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Sobolev functions whose inner trace at the boundary is zero

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Abstract. Let $\Omega\subset\mathbf{R}^n$ be an arbitrary open set. In this paper it is shown that if a Sobolev function $f\in W^{1,p}(\Omega)$ possesses a zero trace (in the sense of Lebesgue points) on $\partial\Omega$, then f is weakly zero on $\partial\Omega$ in the sense that $f\in W_0^{1,p}(\Omega)$.

1. Notation and preliminaries

If $\Omega\subset\mathbf{R}^n$ is an open set, $W^{k,p}(\Omega)$, $p\geq1$, will denote the Sobolev space of functions $f\in L^p(\Omega)$ whose distributional derivatives of order up to and including k are also elements of $L^p(\Omega)$. The norm on $W^{k,p}(\Omega)$ is defined by

$$\|f\|_{W^{k,p}(\Omega)}:=\biggl(\sum_{|\sigma|\leq k}\int_\Omega |D^\sigma f|^p\,dx\biggr)^{1/p}$$

and $W_0^{k,p}(\Omega)$ is defined as the closure in $W^{k,p}(\Omega)$ of the family of C^∞ functions in Ω with compact support. It is well known that the space of Bessel potentials

$$L^{k,p}(\mathbf{R}^n):=\{f:f=G_k*g,\ g\in L^p(\mathbf{R}^n)\}$$

with norm $\|f\|_{k,p}:=\|g\|_p$ is isometric to $W^{k,p}(\mathbf{R}^n)$. For arbitrary $\alpha>0$, the Bessel kernel G_α is that function whose Fourier transform is

$$\widehat{G}_\alpha(x)=(2\pi)^{-n/2}(1+|x|^2)^{-\alpha/2}.$$

The Bessel capacity of an arbitrary set $E\subset\mathbf{R}^n$ is defined as

$$C_{k,p}(E):=\inf\{\|g\|_p:g\in L^p(\mathbf{R}^n),\ g\geq0,\ G_k*g\geq1 \text{ on } E\}.$$

When $k=1$ and $1<p<\infty$, this capacity is equivalent to the p-capacity, γ_p, whose definition is given by

$$\gamma_p(E)=\inf\left\{\int_{\mathbf{R}^n}(|f|^p+|Df|^p)\,dx\right\},$$

where the infimum is taken over all $f\in W^{1,p}(\mathbf{R}^n)$ for which E is contained in the interior of $\{f\geq 1\}$. When $p>n$ the p-capacity of any non-empty set is positive. The Lebesgue measure of a set $E\subset\mathbf{R}^n$ is denoted by $|E|$ and $B(x,r)$ is the open ball of radius r centered at x. The dimension of the Euclidean space on which Lebesgue measure is defined will be clear from the context. Hausdorff $(n-1)$-dimensional measure will be denoted by H^{n-1}. The integral average of a function f over a set E is denoted by

$$\fint_E f=\frac{1}{|E|}\int_E f(x)\,dx.$$

An integrable function f is said to possess a Lebesgue point at x_0 if there is a number $l=l(x_0)$ such that

$$\lim_{r\to 0}\fint_{B(x_0,r)}|f(y)-l|\,dy=0.$$

Recall that $l=f$ almost everywhere. Also, f is said to be approximately continuous at x_0 if there is a measurable set E with metric density one at x_0 such that

$$\lim_{\substack{x\to x_0\\ x\in E}}|f(x)-f(x_0)|=0.$$

Note that if f has a Lebesgue point at x_0 and $l(x_0)=f(x_0)$, then f is approximately continuous at x_0.

If $f\in W_0^{k,p}(\Omega)$, then the function f^* defined as

$$f^*(x):=\begin{cases} f(x) & \text{if } x\in\Omega,\\ 0 & \text{if } x\notin\Omega\end{cases} \tag{1.1}$$

is an element of $W^{k,p}(\mathbf{R}^n)$. It is well known that a Sobolev function $f\in W^{k,p}(\mathbf{R}^n)$ possesses a Lebesgue point everywhere except for a $C_{k,p}$ null set, cf. [Z, Theorem 3.3.3]. Furthermore, if $f\in W_0^{k,p}(\Omega)$, it is not difficult to prove that

$$\lim_{r\to 0}\fint_{B(x,r)}f^*(y)\,dy=\lim_{r\to 0}\frac{1}{|B(x,r)|}\int_{B(x,r)\cap\Omega}f(y)\,dy=0 \tag{1.2}$$

for $C_{k,p}$-q.e. $x\in\mathbf{R}^n\setminus\Omega$, in particular for $C_{k,p}$-q.e. $x\in\partial\Omega$. The converse of this is one of the main results in [AH] which states the following.

