ALMOST PERIODICITY AND GENERAL TRIGONOMETRIC SERIES.[1]

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Preface.

The class of Bohr's [1], [2], [3][2] almost periodic functions may be considered from two different points of view. On the one hand it is the class of continuous functions possessing a certain structural property which is a generalisation of pure periodicity, and on the other it is the class of limit functions of uniformly convergent sequences of finite trigonometric polynomials. The main part of Bohr's theory of a. p. functions of a real variable developed in his first two papers [1], [2] was devoted to the proof of the identity of these two classes.

Further development of the theory of almost periodic functions was directed to generalisations of the theory. Corresponding to the two different points of view of the class of a. p. functions the generalisations went in two different directions. On the one hand there were further structural generalisations of pure periodicity. The first generalisations were very important ones given by W. Stepanoff [1], who succeeded in removing the continuity restrictions, and characterised the generalised almost periodicity not by values of the functions at each point, but by mean values over intervals of fixed length. N. Wiener studied almost periodicity and gave a new proof of the Fundamental Theorem by means of representation of functions by Fourier integrals [1], [2] and independently of Stepanoff he arrived at one of his (Stepanoff's) generalisations.

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[1] The investigations in this paper were completed in a collaboration between the authors several years ago. The final redaction of the paper belongs to A. S. Besicovitch. An account of the principles of this paper was given by H. Bohr at the Congress at Bologna 1928 and in his paper [4].

[2] The list of papers referred to is given at the end of this paper.

H. Weyl [1] also gave a new method in the theory of almost periodic functions based on integral equations. His method led him to a new structural generalisation of almost periodicity, which was wider than one of Stepanoff's types of almost periodicity.

The second direction of generalisations of almost periodicity was that followed by Besicovitch. Corresponding to the definition of almost periodic functions as limits of convergent sequences of trigonometrical polynomials, Besicovitch enlarged the class of almost periodic functions by considering the convergence of sequences in a more general sense than uniform convergence, and by defining almost periodic functions as limits of such sequences of trigonometric polynomials. The purpose of his generalisation was to enlarge the class of almost periodic functions to the extent of existence of the Riesz-Fischer theorem.

However, all these generalisations were not just directed by the idea of a reciprocity between structural properties and the character of convergence of sequences of trigonometric polynomials, though important results were given by S. Bochner [2], H. Weyl [1] and R. Schmidt [1].

We give in this paper a systematic investigation of structural generalisations of almost periodicity and we establish a strict correspondence between various types of almost periodicity and the character of convergence of corresponding sequences of polynomials.[1]

This question, being of interest for the theory of almost periodic functions, acquires also its importance on account of the connection of almost periodic functions with general trigonometric series. The fact is that any a. p. function $f(x)$ has a »Fourier series» in the form of a general trigonometric series $\Sigma a_j e^{i\lambda_j x}$ (λ_j any real numbers), and the sequences of trigonometric polynomials »convergent» to $f(x)$ converge formally to this series (i. e. the coefficients of the polynomials converge to those of the series).

Thus while studying various types of almost periodicity we study at the same time a large class of general trigonometric series (including for instance *all series for which* $\Sigma|a|^2$ *is finite*) with appropriate character of convergence. In order to show clearly the idea of our investigation we shall first illustrate our problem on some known results concerning purely periodic functions.

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[1] In papers on this question by A. S. Besicovitch and H. Bohr [1] and by H. Bohr [4] was given the general idea of the present investigation and the method was carried out on the class of S. a. p. functions. Later on was published a paper by P. Franklin [1] in which he arrived partly at the results published in our paper, and partly at new results.

CHAPTER I.

§ 1. Auxiliary Theorems and Formulae.

In this § we shall quote some theorems and formulae, which will be used later.

If $f(t)$ is given in the interval $-\infty < t < \infty$ then the various limits ($\lim$, $\overline{\lim}$, $\underline{\lim}$) of the expression

$$\frac{1}{2T}\int_{-T}^{+T} f(t)\,dt, \quad \text{as} \quad T \to \infty,$$

are denoted by the symbols

$$M\{f(t)\}, \quad \overline{M}\{f(t)\}, \quad \underline{M}\{f(t)\}$$

and are called *mean value, upper mean value, lower mean value.*

If instead of $f(t)$ we have a function of two or more variables then we indicate the variable with respect to which the mean value is taken by a suffix: we write for instance $M_t\{f(t, x)\}$.

In the same way, if $f(i)$ is a function of an integral variable i given in the interval $-\infty < i < +\infty$, we denote the various limits of the expression

$$\frac{1}{2n+1}\sum_{i=-n}^{i=+n} f(i), \quad \text{as} \quad n \to \infty,$$

by the symbols

$$M\{f(i)\}, \quad \overline{M}\{f(i)\}, \quad \underline{M}\{f(i)\}.$$

If $f(i)$ is defined only for positive integers then we denote by these symbols the limits of the expression

$$\frac{1}{n}\sum_{i=1}^{i=n} f(i), \quad \text{as} \quad n \to \infty.$$

Hölder's Inequalities.

Let p and q be positive numbers satisfying the condition

$$1/p + 1/q = 1$$

and $\varphi(t)$, $\psi(t)$ two non negative functions: then we have

$$\int_a^b \varphi(t)\,\psi(t)\,dt \leqq \left[\int_a^b \{\varphi(t)\}^p\,dt\right]^{1/p}\left[\int_a^b \{\psi(t)\}^q\,dt\right]^{1/q}. \tag{1}$$

Similarly we have

$$\sum_m^n \varphi(i)\,\psi(i) \leqq \left[\sum_m^n \{\varphi(i)\}^p\right]^{1/p}\left[\sum_m^n \{\psi(i)\}^q\right]^{1/q}. \tag{2}$$

From (1), (2) we conclude immediately that

$$M\{\varphi\,.\,\psi\} \leqq [M\{\varphi^p\}]^{1/p}\,[M\{\psi^q\}]^{1/q} \tag{3}$$

or more generally that

$$\underline{M}\{\varphi\,.\,\psi\} \leqq [\underline{M}\{\varphi^p\}]^{1/p}\,[\overline{M}\{\psi^q\}]^{1/q} \tag{4}$$

$$\overline{M}\{\varphi\,.\,\psi\} \leqq [\overline{M}\{\varphi^p\}]^{1/p}\,[\overline{M}\{\psi^q\}]^{1/q}. \tag{5}$$

Formulae (3), (4) and (5) hold whether φ, ψ are functions of a continuous variable t or of an integral variable i.

Fatou's Theorem.

Let $f(t, n)$ be a non negative function given for all t in a finite interval (a, b) and for all positive integral values of n. Then we have

$$\int_a^b \underline{\lim}_{n\to\infty} f(t, n)\,dt \leqq \underline{\lim}_{n\to\infty} \int_a^b f(t, n)\,dt. \tag{6}$$

This formula also holds if n is a continuous variable. As an immediate corollary we have

$$\int_a^b \underline{M}_x\{f(t, x)\}\,dt \leqq \underline{M}_x\left\{\int_a^b f(t, x)\,dt\right\}. \tag{7}$$

Lemma I. *Smoothing an integrable function.*

Let $f(t)$ be a function integrable (L) in a finite interval (a, b) and let

$$f_\delta(t) = \frac{1}{\delta}\int_t^{t+\delta} f(u)\,du \qquad (a < t < b - \delta).$$

Then for any $\beta < b$

$$\lim_{\delta \to 0} \int_a^\beta |f(t) - f_\delta(t)|\, dt = 0. \tag{8}$$

Remark. We observe that if $f(t)$ is periodic then f_δ is continuous and periodic with the same period.

Some Inequalities.

1) *If* $f(t)$ *is a non negative function then*

$$\int_a^b dt \frac{1}{\delta} \int_t^{t+\delta} f(u)\, du \leqq \int_a^{b+\delta} f(t)\, dt. \tag{9}$$

2) *If* $f(t)$ *is a non negative function and if* $p' < p''$, *then*

$$\left[\frac{1}{b-a} \int_a^b \{f(t)\}^{p'} dt\right]^{1/p'} \leqq \left[\frac{1}{b-a} \int_a^b \{f(t)\}^{p''} dt\right]^{1/p''}, \tag{10}$$

$$\overline{M}\{[f(t)]^{p'}\}^{1/p'} \leqq \overline{M}\{[f(t)]^{p''}\}^{1/p''}. \tag{11}$$

3) *If* $\varphi(t)$ *and* $\psi(t)$ *are non negative and if* $p \geqq 1$, *then*

$$\left[\int_a^b (\varphi + \psi)^p dt\right]^{1/p} \leqq \left[\int_a^b \varphi^p dt\right]^{1/p} + \left[\int_a^b \psi^p dt\right]^{1/p}. \tag{12}$$

CHAPTER II.

Purely Periodic Functions.

§ 2. Notation and Problems.

We shall consider functions $f(x)$ given on a circle of radius $r = 1$. We shall always mean by x any amplitude of a point of the circle. Thus the functions $f(x)$ are periodic with period 2π.

We shall talk of functions in terms of geometry. Any function is considered as a point of a »functional space».

We first define the distance between two points (i. e. functions). We give various definitions of the distance and correspondingly we define various »functional spaces».

In the class of continuous functions we define the distance $\boldsymbol{d}[f(x), \varphi(x)]$ between two functions $f(x)$ and $\varphi(x)$ (*the uniform distance*) by the equation

$$\boldsymbol{d}[f(x), \varphi(x)] = \underset{-\pi \leqq x < \pi}{\text{up. b.}} |f(x) - \varphi(x)|.$$

We define further the symbol $\boldsymbol{d}[f(x)]$ by the equation

$$\boldsymbol{d}[f(x)] = \boldsymbol{d}[f(x), 0] = \underset{-\pi \leqq x < \pi}{\text{up. b.}} |f(x)|.$$

It is obvious that in the functional space defined in this way the *Triangle Rule* holds true, i. e.

if $f(x), \varphi(x), \psi(x)$ *are any three functions (points) of this space the inequality*

$$\boldsymbol{d}[f(x), \psi(x)] \leqq \boldsymbol{d}[f(x), \varphi(x)] + \boldsymbol{d}[\varphi(x), \psi(x)]$$

is satisfied.

In the class $L^p (p \geqq 1)$, i. e. in the class of functions which are measurable and whose p-th power of modulus is integrable (L), we define the distance $d^p[f(x), \varphi(x)]$ between two functions $f(x)$, $\varphi(x)$ by the equation

$$d^p[f(x), \varphi(x)] = \left[\frac{1}{2\pi} \int_{-\pi}^{+\pi} |f(x) - \varphi(x)|^p dx\right]^{1/p}.$$

We define further the symbol $d^p[f(x)]$ by the equation

$$d^p[f(x)] = d^p[f(x), 0] = \left[\frac{1}{2\pi} \int_{-\pi}^{+\pi} |f(x)|^p dx\right]^{1/p}.$$

On account of (12) the *Triangle Rule* holds true also in this space, i. e. if $f(x)$, $\varphi(x)$, $\psi(x)$ are any three points of this space we have

$$d^p[f(x), \psi(x)] \leqq d^p[f(x), \varphi(x)] + d^p[\varphi(x), \psi(x)].$$

For the case of $p=1$ we write $d[f(x), \varphi(x)]$, $d[f(x)]$ instead of $d^1[f(x), \varphi(x)]$, $d^1[f(x)]$. For the case of purely periodic functions we shall consider only the above distances, i. e. the distances

$$\boldsymbol{d}[f(x), \varphi(x)],\ d^p[f(x), \varphi(x)]\ (p \geqq 1),\ d[f(x), \varphi(x)].$$

We have the following formulae

$$(13)\qquad \boldsymbol{d}[f(x), \varphi(x)] \geqq d^p[f(x), \varphi(x)], \quad \text{for } 1 \leqq p$$

$$(14)\qquad d^{p'}[f(x), \varphi(x)] \leqq d^{p''}[f(x), \varphi(x)], \quad \text{for } p' < p''$$

the first of these formulae is obvious and the second one follows from (10).

When we wish to speak of any of these distances without specifying a definite kind we shall write $d_g[f(x), \varphi(x)]$.

Thus we have a general formula (*Triangle Rule*)

$$d_g[f(x), \psi(x)] \leqq d_g[f(x), \varphi(x)] + d_g[\varphi(x), \psi(x)].$$

We call a point (a function) $f(x)$ a limit point (limit function) of a sequence of points

$$\{f_n(x)\}$$

if

$$d_g[f(x), f_n(x)] \to 0, \quad \text{as } n \to \infty.$$

We call a point $f(x)$ a limit point of a set $\mathfrak{A}$ of points (functions), if the set $\mathfrak{A}$ contains a sequence of points $\{f_n(x)\}$ such that $f(x)$ is its limit point. A set $\mathfrak{A}$ augmented by the set of all its limit points is called *the closure of the set* $\mathfrak{A}$ and is denoted by

$$c_g(\mathfrak{A}).$$

Corresponding to the various definitions of the distance $\boldsymbol{d}$, d^p, d, we have the closures

$$(15)\qquad \boldsymbol{c}(\mathfrak{A}),\ c^p(\mathfrak{A}),\ c(\mathfrak{A}).$$

The set $\mathfrak{A}$ is called a »base» of each of these closures.

We have now to prove a very simple theorem which is of importance for the further theory.

Theorem on Uniform Closure of the Base.

The closure of a set $\mathfrak{A}$ *and the closure of the set* $\boldsymbol{c}(\mathfrak{A})$ *are identical, or in symbols*

$$c_g(\mathfrak{A}) = c_g(\boldsymbol{c}(\mathfrak{A})).$$

1°. It is obvious that

$$c_g(\mathfrak{A}) \subset c_g(\boldsymbol{c}(\mathfrak{A})).^1 \tag{16}$$

2°. Let now $f(x)$ belong to $c_g(\boldsymbol{c}(\mathfrak{A}))$. If it belongs at the same time to $\boldsymbol{c}(\mathfrak{A})$ then it also belongs to $c_g(\mathfrak{A})$, as $c_g(\mathfrak{A}) \supset \boldsymbol{c}(\mathfrak{A})$. If, however, $f(x)$ does not belong to $\boldsymbol{c}(\mathfrak{A})$ then $\boldsymbol{c}(\mathfrak{A})$ contains a sequence of points $\{\varphi_n(x)\}$ such that

$$d_g[f(x), \varphi_n(x)] \to 0, \quad \text{as } n \to \infty. \tag{17}$$

$\varphi_n(x)$ belonging to $\boldsymbol{c}(\mathfrak{A})$ we conclude that $\mathfrak{A}$ contains a point $f_n(x)$ (which may coincide with $\varphi_n(x)$) such that

$$\boldsymbol{d}[\varphi_n(x), f_n(x)] < \frac{1}{n}$$

and consequently that

$$d_g[\varphi_n(x), f_n(x)] < \frac{1}{n} \cdot \tag{18}$$

By the Triangle Rule

$$d_g[f(x), f_n(x)] \leqq d_g[f(x), \varphi_n(x)] + d_g[\varphi_n(x), f_n(x)]$$

and thus by (17), (18)

$$d_g[f(x), f_n(x)] \to 0, \quad \text{as } n \to \infty.$$

$f_n(x)$ belonging to $\mathfrak{A}$ we conclude that $f(x)$ belongs to $c_g(\mathfrak{A})$.

Thus in either case, whether $f(x)$ belongs to $\boldsymbol{c}(\mathfrak{A})$ or not, it belongs to $c_g(\mathfrak{A})$. Consequently

$$c_g(\boldsymbol{c}(\mathfrak{A})) \subset c_g(\mathfrak{A}). \tag{19}$$

By (16), (19) the theorem has been proved.

After having introduced the above notation and ideas we pass to our problems for the case of purely periodic functions. We have two different problems:

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[1] The equations $\mathfrak{A} \supset \mathfrak{B}$, $\mathfrak{A} \subset \mathfrak{B}$ express resp. that $\mathfrak{A}$ contains $\mathfrak{B}$, and that $\mathfrak{A}$ is contained in $\mathfrak{B}$.

Problem I. *To characterise the classes of functions which can be approximated in some of the above mentioned ways by finite harmonic polynomials, i. e. by finite sums of the form*

$$s(x) = \Sigma a_j e^{ijx} \tag{20}$$

where j are any positive or negative integers and a_j any real or complex numbers.

Problem II. *To find an algorithm for the definition of functions $s(x)$ approximating a given function of each of the above classes.*

In this case we take for the class $\mathfrak{A}$ the class of all finite harmonic polynomials, i. e. the class of functions $s(x)$ defined by (20). We denote this class by $A_{\odot}$. Then our Problem I may be formulated in the following way:

I. *To characterise the closures*

$$\boldsymbol{c}(A_{\odot}),\quad c(A_{\odot}),\quad c^p(A_{\odot})\qquad (p > 1).$$

We shall consider each of these closures consecutively.

§ 3. Problem I.

Denote the class of all continuous functions (on a circle of radius $r = 1$, i. e. continuous functions with period 2π) by the symbol $\{c.f.\}$. We have the theorems:

Theorem I, 1. *The closure $\boldsymbol{c}(A_{\odot})$ is identical with $\{c.f.\}$, i. e.*

$$\boldsymbol{c}(A_{\odot}) = \{c.f.\}.$$

1°. The fact that

$$\boldsymbol{c}(A_{\odot}) \subset \{c.f.\} \tag{21}$$

is obvious.

2°. The formula

$$\{c.f.\} \subset \boldsymbol{c}(A_{\odot}) \tag{22}$$

expresses the famous theorem of Weierstrass that any continuous periodic function can be uniformly approximated by harmonic polynomials.

Corollary. For the case $\mathfrak{A} = A_{\odot}$ the theorem on uniform closure of the base may be expressed by the formula

$$c_g(A_{\odot}) = c_g(\{c.f.\}). \tag{23}$$

Theorem I, 2. *The closure* $c(A_{\odot})$ *is identical with the class* L (*the class of integrable* (L) *functions defined on a circle of radius* 1), *i. e.*

$$c(A_{\odot}) = L. \tag{24}$$

1°. Let $f(x)$ be any function belonging to $c(A_{\odot})$. Then there exists a sequence $\{f_n(x)\}$ of functions of $A_{\odot}$ such that

$$d[f(x), f_n(x)] \to 0, \quad \text{as} \quad n \to \infty,$$

i. e. such that

$$\frac{1}{2\pi}\int_{-\pi}^{+\pi} |f(x) - f_n(x)|\, dx \to 0, \quad \text{as} \quad n \to 0. \tag{25}$$

From the limiting equation (25) it follows that $f(x)$ is a measurable function, and then from the existence of the integral (25) follows that $f(x)$ is integrable (L). Thus

$$c(A_{\odot}) \subset L. \tag{26}$$

2°. Let now $f(x)$ be any function of the class L. Then on account of the remark to the lemma I of § 1 there can be found a continuous function $f_\delta(x)$ such that $d[f(x), f_\delta(x)]$ is as small as we please, which shows that $f(x)$ belongs to the closure $c(\{c.f.\})$ and consequently on account of (23) to the closure $c(A_{\odot})$. Thus

$$L \subset c(A_{\odot}). \tag{27}$$

By (26), (27) the theorem has been proved.

Theorem I, 3. *The closure* $c^p(A_{\odot})$ *is identical with the class* L^p, *i. e.*

$$c^p(A_{\odot}) = L^p.$$

1°. In the same way as in Theorem I, 2 we prove that

$$c^p(A_\odot) \subset L^p. \tag{28}$$

2°. Let now $f(x)$ be any function of the class L^p and Q a positive number. Define the function $f_Q(x)$ by the equations

$$f_Q(x) = f(x) \qquad \text{if } |f(x)| \leqq Q$$

and

$$f_Q(x) = Q\frac{f(x)}{|f(x)|} \quad \text{if } |f(x)| > Q.$$

To any positive number ε corresponds a value of Q such that

$$d^p[f(x), f_Q(x)] < \frac{\varepsilon}{2}\cdot \tag{29}$$

$f_Q(x)$ belonging to the class L there exists a continuous function $\varphi(x)$ such that $|\varphi(x)| \leqq Q$ and that

$$d[f_Q(x), \varphi(x)] < \left(\frac{\varepsilon}{2}\right)^p (2Q)^{1-p}. \tag{30}$$

We have

$$d^p[f_Q(x), \varphi(x)] \leqq \{(2Q)^{p-1} d[f_Q(x), \varphi(x)]\}^{1/p}$$

and consequently by (30)

$$d^p[f_Q(x), \varphi(x)] < \frac{\varepsilon}{2}\cdot \tag{31}$$

By (29), (31) and by the Triangle Rule

$$d^p[f(x), \varphi(x)] < \varepsilon$$

which shows that $f(x)$ belongs to $c^p(\{c.f.\})$ and on account of (23) to $c^p(A_\odot)$. Thus

$$L^p \subset c^p(A_\odot). \tag{32}$$

By (28), (32) the theorem has been proved.

§ 4. Problem II.

By the theorems I, 1; I, 2; I, 3 it has been proved that any function of each of the classes $\{c.f.\}$, L, $L^p\,(p > 1)$ can be approximated in a definite way by harmonic polynomials. Now we pass to Problem II of the definition of an algorithm for the construction of these approximations. For all our classes this will be given by *Fejér sums*.

Let $f(x)$ be any function of one of our classes. Take its Fourier series

$$f(x) \sim \sum_{\nu=-\infty}^{+\infty} A_\nu e^{i\nu x}, \quad A_\nu = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(t) e^{-i\nu t}\, dt. \tag{33}$$

The Fejér sums $\sigma_N(x)$ (N positive integers) are given by the expressions

$$\sigma_N(x) = \sum_{\nu=-N}^{+N} A_\nu \left(1 - \frac{|\nu|}{N}\right) e^{i\nu x} \tag{34}$$

or by the equivalent integral expressions

$$\sigma_N(x) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(x+t)\, K_N(t)\, dt \tag{35}$$

where the »kernel» $K_N(t)$ is given by

$$K_N(t) = \sum_{\nu=-N}^{+N} \left(1 - \frac{|\nu|}{N}\right) e^{-i\nu t} = \frac{1}{N}\left(\frac{\sin {}^1/_2\, N t}{\sin {}^1/_2\, t}\right)^2.$$

We shall prove that the functions $\sigma_N(x)$ give required approximations for all our classes.

Theorem II, 1. *If a function $f(x)$ belongs to the class $\{c.f.\}$ then*

$$\boldsymbol{d}\,[f(x), \sigma_N(x)] \to 0, \quad \text{as} \quad N \to \infty.$$

This is Fejér's well known theorem; we shall not dwell on its proof.

