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Mutually stationary sequences of sets and the non-saturation of the non-stationary ideal on $P_\varkappa(\lambda)$

by

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1. Introduction and preliminary remarks

An important combinatorial property of an ideal I on a set Z is the *saturation* of the ideal, i.e. the least upper bound on the cardinality of a well-ordered chain in the completion of the Boolean algebra $P(Z)/I$. This is particularly interesting in the case where I is a naturally defined ideal, such as the non-stationary ideal.

It is known to be consistent for the non-stationary ideal on ω_1 to be ω_2-saturated. This was shown first from strong determinacy hypotheses by Steel and Van Wesep [SV], and later from large cardinals by Foreman, Magidor and Shelah [FMS]. Shelah has the optimal result, showing this property consistent relative to the existence of a Woodin cardinal.

Shelah has shown that the non-stationary ideal on a successor cardinal $\varkappa>\omega_1$ can never be saturated, by showing that any saturated ideal on a successor cardinal must concentrate on a critical cofinality. Further, Gitik and Shelah (extending earlier work of Shelah) have shown that for $\varkappa$ a successor of a singular cardinal, the non-stationary ideal cannot be $\varkappa^+$-saturated even when restricted to the critical cofinality. They have also shown that the non-stationary ideal on an inaccessible cardinal can never be saturated.

Similar questions arise for the non-stationary ideal on $P_\varkappa(\lambda)$. Burke and Matsubara, using work of Cummings, Gitik and Shelah, were able to establish that the non-stationary ideal on $P_\varkappa(\lambda)$ is not λ^+-saturated in the cases where $\mathrm{cof}(\lambda)\neq\varkappa$, and when $\varkappa>\omega_1$ is a

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successor cardinal. Shelah established the non-saturation of the non-stationary ideal on $P_{\omega_1}(\mu^+)$ for those μ where $\mu^\omega=\mu$.

The main result of this paper is that the non-stationary ideal on $P_\varkappa(\lambda)$ cannot be saturated unless $\varkappa=\lambda=\omega_1$. We give a complete proof of this theorem in the paper.

The structure of the paper is as follows: In §2 we state important results of Shelah and Cummings that we will use in the sequel. For completeness we use these results to prove some of the theorems of Burke and Matsubara [BMt].

In §3, we handle the case of $\varkappa=\omega_1$. In this section we define the notion of *mutually stationary sets*, which we believe to be of independent interest. The crux of §3 is the result that every sequence of stationary sets of points of cofinality ω is mutually stationary.

In §4, we show that if the cofinality of λ is less than $\varkappa$, the non-stationary ideal on $P_\varkappa(\lambda)$ is not even λ^{++}-saturated. These results strengthen the results of Burke and Matsubara.

In §5, we show the non-saturation result in the case where $\varkappa$ is a regular limit cardinal.

In §6, we will give examples showing that our results are sharp, or nearly sharp in several cases. For example, we force over L to give the consistency of the statement that the non-stationary ideal on $P_\varkappa(\lambda)$ is $|P_\varkappa(\lambda)|$-saturated. Thus one cannot always prove that there is an antichain in $P_\varkappa(\lambda)$ modulo the non-stationary ideal that has size $[\lambda]^{<\varkappa}$.

In §7, we discuss mutually stationary sets abstractly, prove some splitting results and give an example in L of a sequence of stationary subsets of the ω_n's that is not mutually stationary.

The rest of this section is devoted to a cursory discussion of background information assumed by the paper and of the notation used in the paper.

For the most part the notation of this paper is standard. We denote the ordinals of cofinality $\varkappa$ by $\operatorname{cof}(\varkappa)$, and the cofinality of a particular ordinal α by $\operatorname{cf}(\alpha)$. We will use the abbreviation NS for the non-stationary ideal, usually on $P_\varkappa(\lambda)$, the collection of subsets of λ of cardinality less than $\varkappa$. (We will use the definition of stationarity given by Jech in [J].) We adopt the convention that, unless we state otherwise, $x\in P_\varkappa(\lambda)$ implies that $x\cap\varkappa\in\varkappa$. We will also concentrate our attention, without further notice, on those x where $|x|=|x\cap\varkappa|$.