1.1. Theorem. ([AH, Theorem 9.1.3]) *Let k be a positive integer, let $1<p<\infty$ and let $f\in W^{k,p}(\mathbf{R}^n)$. If $\Omega\subset\mathbf{R}^n$ is an arbitrary open set, then $f\in W_0^{k,p}(\Omega)$ if and only if*

$$\lim_{r\to 0} \fint_{B(x,r)} |D^\beta f(y)|\,dy=0 \tag{1.3}$$

for $C_{k-|\beta|,p}$-q.e. $x\in\mathbf{R}^n\setminus\Omega$ and for all multiindices β, $0\le|\beta|\le k-1$.

For $W^{1,p}(\mathbf{R}^n)$, $1<p<\infty$, this result is due to Havin [H] and Bagby [B].

A natural question arises whether the assumption that $f\in W^{k,p}(\mathbf{R}^n)$ can be replaced by the weaker one, $f\in W^{k,p}(\Omega)$, in which case (1.3) would have to be replaced by

$$\lim_{r\to 0}\frac{1}{r^n}\int_{B(x,r)\cap\Omega}|D^\beta f(y)|\,dy=0.$$

A similar question is raised in [AH, Section 9.12.1] concerning a different result. The purpose of this note is to provide an affirmative answer to this question.

In the course of this development, we will utilize the space BV, the class of functions of bounded variation.

1.2. *Definitions.* The space $\mathrm{BV}(\Omega)$ consists of all real-valued integrable functions f defined on Ω with the property that the distributional partial derivatives of f are totally finite Radon measures. The total variation measure of the vector valued measure associated with the gradient of f is denoted by $\|Df\|$. When viewed as a linear functional, its value on a nonnegative real-valued continuous function g supported in Ω is

$$\|Df\|(g)=\sup\left\{\int_\Omega g\operatorname{div} v\,dx : v\in C_c^\infty(\Omega;\mathbf{R}^n),\ |v(x)|\le f(x),\ x\in\Omega\right\},$$

and its value on a set E is $\|Df\|(E)$. The space $\mathrm{BV}_{\mathrm{loc}}(\Omega)$ consists of all functions f defined on Ω with the property that $f\in\mathrm{BV}(\Omega')$ for every open set Ω' compactly contained in Ω. The *measure theoretic boundary* of a set $E\subset\mathbf{R}^n$ is defined as

$$\partial_m E=\left\{x:0<\limsup_{r\to 0}\frac{|E\cap B(x,r)|}{|B(x,r)|}\right\}\cap\left\{x:\liminf_{r\to 0}\frac{|E\cap B(x,r)|}{|B(x,r)|}<1\right\}.$$

If $H^{n-1}(\partial_m E\cap\Omega)<\infty$, then E is said to have *finite perimeter in Ω*.

Functions in $\mathrm{BV}(\mathbf{R}^n)$ can be characterized in terms of their behavior as functions of one variable. For this, consider a real valued function g defined on the interval $[a,b]$. The *essential variation* of g on $[a,b]$ is defined as

$$\operatorname{ess} V_a^b(g):=\sup\left\{\sum_{i=1}^k |g(t_i)-g(t_{i-1})|\right\},$$

where the supremum is taken over all finite partitions of $[a,b]$ induced by $a<t_0<t_1<t_2<\dots<t_k<b$, where g is approximately continuous at each point of $\{t_0,t_1,\dots,t_k\}$.

Now consider $f\in\mathrm{BV}(\mathbf{R}^n)$ as a function of a single variable x_n while keeping the remaining $(n-1)$ variables fixed. Thus, let $\hat{x}_n:=(x_1,x_2,\dots,x_{n-1})$ and define $f_{\hat{x}_n}(t):=f(\hat{x}_n,t)$. In a similar manner, we can define the remaining functions $f_{\hat{x}_1},f_{\hat{x}_2},\dots,f_{\hat{x}_{n-1}}$. A function $f\in\mathrm{BV}_{\mathrm{loc}}(\mathbf{R}^n)$ if and only if for almost every $\hat{x}_k\in\mathbf{R}^{n-1}$, $\operatorname{ess} V_{a_k}^{b_k} f_{\hat{x}_k}(\,\cdot\,)<\infty$ and

$$\int_R \operatorname{ess} V_{a_k}^{b_k} f_{\hat{x}_k}(\,\cdot\,)\,d\hat{x}_k<\infty \tag{1.4}$$

for each rectangular cell $R\subset\mathbf{R}^{n-1}$, $k\in\{1,2,\dots,n\}$, and $-\infty<a_k<b_k<\infty$.