Uniqueness Theorem. *If two functions $f(x)$, $g(x)$ belong to the class $\{c.f.\}$ and have the same Fourier series, then*

$$\boldsymbol{d}\,[f(x),\ g(x)] = 0.$$

Denote by $\sigma_N^f(x)$, $\sigma_N^g(x)$ the Fejér sums for the functions $f(x)$, $g(x)$. The functions $f(x)$, $g(x)$ having the same Fourier series we conclude on account of (34) that $\sigma_N^f(x)$, $\sigma_N^g(x)$ are identical, i. e.

$$\sigma_N^f(x) = \sigma_N^g(x) = \sigma_N(x).$$

By Theorem II, 1

$$\boldsymbol{d}\,[f(x),\,\sigma_N(x)] \to 0,\quad \boldsymbol{d}\,[g(x),\,\sigma_N(x)] \to 0,\quad \text{as}\quad N \to \infty.$$

Consequently

$$\boldsymbol{d}\,[f(x),\,g(x)] \leqq \boldsymbol{d}\,[f(x),\,\sigma_N(x)] + \boldsymbol{d}\,[g(x),\,\sigma_N(x)] \to 0,\quad \text{as}\quad N \to \infty,$$

whence

$$\boldsymbol{d}\,[f(x),\,g(x)] = 0$$

which proves the theorem.

Theorem II, 2. *If a function $f(x)$ belongs to the class L then*

$$d\,[f(x),\,\sigma_N(x)] \to 0,\quad \textit{as}\quad N \to \infty.$$

We prove first the following auxiliary inequality.

If $f(x)$ is any function of the class L, then

$$d\,[\sigma_N(x)] \leqq d\,[f(x)]. \tag{36}$$

For

$$\sigma_N(x) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(x+t)\,K_N(t)\,dt,$$

whence

$$d\,[\sigma_N(x)] = \frac{1}{2\pi}\int_{-\pi}^{+\pi} |\,\sigma_N(x)\,|\,dx \leqq$$

$$\leqq \frac{1}{2\pi}\int_{-\pi}^{+\pi} dx\,\frac{1}{2\pi}\int_{-\pi}^{+\pi} |f(x+t)|\,K_N(t)\,dt =$$

$$= \frac{1}{2\pi}\int_{-\pi}^{+\pi} K_N(t)\,dt \frac{1}{2\pi}\int_{-\pi}^{+\pi} |f(x+t)|\,dx =$$

$$= d[f(x)] \frac{1}{2\pi}\int_{-\pi}^{+\pi} K_N(t)\,dt$$

which proves the formula (36), since on account of a well known property of Fejér's kernel we have

$$\frac{1}{2\pi}\int_{-\pi}^{+\pi} K_n(t)\,dt = 1.$$

From (36) we obtain the following inequality:

If $f(x)$ and $g(x)$ are any two functions of the class L then

$$d[\sigma_N^f(x),\, \sigma_N^g(x)] \leqq d[f(x),\, g(x)]. \tag{37}$$

In order to prove the formula (37) we have only to apply the formula (36) to the function $f(x)-g(x)$ and to observe that $\sigma_N^{f-g}(x) = \sigma_N^f(x) - \sigma_N^g(x)$ and that

$$d[\sigma_N^{f-g}(x)] = d[\sigma_N^f(x),\, \sigma_N^g(x)], \quad d[f(x)-g(x)] = d[f(x),\, g(x)].$$

We shall now prove Theorem II, 2 by showing that corresponding to any fixed positive number ε we can choose an integer N_0 such that

$$d[f(x),\, \sigma_N^f(x)] < \varepsilon \qquad \text{for all } N > N_0. \tag{38}$$

For $f(x)$ belonging to L we can (on account of the Remark to Lemma I of § 1) find a continuous function $\varphi(x)$ such that

$$d[f(x),\, \varphi(x)] < \frac{\varepsilon}{3} \tag{39}$$

and an integer N_0 such that

$$d[\varphi(x),\, \sigma_N^\varphi(x)] < \frac{\varepsilon}{3} \qquad \text{for all } N > N_0. \tag{40}$$

We have then for $N > N_0$

$$d[f(x), \sigma_N^f(x)] \leqq d[f(x), \varphi(x)] + d[\varphi(x), \sigma_N^\varphi(x)] +$$
$$+ d[\sigma_N^\varphi(x), \sigma_N^f(x)] \leqq 2d[f(x), \varphi(x)] + d[\varphi(x), \sigma_N^\varphi(x)]$$

whence on account of (39), (40)

$$d[f(x), \sigma_N^f(x)] < \varepsilon$$

and thus Theorem II, 2 has been proved.

Theorem II, 3. *If a function $f(x)$ belongs to the class L^p $(p > 1)$ then*

$$d^p[f(x), \sigma_N(x)] \to 0, \quad \textit{as} \quad N \to \infty.$$

As in the preceeding case we first prove an auxiliary inequality:

If $f(x)$ is any function of the class L^p then

(41) $$d^p[\sigma_N(x)] \leqq d^p[f(x)].$$

We write

(42) $$|\sigma_N(x)| \leqq \frac{1}{2\pi} \int_{-\pi}^{+\pi} |f(x+t)| K_N(t)\, dt.$$

By Hölder's inequality

$$\int_{-\pi}^{+\pi} |f(x+t)| K_N(t)\, dt \leqq \left\{ \int_{-\pi}^{+\pi} |f(x+t)|^p K_N(t)\, dt \right\}^{1/p} \left\{ \int_{-\pi}^{+\pi} K_N(t)\, dt \right\}^{\frac{p-1}{p}} =$$
$$= (2\pi)^{\frac{p-1}{p}} \left\{ \int_{-\pi}^{+\pi} |f(x+t)|^p K_N(t)\, dt \right\}^{1/p}$$

and thus by (42)

$$|\sigma_N(x)|^p \leqq \frac{1}{2\pi} \int_{-\pi}^{+\pi} |f(x+t)|^p K_N(t)\, dt$$

whence

$$d^p[\sigma_N(x)] = \left[\frac{1}{2\pi}\int_{-\pi}^{+\pi} |\sigma_N(x)|^p\,dx\right]^{1/p} \leqq \left[\frac{1}{2\pi}\int_{-\pi}^{+\pi} K_N(t)\,dt\,\frac{1}{2\pi}\int_{-\pi}^{+\pi} |f(x+t)|^p\,dx\right]^{1/p} =$$

$$= d^p[f(x)],$$

which proves (41).

Now we pass to the proof of Theorem II, 3. Let Q be a positive number. We write

$$f(x) = f_Q(x) + R_Q(x)$$

where $f_Q(x)$ is defined by the equations

$$f_Q(x) = f(x) \quad \text{if} \quad |f(x)| \leqq Q$$

and

$$f_Q(x) = Q\,\frac{f(x)}{|f(x)|} \quad \text{if} \quad |f(x)| > Q.$$

Given a positive number ε we can always find a number Q such that

$$d^p[R_Q(x)] < \frac{\varepsilon}{3} \tag{43}$$

and then on account of Theorem II, 2 an integer N_0 such that for $N > N_0$

$$d[f_Q(x),\ \sigma_N^{f_Q}(x)] < \left(\frac{\varepsilon}{3}\right)^p (2\,Q)^{1-p}. \tag{44}$$

From the inequality $|\sigma_N^{f_Q}(x)| \leqq Q$ we conclude

$$d^p[f_Q(x),\ \sigma_N^{f_Q}(x)] \leqq \{(2\,Q)^{p-1} d\,[f_Q(x),\ \sigma_N^{f_Q}(x)]\}^{1/p}. \tag{45}$$

By (44), (45)

$$d^p[f_Q(x),\ \sigma_N^{f_Q}(x)] < \frac{\varepsilon}{3}\cdot \tag{46}$$

Observing that

$$\sigma_N^f(x) = \sigma_N^{f_Q}(x) + \sigma_N^{R_Q}(x)$$

we write

$$d^p[f(x), \sigma_N^f(x)] \leqq d^p[f(x), f_Q(x)] + d^p[f_Q(x), \sigma_N^{f_Q}(x)] + d^p[\sigma_N^{f_Q}(x), \sigma_N^f(x)] =$$
$$= d^p[R_Q(x)] + d^p[f_Q(x), \sigma_N^{f_Q}(x)] + d^p[\sigma_N^{R_Q}(x)] \leqq$$
$$\leqq 2\, d^p[R_Q(x)] + d^p[f_Q(x), \sigma_N^{f_Q}(x)].$$

By (43), (46)

$$d^p[f(x), \sigma_N^f(x)] < \varepsilon$$

which proves the theorem.

Uniqueness Theorem in the Class L^p ($p \geqq 1$). *If two functions $f(x)$ and $g(x)$ of the class L^p ($p \geqq 1$) have the same Fourier series then*

$$d^p[f(x), g(x)] = 0.$$

The proof is identical with that for the class $\{c.f.\}$.

CHAPTER III.

§ 5. General Closures and General Almost Periodicity.

We now pass to our main problem, i. e. to the investigation of various classes of functions given in the whole interval $-\infty < x < +\infty$ which can be approximated in some way or other by finite trigonometric polynomials

$$s(x) = \Sigma a_\nu e^{i\lambda_\nu x} \tag{47}$$

where the exponents λ_ν are any real numbers, and the coefficients a_ν any real or complex numbers. We denote the class of all polynomials (47) by A. We shall consider only those approximations which preserve the main characteristic properties of the functions $s(x)$ (those relating to oscillations).

Then we have to consider only approximations *which involve some sort of uniformity in the whole interval* $-\infty < x < +\infty$.

For, as Besicovitch has shown [1], the class of functions which can be approximated by a sequence $\{s_n(x)\}$ of polynomials (47), even uniformly in any finite interval, is too wide: this class contains in fact all continuous functions.

As in Chapter II we shall talk of functions in terms of geometry. We call a class of functions a functional space, and any function of this class a point of the functional space. We first define the notion of the distance of two points. Corresponding to various definitions of the distance we define various »functional spaces».

We introduce the following definitions of the distance between two points (functions) $f(x)$ and $\varphi(x)$.

1°. We define the distance $\boldsymbol{D}[f(x), \varphi(x)]$ by the equation

$$\boldsymbol{D}[f(x), \varphi(x)] = \underset{-\infty < x < +\infty}{\text{up. b.}} |f(x) - \varphi(x)|. \tag{48}$$

We define further the symbol $\boldsymbol{D}[f(x)]$ by the equation

$$\boldsymbol{D}[f(x)] = \boldsymbol{D}[f(x), 0] = \underset{-\infty < x < +\infty}{\text{up. b.}} |f(x)|. \tag{49}$$

2°. We define S. distance of the class p ($p \geqq 1$) relating to the length l $D_{S_l^p}[f(x), \varphi(x)]$ by the equation

$$D_{S_l^p}[f(x), \varphi(x)] = \underset{-\infty < x < +\infty}{\text{up. b.}} \left[\frac{1}{l}\int_x^{x+l} |f(t) - \varphi(t)|^p dt\right]^{1/p}. \tag{50}$$

We define the symbol $D_{S_l^p}[f(x)]$ by the equation

$$D_{S_l^p}[f(x)] = D_{S_l^p}[f(x), 0] = \underset{-\infty < x < +\infty}{\text{up. b.}} \left[\frac{1}{l}\int_x^{x+l} |f(x)|^p dx\right]^{1/p}. \tag{51}$$

When any of the numbers p, l is equal to 1 we drop it in our notation. Thus we write D_{S_l}, D_{S^p}, D_S instead of $D_{S_l^1}$, $D_{S_1^p}$, $D_{S_1^1}$.

If $p' < p''$ we have an account of (10)

$$D_{S_l^{p''}}[f(x), \varphi(x)] \geqq D_{S_l^{p'}}[f(x), \varphi(x)]. \tag{52}$$

3°. We define W. distance of class p $D_{W^p}[f(x), \varphi(x)]$ by the equation

$$D_{W^p}[f(x), \varphi(x)] = \lim D_{S_l^p}[f(x), \varphi(x)], \quad \text{as } l \to \infty \tag{53}$$

and we write

$$D_{W^p}[f(x)] = D_{W^p}[f(x), 0]. \tag{54}$$

For the validity of this definition we have to prove the existence of the limit on the right hand side of (53). Obviously it is enough to prove the existence for the case of $\varphi(x)=0$ and $p=1$, i. e. to prove the existence of the limit

(55) $$\lim D_{S_l}[f(x)], \quad \text{as} \quad l\to\infty.$$

If $D_{S_l}[f(x)]$ is infinite for one value of l then it is also infinite for all others and thus in this case the limit (55) exists. Suppose now that $D_{S_l}[f(x)]$ is finite for all values of l. Let l_0, l be any two positive numbers and n the positive integer such that

(56) $$(n-1)\,l_0 < l \leqq n l_0.$$

We have

$$\frac{1}{l}\int\limits_x^{x+l} |f(x)|\,dx \leqq \frac{n l_0}{l}\,\frac{1}{n l_0}\int\limits_x^{x+n l_0} |f(x)|\,dx$$

and thus

(57) $$D_{S_l}[f(x)] \leqq \frac{n l_0}{l}\, D_{S_{n l_0}}[f(x)] < \frac{l+l_0}{l}\, D_{S_{nl}}[f(x)].$$

We have evidently

(58) $$D_{S_{n l_0}}[f(x)] \leqq D_{S_{l_0}}[f(x)].$$

By (57), (58)

(59) $$D_{S_l}[f(x)] < \left(1+\frac{l_0}{l}\right) D_{S_{l_0}}[f(x)],$$

from which we conclude

(60) $$\overline{\lim_{l\to\infty}}\, D_{S_l}[f(x)] \leqq D_{S_{l_0}}[f(x)].$$

(60) being true for all values of l_0 we conclude that

(61) $$\overline{\lim_{l\to\infty}}\, D_{S_l}[f(x)] \leqq \underline{\lim_{l_0\to\infty}}\, D_{S_{l_0}}[f(x)] = \underline{\lim_{l\to\infty}}\, D_{S_l}[f(x)],$$

which proves the existence of the limit (55) and thus the existence of the limit on the right hand side of (53).

In the same way as in 2° we write D_W instead of D_{W^1}.

4°. We define B.distance of class p ($p \geqq 1$) $D_{B^p}[f(x), \varphi(x)]$ by the equation

(62) $$D_{B^p}[f(x), \varphi(x)] = [\bar{M}\{|f(x) - \varphi(x)|^p\}]^{1/p}.$$

In the same way as before we write

(63) $$D_{B^p}[f(x)] = D_{B^p}[f(x), 0].$$

If the number p is equal to one we drop it in our notation.

Now we easily find

$$D_{S_l^p} \geqq D_{B^p}$$

and hence by definition

$$D_{W^p} \geqq D_{B^p}.$$

Thus by (60) we have

(63, 1) $$\boldsymbol{D}[f(x), \varphi(x)] \geqq D_{S_l^p} \geqq D_{W^p} \geqq D_{B^p} \qquad p \geqq 1.$$

By (10), (11)

(63, 2) $$D_{S_l^{p'}} \leqq D_{S_l^{p''}}, \; D_{W^{p'}} \leqq D_{W^{p''}}, \; D_{B^{p'}} \leqq D_{B^{p''}} \quad \text{for } p' < p''.$$

These are the only distances which we shall consider in our general investigation. When we wish to speak of one of these distances without specifying a definite kind we shall write

$$D_G[f(x), \varphi(x)].$$

It can be readily seen that the *Triangle Rule* holds in the general case, i. e. if $f(x)$, $\varphi(x)$, $\psi(x)$ are any three functions then

(64) $$D_G[f(x), \psi(x)] \leqq D_G[f(x), \varphi(x)] + D_G[\varphi(x), \psi(x)].$$

If a function $f(x)$ and a sequence $\{f_n(x)\}$ of functions are such that

(65) $$D_G[f(x), f_n(x)] \to 0, \text{ as } n \to \infty,$$

then we say that the function $f(x)$ is a G.limit of the sequence $\{f_n(x)\}$.

Corresponding to the particular meanings of $D_G[f(x), f_n(x)]$, namely

$$\boldsymbol{D},\ D_{S_l^p},\ D_{W^p},\ D_{B^p},$$

we write

$$f(x) = \mathbf{lim}\, f_n(x),\ = S_l^p.\lim f_n(x),\ = W^p.\lim f_n(x),\ = B^p.\lim f_n(x).$$

It is easy to see that if $S_l^p.\lim f_n(x)$ exists, then $S^p.\lim f_n(x)$ also exists and we have $S_l^p.\lim f_n(x) = S^p.\lim f_n(x)$.

Remark. We shall indicate the kind of uniqueness which lies in the definition of a limit function of a sequence $\{f_n(x)\}$ of functions. We conclude from (64) that if $f(x)$ is a G.limit function of the sequence $\{f_n(x)\}$ then every function $f'(x)$ which satisfies the condition

(66) $$D_G[f(x), f'(x)] = 0$$

is also a G.limit function of the sequence $\{f_n(x)\}$ and no other function can be a G.limit of this sequence. If

$$\boldsymbol{D}[f(x), f'(x)] = 0$$

then

$$f(x) = f'(x) \text{ for all } x.$$

If

$$D_{S^p}[f(x), f'(x)] = 0$$

then $f'(x)$ may be any function which is equal to $f(x)$ at almost all points. In the cases when

$$D_{W^p}[f(x), f'(x)] = 0, \quad \text{or} \quad D_{B^p}[f(x), f'(x)] = 0$$

the functions $f(x)$, $f'(x)$ may differ at a set of points of finite and even of infinite measure.

Thus, $\mathbf{lim}\, f_n(x)$ *is defined in a completely unique way; two determinations of* $S^p.\lim f_n(x)$ *differ from one another only in a set of measure zero; but two determinations of* $W^p.\lim f_n(x)$, *and a fortiori also of* $B^p.\lim f_n(x)$, *may differ from one another at a set of positive and indeed infinite measure.*

We call a function $f(x)$ a limit function of a set $\mathfrak{A}$ of functions if $f(x)$ is a limit function of a sequence $\{f_n(x)\}$ contained in $\mathfrak{A}$. The set $\mathfrak{A}$ with the set of all its limit functions is called *the closure of the set* $\mathfrak{A}$ and is denoted by

$$C_G(\mathfrak{A}).$$

Corresponding to various definitions of the distance we have the closures

$$(67) \qquad \boldsymbol{C}(\mathfrak{A}),\ C_{S^p}(\mathfrak{A}),\ C_{W^p}(\mathfrak{A}),\ C_{B^p}(\mathfrak{A}) \qquad (p \geqq 1).$$

The set $\mathfrak{A}$ is called a base of the closure $C_G(\mathfrak{A})$. In the same way as in Chapter II we can prove a

Theorem on Uniform Closure of the Base: *The closure of the set* $\mathfrak{A}$ *and the closure of the set* $\boldsymbol{C}(\mathfrak{A})$ *are identical, or in symbols*

$$(68) \qquad C_G(\mathfrak{A}) = C_G[\boldsymbol{C}(\mathfrak{A})].$$

At the beginning of this chapter we indicated the general nature of our investigation: we can now define their scope precisely by means of the symbols that have been introduced.

We shall study the closures

$$(69) \qquad \boldsymbol{C}(A),\ C_{S^p}(A),\ C_{W^p}(A),\ C_{B^p}(A)$$

where $p \geqq 1$ *and* A *is the class of all polynomials* (47).

We have

$$(69,1) \qquad \boldsymbol{C}(A) \subset C_{S^p}(A) \subset C_{W^p}(A) \subset C_{B^p}(A), \quad p \geqq 1$$

and if $p' < p''$

$$(69,2) \qquad C_{S^{p''}} \subset C_{S^{p'}},\ C_{W^{p''}} \subset C_{W^{p'}},\ C_{B^{p''}} \subset C_{B^{p'}}.$$

As in the case of periodic functions our task falls into two parts:

Problem I. *To characterise the closures* (69) *by structural properties of functions.*

Problem II. *To find an algorithm for the construction of proper approximations to functions of various closures.*

The solution of these two problems for the general case of the closures

$$C_{S^p}(A),\ C_{W^p}(A),\ C_{B^p}(A) \qquad (p \geqq 1)$$

will be based on the solution for the special case of the closure $\boldsymbol{C}(A)$.

As in the case of periodic functions we shall omit the investigation of this fundamental case and merely quote its results.

We have first to give the definition of *almost periodic functions.*

Let $f(x)$ be a function, continuous in the whole interval $-\infty < x < +\infty$, and ε a positive number. If a real number τ satisfies the condition

$$(70) \qquad \boldsymbol{D}\left[f(x+\tau), f(x)\right] \leqq \varepsilon$$

then we call the number τ a (uniform) *translation number of the function* $f(x)$ *belonging to* ε. Denote by $\boldsymbol{E}_\varepsilon[f(x)]$ the set of all such numbers τ.

Before giving the definition of almost periodic functions we have to give an auxiliary definition of a property of numerical sets.

If to a set E of real numbers corresponds a positive number l such that every interval (a, b) of length l contains at least one number of the set E then we say that the set E is relatively dense.

Definition of Uniformly Almost Periodic Functions.

*If the set $\boldsymbol{E}_\varepsilon[f(x)]$ is relatively dense for all positive values of ε then we say that the function $f(x)$ is a uniformly almost periodic function (**a.p.** function).*

We shall denote the class of all **a.p.** functions by the symbol $\{\boldsymbol{a.p.}\}$.

The theory of **a.p.** functions was created by H. Bohr and has been developed in his three papers in Acta math. [1], [2], [3] and later in a series of his own and other author's papers. We shall not enter here into the theory of **a.p.** functions but assume its fundamental results to be known to the reader.

The main result of Bohr's first two papers gives the solution of Problem I and Problem II for the case of the closure $\boldsymbol{C}(A)$.[1]

The result concerning Problem I can be expressed as follows:

[1] New methods in Problem I and in Problem II have been given by S. Bochner [1], N. Wiener [1], [2], H. Weyl [1], C. de la Vallée Poussin [1].

Theorem I $C(A)$. *The closure* $C(A)$ *is identical with* $\{a.p.\}$, *i. e.*

$$C(A) = \{a.p.\}. \tag{71}$$

Corollary. *For all closures which are considered in this paper we have*

$$C_G(A) = C_G[\{a.p.\}]. \tag{72}$$

The proof follows from (68), (71).