An important fact for this paper is that the non-stationary sets in $P_\varkappa(\lambda)$ form a *tower* as λ ranges over cardinals greater than or equal to $\varkappa$. In other words, if $\lambda<\lambda'$ and $\pi\colon P_\varkappa(\lambda')\to P_\varkappa(\lambda)$ by $\pi(x)=x\cap\lambda$, then π induces a map from $P(P_\varkappa(\lambda'))$ to $P(P_\varkappa(\lambda))$ which we will also call π in an abuse of notation. The assertion that the stationary sets form a tower is the statement that for all stationary $S\subset P_\varkappa(\lambda')$, $\pi(S)$ is a stationary subset of $P_\varkappa(\lambda)$; and that for all stationary $T\subset P_\varkappa(\lambda)$, $\pi^{-1}(T)$ is a stationary subset of $P_\varkappa(\lambda')$.

The map $\pi^{-1}: P(P_\varkappa(\lambda)) \to P(P_\varkappa(\lambda'))$ preserves order and incompatibility, although, as we show, it is never a neat embedding. Thus we conclude that the saturation of the non-stationary ideal on $P_\varkappa(\lambda)$ is a monotone function of λ.

We will frequently casually refer to "$H(\theta)$", without much further explanation. This is in the spirit of Shelah's *Proper Forcing.* The structure $H(\theta)$ refers to a structure in a (usually) countable language whose domain is the collection of sets of hereditary cardinality less than θ for some regular cardinal θ, large enough to contain all of the relevant information. We assume that the structure had $\in$ in the language as well as a well-ordering Δ of $H(\theta)$. We will also assume that all of the relevant sets in the argument at hand are named in $H(\theta)$.

Without mention, we will use the fact that closed unbounded subsets of $P_\varkappa(\lambda)$ are determined by algebras on λ, where an *algebra* on λ is a structure $\langle \lambda, f_i \rangle_{i\in\omega}$. Using Skolem functions we can also take closed unbounded sets to be determined by any algebra on a set X with $\lambda \subset X$. We frequently will be considering closed unbounded sets $\{N \cap \lambda : N \prec H(\theta)\}$.

For the purposes of this paper, *inaccessible* means weakly inaccessible.

If I is an ideal on a set Z, then we can form the Boolean algebra $P(Z)/I$ and force with it. If G is the resulting ultrafilter, then we can form the ultraproduct of V by G, using functions lying in V. The result V^Z/G is occasionally well-founded. In this case the ideal I is called *precipitous.* If I is precipitous, then we automatically replace V^Z/G by its transitive collapse M. We then have a canonical elementary embedding $j: V \to M$.

It is well known that if a normal ideal I on $P_\varkappa(\lambda)$ is λ^+-saturated, then it is precipitous, and moreover, the generic ultrapower M is closed under λ-sequences (e.g. $M^\lambda \cap M = M^\lambda \cap V[G]$). Further, if $S \notin I$ and $S \Vdash \dot{x} \in M$, there is a function $f: P_\varkappa(\lambda) \to V$, $f \in V$, such that $S \Vdash [f]_M = \dot{x}$.

The notion of an internally approachable set is quite useful in this paper. It was first formally defined in [FMS] and exposited in some detail in [FM1]. We give the definition here: a set N is said to be *internally approachable* of length μ (or in $\mathrm{IA}(\mu)$) if and only if there is a sequence $\langle N_\alpha : \alpha < \mu \rangle$ such that $N = \bigcup_{\alpha<\mu} N_\alpha$, and for all $\beta < \mu$, $\langle N_\alpha : \alpha < \beta \rangle \in N$.

If $N \prec \langle H(\theta), \in, \Delta \rangle$ is internally approachable of length an ordinal of uncountable cofinality, then $N \cap OR$ is countably closed. Suppose now that N and M are internally approachable of length some uncountable regular cardinal μ, and $N \cap \lambda = M \cap \lambda$ and $\sup(N \cap \lambda^+) = \sup(M \cap \lambda^+)$. We must have that $M \cap N$ is cofinal in $M \cap \lambda^+$, and vice versa. Since M and N are elementary substructures of $H(\theta)$, we deduce that $M \cap \lambda^+ = N \cap \lambda^+$.

PCF Theory. In §§4 and 5 we will make heavy use of the "PCF" theory of Shelah. The primary reference for this theory is Shelah's book [Sh2]; however, [BMg] gives an expository account.

