1}:\exp[s\xi]|_{[0,r]}\text{ is minimal and }\exp[s\xi]\in M,\ s\in[D,r]\}.$$

Then Ξ_r^D is nonincreasing in r. Let $\nu = d\xi$ be the standard measure on $\mathbf{S}^{d-1}$, the assumption of Corollary 1.4(2) can be replaced by the assumption that there exists $D>0$ such that

$$\lim_{r\to\infty} \nu(\Xi_r^D)\exp[\varepsilon r] = \infty \quad \text{for any } \varepsilon > 0. \tag{2.9}$$

3. Spectral gap for elliptic operators on $\mathbf{R}^d$

This section is a continuation of [2] and [4] in which the lower bound estimates are studied for the spectral gap of elliptic operators on $\mathbf{R}^d$.

Consider the operator $L=\sum_{i,j=1}^d a_{ij}(x)\partial_i\partial_j+\sum_{i=1}^d b_i(x)\partial_i$, where $\partial_i=\partial/\partial x_i$, $a(x):=(a_{ij}(x))$ is positively definite, $a_{ij}\in C^2(\mathbf{R}^d)$ and $b_i=\sum_{j=1}^d(a_{ij}\partial_j V+\partial_j a_{ij})$ for some $V\in C^2(\mathbf{R}^d)$ with $Z:=\int \exp[V]\,dx<\infty$. The specific form of b implies that L is symmetric with respect to $d\mu=Z^{-1}\exp[V]\,dx$. In the present setting, the spectral gap of L is described as

$$\lambda_1(a,V)=\inf\{\mu(\langle a\nabla f,\nabla f\rangle): f\in C^1(\mathbf{R}^d),\ \mu(f)=0,\ \mu(f^2)=1\}. \tag{3.1}$$

Moreover, we assume that L is regular in the sense that $C_0^\infty(\mathbf{R}^d)$ is dense in $W^{1,2}(\mathbf{R}^d,d\mu)$ with the Sobolev norm $\|\cdot\|_{L^2(\mu)}+\|\sqrt{\langle a\nabla\cdot,\nabla\cdot\rangle}\|_{L^2(\mu)}$.

Obviously, if $a\geq \alpha I$ for some constant $\alpha>0$, then $\lambda_1(a,V)\geq\alpha\lambda_1(I,V)$. From this one may transform the present setting to the manifold case. But this comparison only works for the case when a is uniformly positively definite, and it will lead to some loss if a is very different from I, see e.g. Examples 3.1 and 3.2 below. Hence, it should be worthy to study L directly as in previous sections.

Define for $r>0$,

$$\begin{aligned}
\gamma(r) &= \sup_{|x|=r}\frac{r(\operatorname{tr}(a(x))+\langle b(x),x\rangle)}{\langle a(x)x,x\rangle}-\frac{1}{r},\\
C(r) &= \int_1^r \gamma(s)\,ds,\\
\alpha(r) &= \inf_{|x|=r}\frac{1}{r^2}\langle a(x)x,x\rangle.
\end{aligned}$$

The main result in this section is the following.

Theorem 3.1. *If there exists a positive* $f\in C[1,\infty)$ *such that*

$$\sup_{t\geq 1}\frac{1}{f(t)}\int_1^t \exp[-C(r)]\,dr\int_r^\infty \exp[C(s)]\frac{f(s)}{\alpha(s)}\,ds<\infty, \tag{3.2}$$

then $\lambda_1>0$.

Proof. For $g\in C^2(\mathbf{R})$ and $|x|>0$, we have

$$Lg(|x|)=\frac{(|x|^2\operatorname{tr}(a(x))+|x|^2\langle b(x),x\rangle-\langle a(x)x,x\rangle)g'(|x|)}{|x|^3}+\frac{\langle a(x)x,x\rangle g''(|x|)}{|x|^2}.$$

For positive $f\in C[1,\infty)$ with

$$\int_1^\infty \exp[C(r)]\frac{f(r)}{\alpha(r)}\,dr<\infty,$$

let

$$g(t)=\int_1^t \exp[-C(r)]\,dr\int_r^\infty \exp[C(s)]\frac{f(s)}{\alpha(s)}\,ds.$$

Then

$$Lg(|x|)\leq -f(|x|),\quad |x|\geq 1.$$

Therefore, the proof of Theorem 1.2 implies that $\lambda^c(1)>0$ provided (3.2) holds, thus $\lambda_1>0$. □

Remark. Theorem 3.1 remains true also for unbounded regular domains with bounded boundary. As for the unbounded boundary case, for the estimation of $\lambda^c(r)$, one has to consider the normal vector field induced by the metric $\langle\partial_i,\partial_j\rangle=(a^{-1})_{ij}$, this will cause difficulty for general a.