The study of the general closures $c(A_\odot)$ and $c^p(A_\odot)$ in the case of purely periodic functions was based on the results concerning the classical closure $c(A_\odot)$. Similarly our investigation of the general closures

$$C_{S^p}(A),\ C_{W^p}(A),\ C_{B^p}(A) \tag{73}$$

will be based on Theorem I $C(A)$.

Each of the classes (73) will be characterised by some kind of »*almost periodic*» property.

For this purpose we introduce the following definitions.

S_l^p. almost Periodic Functions.

If for a given function $f(x)$ a real number τ satisfies the condition

$$D_{S_l^p}[f(x+\tau), f(x)] \leqq \varepsilon \tag{74}$$

then we call the number τ an S_l^p. *translation number belonging to* ε. Denote by $S_l^p.E_\varepsilon[f(x)]$ the set of all these numbers.

If the set $S_l^p.E_\varepsilon[f(x)]$ *is relatively dense for all positive values of* ε *then we say that the function* $f(x)$ *is* S_l^p. *almost periodic* (S_l^p. *a.p.*).

It can be easily seen that *for any* l *an* S_l^p. *a.p. function is also an* S^p. *a.p. function.* Therefore we shall in future speak merely of S^p. *a.p.* functions.

W^p. almost Periodic Functions.

If a function $f(x)$ *is such that to any* $\varepsilon > 0$ *corresponds an* $l = l(\varepsilon)$, *for which the set* $S_l^p.E_\varepsilon[f(x)]$ *is relatively dense, then we say that the function* $f(x)$ *is* W^p. *almost periodic* (W^p. *a.p.*).

Thus the difference between $S^p_l.a.p.$ functions and $W^p.a.p.$ functions is that in the latter class l varies with ε.

It is easy to see that *if* $\overline{\lim\limits_{\varepsilon \to 0}}\, l(\varepsilon)$ *is finite then a* $W^p.a.p.$ *function is an* $S^p.a.p.$ *function.*

Remark. We shall always assume that $S^p.a.p.$ functions and $W^p.a.p.$ functions belong to the Lebesgue class L^p.

B^p. almost Periodic Functions.

We shall first give a definition of a property of numerical sets.

A set E of real numbers is said to be a satisfactorily uniform if there exists a positive number l such that the ratio of the maximum number of terms of E included in an interval of length l to the minimum number is less than 2.

It is obvious that a satisfactorily uniform set is relatively dense.

We say that a function $f(x)$ of the class L^p is B^p. almost periodic ($B^p.a.p.$) if to any $\varepsilon > 0$ corresponds a satisfactorily uniform set of numbers

$$\cdots \tau_{-2} < \tau_{-1} < \tau_0 < \tau_1 < \tau_2 < \cdots \tag{75}$$

such that for each i

$$\overline{M}_x \{|f(x+\tau_i) - f(x)|^p\} < \varepsilon^p \tag{76}$$

and that for every $c > 0$

$$\overline{M}_x \overline{M}_i \frac{1}{c} \int\limits_x^{x+c} |f(x+\tau_i) - f(x)|^p \, dx < \varepsilon^p. \tag{77}$$

We call the numbers τ_i of (75) B^p. translation numbers of the function $f(x)$ belonging to ε. It may seem to be more natural to replace the condition (77) by the condition

$$\overline{M}_x \overline{M}_i \{|f(x+\tau_i) - f(x)|^p\} < \varepsilon^p. \tag{78}$$

We shall further investigate also this kind of almost periodicity, but as in the conditions (77), (78) we use *upper* mean values the smoothing process by which the condition (77) differs from the condition (78) appears to be of importance.

We shall denote by the symbols $\{S^p.a.p.\}$, $\{W^p.a.p.\}$, $\{B^p.a.p.\}$ the classes of all $S^p.a.p.$ functions, $W^p.a.p.$ functions, $B^p.a.p.$ functions.

When we wish to speak of a function of one of these classes without specifying a definite one, we shall call it a $G.a.p.$ function.

Thus we have introduced three kinds of almost periodicity beside the uniform one. $S^p.$ almost periodicity is the nearest to the uniform one. It restricts the class L^p as uniform almost periodicity restricts the class $\{c.f.\}$. The typical property of imitation of values of functions, for values of argument increased by translation numbers, is substantially maintained. But the imitation at each point characterising **a. p.** functions, is replaced by »*integral imitation*» of values over an interval of a fixed length. In other words, the uniformity of imitation belongs not to particular values of the argument but to intervals of a definite length. $B^p.$ almost periodicity is as will be shewn the widest generalisation of the uniform one. The imitation due to this class of almost periodicity appears only as a general effect of the whole class of translation numbers and over the whole range of values of x. When we study this class of functions in connection with Fourier series we shall see that $B^p.$ almost periodicity is probably the generalisation of almost periodicity to its natural bounds.

$W^p.$ almost periodicity is intermediate between the S^p and B^p kinds.

CHAPTER IV.

$S^p.a.p.$ Functions and $W^p.a.p.$ Functions.

§ 6. $S.a.p.$ Functions.

Theorem I $C_S(A)$. *The closure* $C_S(A)$ *is identical with* $\{S.a.p.\}$, *i.e.*

(79) $$C_S(A) = \{S.a.p.\}.$$

1°. We shall first prove that

(80) $$C_S(A) \subset \{S.a.p.\}.$$

Let $f(x)$ be any function of $C_S(A)$ and ε any positive number. In order to prove (80) we have only to prove that the set $S.\,E_\varepsilon[f(x)]$ is always relatively dense. Take a function $s(x)$ of A such that

(81) $$D_S[f(x),\, s(x)] < \frac{\varepsilon}{3}\cdot$$

Let τ be any number of the set $\boldsymbol{E}_{\frac{\varepsilon}{3}}[s(x)]$. We have

(82) $$\begin{aligned} D_S[f(x+\tau), f(x)] &\leqq D_S[f(x+\tau),\, s(x+\tau)] + D_S[s(x+\tau),\, s(x)] + \\ &+ D_S[s(x), f(x)] \leqq 2\,D_S[f(x),\, s(x)] + \boldsymbol{D}[s(x+\tau),\, s(x)] < \varepsilon \end{aligned}$$

which proves that

(83) $$\boldsymbol{E}_{\frac{\varepsilon}{3}}[s(x)] \subset S.\,E_\varepsilon[f(x)].$$

The set $\boldsymbol{E}_{\frac{\varepsilon}{3}}[s(x)]$ being relatively dense we conclude that so is the set $S.\,E_\varepsilon[f(x)]$, and thus (80) has been proved.

2°. We shall now prove that

(84) $$\{S.\,a.\,p.\} \subset C_S(A).$$

Let $f(x)$ be $S.\,a.\,p.$ and consider the functions

(85) $$f_\delta(x) = \frac{1}{\delta}\int_x^{x+\delta} f(t)\,dt \qquad \text{for } 0 < \delta < 1$$

already studied by Stepanoff. Given $\varepsilon > 0$, let τ be any number of the set $S.\,E_{\varepsilon\delta}[f(x)]$. We have

(86) $$\begin{aligned} |f_\delta(x+\tau) - f_\delta(x)| &= \frac{1}{\delta}\left|\int_x^{x+\delta} [f(t+\tau) - f(t)]\,dt\right| \\ &\leqq \frac{1}{\delta}\int_x^{x+\delta} |f(t+\tau) - f(t)|\,dt \leqq \varepsilon \end{aligned}$$

whence

(87) $$S.\,E_{\varepsilon\delta}[f(x)] \subset \boldsymbol{E}_\varepsilon[f_\delta(x)]$$

which proves that the functions $f_\delta(x)$ are $\boldsymbol{a.p.}$ functions.

We shall now prove that

$$f(x) = S. \lim f_\delta(x), \quad \text{as} \quad \delta \to 0. \tag{88}$$

Observe that for any x_0 and for any $\varepsilon > 0$ and $\tau \subset S. E_\varepsilon[f(x)]$ we have

$$\begin{aligned} \int_{x_0}^{x_0+1} |f_\delta(x+\tau) - f_\delta(x)|\, dx &\leqq \frac{1}{\delta}\int_{x_0}^{x_0+1} dx \int_{x}^{x+\delta} |f(\xi+\tau) - f(\xi)|\, d\xi \\ &\leqq \frac{1}{\delta}\int_{x_0}^{x_0+1+\delta} |f(\xi+\tau) - f(\xi)|\, d\xi \int_{\xi-\delta}^{\xi} dx \leqq \int_{x_0}^{x_0+2} |f(\xi+\tau) - f(\xi)|\, d\xi \leqq 2\varepsilon \end{aligned} \tag{89}$$

whence

$$S. E_\varepsilon[f(x)] \subset S. E_{2\varepsilon}[f_\delta(x)]. \tag{90}$$

Given $\eta > 0$, let $l = l\left(\frac{\eta}{4}\right)$ be a number such that every interval of length l contains an $S.$ translation number of $f(x)$ belonging to $\frac{\eta}{4}$ and consequently (on account of (90)) an $S.$ translation number of $f_\delta(x)$ belonging to $\frac{1}{2}\eta$. Corresponding to any number x_0 we can determine a number

$$\tau \subset S. E_{\frac{\eta}{4}}[f(x)] \subset S. E_{\frac{\eta}{2}}[f_\delta(x)]$$

such that the point $x_0 + \tau$ lies in the interval $(0, l)$. Then

$$\begin{aligned} \int_{x_0}^{x_0+1} |f(x) - f_\delta(x)|\, dx &\leqq \int_{x_0}^{x_0+1} |f(x) - f(x+\tau)|\, dx + \\ &+ \int_{x_0}^{x_0+1} |f(x+\tau) - f_\delta(x+\tau)|\, dx + \int_{x_0}^{x_0+1} |f_\delta(x+\tau) - f_\delta(x)|\, dx \\ &\leqq \frac{1}{4}\eta + \int_{x_0+\tau}^{x_0+\tau+1} |f(x) - f_\delta(x)|\, dx + \frac{1}{2}\eta \leqq \frac{3}{4}\eta + \int_{0}^{l+1} |f(x) - f_\delta(x)|\, dx. \end{aligned} \tag{91}$$

By Lemma I of § 1 there exists a number δ_0 such that

$$(92) \qquad \int_0^{l+1} |f(x) - f_\delta(x)|\, dx < \frac{1}{4}\eta \quad \text{for all } \delta < \delta_0.$$

From (91), (92) we conclude that given $\eta > 0$ there always exists a positive number δ_0 such that for all $\delta < \delta_0$ and for all x_0

$$\int_{x_0}^{x_0+1} |f(x) - f_\delta(x)|\, dx < \eta$$

i. e.

$$D_S[f(x), f_\delta(x)] < \eta$$

which proves (88). From (88) we conclude that

$$\{S.\,a.\,p.\} \subset C_S[\{\boldsymbol{a.\,p.}\}],$$

and (84) follows on account of (72). The Theorem I $C_S(A)$ follows from (80) and (84).

§ 7. *W. a. p.* Functions.

Theorem I $C_W(A)$. *The closure $C_W(A)$ is identical with $\{W.\,a.\,p.\}$, i. e.*

$$(93) \qquad C_W(A) = \{W.\,a.\,p.\}.$$

1°. We shall first prove that

$$(94) \qquad C_W(A) \subset \{W.\,a.\,p.\}.$$

Let $f(x)$ be any function of $C_W(A)$. In order to prove (94) we have only to show that to any positive ε corresponds an l such that the set $S_l.\, E_\varepsilon[f(x)]$ is relatively dense.

On account of the definition of $C_W(A)$ there exists a sequence $\{s_n(x)\}$ of functions of A such that

$$(95) \qquad D_W[f(x), s_n(x)] \to 0, \quad \text{as } n \to \infty.$$

D_W being the limit of D_{S_l}, as $l \to \infty$, we can say that to the function $f(x)$ corresponds a sequence $\{s_n(x)\}$ of functions of A and a sequence $\{l_n\}$ of positive numbers such that

$$D_{S_{l_n}}[f(x), s_n(x)] \to 0, \quad \text{as} \quad n \to \infty. \tag{96}$$

Then to any $\varepsilon > 0$ corresponds an n such that

$$D_{S_{l_n}}[f(x), s_n(x)] < \frac{\varepsilon}{3}. \tag{97}$$

Now in the same way as in the case of Theorem I $C_S(A)$ we see that

$$\boldsymbol{E}_{\frac{\varepsilon}{3}}[s_n(x)] \subset S_{l_n}. E_\varepsilon[f(x)] \tag{98}$$

and consequently that the set $S_{l_n}. E_\varepsilon[f(x)]$ is relatively dense, which proves (94).

2°. We shall now prove the converse, i. e.

$$\{W.a.p.\} \subset C_W(A). \tag{99}$$

The proof of this part of the theorem in the case of the closure $C_W(A)$ is considerably more difficult than in the case of the closure $C_S(A)$. We have first to prove two lemmas.

Lemma 1. *W. a. p. functions are »W-bounded», i. e. for any W. a. p. function $f(x)$ $D_W[f(x)]$ is finite.*

Evidently it is enough to show that there exist two positive numbers L and Q such that

$$D_{S_L}[f(x)] \leqq Q. \tag{100}$$

Define L by the condition that when $\varepsilon = 1$ the set

$$S_L. E_\varepsilon[f(x)] \tag{101}$$

is relatively dense. Let $l > 0$ be a number such that every interval of length l contains at least one number of this set. To any real x_0 corresponds a number τ of the set such that the number $x_0 + \tau$ belongs to the interval $(0, l)$. We have then for any x_0

$$\frac{1}{L}\int\limits_{x_0}^{x_0+L}|f(x)|\,dx \leqq \frac{1}{L}\int\limits_{x_0}^{x_0+L}|f(x)-f(x+\tau)|\,dx + \frac{1}{L}\int\limits_{x_0}^{x_0+L}|f(x+\tau)|\,dx$$
(102)
$$\leqq 1 + \frac{1}{L}\int\limits_{x_0+\tau}^{x_0+\tau+L}|f(x)|\,dx \leqq 1 + \frac{1}{L}\int\limits_{0}^{l+L}|f(x)|\,dx = Q,$$

which proves (100).

Lemma 2. *W. a. p. functions are »W-uniformly continuous», i. e. to any W. a. p. function $f(x)$ and to any $\varepsilon > 0$ there exist numbers $L > 0$ and $\delta_0 > 0$ such that*

(103)
$$D_{S_L}[f(x+\delta), f(x)] < \varepsilon \quad \textit{for all } \delta < \delta_0.$$

Define the numbers L and l under the conditions that the set

(104)
$$S_L . E_{\frac{\varepsilon}{3}}[f(x)]$$

is relatively dense and that every interval of length l contains at least one number of the set (104). To any x_0 corresponds a number τ of the set (104) such that the number $x_0 + \tau$ belongs to the interval $(0, l)$.

We have for any $\delta > 0$

$$\frac{1}{L}\int\limits_{x_0}^{x_0+L}|f(x+\delta)-f(x)|\,dx \leqq \frac{1}{L}\int\limits_{x_0}^{x_0+L}|f(x+\delta)-f(x+\delta+\tau)|\,dx$$
(105)
$$+ \frac{1}{L}\int\limits_{x_0}^{x_0+L}|f(x+\delta+\tau)-f(x+\tau)|\,dx + \frac{1}{L}\int\limits_{x_0}^{x_0+L}|f(x+\tau)-f(x)|\,dx$$
$$\leqq \frac{\varepsilon}{3} + \frac{1}{L}\int\limits_{x_0+\tau}^{x_0+L+\tau}|f(x+\delta)-f(x)|\,dx + \frac{\varepsilon}{3}$$
$$\leqq \frac{2}{3}\varepsilon + \frac{1}{L}\int\limits_{0}^{l+L}|f(x+\delta)-f(x)|\,dx.$$

Obviously we can define $\delta_0 > 0$ such that for any $\delta < \delta_0$

(106) $$\frac{1}{L}\int_{0}^{l+L} |f(x+\delta)-f(x)|\,dx < \frac{\varepsilon}{3}\cdot$$

By (105), (106) the lemma is proved.

Now we shall proceed to the proof of (99) or of the following statement equivalent to (99)

(107) $$\{W.\,a.\,p.\} \subset C_W[\{a.\,p.\}].$$

In order to prove (107) we have to prove that to any $W.\,a.\,p.$ function $f(x)$ and to any $\varepsilon > 0$ corresponds an $a.\,p.$ function $\varphi(x)$ such that

(108) $$D_W[f(x),\, \varphi(x)] < \varepsilon.$$

We shall construct a *kernel* $K(t)$. We first choose the numbers L_1, l so that the set

(109) $$S_{L_1}.\, E_{\frac{\varepsilon}{4}}[f(x)]$$

is relatively dense, and that any interval of length l contains at least one number of the set (109). Define now the numbers δ_0 and L_2 so that for any $|\delta| < \delta_0$

(110) $$D_{S_{L_2}}[f(x+\delta), f(x)] < \frac{\varepsilon}{4}\cdot$$

Now, by (59), for any function $p(x)$ we have

(111) $$D_{S_{L'}}[p(x)] \leqq 2\, D_{S_{L''}}[p(x)] \quad \text{if } L' \geqq L''$$

and hence when $L \geqq \max(L_1, L_2)$ we have

(112) $$D_{S_L}[f(x+\delta), f(x)] < \frac{\varepsilon}{2} \quad \text{for any } \delta < \delta_0$$

and

(113) $$D_{S_L}[f(x+\tau), f(x)] < \frac{\varepsilon}{2} \quad \text{for any } \tau \text{ of the set (109).}$$

By (112), (113)

(114) $$D_{S_L}[f(x+\tau+\delta), f(x)] < \varepsilon.$$

We construct the intervals I_n $(n=0, \pm 1, \pm 2, \ldots)$ of length $2\delta_0$

$$(115) \qquad \tau_n - \delta_0 < x < \tau_n + \delta_0$$

so that τ_n belongs to the set (109) and lies between the numbers $n(l+2\delta_0) - \frac{1}{2}l$ and $n(l+2\delta_0) + \frac{1}{2}l$. The intervals I_n do not overlap because

$$(116) \qquad n(l+2\delta_0) + \frac{1}{2}l + \delta_0 = (n+1)(l+2\delta_0) - \frac{1}{2}l - \delta_0.$$

We now define the kernel $K(t)$ by the equations

$$K(t) = c = \frac{1}{2\delta_0}(l+2\delta_0) \quad \text{for } t \subset I_n \ (n=0, \pm 1, \pm 2, \ldots)$$

$$K(t) = 0 \quad \text{for all other values of } t.$$

Evidently

$$(117) \qquad \frac{1}{T}\int_{\gamma}^{\gamma+T} K(t)\,dt \to 1, \text{ as } T \to \infty,$$

uniformly in γ.

We shall now prove that *there exists a sequence*

$$(118) \qquad 1 < T_1 < T_2 < \cdots < T_n \to \infty$$

such that the mean value

$$\lim_{n\to\infty} \frac{1}{2T_n}\int_{-T_n}^{+T_n} f(x+t)\,K(t)\,dt$$

exists for all values of x *in the whole interval* $-\infty < x < +\infty$.

By Lemma 1 $D_W[f(x)]$ is finite. Consequently there exists a number $k > 0$ such, that $D_{S_\lambda}[f(x)] < k$ for all $\lambda > 1$, whence

$$(119) \qquad \left|\frac{1}{2T}\int_{-T}^{+T} f(x+t)\,K(t)\,dt\right| < ck \quad \text{for } T > \frac{1}{2} \text{ and for all } x.$$

Take an enumerable set of numbers $x_1, x_2, \ldots$ which is everywhere dense in the whole interval $-\infty < x < +\infty$. By means of the »diagonal argument» we can construct a set

$$1 < T_1 < T_2 < T_3 \cdots < T_n \to \infty$$

such that the limit

$$\lim_{n\to\infty} \frac{1}{2T_n} \int_{-T_n}^{+T_n} f(x_m + t)\, K(t)\, dt \tag{120}$$

exists for every x_m. We shall now prove that the mean value

$$\lim \frac{1}{2T_n} \int_{-T_n}^{+T_n} f(x + t)\, K(t)\, dt \tag{121}$$

exists for all values of x. Given η there exists $\delta_1 = \delta_1(\eta) > 0$ such that

$$D_W[f(t+\delta), f(t)] < \frac{\eta}{2c} \quad \text{for all } \delta < \delta_1 \tag{122}$$

(on account of Lemma 2).

Let x be any real number: take a number x_m so that $|x - x_m| < \delta_1$. We conclude on account of (122) and of the definition of D_W that

$$\frac{1}{2T_n} \int_{-T_n}^{+T_n} |f(x+t) - f(x_m + t)|\, dt < D_W[f(x+t), f(x_m+t)] + \frac{\eta}{2c} < \frac{\eta}{c} \tag{123}$$

for all sufficiently large n, whence

$$\left| \frac{1}{2T_n} \int_{-T_n}^{+T_n} f(x+t)\, K(t)\, dt - \frac{1}{2T_n} \int_{-T_n}^{+T_n} f(x_m+t)\, K(t)\, dt \right| \leqq c \frac{1}{2T_n} \int_{-T_n}^{+T_n} |f(x+t) - f(x_m+t)|\, dt < \eta \tag{124}$$

for all sufficiently large n. η being arbitrary we conclude that the limit (121) exists with the limit (120).

We write

$$(125)\qquad \varphi(x) = \lim_{n\to\infty} \frac{1}{2T_n} \int_{-T_n}^{+T_n} f(x+t)\, K(t)\, dt.$$

We have

$$(126)\qquad \begin{aligned} |\varphi(x+\delta) - \varphi(x)| &= \lim_{n\to\infty} \left| \frac{1}{2T_n} \int_{-T_n}^{+T_n} \{f(x+\delta+t) - f(x+t)\}\, K(t)\, dt \right| \\ &\leqq \overline{\lim_{n\to\infty}}\, c \frac{1}{2T_n} \int_{-T_n}^{+T_n} |f(x+\delta+t) - f(x+t)|\, dt \leqq c\, D_W[f(x+\delta), f(x)] \end{aligned}$$

which proves (on account of Lemma 2) that $\varphi(x)$ *is continuous.* Next, to $\frac{\eta}{c}$ there exists a length l_0 such that the set $S_{l_0}.E_{\frac{\eta}{c}}[f(x)]$ is relatively dense. For each number τ belonging to this set we get, applying (126) to the case $\delta = \tau$,

$$|\varphi(x+\tau) - \varphi(x)| \leqq c\, D_W[f(x+\tau), f(x)] \leqq c\, D_{S_{l_0}}[f(x+\tau), f(x)] \leqq \eta$$

which proves that $\varphi(x)$ is an ***a.p.*** function.