A basic notion in the PCF theory is that of a scale. If λ is a singular cardinal and λ_i ($i\in\mathrm{cf}(\lambda)$) is an increasing cofinal sequence of regular cardinals, and I is an ideal on $\mathrm{cf}(\lambda)$, a *scale* in $\prod_{i\in\mathrm{cf}(\lambda)}\lambda_i/I$ is a sequence $\langle f_\alpha:\alpha<\eta\rangle$ such that for $\alpha<\alpha'$, $\{i:f_\alpha(i)\geqslant f_{\alpha'}(i)\}\in I$ (i.e. the sequence is *increasing*) and for all $g\in\prod_{i\in\mathrm{cf}(\lambda)}\lambda_i$, there is an α, $\{i:g(i)\geqslant f_\alpha(i)\}\in I$ (i.e. the sequence is cofinal).

One of the main results of Shelah is that for all singular cardinals λ there is a cofinal sequence $\langle\lambda_i:i\in\mathrm{cf}(\lambda)\rangle$ such that if I is the ideal of bounded sets on $\mathrm{cf}(\lambda)$ then there is a scale in $\prod_{i\in\mathrm{cf}(\lambda)}\lambda_i/I$ of length λ^+. In this case we will say that $\prod_{i\in\mathrm{cf}(\lambda)}\lambda_i/I$ has *true cofinality* λ^+.

Given a sequence of functions $\langle f_\alpha:\alpha<\eta\rangle$, a function g is called an *exact upper bound* for this sequence if and only if for all $\alpha<\eta$, $\{i:f_\alpha(i)\geqslant g(i)\}\in I$, and if $h\in\prod_{i\in\mathrm{cf}(\lambda)}\lambda_i$ is such that $\{i:h(i)\geqslant g(i)\}\in I$, then there is an α, $\{i:h(i)\geqslant f_\alpha(i)\}\in I$. Note that if there is an exact upper bound then its I-equivalence class is unique.

A scale $\langle f_\alpha:\alpha<\eta\rangle$ is called *continuous* if and only if for all β whenever there is an exact upper bound for $\langle f_\alpha:\alpha<\beta\rangle$, the function f_β is the exact upper bound.

If we are given a scale $\langle f_\alpha:\alpha<\eta\rangle$, a point β is *good* if and only if there is a set $B=\{h_\xi:\xi<\mathrm{cf}(\beta)\}\subset\prod_{i\in\mathrm{cf}(\lambda)}\lambda_i$ and a set $S\in I$ such that

(1) for all $\xi<\eta<\mathrm{cf}(\beta)$, $i\in\mathrm{cf}(\lambda)\backslash S$, $h_\xi(i)<h_\eta(i)$,

(2) for all $h\in B$ there is an α such that $\{i:h(i)\geqslant f_\alpha(i)\}\in I$,

(3) for all $\alpha<\beta$ there is an $h\in B$ such that $\{i:f_\alpha(i)\geqslant h(i)\}\in I$.

If I is generated by less than $\mathrm{cf}(\beta)$ sets this is equivalent to the statement that there is a cofinal set $A\subset\beta$ and a set $S\in I$ such that for all $j\in\mathrm{cf}(\lambda)\backslash S$, $\langle f_\alpha(j):\alpha\in A\rangle$ is strictly increasing. If the ideal I is $\mathrm{cf}(\lambda)$-complete, then every point of cofinality less than $\mathrm{cf}(\lambda)$ is good. We will assume of all of our scales that if I is $\mathrm{cf}(\lambda)$-complete, and $\mathrm{cf}(\beta)<\mathrm{cf}(\lambda)$, then there is a cofinal subset of β, A such that $f_\beta(i)=\sup\{f_\alpha(i):\alpha\in A\}$ for all $i\in\mathrm{cf}(\lambda)$.

For points β of cofinality larger than $\mathrm{cf}(\lambda)$, being good is equivalent to the statement that there is an exact upper bound g for $\langle f_\alpha:\alpha<\beta\rangle$ such that $\{i:\mathrm{cf}(g(i))\neq\mathrm{cf}(\beta)\}\in I$. Shelah has shown that there is always a stationary set of good points (see [Sh2] or [FM2]).

Given a sequence $\langle\lambda_i\rangle$ cofinal in λ and a set N, we define $\chi_N\in\prod_{i\in\mathrm{cf}(\lambda)}\lambda_i$, by setting $\chi_N(i)=\sup(N\cap\lambda_i)$. This χ_N is called the *characteristic function of* N with respect to the sequence $\langle\lambda_i\rangle$.

