REMARKS ON A PAPER OF DR. TAUTZ.

BY

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After reading the paper of Dr. Tautz »Über die Existenz von Eigenfunktionen einer reellen Variabeln bei linearen homogenen Differentialgleichungen zweiter Ordnung und Randbedingung mit komplexen Koeffizienten; Entwicklung willkürlicher Funktionen und Anwendung auf partielle Differentialgleichungen» which appeared in the Acta Mathematica, 56 (1930), pp. 23—148, we feel that it is desirable to record briefly certain historical and mathematical comments on the subject-matter of his paper. It is our opinion that Dr. Tautz has indicated in too sparing a fashion the results established since Birkhoff's fundamental memoirs on expansion problems and has thus conveyed to the general reader an erroneous impression concerning the recent development and present status of the theory of such problems. On this account we wish to present our own interpretation of the facts, with appropriate emphasis upon the points at which we are unable to adopt the views expressed or implied by Dr. Tautz.

We commence with a brief list of papers to which we shall refer subsequently by number. This list is arranged in chronological order, according to dates of publication.

(1) Birkhoff, Transactions of the American Mathematical Society, 9 (1908), pp. 219—231.
(2) Birkhoff, ibid., pp. 373—395.
(3) Tamarkin, Rendiconti del Circolo Matematico di Palermo, 34 (1912), pp. 345—382.
(4) Birkhoff, ibid., 36 (1913), pp. 115—126.

(5) Tamarkin, О нѣкоторыхъ общихъ задачахъ теорій обыкновенныхъ линейныхъ дифференціальныхъ уравненій и о разложеній произвольныхъ функцій въ ряды, Петроградъ, 1917. (On certain general problems in the theory of ordinary linear differential equations and the development of arbitrary functions in series) xiv + 308 pages.
(6) Stone, Transactions of the American Mathematical Society, 28 (1926), pp. 695—761.
(7) Tamarkin, Mathematische Zeitschrift, 27 (1927), pp. 1—54; a condensed and modified version in English of (5).
(8) Stone, Transactions of the American Mathematical Society, 29 (1927), pp. 23—53.
(9) Stone, ibid., pp. 826—844.

We do not pretend, of course, that this list has any claims to completeness; in fact, if we were engaged in a detailed historical survey we should include in our bibliography a great number of other papers by such authors as Birkhoff, Bliss, Bôcher, Ettlinger, Geppert, Haar, Hilb, Hopkins, Jackson, Kneser, Langer, Lichtenstein, Milne, Pollaczek-Geiringer, A. Schur, Stekloff, Stone, Tamarkin, Walsh, Ward, and Wilder. We have confined our list to papers which have a direct and unequivocal bearing upon the problems considered by Dr. Tautz. These papers deal with boundary value and expansion problems in the theory of ordinary linear differential equations of the nth order and attain by systematic methods the general results which we must cite in the sequel. We observe that Dr. Tautz mentions only the first two papers of the list.

Dr. Tautz considers the boundary value and expansion problems defined by the self-adjoint differential system

$$
\begin{gathered}
u'' + (\lambda - L(x))u = 0, \qquad 0 \leq x \leq a, \\
a_1 u(0) + a_2 u'(0) + a_3 u(a) + a_4 u'(a) = 0, \\
b_1 u(0) + b_2 u'(0) + b_3 u(a) + b_4 u'(b) = 0,
\end{gathered}
$$

where λ is a complex parameter, $L(x)$ is a complex-valued continuous function, and the coefficients in the boundary conditions are complex constants. With one exception, such systems are regular according to Birkhoff's definition of the term (2, pp. 382—383). Differential systems which include these as special cases have been examined quite exhaustively by Tamarkin (5, 7) and Stone (6, 8, 9). Tamarkin, in particular, has investigated general systems in which the parameter enters the boundary conditions as well as the differential equation itself.