Finally we prove that $\varphi(x)$ satisfies (108) and in this way complete the theorem. By (117)

$$(127)\qquad f(x) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} f(x)\, K(t)\, dt$$

and thus

$$\begin{aligned} |\varphi(x) - f(x)| &= \lim_{n\to\infty} \left| \frac{1}{2T_n} \int_{-T_n}^{+T_n} [f(x+t) - f(x)]\, K(t)\, dt \right| \\ &\leqq \underline{\lim_{n\to\infty}}\, \frac{1}{2T_n} \int_{-T_n}^{+T_n} |f(x+t) - f(x)|\, K(t)\, dt \end{aligned}$$

whence

$$(128)\qquad \frac{1}{L}\int_{x}^{x+L} |f(\xi)-\varphi(\xi)|\,d\xi \leqq \frac{1}{L}\int_{x}^{x+L} d\xi \varliminf_{n\to\infty} \frac{1}{2T_n}\int_{-T_n}^{+T_n} |f(\xi+t)-f(\xi)|\,K(t)\,dt.$$

Applying Fatou's theorem and then reversing the order of integration we have

$$\frac{1}{L}\int_{x}^{x+L} |f(\xi)-\varphi(\xi)|\,d\xi \leqq \varliminf_{n\to\infty} \frac{1}{2T_n}\int_{-T_n}^{+T_n} K(t)\,dt \frac{1}{L}\int_{x}^{x+L} |f(\xi+t)-f(\xi)|\,d\xi$$

$$\leqq \overline{M}_t\{D_{S_L}[f(x+t), f(x)]\,K(t)\}.$$

Assuming now that L has been chosen as in (112), (113), (114), we have

$$D_{S_L}[f(x+t), f(x)] < \varepsilon$$

for all values of t for which $K(t)$ differs from zero, whence

$$\frac{1}{L}\int_{x}^{x+L} |f(\xi)-\varphi(\xi)|\,d\xi \leqq \varepsilon\, M\{K(t)\} = \varepsilon$$

which proves (108) and consequently (99). Theorem I $C_W(A)$ follows from (94) and (99).

Remark. We conclude from (117) and (125) that if $|f(x)| \leqq Q$ for all x then also $|\varphi(x)| \leqq Q$.

§ 8. $S^p.\,a.p.$ and $W^p.\,a.p.$ Functions, $p > 1$.

Theorems I $C_{S^p}(A)$ *and* **I** $C_{W^p}(A)$ $(p > 1)$. *The closure* $C_{S^p}(A)$ *is identical with* $\{S^p.\,a.p.\}$ *and the closure* $C_{W^p}(A)$ *is identical with* $\{W^p.\,a.p.\}$, *i. e.*

$$(129)\qquad C_{S^p}(A) = \{S^p.\,a.p.\},$$

$$(130)\qquad C_{W^p}(A) = \{W^p.\,a.p.\}.$$

1°. We have first to prove that

(131) $$C_{S^p}(A) \subset \{S^p. a. p.\},$$

(132) $$C_{W^p}(A) \subset \{W^p. a. p.\}.$$

Let $f(x)$ be a function of the closure $C_{W^p}(A)$. Corresponding to any $\varepsilon > 0$ there exists a number $L > 0$ and a function $s(x)$ of A such that

(133) $$D_{S^p_L}[f(x), s(x)] < \frac{\varepsilon}{3}\cdot$$

Further we have for any

(134) $$\tau \subset \boldsymbol{E}_{\frac{\varepsilon}{3}}[s(x)]$$

(135) $$\begin{aligned} D_{S^p_L}[f(x+\tau), f(x)] &\leqq D_{S^p_L}[f(x+\tau), s(x+\tau)] + \\ &+ D_{S^p_L}[s(x+\tau), s(x)] + D_{S^p_L}[s(x), f(x)] \\ &< \frac{\varepsilon}{3} + \boldsymbol{D}[s(x+\tau), s(x)] + \frac{\varepsilon}{3} < \varepsilon, \end{aligned}$$

which shows that

(136) $$\boldsymbol{E}_{\frac{\varepsilon}{3}}[s(x)] \subset S^p_L. E_\varepsilon[f(x)].$$

Thus the set $S^p_L. E_\varepsilon[f(x)]$ is relatively dense, which proves (132). If $f(x)$ is a function of the closure $C_{S^p}(A)$ then we can put $L = 1$ in the formulae (133), (135), (136) and so obtain the result that $f(x)$ belongs to $\{S^p. a. p.\}$: this proves (131).

2°. We have now to prove the converse

(137) $$\{S^p. a. p.\} \subset C_{S^p}(A),$$

(138) $$\{W^p. a. p.\} \subset C_{W^p}(A).$$

Lemma. *To any $W^p. a. p.$ function ($S^p. a. p.$ function) $f(x)$ and any $\varepsilon > 0$ corresponds a bounded $W^p. a. p.$ function ($S^p. a. p.$ function) $g(x)$ such that*

(139) $$D_{W^p}[f(x), g(x)] < \varepsilon \quad (D_{S^p}[f(x), g(x)] < \varepsilon).$$

Let N be any positive number. We define the function $f_N(x)$ by the equations

(140) $$f_N(x) = f(x) \quad \text{when} \quad |f(x)| \leqq N,$$

(141) $$f_N(x) = N\frac{f(x)}{|f(x)|} \quad \text{when} \quad |f(x)| > N.$$

We have for any pair x_1, x_2 of real numbers

$$|f_N(x_1) - f_N(x_2)| \leqq |f(x_1) - f(x_2)|$$

from which we conclude that if $f(x)$ is a $W^p.\,a.\,p.$ function ($S^p.\,a.\,p.$ function) then so is $f_N(x)$.

By the definition of $W^p.\,a.\,p.$ functions, to any ε there corresponds an $L > 0$ such that the set

(142) $$S_L^p.\,E_{\frac{\varepsilon}{3}}[f(x)]$$

is relatively dense. Let $l > 0$ be a number such that any interval of length l contains at least one number of the set (142). Then to any real x_0 corresponds a number τ of the set (142) such that the number $x_0 + \tau$ belongs to the interval $(0, l)$. We have

(143) $$\begin{aligned} \left[\frac{1}{L}\int_{x_0}^{x_0+L} |f(x) - f_N(x)|^p dx\right]^{1/p} &\leqq \left[\frac{1}{L}\int_{x_0}^{x_0+L} |f(x) - f(x+\tau)|^p dx\right]^{1/p} + \\ &+ \left[\frac{1}{L}\int_{x_0}^{x_0+L} |f(x+\tau) - f_N(x+\tau)|^p dx\right]^{1/p} + \left[\frac{1}{L}\int_{x_0}^{x_0+L} |f_N(x+\tau) - f_N(x)|^p dx\right]^{1/p} \\ &\leqq \frac{\varepsilon}{3} + \left[\frac{1}{L}\int_{0}^{l+L} |f(x) - f_N(x)|^p dx\right]^{1/p} + \frac{\varepsilon}{3}. \end{aligned}$$

Given the numbers ε, l, L we can chose N so that

(144) $$\left[\frac{1}{L}\int_{0}^{l+L} |f(x) - f_N(x)|^p dx\right]^{1/p} < \frac{\varepsilon}{3}.$$

By (143), (144)

(145) $$D_{S_L^p}[f(x), f_N(x)] < \varepsilon.$$

If $f(x)$ is an $S^p.a.p.$ function then we can put $L=1$ in all above formulae. Thus we see that (139) will be satisfied by taking $g(x)=f_N(x)$.

After the Lemma has been proved the proof of both statements (137), (138) presents no difficulty. The proof being identical in the two cases we shall give it for the second case only. Given a $W^p.a.p.$ function $f(x)$ and a number $\varepsilon>0$ we define an $N>0$ so that

$$(146)\qquad D_{W^p}[f(x), f_N(x)] < \frac{\varepsilon}{2}$$

where $f_N(x)$ is defined by (140), (141) and is a $W^p.a.p.$ function, and therefore also a $W.a.p.$ function. On account of Theorem I $C_W(A)$ and of the remark at the end of § 7 to any $\varepsilon>0$ there corresponds an $\boldsymbol{a.p.}$ function $\varphi(x)$ and an $L>0$ such that the conditions

$$(147)\qquad D_{S_L}[f_N(x), \varphi(x)] < \left(\frac{\varepsilon}{2}\right)^p (2N)^{1-p}$$

$$|\varphi(x)| \leqq N$$

are satisfied. We have

$$(148)\qquad \left\{\frac{1}{L}\int_{x_0}^{x_0+L} |f_N(x)-\varphi(x)|^p\, dx\right\}^{1/p} \leqq (2N)^{1-\frac{1}{p}} \left\{\frac{1}{L}\int_{x_0}^{x_0+L} |f_N(x)-\varphi(x)|\, dx\right\}^{1/p}.$$

By (147), (148)

$$(149)\qquad D_{S_L^p}[f_N(x), \varphi(x)] < \frac{\varepsilon}{2}\cdot$$

By (146), (149)

$$(150)\qquad D_{W^p}[f(x), \varphi(x)] < \varepsilon.$$

The number $\varepsilon>0$ and the $W^p.a.p.$ function $f(x)$ being arbitrary in (150) we have proved (138). In the same way can be proved (137). By (131), (132), (137), (138) Theorems I $C_{S^p}(A)$ and I $C_{W^p}(A)$ have been proved.

CHAPTER V.[1]

B^p. *a. p.* Functions.

§ 9. Some Notions and Theorems on Translation Numbers.

We pass now to the investigation of the general closure $C_{B^p}(A)$ and of the class of B^p. *a. p.* functions ($p \geqq 1$). But we have first to quote some theorems and notions concerning translation numbers of **a.p.** functions.

Let $f(x)$ be an **a.p.** function. We denote as usual by

(151) $$\boldsymbol{E}_\varepsilon [f(x)]$$

the set of all translation numbers of $f(x)$ belonging to ε. We denote by

(152) $$\bar{\boldsymbol{E}}_\varepsilon [f(x)]$$

the set of all integers of the set (151). Instead of the symbol (152) we shall sometimes write only $\bar{\boldsymbol{E}}_\varepsilon$ omitting the sign of the function. Evidently *the set* $\bar{\boldsymbol{E}}_\varepsilon$ *is symmetrical with respect to the origine.*

We have the following theorems.

Theorem 1.[2] *For any* **a.p.** *function* $f(x)$ *and for any* $\varepsilon > 0$ *the set*

$$\bar{\boldsymbol{E}}_\varepsilon [f(x)]$$

is relatively dense.

We say that the set $\bar{\boldsymbol{E}}_\varepsilon [f(x)]$ is almost periodic with an error $\leqq \eta (> 0)$ if there exists a positive $\varrho < \varepsilon$ and a positive I_0 such that, (a, b) being any interval of length $> I_0$, the points of

$$\bar{\boldsymbol{E}}_\varepsilon [f(x)] \times (a, b)$$

translated by any number of $\bar{\boldsymbol{E}}_\varrho [f(x)]$ go over on to points of $\boldsymbol{E}_\varepsilon[f(x)]$ again, with the exception of at most $\eta(b-a)$ of them. We shall express this condition otherwise by saying that the points of $\bar{\boldsymbol{E}}_\varepsilon [f(x)] \times (a, b)$ translated by any number of $\bar{\boldsymbol{E}}_\varrho [f(x)]$ have a »relative loss» $\leqq \eta$.

If a set $\bar{\boldsymbol{E}}_\varepsilon$ *is almost periodic with an error as small as we please then we say that the set* $\bar{\boldsymbol{E}}_\varepsilon$ *is almost periodic* (*a.p. set*).

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[1] We are indebted to Mr. H. D. Ursell for valuable simplifications of some of the proofs of this Chapter.

[2] H. Bohr [1].

In other words:

A set $\bar{E}_\varepsilon$ *is a.p. if to any* $\eta > 0$ *there corresponds a positive* $\varrho < \varepsilon$ *and a positive* I_0 *such that,* (a, b) *being any interval of length* $> I_0$, *the points of* $\bar{E}_\varepsilon \times (a, b)$ *translated by any number of* $\bar{E}_\varrho$ *go over on to the points of* $\bar{E}_\varepsilon$ *again, with the exception of at most* $\eta(b-a)$ *of them.*

It is easily seen that an *a. p.* set $\bar{E}_\varepsilon$ is always *satisfactorily uniform.*

We have

Theorem 2.[1] *For any* **a.p.** *function* $f(x)$ *the set*

$$\bar{E}_\varepsilon [f(x)]$$

is a. p. for almost all $\varepsilon > 0$.

§ 10. *B. a. p.* Functions.

Given any *a.p.* set $\bar{E}_\varepsilon$ of numbers τ_i we define the function $K(t) = K(t, \delta)$ $(\delta < 1)$ by the conditions

$$K(t) = 1 \quad \text{for all intervals } \tau_i \leqq t \leqq \tau_i + \delta,$$
$$= 0 \quad \text{otherwise.}$$

Lemma 1. *The function* $K(t)$ *is* $W. a. p.$

For given $\eta > 0$ there exist positive numbers ϱ and $L > 1$ such that the points of $\bar{E}_\varepsilon \times (c, c + L)$ for any c translated by any number of $\bar{E}_\varrho$ have a »relative loss» $\leqq \eta$. Then for any τ belonging to $\bar{E}_\varrho$

$$\frac{1}{L}\int_c^{c+L} |K(t+\tau) - K(t)|\, dt \leqq \frac{2\eta L + 1}{L} \cdot \delta < 3\eta$$

which proves the lemma, since $\bar{E}_\varrho$ is relatively dense.

Lemma 2. *The product* $u(t)v(t)$ *of a trigonometric polynomial* $u(t)$ *and a* $W. a. p.$ *function* $v(t)$ *is a* $W. a. p.$ *function.*

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[1] A. Besicovitch and H. Bohr [2].

Let $D[u(t)] = M$. Given ε we can find an $l > 0$ and a trigonometric polynomial $s(t)$ such that

$$D_{S_l}[v(t), s(t)] < \frac{\varepsilon}{M}$$

since $v(t)$ belongs to $C_W(A)$. We obviously have

$$D_{S_l}[u(t)v(t), u(t)s(t)]$$
$$\leqq M D_{S_l}[v(t), s(t)] < \varepsilon$$

which proves that $u(t)v(t)$ is $W.a.p.$ since $u(t)s(t)$ is a trigonometric polynomial.

Lemma 3. *If $f(t) \subset C_B(A)$ then also $f(t)K(t) \subset C_B(A)$.*

Given ε there exists a trigonometric polynomial $\sigma(t)$ such that

$$D_B[f(t), \sigma(t)] < \frac{\varepsilon}{2}$$

consequently

$$(153) \qquad D_B[f(t)K(t), \sigma(t)K(t)] < \frac{\varepsilon}{2} \cdot$$

On account of Lemma 2 $\sigma(t)K(t)$ is $W.a.p.$ Consequently there exists a polynomial $s(t)$ such that

$$(154) \qquad D_W[\sigma(t)K(t), s(t)] < \frac{\varepsilon}{2} \cdot$$

By (153), (154)

$$D_B[f(t)K(t), s(t)] < \varepsilon$$

which proves the lemma.

Lemma 4. *If $f(t)$ belongs to the closure $C_B(A)$ then $M\{f(t)\}$ exists.*

Given $\varepsilon > 0$ we can always write

$$f(t) = s(t) + \theta(t)$$

where $s(t)$ is a trigonometric polynomial and $\overline{M}\{|\theta(t)|\} < \frac{\varepsilon}{4} \cdot$

There exists $T_0 > 0$ such that

$$\frac{1}{2T}\int_{-T}^{+T} |\theta(t)|\, dt < \frac{\varepsilon}{4} \quad \text{for all } T > T_0$$

and

$$\left|\frac{1}{2T}\int_{-T}^{+T} s(t)\, dt - M\{s(t)\}\right| < \frac{\varepsilon}{4} \quad \text{for all } T > T_0.$$

Consequently

$$\left|\frac{1}{2T}\int_{-T}^{+T} f(t)\, dt - M\{s(t)\}\right| < \frac{\varepsilon}{2} \quad \text{for all } T > T_0.$$

Thus for any pair of numbers T', T'' each of which is greater than T_0 we have

$$\left|\frac{1}{2T'}\int_{-T'}^{+T'} f(t)\, dt - \frac{1}{2T''}\int_{-T''}^{+T''} f(t)\, dt\right| < \varepsilon$$

which proves the lemma.

Corollary 1. *$M\{f(t)\}$ exists for any S. a. p. and for any W. a. p. function $f(t)$.*

Corollary 2. *If $\{\tau_i\}$ $(i = \ldots, -2, -1, 0, 1, 2, \ldots;\ \tau_0 = 0,\ \tau_i = -\tau_{-i})$ is an a. p. set then the limit*

$$\lim_{i \to \infty} \frac{i}{\tau_i} = p$$

exists (and evidently is > 0).

For

$$M\{K(t)\} = \lim \frac{2i\delta}{2\tau_i}, \quad \text{as } i \to \infty$$

exists, since $K(t)$ is *W. a. p.*

Remark. An *a. p.* set $\{\tau_i\}$ being satisfactorily uniform there exists a number b such that $\nu(b) < 2\mu(b)$ where $\mu(b)$, $\nu(b)$ are the minimum and the maximum number of numbers τ_i on any interval of length b. We obviously have

$$\frac{\mu(b)}{b} \leqq p \leqq \frac{\nu(b)}{b} \leqq 2p,$$

where p has the same meaning as in Cor. 2. Denoting by $n(t, T)$ the number of τ's in the interval $(t-T, t+T)$ we have for $T > b$

$$\frac{n(t, T)}{2T} < \left(\frac{2T}{b} + 1\right)\frac{\nu(b)}{2T} < 3p. \tag{155}$$

Lemma 5. *If $f(t)$ belongs to $C_B(A)$ then so does $|f(t)|$.*

Given $\varepsilon > 0$ there exists a trigonometric polynomial $\sigma(t)$ such that

$$D_B[f, \sigma] < \frac{\varepsilon}{2}.$$

Evidently

$$D_B[|f|, |\sigma|] < \frac{\varepsilon}{2}.$$

As $|\sigma(t)|$ is $\boldsymbol{a.p.}$ there exists a trigonometric polynomial $s(t)$ such that

$$\boldsymbol{D}[|\sigma(t)|, s(t)] < \frac{\varepsilon}{2}.$$

Thus

$$D_B[|f(t)|, s(t)] < \varepsilon$$

which proves the lemma.

Lemma 6. *If $f(t)$ belongs to $C_B(A)$ and if $\{\tau_i\}$ $(-\infty < i < +\infty,\ \tau_0 = 0,\ \tau_{-i} = -\tau_i)$ is an arbitrary a.p. set (i.e. an a.p. set belonging to an arbitrary $\boldsymbol{a.p.}$ function) then*

$$M_i\left\{\int_x^{x+\delta} |f(t+\tau_i) - f(t)|\, dt\right\}$$

exists for every x and every $\delta > 0$.

Evidently we may assume $\delta < 1$. Define a purely periodic function $p(t)$ with period 1 by the condition

$$p(t) = f(t) \quad \text{for} \quad x \leqq t < x + 1.$$

The function $p(t)$ obviously belongs to $C_B(A)$ and consequently $f(t) - p(t)$ also belongs to $C_B(A)$. On account of Lemma 5 the function $|f(t+x) - p(t+x)|$ (as a function of t) also belongs to $C_B(A)$.

On account of Lemmas 3 and 4 the mean value

$$M_t\{|f(t+x)-p(t+x)|\,K(t)\}$$

exists. But

$$\begin{aligned} &M_t\{|f(t+x)-p(t+x)|\,K(t)\}\\ &\quad=\lim\frac{1}{2\tau_n}\int_{-\tau_n}^{\tau_n+\delta}|f(t+x)-p(t+x)|\,K(t)\,dt\\ &\quad=\lim\frac{2n+1}{2\tau_n}\cdot\frac{1}{2n+1}\sum_{i=-n}^{+n}\int_{\tau_i}^{\tau_i+\delta}|f(t+x)-p(t+x)|\,dt\\ &\quad=\lim\frac{2n+1}{2\tau_n}\cdot\frac{1}{2n+1}\sum_{i=-n}^{+n}\int_{x}^{x+\delta}|f(t+\tau_i)-f(t)|\,dt. \end{aligned}$$

By Lemma 4, Cor. 2

$$\lim\frac{2n+1}{2\tau_n},\quad \text{as } n\to\infty$$

exists (and is $\neq 0$) and consequently the limit

$$\lim\frac{1}{2n+1}\sum_{i=-n}^{n}\int_{x}^{x+\delta}|f(t+\tau_i)-f(t)|\,dt$$

also exists, which proves the lemma.

Theorem I $C_B(A)$. *The closure* $C_B(A)$ *is identical with the class* $\{B.\,a.\,p.\}$, *i. e.* $C_B(A)=\{B.\,a.\,p.\}$.