Another characterization of $\mathrm{BV}(\Omega)$ is due to Fleming and Rishel [FR], and its statement most suitable for our purposes can be found in [Z, Theorem 5.4.4].

1.3. Theorem. *If $\Omega\subset\mathbf{R}^n$ is open and $f\in\mathrm{BV}(\Omega)$, then*

$$\|Df\|(\Omega)=\int_{\mathbf{R}^1} H^{n-1}(\partial_m A_t\cap\Omega)\,dt, \tag{1.5}$$

where $A_t:=\{x:f(x)>t\}$. Conversely, if $f\in L^1(\Omega)$ and A_t has finite perimeter in Ω for almost all t with

$$\int_{\mathbf{R}^1} H^{n-1}(\partial_m A_t\cap\Omega)\,dt<\infty, \tag{1.6}$$

then $f\in\mathrm{BV}(\Omega)$.

In addition we will need the following known results concerning BV and Sobolev functions.

1.4. Theorem. ([F, Theorem 4.5.9(29)]) *If $f\in\mathrm{BV}(\mathbf{R}^n)$ is approximately continuous at H^{n-1}-almost all points of $\mathbf{R}^n$, then f is continuous on almost all lines parallel to the coordinate axes.*

1.5. Theorem. ([GZ, Theorem 7.45]) *A function f defined on $[a,b]$ is absolutely continuous if and only if f is of bounded variation, continuous, and carries sets of measure zero into sets of measure zero.*

1.6. Theorem. ([Z, Theorem 2.1.4]) *Suppose $f\in W^{1,p}(\Omega)$, $p\geq 1$. Let $\Omega'\subset\subset\Omega$. Then f has a representative $\tilde{f}$ that is absolutely continuous on almost all line segments of Ω' that are parallel to the coordinate axes, and the classical partial derivatives of $\tilde{f}$ agree almost everywhere with the distributional derivatives of u. Conversely, if f has such a representative and the classical partial derivatives $D_1f,\dots,D_nf$ together with f are in $L^p(\Omega')$ then $f\in W^{1,p}(\Omega')$.*

2. The main result

We are now in a position to prove our theorem.

2.1. Theorem. *Let $\Omega\subset\mathbf{R}^n$ be an arbitrary open set and let f be a function defined on Ω with the property that $f\in\mathrm{BV}(\Omega')$ for every open bounded subset $\Omega'\subset\Omega$. If f^* is approximately continuous H^{n-1}-a.e. in $\mathbf{R}^n$, then $f^*\in\mathrm{BV}_{\mathrm{loc}}(\mathbf{R}^n)$.*

Proof. Let $A_t:=\{f>t\}$ and $A_t^*:=\{f^*>t\}$. We claim that $H^{n-1}[\partial_m A_t^*\setminus\Omega]=0$ for each $t\neq 0$. For this purpose, let $x_0\in\mathbf{R}^n\setminus\Omega$ be a point of approximate continuity of f^*. Then $f^*(x_0)=0$ and

$$\lim_{\substack{x\to x_0\\ x\in E}} f^*(x)=0 \tag{2.1}$$

for some set $E\subset\mathbf{R}^n$ whose metric density is one at x_0. If $t>0$ this implies that

$$\lim_{r\to 0}\frac{|A_t^*\cap B(x_0,r)|}{|B(x_0,r)|}=0$$

and therefore that $x_0\notin\partial_m A_t^*$. Similarly, if $t<0$ let $B_t^*:=\{f^*<t\}$. Then equation (2.1) implies that

$$\lim_{r\to 0}\frac{|B_t^*\cap B(x_0,r)|}{|B(x_0,r)|}=0 \quad\text{and therefore}\quad \lim_{r\to 0}\frac{|A_t^*\cap B(x_0,r)|}{|B(x_0,r)|}=1,$$

thus showing that $x_0\notin\partial_m A_t^*$. Since H^{n-1}-a.e. point of $\mathbf{R}^n\setminus\Omega$ is a point of approximate continuity of f^*, this shows that $H^{n-1}[\partial_m A_t^*\setminus\Omega]=0$ for all $t\neq 0$.