The boundary value problem — that is, the problem of determining the characteristic values and functions of the system — must be regarded as solved for all regular systems by the work of Birkhoff (1, 2, 4) and Tamarkin (3, 5, 7). Theorems 1, 6 (in so far as it is correct), 11, 15 of Dr. Tautz are special cases of their results. A general theory of the irregular systems of the second order, including the one discussed by Dr. Tautz on pages 133—141 of his paper, has been given by Stone (8). Theorem 20 of Dr. Tautz is obtained there. An important aspect of the boundary value problem is the investigation of the Green's function $G(x, y; \lambda)$ associated with the differential system. Since this function is meromorphic in λ, careful studies of its behavior at its poles have been made, the most general results being those of Tamarkin (5, 7). Dr. Tautz has attempted in Theorem 5 of his paper to show that the Green's function has only simple poles, in the case of an arbitrary linear differential system of the nth order with constant coefficients in the boundary conditions. For real self-adjoint systems, the theorem has already been established; but in the general case it unfortunately does not hold. Tamarkin (5, p. 124; cf. 7, p. 16, footnote 1) has given an example of a Green's function with double poles: the Green's function of the system

$$u'' + \lambda u = 0, \quad 0 \leq x \leq 1,$$
$$u'(0) - u'(1) = 0,$$
$$2u(0) + u(1) = 0,$$

has poles at the points $\lambda = 4k^2\pi^2$, $k = 0, 1, 2, \ldots$, the principal part at $\lambda = 0$ being $2(3x - 1)/\lambda$ and that at $\lambda = 4k^2\pi^2$, $k = 1, 2, 3, \ldots$, being

$$\frac{48 k\pi u(x) v(y)}{(\lambda - 4k^2\pi^2)^2} + \frac{4[(3x-1)v(x)v(y) + (2-3y)u(x)u(y)]}{\lambda - 4k^2\pi^2}$$

where $u(x) = \sin 2k\pi x$, $v(x) = \cos 2k\pi x$. Further examples can be constructed on the basis of computations reported by Stone (9, p. 841). The unsatisfactory point of Dr. Tautz's argument occurs in lines $1-4$ on page 58. In subsequent developments — for example, in Theorem 7 — Dr. Tautz has applied the property of the Green's function asserted in Theorem 5; all results which are dependent upon this property for their validity must therefore be rejected.

The expansion problem — that is, the problem of representing an arbitrary function by a development in series in terms of the residues of the Green's function — must also be regarded as solved in papers (1) — (9). For regular systems, Birkhoff (1, 2, 4) has considered the expansion of an arbitrary function piecewise continuous with piecewise continuous first derivative; Tamarkin (3) has extended this investigation to the case of an arbitrary Riemann integrable function; and Tamarkin (5, 7) and Stone (6) have considered the case of a Lebesgue integrable function. For regular second order systems, Stone (9) has studied the expansions of functions which are Denjoy integrable. The method of Tamarkin (5, 7) and Stone (6) consists in proving equiconvergence theorems relating the formal series for an arbitrary Lebesgue integrable function to its formal Fourier series or to some other easily treated trigonometric representation. All convergence and regular summability criteria for such trigonometric developments can then be applied directly to the formal series under consideration; in particular, the behavior of the formal series at the end-points $x=0$, a and at points of discontinuity of the function represented (Gibbs's phenomenon) is automatically ascertained by this procedure. Thus Theorems 2, 3, 12, 16 and part of Theorem 19 of Dr. Tautz's paper cannot be considered as new. Dr. Tautz does not study the expansion problem arising from the irregular system which he encounters; but the general theory of irregular second order systems given by Stone (8) treats the problem by means of Riesz summation of the formal series and leads to some indications concerning ordinary convergence. On the other hand, no general study of the absolute convergence of the formal series prior to Dr. Tautz's paper is known to the writers.[1] A special aspect of the expansion problem which is of interest is the investigation of the development of the Green's function in terms of its principal parts. Tamarkin (5, 7) has established a general theory of this representation; his results include those which Dr. Tautz has given for Green's functions with simple poles.

In his paper Dr. Tautz examines the convergence of the series obtained from the formal expansions by term-wise differentiation. Many of these results appear to be new, particularly insofar as they concern complex systems; but it should be remarked that for real systems results along the same lines have been

[1] Walsh, Annals of Mathematics, 24 (1923), pp. 109—120, has discussed a special second order system from this point of view, establishing a general equiconvergence theorem; other special results can be found in the literature.

obtained by Stekloff.[1] A general theory of the Riesz summation of the differentiated series has been given by Stone (6) for nth order systems. Dr. Tautz also applies his study of the differentiated series to boundary value problems for partial differential equations. Again, his conclusions seem to be new for complex systems, but the work of Stekloff for the real case, which is of primary interest, should be cited here.[2].

[1] Stekloff, Основные задачи математической физики, том 1, Петербург, 1922, IV + 285 pp. Chapters 9, 10.

[2] Stekloff, loc. cit., Ch. 11.