1°. We shall first prove that

$$C_B(A)\subset\{B.\,a.\,p.\}.$$

Let $f(x)$ be a function of $C_B(A)$ and ε_0 any positive number. We put

$$f(x)=s(x)+\theta(x)$$

where $s(x)$ belongs to A and

$$\bar{M}\{|\theta(x)|\}<\varepsilon_0.$$

Choose $\varepsilon < \varepsilon_0$ so that the set $\bar{E}_\varepsilon\{s(x)\}$ is *a.p.*, and let its members be written τ_i. Then the condition expressed by (76) is evidently fullfilled; in fact we have for each i

$$(156) \qquad \bar{M}_t\{|f(t) - f(t+\tau_i)|\} \leqq M_t\{|s(t) - s(t+\tau_i)|\} + \bar{M}_t\{|\theta(t)|\} + \bar{M}_t\{|\theta(t+\tau_i)|\} < 3\varepsilon_0.$$

Wo now proceed to prove the condition (77). By Lemma 6

$$M_i\left\{\frac{1}{c}\int_x^{x+c} |f(t) - f(t+\tau_i)|\,dt\right\}$$

exists. By (6) and (9)

$$\frac{1}{2T}\int_{-T}^{+T} M_i\left\{\frac{1}{c}\int_x^{x+c} |f(t) - f(t+\tau_i)|\,dt\right\}dx$$

$$\leqq \bar{M}_i \frac{1}{2T}\int_{-T}^{+T}\left\{\frac{1}{c}\int_x^{x+c} |f(t) - f(t+\tau_i)|\,dt\right\}dx$$

$$\leqq M_i \frac{1}{2T}\int_{-T}^{T+c} |f(t) - f(t+\tau_i)|\,dt.$$

Hence for every $c > 0$ we have

$$\bar{M}_x M_i\left\{\frac{1}{c}\int_x^{x+c} |f(t) - f(t+\tau_i)|\,dt\right\}$$

$$\leqq \overline{\lim_{T\to\infty}}\, M_i \frac{1}{2T}\int_{-T}^{+T} |f(t) - f(t+\tau_i)|\,dt.$$

In this relation we write

$$|f(t) - f(t+\tau_i)| \leqq \varepsilon + |\theta(t)| + |\theta(t+\tau_i)|$$

and so obtain

$$(157)\qquad \bar{M}_x M_i\left\{\frac{1}{c}\int_x^{x+c}|f(t)-f(t+\tau_i)|\,dt\right\}$$

$$\leqq \varepsilon + \varepsilon_0 + \overline{\lim_{T\to\infty}}\,\bar{M}_i\left\{\frac{1}{2T}\int_{-T}^{+T}|\theta(t+\tau_i)|\,dt\right\}.$$

Now

$$\frac{1}{2n+1}\sum_{i=-n}^{+n}\frac{1}{2T}\int_{-T}^{+T}|\theta(t+\tau_i)|\,dt \leqq \frac{1}{2T(2n+1)}\int_{\tau_{-n}-T}^{\tau_n+T}|\theta(t)|\,n(t,T)\,dt.$$

By (155)

$$(158)\qquad \frac{1}{2n+1}\sum_{i=-n}^{+n}\frac{1}{2T}\int_{-T}^{+T}|\theta(t+\tau_i)|\,dt < 3p\,\frac{1}{2n+1}\int_{\tau_{-n}-T}^{\tau_n+T}|\theta(t)|\,dt$$

and by Lemma 4, Cor. 2

$$(159)\qquad \overline{\lim_{n\to\infty}}\,\frac{1}{2n+1}\int_{\tau_{-n}-T}^{\tau_n+T}|\theta(t)|\,dt = \frac{1}{p}\bar{M}\{|\theta(t)|\}.$$

By (158), (159)

$$(160)\qquad \overline{\lim_{T\to\infty}}\,\bar{M}_i\frac{1}{2T}\int_{-T}^{+T}|\theta(t+\tau_i)|\,dt \leqq 3\,\bar{M}\{|\theta(t)|\} < 3\,\varepsilon_0$$

and so finally by (157), (160)

$$(161)\qquad \bar{M}_x M_i\left\{\frac{1}{c}\int_x^{x+c}|f(t)-f(t+\tau_i)|\,dt\right\} < 5\,\varepsilon_0.$$

Since ε_0 is arbitrary, (156) and (161) show that $f(x)$ is $B.a.p.$

2°. We have now to prove that $\{B.a.p.\}\subset C_B(A)$. We commence with an important lemma

Lemma 7. *If $f(t)$ is a B.a.p. function and*

$$f_\delta(t) = \frac{1}{\delta}\int_t^{t+\delta} f(u)\,du,$$

then

$$\bar{M}_t\{|f(t) - f_\delta(t)|\} \to 0, \text{ as } \delta \to 0.$$

The function $f(t)$ being $B.a.p.$, to any $\varepsilon > 0$ corresponds a satisfactorily uniform sequence $\{\tau_n\}$ such that the inequalities (76), (77) hold with $p = 1$. Denote by $\mu(b)$, $\nu(b)$ respectively the minimum and maximum number of numbers τ_i in an interval of length b: we choose b (as we may) so large that

$$\nu(b) < 2\mu(b). \tag{162}$$

The fact that the function f_δ approximates the function f in mean in any fixed interval arbitrarily closely is stated by Lemma I of § 1. Let η be any positive number less than b and put $c = b + \eta$ in (77). It follows that we can choose a so that

$$\bar{M}_i \frac{1}{b+\eta}\int_a^{a+b+\eta} |f(t) - f(t+\tau_i)|\,dt < \varepsilon. \tag{163}$$

We shall deduce from (163) that both f and f_δ »imitate», in mean over the whole line from $-\infty$ to $+\infty$, their values in the interval $(a, a+b)$ with an approximation arbitrarily small with ε. Combining this result with lemma I of § 1 we shall obtain the result desired. We have

$$\bar{M}_i \int_a^{a+b} |f_\delta(t+\tau_i) - f(t+\tau_i)|\,dt \leqq \bar{M}_i \int_a^{a+b} |f_\delta(t+\tau_i) - f_\delta(t)|\,dt$$

$$+ \bar{M}_i \int_a^{a+b} |f(t+\tau_i) - f(t)|\,dt \quad + \int_a^{a+b} |f_\delta(t) - f(t)|\,dt$$

$$\leqq 2\bar{M}_i \int_a^{a+b+\delta} |f(t+\tau_i) - f(t)|\,dt + \int_a^{a+b} |f_\delta(t) - f(t)|\,dt$$

by (9). If $\delta < \eta$ is so small that the last integral is less than $b\varepsilon$ we at once obtain

$$\bar{M}_i \int_a^{a+b} |f_\delta(t+\tau_i) - f(t+\tau_i)|\,dt < b\varepsilon + 2(b+\eta)\,\varepsilon < 5\,b\varepsilon.$$

Thus for n sufficiently large we have

$$\frac{1}{2n+1} \sum_{i=-n}^{n} \int_a^{a+b} |f_\delta(t+\tau_i) - f(t+\tau_i)|\,dt < 5\,b\varepsilon.$$

Writing this in the form

$$(164) \qquad \frac{1}{2n+1} \int_{a+\tau_{-n}}^{a+b+\tau_n} \lambda(t)\,|f_\delta(t) - f(t)|\,dt < 5\,b\varepsilon$$

we see that in the interval $(a+\tau_{-n}+b,\ a+\tau_n)$ the factor $\lambda(t)$ lies between $\mu(b)$ and $\nu(b)$ and that in the rest of the range of integration, it lies between 0 and $\nu(b)$. Write

$$T_n = \min(a+\tau_n,\ -a-b-\tau_{-n});$$

for n sufficiently large both the terms in the bracket are positive. Then (164) gives

$$\frac{\mu(b)}{2n+1} \int_{-T_n}^{T_n} |f_\delta(t) - f(t)|\,dt < 5\,b\varepsilon.$$

Writing this formula for $n+1$ we have

$$\int_{-T_{n+1}}^{+T_{n+1}} |f_\delta(t) - f(t)|\,dt < \frac{5\,b\varepsilon(2n+3)}{\mu(b)}.$$

Hence for $T_n \leqq T \leqq T_{n+1}$

$$\frac{1}{2T} \int_{-T}^{+T} |f_\delta(t) - f(t)|\,dt < \frac{1}{2T_n} \int_{-T_{n+1}}^{+T_{n+1}} |f_\delta(t) - f(t)|\,dt < \frac{5\,b\varepsilon(2n+3)}{2\,T_n\,\mu(b)}$$

and thus

$$(165)\qquad \bar{M}_t\{|f_\delta(t)-f(t)|\} < \overline{\lim_{n\to\infty}}\,\frac{5\,b\,\varepsilon(2n+3)}{2\,T_n\,\mu(b)} \leqq \frac{5\,\varepsilon\,\nu(b)}{\mu(b)} \leqq 10\,\varepsilon.$$

Thus the left hand side is arbitrarily small with δ: this proves the lemma.

We now suppose as we may $f(t)$ is real and positive and define a function $\varphi_\delta(t)$ by putting

$$\varphi_\delta(t) = \bar{M}_i \frac{1}{\delta}\int_t^{t+\delta} f(u+\tau_i)\,du.$$

Then

$$|f_\delta(t)-\varphi_\delta(t)| = \left|\frac{1}{\delta}\int_t^{t+\delta} f(u)\,du - \bar{M}_i\frac{1}{\delta}\int_t^{t+\delta} f(u+\tau_i)\,du\right| \leqq \bar{M}_i\frac{1}{\delta}\int_t^{t+\delta}|f(u+\tau_i)-f(u)|\,du.$$

By (77)

$$\bar{M}_t|f_\delta(t)-\varphi_\delta(t)| \leqq \varepsilon,$$

in particular $\varphi_\delta(t)$ is summable with the continuous function $f_\delta(t)$. By (165) we now have

$$(166)\qquad \bar{M}_t|f(t)-\varphi_\delta(t)| \leqq 11\,\varepsilon.$$

But if τ be any real number we have

$$|\varphi_\delta(t+\tau)-\varphi_\delta(t)| \leqq \bar{M}_i\frac{1}{\delta}\int_{t+\tau_i}^{t+\delta+\tau_i}|f(u+\tau)-f(u)|\,du$$

$$\leqq \frac{1}{\delta}\,\overline{\lim_{n\to\infty}}\,\frac{1}{2n+1}\int_{t+\tau_{-n}}^{t+\tau_n+\delta}|f(u+\tau)-f(u)|\,du$$

$$\leqq \frac{1}{\delta}\bar{M}_u\{|f(u+\tau)-f(u)|\}\,\overline{\lim_{|n|\to\infty}}\,\frac{\tau_n}{n}.$$

If k be the value of $\overline{\lim_{|n|\to\infty}}\,\frac{\tau_n}{n}$ then every B. translation number of $f(t)$ belonging to $\frac{\eta\delta}{k}$ is a uniform translation number, *a fortiori* an S. translation number, of $\varphi_\delta(t)$ belonging to η. It follows that $\varphi_\delta(t)$ is an $S.a.p.$ function (we do not assert that it is an $\boldsymbol{a.p.}$ function because we have not proved it continuous). Thus we can find a polynomial $s(t)$ such that

$$D_S[s(t)-\varphi_\delta(t)] \leqq \varepsilon$$

and *a fortiori* that

$$\bar{M}_t\{|s(t)-\varphi_\delta(t)|\} \leqq \varepsilon.$$

Combining this with (166) we obtain

$$\bar{M}_t\{|f-s|\} \leqq 12\,\varepsilon$$

and since ε is arbitrary $f(t)$ belongs to $C_B(A)$.

§ 11. $B^p.a.p.$ Functions.

Theorem I $C_{B^p}(A)$. *The closure $C_{B^p}(A)$ is identical with the class $\{B^p.a.p.\}$, i.e. $C_{B^p}(A)=\{B^p.a.p.\}$.*

Given a function $f(x)$ we define $f_N(x)$ by the equations

$$f_N(x)=f(x) \quad \text{when} \quad |f(x)|\leqq N$$

$$f_N(x)=N\frac{f(x)}{|f(x)|} \quad \text{when} \quad |f(x)|>N$$

and we write

$$f(x)=f_N(x)+R_N(x).$$

1°. **Lemma 8.** *If $f(x)\subset C_{B^p}(A)$ then*

$$D_{B^p}[R_N(x)]\to 0 \quad \textit{as} \quad N\to\infty.$$

Given ε we can find a polynomial $s(x)$ such that

$$D_{B^p}[f-s]\leqq\varepsilon.$$

For any $N\geqq \boldsymbol{D}[s]$ and for all x we have

$$|R_N(x)|=|f(x)-f_N(x)|\leqq|f(x)-s(x)|$$

and therefore

$$D_{B^p}[f-f_N] \leqq D_{B^p}[f-s] \leqq \varepsilon,$$

which proves the lemma.

Since $f(x)$ belongs to $C_{B^p}(A)$, it belongs also to $C_B(A)$ and so does $f_N(x)$. Now in proving that $C_B(A) \subset \{B.\,a.\,p.\}$ we showed that, given a function f_N of $C_B(A)$ we can find corresponding to any $\eta > 0$ an $a.\,p.$ sequence $\{\tau_i\}$ (where i runs from $-\infty$ to $+\infty$) such that

$$\bar{M}_t\{|f_N(t+\tau_i) - f_N(t)|\} \leqq \eta \quad \text{for all } i$$

and

$$\bar{M}_x \bar{M}_i \frac{1}{c} \int_x^{x+c} |f_N(t+\tau_i) - f_N(t)|\, dt \leqq \eta \quad \text{for all } c > 0.$$

We choose $\eta = \varepsilon^p (2N)^{1-p}$ and find for all i

$$D_{B^p}[f_N(t+\tau_i) - f_N(t)] \leqq \varepsilon,$$

(167) $$D_{B^p}[f(t+\tau_i) - f(t)] \leqq \varepsilon + 2 D_{B^p}[R_N(t)] \leqq 3\varepsilon.$$

We also find for any $c > 0$

$$\left\{\bar{M}_x \bar{M}_i \frac{1}{c} \int_x^{x+c} |f(t+\tau_i) - f(t)|^p\, dt\right\}^{1/p}$$

$$\leqq \left\{\bar{M}_x \bar{M}_i \frac{1}{c} \int_x^{x+c} |R_N(t+\tau_i)|^p\, dt\right\}^{1/p} + \left\{\bar{M}_x \bar{M}_i \frac{1}{c} \int_x^{x+c} |f_N(t+\tau_i) - f_N(t)|^p\, dt\right\}^{1/p}$$

$$+ \left\{\bar{M}_x \bar{M}_i \frac{1}{c} \int_x^{x+c} |R_N(t)|^p\, dt\right\}^{1/p}$$

$$\leqq 2\varepsilon + \left\{\bar{M}_x \bar{M}_i \frac{1}{c} \int_x^{x+c} |R_N(t+\tau_i)|^p\, dt\right\}^{1/p},$$

since it is easily seen that

$$\overline{M}_x\left\{\frac{1}{c}\int_x^{x+c}|R_N(t)|^p\,dt\right\}=\overline{M}_t\{|R_N(t)|^p\}.$$

Now $R_N(t)$ belongs to $C_{B^p}(A)$ and hence

$$|R_N(t)|^p$$

belongs to $C_B(A)$.

As in the proof of Lemma 6 of § 10 it follows that

$$M_i\frac{1}{c}\int_x^{x+c}|R_N(t+\tau_i)|^p\,dt$$

exists, and hence by Fatou's theorem

$$\frac{1}{2T}\int_{-T}^{+T}M_i\left\{\frac{1}{c}\int_x^{x+c}|R_N(t+\tau_i)|^p\,dt\right\}dx\leqq \underline{M}_i\left\{\frac{1}{2T}\int_{-T}^{+T}\frac{dx}{c}\int_x^{x+c}|R_N(t+\tau_i)|^p\,dt\right\}$$

$$\leqq \underline{M}_i\frac{1}{2T}\int_{-T}^{T+c}|R_N(t+\tau_i)|^p\,dt$$

$$=\underline{\lim}_{n\to\infty}\frac{1}{2n+1}\int_{\tau_{-n}-T}^{\tau_n+T+c}|R_N(t)|^p\frac{\lambda(t)}{2T}\,dt$$

where $\lambda(t)$ is non-negative and does not exceed the number of numbers τ_i in the interval $(t-T-c,\ t+T)$. Choose b as in § 10, (162) so that $\nu(b)<2\mu(b)$. Then

$$\lambda(t)<\left(\frac{2T+c}{b}+1\right)\nu(b)$$

and

$$\overline{M}_x\overline{M}_i\frac{1}{c}\int_x^{x+c}|R_N(t+\tau_i)|^p\,dt\leqq\frac{\nu(b)}{b}\lim_{n\to\infty}\frac{\tau_n}{n}\underline{M}_x\{|R_N(x)|^p\}\leqq 2\underline{M}_x\{|R_N(x)|^p\},$$

the argument proceeding exactly as in Lemma 7 of § 10. We now have

$$\left[\bar{M}_x \bar{M}_i \left\{\frac{1}{c}\int_x^{x+c} |f(t+\tau_i) - f(t)|^p\, dt\right\}\right]^{1/p} \leqq (2 + 2^{1/p})\,\varepsilon \leqq 4\varepsilon.$$

Since ε is arbitrary these results combined with (167) show that $f(x)$ is B^p. *a. p.*

2°. **Lemma 9.** *If $f(t)$ is B^p. a. p. then*

$$D_{B^p}[f - f_N] \to 0, \text{ as } N \to \infty.$$

Given any $\varepsilon > 0$ we can find a satisfactorily uniform sequence $\{\tau_i\}$ such that for every $b > 0$

$$\bar{M}_x \bar{M}_i \frac{1}{b}\int_x^{x+b} |f(t+\tau_i) - f(t)|^p\, dt < \varepsilon^p.$$

As in Lemma 7 we choose b so large that $\nu(b) < 2\mu(b)$: we can then find an a such that

$$\bar{M}_i \frac{1}{b}\int_a^{a+b} |f(t+\tau_i) - f(t)|^p\, dt < \varepsilon^p.$$

Hence also

$$\bar{M}_i \frac{1}{b}\int_a^{a+b} |f_N(t+\tau_i) - f_N(t)|^p\, dt < \varepsilon^p \text{ for every } N.$$

Take now N so large that

$$\frac{1}{b}\int_a^{a+b} |f(t) - f_N(t)|^p\, dt < \varepsilon^p.$$

It follows that

$$(168) \qquad \bar{M}_i \frac{1}{b}\int_a^{a+b} |f(t+\tau_i) - f_N(t+\tau_i)|^p\, dt < (3\varepsilon)^p.$$

Now

$$\frac{1}{2n+1}\sum_{i=-n}^{n}\frac{1}{b}\int_{a}^{a+b}|f(t+\tau_i)-f_N(t+\tau_i)|^p\,dt$$

$$=\frac{1}{b(2n+1)}\int_{a+\tau_{-n}}^{a+b+\tau_n}|f(t)-f_N(t)|^p\,\lambda(t)\,dt$$

where $\lambda(t)\geqq\mu(b)$ in the interval $(a+b+\tau_{-n},\ a+\tau_n)$. Hence

$$\bar{M}_i\frac{1}{b}\int_{a}^{a+b}|f(t+\tau_i)-f_N(t+\tau_i)|^p\,dt\geqq\frac{\mu(b)}{b}\varliminf_{|n|\to\infty}\frac{\tau_n}{n}\cdot\bar{M}_t\{|f(t)-f_N(t)|^p\}$$

$$\geqq\frac{1}{2}\bar{M}_t\{|f(t)-f_N(t)|^p\}$$

and so by (168) $$\bar{M}_t\{|f(t)-f_N(t)|^p\}\leqq 2(3\varepsilon)^p.$$

Since this holds for all N sufficiently large, the lemma is proved.

Now f_N being $B^p.a.p.$ is clearly also $B.a.p.$ and therefore belongs to $C_B(A)$. Being bounded it also belongs to $C_{B^p}(A)$. Thus we can find a polynomial $s(t)$ of A such that $D_{B^p}[f_N-s]$ is arbitrarily small: since, by Lemma 9, $D_{B^p}[f-f_N]$ is arbitrarily small so is $D_{B^p}[f-s]$, and thus f belongs to $C_{B^p}(A)$. Combining 1° and 2° we obtain the Theorem.

CHAPTER VI.

Algorithm for Polynomial Approximation.

§ 12. Problem II in the Case of $a.p.$ Functions.

In Chapters IV and V we have given a solution of Problem I of Chapter III. Let us now pass to Problem II, i. e. to the construction of an algorithm for the approximation by finite trigonometrical polynomials to functions of various types of almost periodicity.

For the solution of this problem in the case of purely periodic functions we started from Fourier series. The required approximations were given by Fejér sums.

The solution of Problem II for various types of $a.p.$ functions will be based on that for the class of $\boldsymbol{a.p.}$ functions. In the theory of $\boldsymbol{a.p.}$ functions the notion of Fourier series is a fundamental one. It has the following meaning. If a function $f(x)$ is $\boldsymbol{a.p.}$ then it can be shewn that the mean value

$$M\{f(t)\,e^{-i\lambda t}\}$$

exists for all real values of λ, and that it may differ from 0 for at most an enumerable set of values of λ

$$\Lambda_1,\ \Lambda_2,\ \ldots$$

Writing

$$M\{f(t)\,e^{-i\Lambda_\nu t}\} = A_\nu$$

we call the series

$$\Sigma\, A_\nu e^{i\Lambda_\nu x}$$

the Fourier series of $f(x)$ and write

$$f(x) \sim \Sigma\, A_\nu e^{i\Lambda_\nu x}.$$

It will be shewn further that the Fourier series exists in the same sense for all types of $a.p.$ functions, which have been considered.

In the case of $\boldsymbol{a.p.}$ functions a solution of Problem II was given by H. Bohr [2]. In this case it is a problem of a construction of a sequence $\{s_n(x)\}$ of finite trigonometrical polynomials, which approximate the function $f(x)$ uniformly in the whole interval $-\infty < x < +\infty$. Bohr's sums $s_n(x)$ contained as exponents only the Fourier exponents of $f(x)$, a fact of importance for the extension of the theory to the case of functions of a complex variable. An essentially simpler method of obtaining such approximation functions $s_n(x)$ was given by Bochner [1], who succeeded in extending the Fejér summation method of classical Fourier series to the class of $\boldsymbol{a.p.}$ functions. In a later paper ([3] p. 205 footnote) he extended the Fejér summation also to the class of $S.\,a.p.$ functions, giving thus a solution of Problem II for this class. Like Bohr, he started from the representation of the »Fourier exponents» Λ_ν with the help of a »base» $\alpha_1, \alpha_2, \ldots$ By a base we mean a sequence of linearly independent[1] positive numbers $\alpha_1, \alpha_2, \ldots$ (which generally is infinite but in particular cases may be finite) such that every exponent Λ_ν may be expressed as a finite linear form in the α's with rational coefficients,

$$\Lambda_\nu = r_{\nu,1}\alpha_1 + r_{\nu,2}\alpha_2 + \cdots + r_{\nu,q_\nu}\alpha_{q_\nu}.$$

As was explained in Chapter II, Fejér in his summation of Fourier series of purely periodic functions $f(x)$, with period 2π, used as approximation sums the expressions

$$(169) \qquad \sigma_n(x) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(x+t)\,K_n(t)\,dt = M\{f(x+t)\,K_n(t)\}$$

where the »kernel» was given by

$$K_n(t) = \sum_{\nu=-n}^{\nu=+n}\left(1 - \frac{|\nu|}{n}\right) e^{-i\nu t} = \frac{1}{n}\left(\frac{\sin n\frac{t}{2}}{\sin\frac{t}{2}}\right)^2.$$

Bochner replaced Fejér's simple kernel by a finite product of such kernels

$$(170) \qquad K(t) = K_{\binom{n_1, n_2, \ldots, n_p}{\beta_1, \beta_2, \ldots, \beta_p}}(t) = K_{n_1}(\beta_1 t)\cdots K_{n_p}(\beta_p t) =$$

$$= \sum_{\substack{-n_1 \leqq \nu_1 \leqq +n_1 \\ \cdots\cdots\cdots \\ -n_p \leqq \nu_p < +n_p}} \left(1 - \frac{|\nu_1|}{n_1}\right)\cdots\left(1 - \frac{|\nu_p|}{n_p}\right) e^{-i(\nu_1\beta_1 + \cdots + \nu_p\beta_p)t}$$

where the β's are linearly independent numbers. This composite kernel has the same characteristic properties as the Fejér kernel: it is always positive and its mean value is equal to 1 (the constant term in the polynomial expansion of $K(t)$ being 1 on account of the linear independence of the β's).