Having established our claim, it follows that for any bounded open set $U\subset\mathbf{R}^n$,

$$\begin{aligned}\int_{-\infty}^{\infty} H^{n-1}(\partial_m A_t^*\cap U)\,dt &= \int_{-\infty}^{\infty} H^{n-1}(\partial_m A_t^*\cap\Omega\cap U)\,dt\\ &= \int_{-\infty}^{\infty} H^{n-1}(\partial_m A_t\cap\Omega\cap U)\,dt = \|Df\|(\Omega\cap U)<\infty,\end{aligned}$$

where the third equality is implied by (1.5) and is finite by the assumption that $f\in\mathrm{BV}(\Omega\cap U)$. That $f^*\in\mathrm{BV}(U)$ now follows from the first equality and (1.6). Since U is arbitrary, we conclude that $f^*\in\mathrm{BV}_{\mathrm{loc}}(\mathbf{R}^n)$, as desired. □

2.2. Theorem. *Let $\Omega\subset\mathbf{R}^n$ be an arbitrary open set and assume $f\in W^{1,p}(\Omega)$, $1<p<\infty$, has the property that*

$$\lim_{r\to 0}\frac{1}{r^n}\int_{B(x,r)\cap\Omega}|f(y)|\,dy=0 \tag{2.2}$$

for γ_p*-q.e.* $x\in\partial\Omega$*. Then* $f\in W_0^{1,p}(\Omega)$.

Except for a factor of $\frac{1}{2}$, the left side of (2.2) could be interpreted as the *inner trace* of f on domains with sufficient regularity, for example, on domains of finite perimeter. Thus our theorem states that if the inner trace of f is zero γ_p-q.e. on $\partial\Omega$, then $f\in W_0^{1,p}(\Omega)$.

Proof. Define f^* as in (1.1). The proof consists of the following steps.

Step 1. The function f^* is approximately continuous H^{n-1}-a.e. in $\mathbf{R}^n$.

Recall that f has a Lebesgue point at γ_p-q.e. point in Ω. Furthermore, for any set E, $\gamma_p(E)=0$ implies $H^{n-p+\varepsilon}(E)=0$ for all $\varepsilon>0$, cf. [Z, Theorem 2.6.16]. In particular, $H^{n-1}(E)=0$. Consequently, f^* has a Lebesgue point at H^{n-1}-almost all points in Ω. Furthermore, for γ_p-q.e. $x\in\partial\Omega$, we know that

$$\lim_{r\to 0} \fint_{B(x,r)} |f^*(y)|\,dy = \lim_{r\to 0}\frac{1}{r^n}\int_{B(x,r)\cap\Omega} |f(y)|\,dy = 0,$$

so f^* has a Lebesgue point at H^{n-1}-a.e. point in $\partial\Omega$. Finally, f^* is identically zero on $\mathbf{R}^n\setminus\Omega$ and therefore we conclude that f^* is approximately continuous at H^{n-1}-a.e. on $\mathbf{R}^n$.

Step 2. We know from Theorem 2.1 that $f^*\in\mathrm{BV}_{\mathrm{loc}}(\mathbf{R}^n)$.

Step 3. The function f^* is continuous on almost all line segments parallel to the coordinate axes.

This follows from Steps 1, 2 and Theorem 1.4.

Step 4. The function f^* is of bounded variation on each bounded interval of almost all lines parallel to the coordinate axes.

This follows from Step 2 and (1.4).

Step 5. The function f^* is absolutely continuous on almost all line segments parallel to the coordinate axes.

In view of Theorem 1.5 we must show that on almost all line segments parallel to the coordinate axes, f^* (as a function of one variable) carries sets of Lebesgue measure zero (linear measure zero) into sets of Lebesgue measure zero. For this, consider for example a line segment λ parallel to the n^{th} coordinate axis passing through the point $x=(\hat{x},x_n)$ with the property that $f^*(\hat{x},\cdot)$ is continuous and of bounded variation and that $f(\hat{x},\cdot)$ is absolutely continuous on each bounded interval contained in $\lambda\cap\Omega$. Recall from Steps 3 and 4 and Theorem 1.6 that almost all $\hat{x}$ in $\mathbf{R}^{n-1}$ have this property. Let $E\subset\lambda$ be a set of linear measure zero and let I be any bounded, open interval of $\lambda\cap\Omega$. For any closed interval $J\subset I$, it follows from

Theorem 1.6 that $f^*(J\cap E)$ is of measure zero and therefore, by a limiting process, $f^*(I\cap E)$ is of measure zero. Hence, $E\cap\lambda\cap\Omega$ is carried into a set of measure zero. Finally, f^* is constantly zero on $E\cap\lambda\cap(\mathbf{R}^n\setminus\Omega)$, and so f^* carries sets of measure zero into measure zero.