For the case $M=[0,\infty)$, one has $\gamma=b/a$, $\alpha=a$. Then, by Theorem 3.1, we have $\lambda_1>0$ if there exists a positive $f\in C[1,\infty)$ with $f'<0$ such that

$$\sup_{t>1}\frac{1}{f'(t)}\exp[-C(t)]\int_t^\infty \exp[C(s)]\frac{f(s)}{\alpha(s)}\,ds<\infty. \tag{3.3}$$

This is just the condition in [4, Theorem 2.1]. Therefore, Theorem 3.1 is the exact extension of [4, Theorem 2.1] to high dimensions.

Next, the following examples show that Theorem 3.1 can be better than comparing a with a constant matrix.

Example 3.1. Take $a(x)=(1+|x|^2)^\alpha I$, $b(x)=0$, $\alpha\geq\frac{1}{2}(1+d)$. Then L is regular. It is easy to see that $V=-\alpha\log(1+|x|^2)$. Noting that $a\geq I$, by the comparison procedure, we may consider the operator $\bar{L}=\Delta-\nabla V$. But by (1.7) the spectral gap of $\bar{L}$ does not exist since $\mu(\exp[\varepsilon|x|])=\infty$ for any $\varepsilon>0$. Hence the comparison procedure does not work for this example.

On the other hand, one has $\alpha(r)=(1+r^2)^\alpha$, $\gamma(r)=(d-1)/r$, $C(r)=r^{d-1}$. Taking $f=\sqrt{t}$ we obtain

$$\int_1^t \exp[-C(r)]\,dr\int_r^\infty \exp[C(s)]\frac{f(s)}{\alpha(s)}\,ds\leq 2\int_1^t r^{3/2-2\alpha}\,dr\leq 4f(t)$$

since $2\alpha\geq 1+d$. By Theorem 3.1 we have $\lambda_1>0$.

Example 3.2. Take $a(x)=I/(|x|+1)$, $V(x)=-|x|^2$ for $|x|\geq 1$. Then the comparison procedure does not apply. Now we go to check the condition of Theorem 3.1. Obviously, $\alpha(r)=1/(1+r)$, $\langle b(x),x\rangle=-r/(1+r)^2-2r^2/(1+r)$. Then

$$\gamma(r)=\frac{d-1}{r}-\frac{1}{1+r}-2r,\quad \exp[C(r)]=\frac{c_1r^{d-1}}{r+1}\exp[-4r^2],\quad r\geq 1,$$

for some $c_1>0$. Take $f(r)=r^{1-d}\exp[-r+4r^2]$, then there exists $c_2>0$ such that

$$\frac{1}{f(t)}\int_1^t \exp[-C(r)]\,dr\int_r^\infty \exp[C(s)]f(s)\,ds\leq c_2\frac{1}{f(t)}\int_1^t \exp[-r+4r^2]\frac{1+r}{r^{d-1}}\,dr$$

which goes to $\frac{1}{8}c_2$, as $t\to\infty$. Therefore, Theorem 3.1 implies that $\lambda_1>0$.

Finally, we present an upper bound estimate like (1.7).

Theorem 3.2. *Let* $\beta(r)=\sup_{|x|=r}(1/r^2)\langle a(x)x,x\rangle$, *we have*

$$\lambda_1\leq\frac{1}{4}\sup\left\{\varepsilon^2:\mu\left(\exp\left[\varepsilon\int_0^{|x|}\frac{1}{\sqrt{\beta(r)}}\,dr\right]\right)<\infty\right\}.$$

Proof. Let

$$h(r)=\int_0^r\frac{1}{\sqrt{\beta(s)}}\,ds.$$

If $\mu(\exp[\varepsilon h(|x|)])=\infty$, we go to prove that $\lambda_1\leq\frac{1}{4}\varepsilon^2$. Let $f(x)=\exp\left[\frac{1}{2}\varepsilon(h(|x|)\wedge n)\right]$, $n\geq 1$. By (3.1), we have

$$\lambda_1\leq\frac{\varepsilon^2\mu(f^2)}{4(\mu(f^2)-\mu(f)^2)}. \tag{3.4}$$

Next, for any $m>1$ choose $r_m>0$ such that $\mu(\{h(|x|)\geq r_m\})=1/m$, we have

$$\mu(1_{\{|x|\geq r_m\}}f^2)^{1/2}\geq\sqrt{m}\,\mu(1_{\{|x|\geq r_m\}}f)\geq\sqrt{m}\,\mu(f)-\sqrt{m}\,\exp\big[\tfrac{1}{2}\varepsilon h(r_m)\big].$$

Then

$$\mu(f)^2\leq\left(\frac{\sqrt{\mu(f^2)}}{\sqrt{m}}+\exp\big[\tfrac{1}{2}\varepsilon h(r_m)\big]\right)^2. \tag{3.5}$$

Noting that $\mu(f^2)\to\infty$ as $n\to\infty$, by combining (3.4) with (3.5), we obtain

$$\lambda_1\leq\frac{\varepsilon^2}{4(1-1/m)},\quad m>1.$$

Therefore, $\lambda_1\leq\frac{1}{4}\varepsilon^2$ since m is arbitrary. □

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