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[1] $\alpha_1, \alpha_2, \ldots$ are said to be linearly independent if no equation of the form

$$r_1\alpha_1 + r_2\alpha_2 + \cdots + r_N\alpha_N = 0$$

holds in which $N \geqq 1$ and the r_n are rational and not all zero.

We form an expression similar to (169)

$$(171)\qquad \sigma(x) = \sigma_{\binom{n_1, n_2, \ldots, n_p}{\beta_1, \beta_2, \ldots, \beta_p}}(x) = M\{f(x+t)\, K_{\binom{n_1, n_2, \ldots, n_p}{\beta_1, \beta_2, \ldots, \beta_p}}(t)\}.$$

We have

$$f(x+t) \sim \Sigma A_\nu e^{i\Lambda_\nu x} e^{i\Lambda_\nu t}.$$

Multiplying $f(x+t)$ by each term of the right hand side of (170) and taking mean value we find

$$(172)\qquad \sigma_{\binom{n_1, n_2, \ldots, n_p}{\beta_1, \beta_2, \ldots, \beta_p}}(x) = \sum_{\substack{-n_1 \leqq \nu_1 \leqq n_1 \\ \cdots\cdots \\ -n_p \leqq \nu_p \leqq n_p}} \left(1 - \frac{|\nu_1|}{n_1}\right) \cdots \left(1 - \frac{|\nu_p|}{n_p}\right) A_\nu e^{i\Lambda_\nu x}$$

where

$$(173)\qquad \Lambda_\nu = \nu_1\beta_1 + \cdots + \nu_p\beta_p$$

and A_ν is to be interpreted as zero when the linear combination (173) of β's is not an exponent in the Fourier series of $f(x)$.

We call the kernel (170) »Bochner-Fejér kernel» and the polynomial (172) »Bochner-Fejér polynomial». We call the numbers $\beta_1, \beta_2, \ldots, \beta_p$ »basic numbers» and the numbers $n_1, n_2, \ldots, n_p$ »indices» of Bochner-Fejér kernel (or polynomial). We shall further use the notation $\sigma_B(x)$ instead of the detailed notation

$$\sigma_{\binom{n_1, n_2, \ldots, n_p}{\beta_1, \beta_2, \ldots, \beta_p}}(x)$$

and we shall use the notation $\sigma_{B_1}(x), \sigma_{B_2}(x), \ldots$ for Bochner-Fejér polynomials corresponding to different systems of basic numbers and indices. Bochner takes as basic numbers for his polynomials, numbers formed from α's (base of $f(x)$). In fact he puts

$$\beta_1 = \frac{\alpha_1}{N_1!}, \quad \beta_2 = \frac{\alpha_2}{N_2!}, \quad \ldots, \quad \beta_p = \frac{\alpha_p}{N_p!}$$

where $N_1, N_2, \ldots, N_p$ are positive integers. His result is

Theorem II C(A). *The sum* $\sigma_B(x)$ *tends uniformly to* $f(x)$, *as* $p \to \infty$, $N_1 \to \infty$, $N_2 \to \infty, \ldots$ *and* $\frac{n_1}{N_1!} \to \infty, \frac{n_2}{N_2!} \to \infty, \ldots$

A sequence of Bochner-Fejér polynomials

$$(174) \qquad \sigma_{B_1}(x),\ \sigma_{B_2}(x),\ \ldots$$

is called »Bochner sequence» if the basic numbers and the indices satisfy the condition of the above theorem.

Remark. From the expression (172) it can easily be seen that

$$(175) \qquad \sigma\begin{pmatrix} n_1, & n_2, & \ldots, & n_p \\ \frac{\alpha_1}{N_1!}, & \frac{\alpha_2}{N_2!}, & \ldots, & \frac{\alpha_p}{N_p!} \end{pmatrix}(x) = \sigma\begin{pmatrix} n_1, & n_2, & \ldots, & n_p, & n_{p+1}, & \ldots, & n_{p+q} \\ \frac{\alpha_1}{N_1!}, & \frac{\alpha_2}{N_2!}, & \ldots, & \frac{\alpha_p}{N_p!}, & \beta_1, & \ldots, & \beta_q \end{pmatrix}(x)$$

if all α's of the base of $f(x)$ and the numbers $\beta_1, \ldots, \beta_q$ form a linearly independent system. Thus the sequence (174) remains unaltered if we add new basic numbers, linearly independent with the base of $f(x)$, and with arbitrary indices, in other words the sequence (174) is identical with the sequence

$$(176) \qquad \sigma_{B_1'}(x),\ \sigma_{B_2'}(x),\ \ldots$$

if for any i B_i' contains all basic numbers of B_i with the same indices plus any number of other basic numbers which form a linearly independent system with all α's.

The notion of a base of the exponents of the Fourier series was introduced by H. Bohr [2] in order to connect the theory of ***a.p.*** functions with that of purely periodic functions of infinitely many variables: it was by means of this connection that he gave a solution of Problem II. It was also used for the investigation of the set of values which an ***a.p.*** function may take. These investigations of Bohr and the above Bochner-Fejér summation show how important is the notion of a base of the Fourier exponents and how close is the connection between an ***a.p.*** function and its Fourier series.

§ 13. Problem II in the General Case.

We shall now show that generally for all types of *a.p.* functions, which have been considered in this paper, a solution of Problem II can be given by Bochner's sequences.

We have first to prove

Theorem I on Existence of Fourier Series.

For all types of a. p. functions the Fourier series exists.

According to the definition of a Fourier series we have to show that

1° for any *a. p.* function $f(x)$ the mean value

$$M\{f(x)e^{-i\lambda x}\} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} f(x)e^{-i\lambda x}\,dx$$

exists for all real values of λ, and that

2° it may differ from nought for at most an enumerable set of values of λ.

As the classes $\{S^p. a.p.\}$, $\{W^p. a.p.\}$ $(p \geqq 1)$ and $\{B^p. a.p.\}$ $(p > 1)$ are included in the class $\{B. a.p.\}$ the proof given for the latter class will be a general proof.

The statement 1° is an immediate consequence of lemma 4, § 10, since for any real λ the function $f(x)e^{-i\lambda x}$ belongs to the class $\{B. a.p.\} = C_B(A)$ together with the function $f(x)$. We now determine a sequence $\{s_n(x)\}$ of finite trigonometrical polynomials such that

$$B. \lim s_n(x) = f(x), \quad \text{as } n \to \infty,$$

whence

$$B. \lim s_n(x)e^{-i\lambda x} = f(x)e^{-i\lambda x}, \quad \text{as } n \to \infty. \tag{177}$$

It follows at once from (177) that

$$\lim M\{s_n(x)e^{-i\lambda x}\} = M\{f(x)e^{-i\lambda x}\}.$$

For any fixed n the number $M\{s_n(x)e^{-i\lambda x}\}$ may differ from 0 only for a finite number of values of λ namely the exponents of $s_n(x)$, from which we conclude that

$$\lim M\{s_n(x)e^{-i\lambda x}\}, \quad \text{as } n \to \infty$$

may differ from 0 for at most an enumerable set of values of λ, which proves the statement 2°.

Theorem II. *If a function $f(x)$ belongs to the closure $C_G(A)$ then any Bochner sequence*

$$\{\sigma^f_{B_n}(x)\}$$

satisfies the condition

$$D_G[f(x),\, \sigma^f_{B_n}(x)] \to 0, \quad as \ n \to \infty. \tag{178}$$

We shall first prove the theorem jointly for the case of the closures $C_{S^p}(A)$, $C_{W^p}(A)$ $(p \geqq 1)$ and then for the case of the closure $C_{B^p}(A)$.

1°. In the same way as in the case of Theorem II, 3 of Chapter II we can prove the following

Auxiliary Inequality.

If $\psi(x)$ is a function of the closure $C_{W^p}(A)$ then for any Bochner-Fejér polynomial $\sigma^\psi_B(x)$ and for any $L > 0$ we have

$$D_{S^p_L}[\sigma^\psi_B(x)] \leqq D_{S^p_L}[\psi(x)]. \tag{179}$$

Let now $f(x)$ be a function of $C_{W^p}(A)$, and let

$$\sigma^f_{B_1}(x),\ \sigma^f_{B_2}(x),\ \ldots \tag{180}$$

be a Bochner sequence. We shall prove the theorem for the case of the closure $C_{W^p}(A)$ by showing that *given ε there exists a number $L > 0$ and an integer n_0 such that*

$$D_{S^p_L}[f(x),\, \sigma^f_{B_n}(x)] < \varepsilon \tag{181}$$

for all $n > n_0$.

We know that to a given ε there corresponds a number L and an ***a.p.*** function $\varphi(x)$ such that

$$D_{S^p_L}[f(x),\, \varphi(x)] < \frac{\varepsilon}{3}\cdot \tag{182}$$

We shall show that we can satisfy (181) by this value of L.

We form a base of $\varphi(x)$ by taking all numbers of the base of $f(x)$ and by adding if necessary some other numbers. We form a Bochner sequence of $\varphi(x)$

$$\sigma^\varphi_{B'_1}(x),\ \sigma^\varphi_{B'_2}(x),\ \ldots \tag{183}$$

in such a way that for all n B'_n consists of all basic numbers contained in B_n with the same indices and possibly of numbers of the base of $\varphi(x)$ which do not belong to the base of $f(x)$. Then on account of the remark to § 12 we have

$$\sigma^f_{B_n}(x) = \sigma^f_{B'_n}(x) \ \text{ for all } n$$

(we shall use this fact also for the proof of the formula (196)).

Putting in (179) $\psi(x)=\varphi(x)-f(x)$, $B=B'_n$ and observing that

$$\sigma_{B'_n}^{\varphi-f}(x) = \sigma_{B'_n}^{\varphi}(x) - \sigma_{B_n}^{f}(x)$$

we shall have on account of (182)

$$(184) \qquad D_{S_L^p}[\sigma_{B'_n}^{\varphi}(x),\ \sigma_{B_n}^{f}(x)] < \frac{\varepsilon}{3}\cdot$$

The sequence (183) being a Bochner sequence and $\varphi(x)$ being an ***a.p.*** function we have

$$(185) \qquad \text{up. b.}\ |\varphi(x) - \sigma_{B'_n}^{\varphi}(x)| < \frac{\varepsilon}{3}$$

for all n greater than a certain integer n_0, whence

$$(186) \qquad D_{S_L^p}[\varphi(x),\ \sigma_{B'_n}^{\varphi}(x)] < \frac{\varepsilon}{3}\cdot$$

By (182), (186), (184) the inequality (181) is proved, which shows that

$$(187) \qquad D_{W^p}[f(x),\ \sigma_{B_n}^{f}(x)] \to 0, \ \text{ as } n \to \infty.$$

In the case when the function $f(x)$ belongs to $C_{S^p}(A)$ we may put in the inequality (182) $L=1$ for any ε and thus we shall have the inequality (181) satisfied by $L=1$ for all values of ε. Consequently in this case

$$(188) \qquad D_{S^p}[f(x),\ \sigma_{B_n}^{f}(x)] \to 0, \ \text{ as } n \to \infty.$$

Thus the theorem II has been proved for the cases of closures C_{S^p}, C_{W^p} $(p \geqq 1)$.

Remark. In the case when the function $f(x)$ belongs to C_{W^p} (but not to C_{S^p}) the inequality (181) is somewhat sharper than the limiting equation (187) which we had to prove. For it shows some feature of uniformity of the approximation to the function $f(x)$ by the functions of the sequence (180). We have really that for a given ε the inequality (181) is satisfied by the same value of L for all $n > n_0$, a fact which cannot be deduced from (187).

In the case of $p=2$ this property of approximation by trigonometrical polynomials was discovered by R. Schmidt [1].

2°. We pass now to the proof of the theorem in the case of the closure C_{B^p} $(p \geqq 1)$. As before we shall first prove the following

Auxiliary Inequality.

If a function $\psi(x)$ *belongs to* $C_{B^p}(A)$ *then for any Bochner-Fejér polynomial* $\sigma_B^{\psi}(x)$ *we have*

$$D_{B^p}[\sigma_B^{\psi}(x)] \leqq D_{B^p}[\psi(x)]. \tag{189}$$

We have

$$\sigma_B^{\psi}(x) = M\{\psi(x+t)\,K_B(t)\},$$

$$|\sigma_B^{\psi}(x)| \leqq M\{|\psi(x+t)|\,K_B(t)\}.$$

By Hölder's theorem

$$|\sigma_B^{\psi}(x)|^p \leqq \underline{M}_t\{|\psi(x+t)|^p\,K_B(t)\}$$

and thus

$$\{D_{B^p}[\sigma_B^{\psi}(x)]\}^p \leqq \overline{M}_x\,\underline{M}_t\{|\psi(x+t)|^p\,K_B(t)\} = \overline{M}_x\,\underline{M}_t\{|\psi(t)|^p\,K_B(t-x)\}. \tag{190}$$

To any $\eta > 0$ correspond values of L as large as we please and such that

$$\overline{M}_x\,M_t\{|\psi(t)|^p\,K_B(t-x)\} < \frac{1}{L}\int_0^L \underline{M}_t\{|\psi(t)|^p\,K_B(t-x)\}\,dx + \eta. \tag{191}$$

By (190), (191) and by Fatou's theorem

$$\{D_{B^p}[\sigma_B^{\psi}(x)]\}^p < \overline{M}_t\left\{|\psi(t)|^p\,\frac{1}{L}\int_0^L K_B(t-x)\,dx\right\} + \eta. \tag{192}$$

But for sufficiently large values of L

$$\frac{1}{L}\int_0^L K_B(t-x)\,dx < M\{K_B(t)\} + \eta = 1 + \eta. \tag{193}$$

By (192), (193)

$$\{D_{B^p}[\sigma_B^{\psi}(x)]\}^p < (1+\eta)\,M_t\{|\psi(t)|^p\} + \eta.$$

η being arbitrary we have

$$\{D_{B^p}[\sigma_B^{\psi}(x)]\}^p \leqq M_t\{|\psi(t)|^p\}$$

which proves (189).

Now we can arrive immediately to the proof of our theorem. We know that to any ε corresponds an $a.p.$ function $\varphi(x)$ such that

$$D_{B^p}[f(x), \varphi(x)] < \frac{\varepsilon}{3}. \tag{194}$$

We define further a Bochner-Fejér polynomial $\sigma_B^{\varphi}(x)$ such that

$$\text{up. b.}\, |\varphi(x) - \sigma_B^{\varphi}(x)| < \frac{\varepsilon}{3}. \tag{195}$$

Putting in (189) $\psi(x) = f(x) - \varphi(x)$ we have

$$D_{B^p}[\sigma_B^{\varphi}(x), \sigma_B^{f}(x)] \leqq D_{B^p}[f(x), \varphi(x)]$$

and by (194)

$$D_{B^p}[\sigma_B^{\varphi}(x), \sigma_B^{f}(x)] < \frac{\varepsilon}{3}. \tag{196}$$

By (194), (195), (196)

$$D_{B^p}[f(x), \sigma_B^{f}(x)] < \varepsilon$$

which proves the theorem.

Uniqueness Theorem.

If two $G.a.p.$ functions $f(x)$, $g(x)$ have the same Fourier series then

$$D_G[f(x), g(x)] = 0.$$

The proof is identical with the one for the class $\{c.f.\}$ of § 4.

Appendix.

$\overline{B}.\,a.p.$ Functions.

When giving the definition of $B.\,a.p.$ functions, we mentioned a possible variation of conditions by which the functions have been defined. By this variation we obtain a new class of $a.p.$ functions, defined as follows.

Definition. *We call a function $f(x)$ a $\overline{B}.\,a.p.$ function if*

$$\underline{M}_x\{|f(x)|\}$$

is finite and if to any $\varepsilon > 0$ corresponds a satisfactorily uniform set of numbers τ_i such that

$$(1) \qquad \overline{M}_x\,|f(x) - f(x+\tau_i)| < \varepsilon$$

for all $-\infty < i < \infty$, and

$$(2) \qquad \overline{M}_x\,\overline{M}_i\,|f(x) - f(x+\tau_i)| < \varepsilon.$$

We call numbers τ_i $\overline{B}$. translation numbers of $f(x)$ belonging to ε.

The class of all $\overline{B}.\,a.p.$ functions is denoted by $\{\overline{B}.\,a.p.\}$. The conditions by which $\overline{B}.\,a.p.$ functions are defined seem to be simpler and more natural than those by which $B.\,a.p.$ functions are defined, as the condition (2) does not involve the smoothing integration in the definition of $B.\,a.p.$ functions, but the class $\{B.\,a.p.\}$ has the advantage of being identical with the closure $C_B(A)$. It will be proved that $\{\overline{B}.\,a.p.\}$ is contained in $\{B.\,a.p.\}$ and that they are very near each other, in fact from some point of view identical to each other. For though $\{B.\,a.p.\}$ is not contained in $\{\overline{B}.\,a.p.\}$, but to every $\overline{B}.\,a.p.$ function corresponds a $B.\,a.p.$ function with the same Fourier series. On account of this connection we consider the study of the class $\{\overline{B}.\,a.p.\}$ as a study of $\{B.\,a.p.\}$ from point of view of $a.p.$ properties given by (2).

Lemma 1. *If $A_1, A_2, \ldots$ is a set of finite real numbers such that $\overline{M}_i A_i > c$ then to any i' corresponds an $i'' > i'$ as large as we please and such that*

$$A_{i'+1} + A_{i'+2} + \cdots + A_{i''} > (i'' - i')\,c.$$

The proof is obvious.

Theorem 1. $\{\bar{B}.\,a.\,p.\} \subset \{B.\,a.\,p.\}$.

In order to prove Theorem 1 we first prove the following lemma.

Lemma 2. *If* $f(x)$ *is a* $\bar{B}.\,a.\,p.$ *function and* $\{\tau_i\}$ *are* $\bar{B}.\,a.\,p.$ *translation numbers of* $f(x)$ *then*

$$(3) \qquad \bar{M}_i \int_0^1 |f(x) - f(x+\tau_i)|\,dx \leqq \int_0^1 \bar{M}_i |f(x) - f(x+\tau_i)|\,dx.$$

Lemma 2 is true when we mean by $\bar{M}_i$ in (3) the upper mean value corresponding either to all τ_i $(-\infty < i < \infty)$ or only to τ_i with positive indices $(0 < i < \infty)$. The proof in both cases is identical, but as the writing of formulae is slightly simpler for the second case we shall prove the lemma for this case.

Proof. Suppose that the lemma is not true and that

$$(4) \qquad \bar{M}_i \int_0^1 |f(x) - f(x+\tau_i)|\,dx = \int_0^1 \bar{M}_i |f(x) - f(x+\tau_i)|\,dx + a$$

where $a > 0$.

We shall first give the idea of the proof. On account of (4) we conclude that to any positive number ε corresponds an integer n as large as we please and such that

$$\int_0^1 \left[\frac{1}{n}\sum_{i=1}^{n} |f(x) - f(x+\tau_i)| - \bar{M}_i\{|f(x) - f(x+\tau_i)|\} - \varepsilon\right] dx > a - 2\varepsilon.$$

For sufficiently large values of n the integrand is negative in the whole range of integration except a set $\mathfrak{E} \subset (0, 1)$ of arbitrarily small measure.

Thus

$$\int_{\mathfrak{E}} \left[\frac{1}{n}\sum_{i=1}^{n} |f(x) - f(x+\tau_i)| - \bar{M}_i\{|f(x) - f(x+\tau_i)|\} - \varepsilon\right] dx > a - 2\varepsilon.$$

The functions

$$\bar{M}_i\{|f(x) - f(x+\tau_i)|\} \text{ and } \frac{1}{n}\sum_{i=1}^{n}|f(x)| = |f(x)|$$

being summable and independent of n we conclude that for sufficiently large values of n

$$\int_{\mathfrak{E}} \frac{1}{n}\sum_{i=1}^{n}|f(x+\tau_i)|\,dx > a - 3\varepsilon.$$

The meaning of this inequality is that we can construct in each of the intervals $(\tau_i, \tau_i + 1)$ $(i = 1, 2, \ldots, n)$ a set $\mathfrak{E}_i$ congruent with $\mathfrak{E}$ and such that

$$\frac{1}{n}\sum_{i=1}^{n}\int_{\mathfrak{E}_i}|f(x)|\,dx > a - 3\varepsilon.$$

By choosing a suitable n we can take for $\mathfrak{E}$ a set of arbitrarily small measure. It follows that we can construct in each of the intervals $(\tau_i, \tau_i + 1)$ $(i = 1, 2, \ldots)$ a set $\mathfrak{E}_i$ such that

$$\bar{M}_i \int_{\mathfrak{E}_i}|f(x)|\,dx > a - 3\varepsilon$$

and that

$$m\mathfrak{E}_i \to 0, \quad \text{as } i \to \infty.$$

These sets naturally are no longer congruent. Let $E = \sum_{i=1}^{\infty}\mathfrak{E}_i$ and let $\varphi(x)$ be the characteristic function of E. We have

$$(5) \qquad M\{\varphi(x)\} = 0, \quad \bar{M}\{\varphi(x)|f(x)|\} > b$$

(we call the second of the above numbers »the upper mean value of $|f(x)|$ along the set E») where $b > 0$ is some constant.

Now we take into account the almost-periodicity condition (2). On account of this condition the values of the function $|f(x)|$ on *almost* any interval are roughly speaking imitated throughout the whole range of values of x.