Step 6. From Step 5 we see that the distributional partial derivatives of f^* are functions and Step 2 implies that $|Df^*|\in L^1_{\mathrm{loc}}(\mathbf{R}^1)$. Since the classical partial derivatives of f^* exist almost everywhere on $\mathbf{R}^n$, we have that $Df^*=0$ a.e. on $\mathbf{R}^n\setminus\Omega$ and that $Df^*=Df$ on Ω. Consequently, $|Df^*|\in L^p(\mathbf{R}^n)$. Theorem 1.6 implies that $f^*\in W^{1,p}(\mathbf{R}^n)$ and since

$$\lim_{r\to 0} \fint_{B(x,r)} |f^*(y)|\,dy=0$$

for γ_p-q.e. $x\in\mathbf{R}^n\setminus\Omega$, it follows from Theorem 1.1 that $f^*\in W_0^{1,p}(\Omega)$. As $f^*=f$ on Ω, it follows that $f\in W_0^{1,p}(\Omega)$ as desired. □

3. Extensions to $W^{k,p}(\Omega)$

As in Theorem 2.2, we address the problem of replacing the requirement that $f\in W^{k,p}(\mathbf{R}^n)$ with $f\in W^{k,p}(\Omega)$. This will be an easy consequence of Theorems 1.1 and 2.1.

For this, we begin with the following observation. If $\Omega\subset\mathbf{R}^n$ is an arbitrary open set and $f\in W_0^{k,p}(\Omega)$, then $f^*\in W^{k,p}(\mathbf{R}^n)$ and

$$D^\alpha f^* = (D^\alpha f)^* \tag{3.1}$$

for each multiindex $0\le|\alpha|\le k$.

We now are in a position to prove the following.

3.1. Theorem. *Let k be a positive integer, let $1<p<\infty$ and let $f\in W^{k,p}(\Omega)$. If $\Omega\subset\mathbf{R}^n$ is an arbitrary open set, then $f\in W_0^{k,p}(\Omega)$ if and only if*

$$\lim_{r\to 0}\frac{1}{r^n}\int_{B(x,r)\cap\Omega} |D^\beta f(y)|\,dy=0 \tag{3.2}$$

for $C_{k-|\beta|,p}$-q.e. $x\in\mathbf{R}^n\setminus\Omega$ and for all multiindices β, $0\le|\beta|\le k-1$.

Proof. The proof of sufficiency is immediate and thus we will consider only necessity. This proceeds by induction on k with the case $k=1$ having been established by Theorem 2.2. Assume that $f\in W^{k,p}(\Omega)$ satisfies condition (3.2). Then $f\in W^{k-1,p}(\Omega)$, and since $C_{k-1-|\beta|,p}\le C_{k-|\beta|,p}$ for every multiindex β, $0\le|\beta|\le k-2$, it follows that f satisfies condition (3.2) as an element of $W^{k-1,p}(\Omega)$. Thus by the induction hypothesis we conclude that $f\in W_0^{k-1,p}(\Omega)$ and hence $f^*\in W^{k-1,p}(\mathbf{R}^n)$.

Let β be a multiindex with $|\beta|=k-1$, and define $g:=D^\beta f$. Then $g\in W^{1,p}(\Omega)$ satisfies the hypotheses of Theorem 2.2, which implies that $g^*\in W^{1,p}(\mathbf{R}^n)$. Thus by (3.1), we have that $D^\beta f^*=(D^\beta f)^*\in W^{1,p}(\mathbf{R}^n)$ whenever $|\beta|=k-1$. It follows that $f^*\in W^{k,p}(\mathbf{R}^n)$. Now we may apply Theorem 1.1 to conclude that $f^*\in W_0^{k,p}(\Omega)$. This yields our desired conclusion since $f^*=f$ on Ω. $\square$

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