By (5) we conclude that we can find an interval (c, d) such that

$$\int_c^d \varphi(x)\,|f(x)|\,dx = \int_{(c,d)\,.\,E} |f(x)|\,dx > b\,(d-c)$$

and that the mean density of E on (c, d) is as small as we please. On account of the above »imitation property» we can construct a set G of arbitrarily small density, imitation of the set (c, d). E in the whole range of values of x, on which the mean density of $|f(x)|$ is greater than b. We define in some way a sequence of non overlapping sets

$$G_1, \; G_2, \ldots$$

similar to G, of decreasing mean density, on each of which the mean density of $|f(x)|$ is greater than a fixed constant, and thus we come to the conclusion that $\underline{M}_x\{|f(x)|\}$ is infinite which is impossible on account of the definition. In this way we shall prove the lemma. We shall prove it in four stages.

1°. We first prove that there exists a set E of values of x such that if $\theta(x)$ is its characteristic function then

$$M_x\{\theta(x)\} = 0 \quad \text{and} \quad \overline{M}_x\{\theta(x)\,|f(x)|\} > \frac{a}{4b} \tag{6}$$

where $b > 0$ is to be defined later.

Let

$$0 < \varepsilon < \frac{a}{4} \tag{7}$$

and let $\eta > 0$ be such that, $\mathfrak{E}$ being any set of points in the interval $(0, 1)$, we have

$$\int_{\mathfrak{E}} |f(x)|\,dx < \varepsilon, \quad \text{if} \quad m\mathfrak{E} < \eta. \tag{8}$$

We can evidently find a number n_0 such that the set $\mathfrak{E}$ of all values of x in $(0, 1)$ for which the inequality

$$\frac{1}{n}\sum_{i=1}^{n} |f(x) - f(x+\tau_i)| > \overline{M}_i |f(x) - f(x+\tau_i)| + \varepsilon \tag{9}$$

is satisfied for at least one value $n \geqq n_0$, is of measure less than η. We have then for any $n \geqq n_0$

$$(10) \qquad \int\limits_{(0,1)-\mathfrak{E}} \frac{1}{n}\sum_{i=1}^{n} |f(x)-f(x+\tau_i)|\,dx \leqq \int\limits_{(0,1)-\mathfrak{E}} \overline{M}_i |f(x)-f(x+\tau_i)|\,dx + \varepsilon$$

$$\leqq \int\limits_{0}^{1} \overline{M}_i |f(x)-f(x+\tau_i)|\,dx + \varepsilon$$

or

$$(11) \qquad \frac{1}{n}\sum_{i=1}^{n} \int\limits_{(0,1)-\mathfrak{E}} |f(x)-f(x+\tau_i)|\,dx \leqq \int\limits_{0}^{1} \overline{M}_i |f(x)-f(x+\tau_i)|\,dx + \varepsilon$$

and consequently

$$(12) \qquad \overline{M}_i \int\limits_{(0,1)-\mathfrak{E}} |f(x)-f(x+\tau_i)|\,dx \leqq \int\limits_{0}^{1} \overline{M}_i |f(x)-f(x+\tau_i)|\,dx + \varepsilon .$$

We have

$$(13) \qquad \overline{M}_i \int\limits_{0}^{1} |f(x)-f(x+\tau_i)|\,dx \leqq \overline{M}_i \int\limits_{(0,1)-\mathfrak{E}} |f(x)-f(x+\tau_i)|\,dx$$

$$+ \overline{M}_i \int\limits_{\mathfrak{E}} |f(x)-f(x+\tau_i)|\,dx .$$

By (4), (12), (13)

$$(14) \qquad \overline{M}_i \int\limits_{\mathfrak{E}} |f(x)-f(x+\tau_i)|\,dx \geqq a-\varepsilon$$

and consequently

$$(15) \qquad \int\limits_{\mathfrak{E}} |f(x)|\,dx + \overline{M}_i \int\limits_{\mathfrak{E}} |f(x+\tau_i)|\,dx \geqq a-\varepsilon .$$

By (7), (8), (15)

$$(16) \qquad \overline{M}_i \int\limits_{\mathfrak{E}} |f(x+\tau_i)|\,dx > \frac{a}{2} \cdot$$

Denote by k_j the number of all τ_i satisfying the inequality

$$(17) \qquad (j-1)\,b \leqq \tau_i < j\,b$$

and suppose that b is so great that the ratio of the greatest of k_j to the smallest is < 2, which is possible, since the sequence of numbers τ_i is satisfactorily uniform. Thus there exists an integer k such that

$$(18) \qquad k \leqq k_j < 2k$$

for all j. We take now those numbers τ_i, which satisfy (17) for an odd j and denote them in order of their greatness by

$$\tau_1', \ \tau_2', \ldots$$

We denote the other τ_i by

$$\tau_1'', \ \tau_2'', \ldots$$

Then at least one of the two inequalities

$$(19) \qquad \bar{M}_i \int_{\mathfrak{E}} |f(x+\tau_i')|\,dx > \frac{a}{2}, \quad \bar{M}_i \int_{\mathfrak{E}} |f(x+\tau_i'')|\,dx > \frac{a}{2}$$

is satisfied. Suppose that it is the first one. Put in (17) $j = 2l-1$ and denote by t_l that one (or one of those) of the numbers τ_i of (17) for which the integral

$$(19,1) \qquad \int_{\mathfrak{E}} |f(x+\tau_i)|\,dx$$

has the maximum value. We shall have by virtue of (18)

$$(20) \qquad 2k \int_{\mathfrak{E}} |f(x+t_l)|\,dx \geqq \sum_{(2l-2)\,b \leqq \tau_i' < (2l-1)\,b} \int_{\mathfrak{E}} |f(x+\tau_i')|\,dx$$

$$(21) \qquad \frac{\sum\limits_{i=1}^{n} \int_{\mathfrak{E}} |f(x+t_i)|\,dx}{n} \geqq \frac{\sum\limits_{0 \leqq \tau_i' < 2nb} \int_{\mathfrak{E}} |f(x+\tau_i')|\,dx}{2kn}$$

$$\geqq \frac{1}{2} \frac{\sum\limits_{0 \leqq \tau_i' < 2nb} \int_{\mathfrak{E}} |f(x+\tau_i')|\,dx}{k_1 + k_3 + \cdots + k_{2n-1}}.$$

But $k_1 + k_3 + \cdots + k_{2n-1}$ is the index of the largest τ_i' which is $< 2nb$. Then we conclude from (21)

$$(22) \qquad \bar{M}_i \int_{\mathfrak{E}} |f(x+t_i)|\,dx \geqq \frac{1}{2} \bar{M}_i \int_{\mathfrak{E}} |f(x+\tau_i')|\,dx > \frac{a}{4}\cdot$$

Thus corresponding to any $\eta > 0$ we have a set $\mathfrak{E} \subset (0, 1)$ of measure $< \eta$, for which (22) is satisfied. Let us give to η a sequence of values

$$(22, 1) \qquad \eta_1 > \eta_2 > \cdots \qquad \eta_n \to 0$$

and let us denote the corresponding sets $\mathfrak{E}$ by

$$\mathfrak{E}_1,\ \mathfrak{E}_2,\ \ldots$$

Denote by $t_i^{(j)}$ the value of t_i corresponding to $\mathfrak{E} = \mathfrak{E}_j$. We have for all j

$$(23) \qquad 2(i-1)b \leqq t_i^{(j)} < 2ib$$

and

$$(24) \qquad \bar{M}_i \int_{\mathfrak{E}_j} |f(x+t_i^{(j)})|\,dx > \frac{1}{4} a.$$

From (24) we conclude on account of Lemma 1 that we can choose numbers $i_1 < i_2 < i_3 < \cdots$ such that

$$(25) \qquad \frac{1}{i_1}\sum_{i=1}^{i_1} \int_{\mathfrak{E}_1} |f(x+t_i^{(1)})|\,dx > \frac{1}{4}a,\quad \frac{1}{i_2 - i_1 - 1}\sum_{i=i_1+2}^{i_2} \int_{\mathfrak{E}_2} |f(x+t_i^{(2)})|\,dx > \frac{1}{4}a,$$

$$\frac{1}{i_3 - i_2 - 1}\sum_{i=i_2+2}^{i_3} \int_{\mathfrak{E}_3} |f(x+t_i^{(3)})|\,dx > \frac{1}{4}a, \ldots$$

and that

$$(26) \qquad i_n - i_{n-1} \to \infty, \quad \text{as } n \to \infty.$$

The difference between any two consecutive numbers of the sequence

$$(27) \qquad t_1^{(1)},\ t_2^{(1)},\ \ldots,\ t_{i_1}^{(1)},\ t_{i_1+2}^{(2)},\ t_{i_1+3}^{(2)},\ \ldots,\ t_{i_2}^{(2)},\ t_{i_2+2}^{(3)},\ \ldots$$

is always greater than b. Denoting by $[\mathfrak{E}+u]$ the set of the numbers of $\mathfrak{E}$ each increased by the number u, and observing that all the sets $\mathfrak{E}_j$ are included in the interval (0, 1), we see that no two sets of the sequence

$$(28) \qquad [\mathfrak{E}_1 + t_1^{(1)}],\ [\mathfrak{E}_1 + t_2^{(1)}],\ \ldots\ [\mathfrak{E}_1 + t_{i_1}^{(1)}],\ [\mathfrak{E}_2 + t_{i_1+2}^{(2)}],\ \ldots$$

have points in common. Denote by E the sum of all sets of (28) and by $\theta(x)$ its characteristic function. Remembering that $m\,\mathfrak{E}_j < \eta_j$ we conclude on account of (22, 1) that

$$M_x\{\theta(x)\} = 0.$$

We conclude further on account of (25), (26), (23) that

$$(29) \qquad \overline{M}_x\{\theta(x)\,|f(x)|\} \geqq \frac{a}{8b}\cdot$$

Definition. *If F is a set of points and $\theta(x)$ its characteristic function then we call the numbers*

$$\overline{M}_x\{\theta(x)\},\quad \underline{M}_x\{\theta(x)\},\quad M_x\{\theta(x)\}$$

the upper density of F, the lower density, and simply density and we denote then by

$$\overline{D}(F),\quad \underline{D}(F),\quad D(F).$$

As we said before we call the number (29) *the upper mean value of $|f(x)|$ on the set E.* In the same way is defined the lower mean value on the set E.

2°. We now proceed to construct a certain class of sets of arbitrarily small density on which the upper mean value of $|f(x)|$ is greater than some fixed number. Let d be an arbitrarily small positive number. Take ε_1

$$(30) \qquad 0 < \varepsilon_1 < \frac{a}{96\,b}$$

and let σ_i be $\overline{B}$. translation numbers of $f(x)$ belonging to $\frac{1}{4}\varepsilon_1$, so that

$$(31) \qquad \overline{M}_x\,\overline{M}_i\,|f(x) - f(x+\sigma_i)| < \varepsilon_1$$

(where $\overline{M}_x$ and $\overline{M}_i$ are taken only over positive values of x and i). We choose a number $c > 0$ satisfying the following conditions

(32) 1)
$$\frac{m\{E.(0,c)\}}{2c} < d$$

(33) 2)
$$\frac{1}{c}\int_0^c \bar{M}_i\{|f(x)-f(x+\sigma_i)|\}\,dx < \varepsilon_1$$

(34) 3)
$$\frac{1}{c}\int_0^c \theta(x)\,|f(x)|\,dx > \frac{a}{8b} - \varepsilon_1$$

4) the ratio of the maximum number of numbers σ_i lying on a segment of length c, to the minimum number is < 2.

Let $\varepsilon_2 > 0$ be such that

(35)
$$\int_{\mathfrak{E}'} \theta(x)\,|f(x)|\,dx < \varepsilon_1$$

for any set $\mathfrak{E}' \subset (0, c)$ of measure less than ε_2. We can find an integer n_0' such that the set $\mathfrak{E}'$ of all values of x of the interval $(0, c)$ for which the inequality

(36)
$$\frac{1}{n}\sum_{i=1}^{n} |f(x)-f(x+\sigma_i)| \geqq \bar{M}_i\,|f(x)-f(x+\sigma_i)| + \varepsilon_1$$

is satisfied for at least one $n \geqq n_0'$, has a measure $< \varepsilon_2$. We have then

(37)
$$\frac{1}{n}\sum_{i=1}^{n}\int_{(0,c)-\mathfrak{E}'} |f(x)-f(x+\sigma_i)|\,dx < \int_{(0,c)-\mathfrak{E}'} \bar{M}_i\,|f(x)-f(x+\sigma_i)|\,dx + \varepsilon_1 c.$$

By (33), (37)

(38)
$$\bar{M}_i\frac{1}{c}\int_{(0,c)-\mathfrak{E}'} |f(x)-f(x+\sigma_i)|\,dx < 2\varepsilon_1$$

and a fortiori

(39)
$$\bar{M}_i\frac{1}{c}\int_{E.(0,c)-\mathfrak{E}'} |f(x)-f(x+\sigma_i)|\,dx < 2\varepsilon_1.$$

In a way analogous to one used in 1° from (17) to (22) for definition of numbers t_l we can find a set of numbers

$$s_1,\ s_2,\ \ldots$$

such that

$$(40)\qquad 2ic \leqq s_i < 2(i+1)c,\quad s_{i+1} - s_i > c \quad\text{for}\quad i = 1, 2, 3, \ldots$$

and that

$$(41)\qquad \overline{M}_i \frac{1}{c} \int\limits_{E.(0,c)-\mathfrak{E}'} |f(x) - f(x+s_i)|\, dx < 4\varepsilon_1$$

(s_l is that of σ_i which renders the integral

$$\int\limits_{E.(0,c)-\mathfrak{E}'} |f(x) - f(x+\sigma_i)|\, dx$$

minimum (and not maximum as in (19, 1)) when i varies in some interval).

Observing that

$$(42)\qquad \underline{M}_i |B_i| \geqq \underline{M}_i |A_i| - \overline{M}_i |A_i - B_i|$$

and putting

$$A_i = \frac{1}{c} \int\limits_{E.(0,c)-\mathfrak{E}'} |f(x)|\, dx,\quad B_i = \frac{1}{c} \int\limits_{E.(0,c)-\mathfrak{E}'} |f(x+s_i)|\, dx$$

we shall have

$$\underline{M}_i \frac{1}{c} \int\limits_{E.(0,c)-\mathfrak{E}'} |f(x+s_i)|\, dx \geqq \frac{1}{c} \int\limits_{E.(0,c)-\mathfrak{E}'} |f(x)|\, dx - \overline{M}_i \left| \frac{1}{c} \int\limits_{E.(0,c)-\mathfrak{E}'} \{|f(x)| - |f(x+s_i)|\}\, dx \right|$$

and a fortiori

$$(43)\qquad \underline{M}_i \frac{1}{c} \int\limits_{E.(0,c)-\mathfrak{E}'} |f(x+s_i)|\, dx \geqq \frac{1}{c} \int\limits_{E.(0,c)} |f(x)|\, dx - \frac{1}{c} \int\limits_{\mathfrak{E}'.E} |f(x)|\, dx$$
$$- \overline{M}_i \frac{1}{c} \int\limits_{E.(0,c)-\mathfrak{E}'} |f(x) - f(x+s_i)|\, dx.$$

By (34), (35), (41), (30)

$$(44)\qquad \underline{M}_i \frac{1}{c} \int\limits_{E.(0,c)} |f(x+s_i)|\,dx > \frac{a}{8b} - 6\varepsilon_1 > \frac{a}{16b}\,.$$

The sets

$$(45)\qquad [E.(0,c)+s_i] \quad i=1,2,3,\ldots$$

are non overlapping. Denoting by G the sum of all sets (45) and by $\varphi(x)$ the characteristic function of G we shall have

$$(46)\qquad M\{\varphi(x)\} = \frac{m\{E.(0,c)\}}{2c} < d$$

and also on account of (44)

$$(47)\qquad \underline{M}_x\{\varphi(x)\,|f(x)|\} > \frac{a}{32b}\,.$$

Thus *corresponding to any number* $d>0$ *we can construct a* (*»segmentwise periodic»*) *set* G *of density* $<d$ *on which the lower mean value of* $|f(x)|$ *is* $> \frac{a}{32b}$.

Remark. Let H be any subset of G. Define the function $f_1(x)$ by putting

$$f_1(x) = f(x-s_i) \qquad x \subset (s_i, s_i+c).H \qquad i=1,2,\ldots$$
$$f_1(x) = f(x) \qquad \text{for all other values of } x.$$

We evidently have for any x in the interval $(0,c)$

$$(48)\qquad |f_1(x) - f_1(x+s_i)| \leqq |f(x) - f(x+s_i)|$$

and consequently the inequality (41) remains true if we put in it $f_1(x)$ instead of $f(x)$. But then (47) remains also true. Thus we have

$$(49)\qquad \underline{M}_x\{\varphi(x)\,|f_1(x)|\} > \frac{a}{32b}\,.$$

3°. We shall now show that corresponding to any $\varepsilon>0$ there exists a number δ such that the lower mean value of $|f(x)|$ on the set $G-H$ is

$> \frac{a}{32b} - \varepsilon$, if H is any subset of G subject to the only condition that $\overline{D}H < \delta$. Given $\varepsilon > 0$, let δ be a number such that for any set $U \subset (0, c)$

(50) $$\int_U |f(x)|\, dx < 2c\varepsilon$$

if only $mU < 2\delta c$. Let $H \subset G$ be a set such that $\overline{D}(H) < \delta$. Define now a function $f_2(x)$ in the following way

$$\begin{aligned} f_2(x) &= f(x - s_i) && s_i \leqq x < s_i + c \quad i = 1, 2, \ldots \\ &= 0 && \text{for all other values of } x. \end{aligned}$$

At all points of H we have

(51) $$f_2(x) = f_1(x)$$

where $f_1(x)$ is the function defined in the above remark. Let n_0'' be such that for all $n \geqq n_0''$

(52) $$m\{H.(0, s_n + c)\} < 2ncd.$$

Consider the integral

$$\int_{H.(0,\, s_n+c)} |f_2(x)|\, dx,$$

the function $|f_2(x)|$ has the same positive values on n intervals $(s_i,\ s_i + c)$ $i = 1, 2, \ldots, n$ (the values of $|f(x)|$ in $(0, c)$) and is zero for other values of x in $(0, s_n + c)$; therefore the above integral is less than n-times the maximum of the integral

$$\int_U |f(x)|\, dx$$

for any set U in $(0, c)$ of measure $\leqq \frac{1}{n} m[H.(0, s_n + c)] < 2c\delta$ and thus on account of (50)

(53) $$\frac{1}{s_n + c} \int_{H.(0,\, s_n+c)} |f_2(x)|\, dx < \frac{2nc\delta}{s_n + c}.$$

Denote by $\psi(x)$ the characteristic function of H. By (53), (51), (40)

$$\overline{M}_x\{\psi(x)|f_1(x)|\} \leqq \varepsilon. \tag{54}$$

By (49)

$$\underline{M}_x\{(\varphi(x)-\psi(x))|f_1(x)|\} > \frac{a}{32b} - \varepsilon. \tag{55}$$

But at the points of $G-H$ $f_1(x)=f(x)$. Therefore

$$\underline{M}_x\{(\varphi(x)-\psi(x))|f(x)|\} > \frac{a}{32b} - \varepsilon > \frac{a}{64b} \text{ if } \varepsilon < \frac{a}{64b}. \tag{56}$$

Thus corresponding to a set G there exists a number δ such that the lower mean value of $|f(x)|$ on any set $G-H$ is greater than $\frac{a}{64b}$ if only H is a subset of G of upper density less than δ.

4°. Let G_1 be one of the above sets and δ_1 the corresponding value of δ. We take a set G_2 of density $d_2<\frac{\delta_1}{2}$ and denote by δ_2 the value of δ corresponding to the set $G=G_2$. We construct further a set G_3 of density $d_3<\frac{\delta_2}{2}$ and so on. Consider now $\underline{M}_x\{|f(x)|\}$. Evidently it is greater than or equal to the lower mean value of $|f(x)|$ on the set

$$G_1+G_2+\cdots+G_s \tag{57}$$

for any s. But the set (57) consists of the s non-overlapping sets

$$\begin{aligned}&G_1-G_1(G_2+G_3+\cdots+G_s),\\&G_2-G_2(G_3+G_4+\cdots+G_s),\\&\cdots\cdots\cdots\cdots\\&\cdots\cdots\cdots\cdots\\&G_{s-1}-G_{s-1}G_s,\\&G_s.\end{aligned} \tag{57, 1}$$

Observing that

$$\overline{D}\, G_i(G_{i+1} + G_{i+2} + \cdots + G_s) \leqq \overline{D}(G_{i+1} + G_{i+2} + \cdots + G_s) \tag{58}$$

$$\leqq d_{i+1} + d_{i+2} + \cdots + d_s < \frac{\delta_i}{2} + \frac{\delta_{i+1}}{2} + \cdots + \frac{\delta_{s-1}}{2}$$

$$< \frac{\delta_i}{2} + \frac{\delta_i}{4} + \frac{\delta_i}{8} + \cdots < \delta_i$$

we conclude that the lower mean value of $|f(x)|$ on any of the sets (57, 1) is greater than $\frac{a}{64\,b}$ and thus the lower mean value on the set (57) is greater than $s\frac{a}{64\,b}$, and consequently

$$\underline{M}_x\{|f(x)|\} > s\frac{a}{64\,b}.$$

s being arbitrary we have

$$\underline{M}_x\{|f(x)|\} = \infty \tag{59}$$

which is impossible since $f(x)$ is a $\overline{B}.a.p.$ function. Thus Lemma 2 is proved.

Evidently we may replace in (3) the limits of integration 0, 1 by any two numbers $\alpha < \beta$, so that the lemma should be formulated as follows.

Lemma 2. *If $f(x)$ is a $\overline{B}.a.p.$ function then for any $\alpha, \beta > \alpha$*

$$\overline{M}_i \int_\alpha^\beta |f(x) - f(x+\tau_i)|\, dx \leqq \int_\alpha^\beta \overline{M}_i\{|f(x) - f(x+\tau_i)|\}\, dx. \tag{60}$$

Proof of Theorem 1. Let $f(x)$ be a $\overline{B}.a.p.$ function. Then to any $\varepsilon > 0$ corresponds a satisfactorily uniform set of numbers τ_i such that

$$\overline{M}_x |f(x) - f(x+\tau_i)| < \varepsilon \text{ for all } i \tag{61}$$

and

$$\overline{M}_x \overline{M}_i\{|f(x) - f(x+\tau_i)|\} < \varepsilon. \tag{62}$$

Remembering that if $\theta(x) \geqq 0$ then

$$\int_a^b dx \frac{1}{c} \int_x^{x+c} \theta(t)\,dt \leqq \int_a^{b+c} \theta(x)\,dx \tag{63}$$

we conclude that

$$\int_{-a}^{+a} \frac{dx}{c} \int_x^{x+c} \overline{M}_i |f(x) - f(x+\tau_i)|\,dx \leqq \int_{-a}^{a+c} \overline{M}_i |f(x) - f(x+\tau_i)|\,dx. \tag{64}$$

Applying Lemma 2 to the left hand side we obtain

$$\frac{1}{2a} \int_{-a}^{+a} dx \left[\overline{M}_i \frac{1}{c} \int_x^{x+c} |f(x) - f(x+\tau_i)|\,dx\right] \leqq \frac{1}{2a} \int_{-a}^{a+c} \overline{M}_i |f(x) - f(x+\tau_i)|\,dx \tag{65}$$

and passing to the limit we have

$$\overline{M}_x \overline{M}_i \frac{1}{c} \int_x^{x+c} |f(x) - f(x+\tau_i)|\,dx \leqq \overline{M}_x \overline{M}_i |f(x) - f(x+\tau_i)|.$$

By (62)

$$\overline{M}_x \overline{M}_i \frac{1}{c} \int_x^{x+c} |f(x) - f(x+\tau_i)|\,dx < \varepsilon \tag{66}$$

for all positive values of c.

The existence of the inequalities (61) and (66) proves Theorem 1.

Theorem 2 (*converse theorem*). *To any B. a. p. function corresponds a $\overline{B}$. a. p. function differing from the first function only by a function the mean value of whose modulus is zero.*

Let $f(x)$ be a $B. a. p.$ function. We consider the set of functions

$$f(x) + \varphi(x)$$

for a given function $f(x)$ and for all functions $\varphi(x)$ which satisfy the condition

$$M\{|\varphi(x)|\} = 0.$$

We call this set a *B. a. p. functional class* (*B. a. p. f. c.*) All functions of a (*B. a. p. f. c.*) are *B. a. p.* functions. Two such classes are either identical or have no function in common. All functions of a *B. a. p. f. c.* have the same Fourier series. Our theorem may be formulated as follows.

Theorem 2. *Any* (*B. a. p. f. c.*) *contains a* $\bar{B}$*. a. p. function.*

Thus on account of this theorem we shall conclude that with respect to Fourier series the classes of *B. a. p.* functions and of $\bar{B}$*. a. p.* functions are identical. We shall give the main idea of the proof without entering into every detail. The proof will be based on the following lemmas.

Let $\psi(x)$ be an ***a. p.*** function and $\bar{E}$ an *a. p.* set of translation numbers τ_i of some function (not necessarily of $\psi(x)$). We denote by the symbol

$$M_i\{\psi(x+\tau_i),\ (\alpha,\beta)\}$$

the mean value of numbers $\psi(x+\tau_i)$ corresponding to all τ_i satisfying the condition

$$\alpha \leqq x+\tau_i \leqq \beta.$$

Lemma 1. $M_i\{\psi(x+\tau_i)\}$ *exists.*

Lemma 2. *The difference*

$$M_i\{\psi(x+\tau_i),\ (\alpha,\beta)\} - M_i\{\psi(x+\tau_i)\} = \varepsilon(x,\alpha,\beta)$$

tends to zero, as $\beta-\alpha\to+\infty$, *uniformly in* x *and* α.

Lemma 3. *To any B. a. p. function* $f_1(x)$ *and to any sequence* $\{\varepsilon_n\}$ *of positive numbers corresponds a series of* ***a. p.*** *functions*

$$\varphi_1(x)+\varphi_2(x)+\varphi_3(x)+\cdots$$

such that

$$M_x\{|f_1(x)-\varphi_1(x)-\varphi_2(x)-\cdots-\varphi_n(x)|\}<\varepsilon_n \tag{67}$$

$$M_x\{|\varphi_n(x)|\}<\varepsilon_n \qquad n>1. \tag{68}$$

Lemma 1 has been proved in the preceding part of the paper. Lemma 3 is quite obvious. The proof of Lemma 2 is similar to those relating to the existence of mean values connected with $a.p.$ sets.

Passing now to the sketch of the proof of Theorem 2 let $f_1(x)$ be a $B.a.p.$ function and

$$\varphi_1(x),\ \varphi_2(x),\ \ldots \tag{69}$$

a series of Lemma 3 corresponding to $f_1(x)$. We assume that *the series* $\Sigma\varepsilon_n$ *is convergent.* We take a sequence of rapidly increasing numbers

$$l_0 = 0 < l_1 < l_2 < \cdots \quad l_n \to \infty. \tag{70}$$

We define a function $f(x)$ by the equations

$$f(x) = \varphi_1(x) + \varphi_2(x) + \cdots + \varphi_n(x) \tag{71}$$

for x belonging to the intervals

$$l_{n-1} \leqq x < l_n, \quad -l_n < x \leqq -l_{n-1}.$$

Let $\{\varepsilon_n'\}$ be another set of positive numbers ($\varepsilon_n' \to 0$, as $n \to \infty$). We study the behaviour of the expression

$$\Phi_n(T) = \frac{1}{2T}\int_{-T}^{+T} |f(x) - \varphi_1(x) - \cdots - \varphi_n(x)|\, dx. \tag{72}$$

It can be shown that if the numbers of (70) increase rapidly enough then for values of T belonging to the interval

$$l_N \leqq T \leqq l_{N+1} \qquad N \geqq n + 1$$

$$|\Phi_n(T) - M\{|\varphi_{n+1}(x) + \cdots + \varphi_N(x) + \theta\varphi_{N+1}(x)|\}| < \varepsilon'_{N+1} \tag{73}$$

where $|\theta| \leqq 1$.[1] Hence

[1] A similar result is proved in detail in Besicovitch's paper [1].

$$(74)\qquad \bar{M}_x\{|f(x)-\varphi_1(x)-\cdots-\varphi_n(x)|\}\leqq\sum_{i=n+1}^{\infty}M_x\{|\varphi_i(x)|\}$$
$$<\sum_{i=n+1}^{\infty}\varepsilon_n.$$

By (67), (74)

$$M_x\{|f_1(x)-f(x)|\}=0.$$

Thus $f(x)$ is a B. a. p. function of the same functional class as $f_1(x)$.

We study now the behaviour of the expression

$$(75)\qquad \Psi(x,T)=M_i\{|f(x)-f(x+\tau_i)|,\ (-T,+T)\}.$$

In a way similar to that which had to be employed for the proof of (73), and on account of Lemmas 1 and 2, it can be shown that, if the numbers of (70) increase rapidly enough, then for values of T in the interval (l_n, l_{n+1}),

$$(76)\qquad |\Psi(x,T)-M_i\{|f(x)-\varphi_1(x+\tau_i)-\cdots-\varphi_n(x+\tau_i)|,\ (-T,+T)\}|$$
$$<\varepsilon'_{n+1}+M_i\{|\varphi_{n+1}(x+\tau_i)|\}.$$

We write (76) for

$$l_N\leqq T\leqq l_{N+1}\qquad (N>n+1)$$

and we conclude that

$$(77)\qquad |\Psi(x,T)-M_i\{|f(x)-\varphi_1(x+\tau_i)-\cdots-\varphi_n(x+\tau_i)|,\ (-T,+T)\}|$$
$$<\varepsilon'_{N+1}+M_i\{|\varphi_{n+1}(x+\tau_i)+\cdots+\varphi_N(x+\tau_i)|,\ (-T,+T)\}$$
$$+M_i\{|\varphi_{N+1}(x+\tau_i)|\}.$$

A certain rapidity of increase of the numbers of (70) can secure the inequality

$$(78)\qquad M_i\{|\varphi_{n+1}(x+\tau_i)+\cdots+\varphi_N(x+\tau_i)|,\ (-T,+T)\}<$$
$$<2M_i\{|\varphi_{n+1}(x+\tau_i)|\}+\cdots+2M_i\{|\varphi_N(x+\tau_i)|\}$$

for any n, N and $T\subset(l_N, l_{N+1})$.

Thus

$$(79)\qquad |\Psi(x,T)-M_i\{|f(x)-\varphi_1(x+\tau_i)-\cdots-\varphi_n(x+\tau_i)|,\ (-T,+T)\}|$$
$$<\varepsilon'_{N+1}+2M_i\{|\varphi_{n+1}(x+\tau_i)|\}+\cdots+2M_i\{|\varphi_{N+1}(x+\tau_i)|\}.$$

Suppose that for a given value of x the series

(80) $$M_i\{|\varphi_1(x+\tau_i)|\} + M_i\{|\varphi_2(x+\tau_i)|\} + \cdots$$

converges. Then, as the second term of the left hand side expression of (79) tends to a limit, as $T\to\infty$, it is not difficult to conclude that $\Psi(x, T)$ *tends to a limit, as* $T\to\infty$.

Take now an interval $(d, d+1)$ of values of x. Consider the question on convergence of the series (80). We have

(81) $$\int_d^{d+1} M_i\{|\varphi_n(x+\tau_i)|\}\,dx \leqq \underline{M}_i \int_d^{d+1} |\varphi_n(x+\tau_i)|\,dx < LM_x\{|\varphi_n(x)|\} < L\varepsilon_n$$

where L denotes the maximum distance between two consecutive τ_i. Denoting by $s_j(x)$ the sum of first j terms of (80) we conclude on account of (81)

(82) $$\int_d^{d+1} s_j(x)dx < L[M_x\{|\varphi_1(x)|\} + \varepsilon_2 + \varepsilon_3 + \cdots] \text{ for any } j.$$

The series $\Sigma\varepsilon_n$ being convergent we conclude that the series (80) converges for almost all values of x, and consequently the limit of $\Psi(x, T)$, as $T\to\infty$, exists for almost all values of T, i. e.

(83) $$M_i\{|f(x)-f(x+\tau_i)|\}$$

exists for almost all values of x.

In Chapter V we have proved that if $f(x)$ is a $B.a.p.$ function then to any $\varepsilon > 0$ corresponds an $a.p.$ set of numbers τ_i such that the condition

(84) $$\overline{M}_x \overline{M}_i \frac{1}{c}\int_x^{x+c} |f(x)-f(x+\tau_i)|\,dx < \varepsilon$$

for all $c > 0$ is satisfied.

On account of the existence of mean value (83) we can apply Fatou's theorem to the inner mean value of (84). We have

(85) $$\frac{1}{c}\int_x^{x+c} M_i\,|f(x)-f(x+\tau_i)|\,dx \leqq \bar{M}_i\frac{1}{c}\int_x^{x+c}|f(x)-f(x+\tau_i)|\,dx.$$

By (84)

$$\bar{M}_x\frac{1}{c}\int_x^{x+c} M_i\,|f(x)-f(x+\tau_i)|\,dx<\varepsilon$$

and consequently[1]

(86) $$\bar{M}_x M_i\,|f(x)-f(x+\tau_i)|<\varepsilon_1$$

which proves that the function $f(x)$ is $\bar{B}.a.p.$ Thus Theorem 2 has been proved.

Example of a $B.a.p.$ Function which is not $\bar{B}.a.p.$

Let a, b, l be positive numbers such that $m=\frac{l}{a+b}$ is an integer. We define the function φ in the following way

(1) $$\begin{aligned}\varphi(x,l,a,b,c) &= c \qquad k(a+b)\leqq x<k(a+b)+a \quad k=0,1,\ldots,m-1\\ &= 0 \qquad \text{for all other values of } x.\end{aligned}$$

Define now the function

(2) $$f(x)=\sum_{n=1}^{\infty}\varphi(x-2^n,\, l_n,\, a_n,\, b_n,\, c_n)$$

where

(3) $l_n=[\sqrt{n}]$ (integral part of $\sqrt{n}$), $\frac{1}{a_n+b_n}=2^{2^n}$, $\frac{a_n}{a_n+b_n}=\frac{1}{n}$, $c_n=2^n\sqrt[4]{n}$.

We have

$$M_x\{f(x)\}=0$$

and thus $f(x)$ *is* a $B.a.p.$ function.

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[1] If $p(x)\geqq 0$ then $\bar{M}_x\left\{\frac{1}{c}\int_x^{x+c}p(x)dx\right\}=\bar{M}_x\{p(x)\}$.

Let now

$$(4) \qquad 0 < \tau_1 < \tau_2 < \cdots$$

be an arbitrary »satisfactorily uniform» (and consequently relatively dense) sequence of numbers.[1]

Then there exists n_0' such that for all $n \geqq n_0'$

$$(5) \qquad nk \leq \tau_n < 2nk$$

where k is some positive members. We shall prove that

$$(6) \qquad \int_d^{d+1} \bar{M}_i |f(x) - f(x+\tau_i)| \, dx = \infty$$

for any value of d.

Let l be a positive number such that any interval of length l contains at least one of the numbers (4).

Let d have a fixed value and let A be an arbitrarily large positive number. Choose $n_0 \geqq n_0'$ such that

$$(7) \qquad \text{i)} \max_{d \leqq x \leqq d+1} |f(x)| + A < \sqrt[4]{n_0}$$

$$(8) \qquad \text{ii)} \quad l_{n_0} = [\sqrt{n_0}] > l + 1 .$$

Then each interval $(2^n, 2^n + l_n)$ for $n \geqq n_0$ contains at least one of the intervals $(\tau_i + d, \tau_i + d + 1)$; denote such an interval (or one of them, if there is more than one) by

$$(\tau_{i_n} + d, \ \tau_{i_n} + d + 1).$$

Let $n_1 \geqq n_0$ and $n_1 > 2$. Consider the values of the functions

$$(9) \qquad f(x + \tau_{i_n}) \text{ for } n = n_1, n_1 + 1, \ldots, 8n_1 - 1$$

in the interval $(d, d+1)$. In the interval $(d, d+1)$ the function $f(x + \tau_{i_n})$ takes the same values as the function

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[1] The meaning of terms »satisfactorily uniform», »relatively dense» for one sided sequences like (4) is clear. Obviously when (6) has been proved for any one-sided s.u. set, it is also proved for any two-sided s.u. set of numbers τ_i.

$$\varphi(x, l_n, a_n, b_n, c_n)$$

does in a certain interval of length 1 belonging to $(0, l_n)$ and thus $f(x + \tau_{i_n})$ is equal either to 0 or to $2^n \sqrt[4]{n}$. Denote by $\mathfrak{E}^{(1)}$ the set of values of x in the interval $(d, d+1)$ for which at least one of the functions (9) differs from zero. We shall prove that

$$\lambda = m\,\mathfrak{E}^{(1)} \geqq \frac{1}{2}. \tag{10}$$

Let $\mathfrak{E}_n$ denote the set of values of x in $(d, d+1)$ for which $f(x + \tau_{i_n}) \neq 0$. Then

$$\mathfrak{E}^{(1)} = \mathfrak{E}_{n_1} + \mathfrak{E}_{n_1+1} + \cdots + \mathfrak{E}_{8n_1-1}. \tag{11}$$

We have

$$m\,\mathfrak{E}_n = \frac{1}{n}.$$

$\mathfrak{E}_n$ consists either of 2^{2^n} equal intervals or of $2^{2^n} + 1$ intervals of which all interior intervals are equal to one another and the sum of the two extreme intervals is equal to the length of an interior interval. Define now

$$m\,[\mathfrak{E}_{n+1} \times (\mathfrak{E}_{n_1} + \mathfrak{E}_{n_1+1} + \cdots + \mathfrak{E}_n)]. \tag{12}$$

The set

$$\mathfrak{E}_{n_1} + \mathfrak{E}_{n_1+1} + \cdots + \mathfrak{E}_n \tag{13}$$

is a set of non overlapping intervals, whose number is $\leqq$

$$2^{2^{n_1}} + 2^{2^{n_1+1}} + \cdots + 2^{2^n} < 2 \,.\, 2^{2^n}. \tag{14}$$

Now if δ is any interval, then it is easy to see that

$$m\,[\mathfrak{E}_{n+1} \times \delta] \leqq \frac{\delta\, a_{n+1}}{a_{n+1} + b_{n+1}} + a_{n+1}. \tag{15}$$

Therefore

$$m\,[\mathfrak{E}_{n+1} \times (\mathfrak{E}_{n_1} + \mathfrak{E}_{n_1+1} + \cdots + \mathfrak{E}_n)] < \frac{[m\,(\mathfrak{E}_{n_1} + \mathfrak{E}_{n_1+1} + \cdots + \mathfrak{E}_n)]\, a_{n+1}}{a_{n+1} + b_{n+1}} + 2 \,.\, 2^{2^n} a_{n+1} \tag{16}$$

and by (11)

$$< \frac{\lambda a_{n+1}}{a_{n+1}+b_{n+1}} + \frac{2 \cdot 2^{2^n}}{(n+1)\, 2^{2^{n+1}}} < \frac{\lambda}{n+1} + \frac{1}{2^{2^n}} \cdot$$

We have

$$(17) \qquad m\,[\mathfrak{E}_{n_1} + \mathfrak{E}_{n_1+1} + \cdots + \mathfrak{E}_n + \mathfrak{E}_{n+1}] =$$

$$= m\,[\mathfrak{E}_{n_1} + \mathfrak{E}_{n_1+1} + \cdots + \mathfrak{E}_n] + m\,\mathfrak{E}_{n+1}$$

$$- m\,[\mathfrak{E}_{n+1} \times (\mathfrak{E}_{n_1} + \mathfrak{E}_{n_1+1} + \cdots + \mathfrak{E}_n)]$$

$$> m\,[\mathfrak{E}_{n_1} + \mathfrak{E}_{n_1+1} + \cdots + \mathfrak{E}_n] + \frac{1-\lambda}{n+1} - \frac{1}{2^{2^n}}$$

$$> (1-\lambda)\left(\frac{1}{n_1} + \frac{1}{n_1+1} + \cdots + \frac{1}{n+1}\right) - \frac{1}{2^{2^{n_1}}} - \frac{1}{2^{2^{n_1+1}}} - \cdots - \frac{1}{2^{2^n}}$$

$$> (1-\lambda)\left(\frac{1}{n_1} + \frac{1}{n_1+1} + \cdots + \frac{1}{n+1}\right) - \frac{2}{2^{2^{n_1}}} \cdot$$

Putting in (17) $n+1 = 8n_1 - 1$ we shall have $\left(\text{observing that } \frac{1}{n_1} + \frac{1}{n_1+1} + \cdots + \frac{1}{8n_1-1} > 2\right)$

$$(18) \qquad \lambda > 2(1-\lambda) - \frac{1}{2}$$

which shows that $\lambda > \frac{1}{2} \cdot$

Now let x be any point of $\mathfrak{E}_n$ for $n \geqq n_0$. We have

$$f(x+\tau_{i_n}) = 2^n \sqrt[4]{n} \geqq 2^n \sqrt[4]{n_0}$$

$$|f(x) - f(x+\tau_{i_n})| \geqq 2^n \sqrt[4]{n_0} - \operatorname*{Max}_{d \leqq x \leqq d+1} |f(x)| \geqq 2^n (\sqrt[4]{n_0} - \operatorname*{Max}_{d \leqq x \leqq d+1} |f(x)|)$$

and by (7)

$$(19) \qquad |f(x) - f(x+\tau_{i_n})| > 2^n A$$

and a fortiori

$$\sum_{i=1}^{i=i_n} |f(x) - f(x+\tau_i)| > 2^n A$$

whence

$$\frac{\sum_{i=1}^{i=i_n} |f(x) - f(x+\tau_i)|}{i_n} > \frac{2^n k}{i_n k} A.$$

By (5)

$$(20) \qquad \frac{\sum_{i=1}^{i=i_n} |f(x) - f(x+\tau_i)|}{i_n} > \frac{2^n k}{\tau_{i_n}} A > \frac{2^n k}{2^n + [\sqrt{n}]} A > \frac{2^n k}{2 \cdot 2^n} A = \frac{k}{2} A.$$

Denote by $M(x, n', n'')$ the maximum of

$$\frac{\sum_{i=1}^{i=n} |f(x) - f(x+\tau_i)|}{n}$$

as n varies from $i_{n'}$ to $i_{n''-1}$. We conclude from (20) that

$$(21) \qquad M(x, n_1, 8n_1) > \frac{k}{2} A$$

for all values of x belonging to $\mathfrak{E}^{(1)}$.

We take now a sequence of integers

$$n_1, \; n_2, \; n_3, \; \ldots$$

such that $n_2 \geqq 8n_1$, $n_3 \geqq 8n_2, \ldots$

Write

$$(22) \qquad \mathfrak{E}^{(j)} = \mathfrak{E}_{n_j} + \mathfrak{E}_{n_j+1} + \cdots + \mathfrak{E}_{8n_j-1} \qquad (j = 1, 2, 3, \ldots).$$

At any point x of $\mathfrak{E}^{(j)}$

$$(23) \qquad M(x, n_j, 8n_j) > \frac{k}{2} A.$$

Let $\mathfrak{E}$ be the upper limit of the sequence of sets

$$(24) \qquad \mathfrak{E}^{(1)}, \; \mathfrak{E}^{(2)}, \; \ldots$$

i. e. the set of points each of which belongs to an infinite number of sets of (24). Then

(25) $$m\mathfrak{E} \geqq \frac{1}{2}\cdot$$

Let x be a point of $\mathfrak{E}$ and let

$$\mathfrak{E}^{(j_1)},\ \mathfrak{E}^{(j_2)},\ \ldots$$

be the sets which contain x; then for any $s = 1, 2, 3, \ldots$

$$M(x, n_{j_s}, 8n_{j_s}) > \frac{k}{2}A$$

i. e.

$$\frac{\sum\limits_{i=1}^{i=n} |f(x) - f(x+\tau_i)|}{n} > \frac{k}{2}A$$

for at least one n in any interval $i_{n_{j_s}} \leqq n \leqq i_{8n_{j_s}-1}$, which proves that

(26) $$\overline{M}_i\{|f(x) - f(x+\tau_i)|\} \geqq \frac{k}{2}A.$$

By (25) and (26)

(27) $$\int\limits_d^{d+1} \overline{M}_i\{|f(x) - f(x+\tau_i)|\}\,dx \geqq \frac{k}{4}A.$$

A being arbitrary we have

(28) $$\int\limits_d^{d+1} \overline{M}_i|f(x) - f(x+\tau_i)|\,dx = \infty,$$

and thus

$$M_x \overline{M}_i\{|f(x) - f(x+\tau_i)|\} = \infty$$

for any satisfactorily uniform set of numbers $\tau_1, \tau_2, \ldots$, which proves that $f(x)$ is not a $\overline{B}.a.p.$ function.

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