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2. Theorems of Shelah, Cummings, Burke and Matsubara

The following theorems, due to Shelah and Cummings, are central to many of the results:

THEOREM 1 (Shelah's Theorem [Sh1]). *Let λ be a regular cardinal and $\mathbf{P}$ a partial ordering that preserves λ^+. Then for all generic $G \subset \mathbf{P}$, $V[G]$ satisfies the statement that* $\mathrm{cf}(|\lambda|) = \mathrm{cf}(\lambda)$.

As a corollary Shelah showed:

THEOREM 2 (Shelah [Sh2]). *Let $\lambda = \varrho^+$ be a regular cardinal. Then for all $\mu \neq \mathrm{cf}(\varrho)$ the non-stationary ideal on λ restricted to cofinality μ is not λ^+-saturated.*

Using Shelah's PCF theory, Cummings found a variation on this theorem for singular cardinals:

THEOREM 3 (Cummings' Theorem [C]). *Suppose that λ is a singular cardinal and $\mathbf{P}$ is a partial ordering that preserves stationary subsets of λ^+. Then for all generic $G \subset \mathbf{P}$, $V[G]$ satisfies the statement that* $\mathrm{cf}(|\lambda|) = \mathrm{cf}(\lambda)$.

(Cummings tells us that this exact version is due to Burke.) We note that any λ^+-chain condition forcing satisfies the hypothesis of both theorems, and hence if the non-stationary ideal on $P_\varkappa(\lambda)$ is saturated, the theorems apply to partial ordering $P(P_\varkappa(\lambda))/\mathrm{NS}$.

For completeness, we now give proofs of the non-saturation in the case where $\mathrm{cf}(\lambda) > \varkappa$ or $\varkappa$ is a successor at least ω_2 and the cofinality of λ is greater than or equal to $\varkappa$. (This is due to Burke and Matsubara.) In the case where $\mathrm{cf}(\lambda) < \varkappa$ we prove in §3 the more general and stronger result that the non-stationary ideal is not even λ^{++}-saturated.

The following lemma is standard (see [Ba] or [FMS] for a proof):

LEMMA 4. *Let $\varkappa$ and λ be cardinals with $\varkappa$ regular.*

(1) *Suppose that $\mathrm{cf}(\lambda) > \varkappa$, and that μ, ν are regular cardinals less than $\varkappa$. Let*

$$S = \{x \in P_\varkappa(\lambda) : |x| = |x \cap \varkappa|,\ \mathrm{cf}(x \cap \varkappa) = \mu \text{ and } \mathrm{cf}(\sup x) = \nu\}.$$

Then S is stationary.

(2) *Suppose that $\varkappa = \varrho^+ \geqslant \omega_2$ and $\mathrm{cf}(\lambda) \geqslant \varkappa$. Then*

$$S = \{x \in P_\varkappa(\lambda) : \mathrm{cf}(x \cap \varkappa) = \mathrm{cf}(\sup x) \neq \mathrm{cf}(\varrho)\}$$

is stationary.

We can now derive many of Burke and Matsubara's results from the lemma and the theorems of Shelah and Cummings.

We begin by remarking that the standard theory of saturated ideals [F] shows that when one forces with a normal λ^+-saturated ideal on $P_\varkappa(\lambda)$ below $S=\{x\in P_\varkappa(\lambda): |x|=|x\cap\varkappa|, \operatorname{cf}(x\cap\varkappa)=\mu \text{ and } \operatorname{cf}(\sup x)=\nu\}$, to get a model $V[G]$, then

$$V[G]\models|\lambda|=|\varkappa|, \quad \operatorname{cf}(\varkappa)=\mu \quad \text{and} \quad \operatorname{cf}(\lambda)=\nu.$$

To see this let $j: V\to M\subset V[G]$ be the generic elementary embedding. Since we forced below S, we know that M satisfies the statements that $|\lambda|=\varkappa$ and $\nu=\operatorname{cf}(\lambda)\neq\operatorname{cf}(\varkappa)=\mu$. Because M is closed under λ-sequences from $V[G]$, we see that $V[G]$ satisfies these statements as well. We note that if $\varkappa$ is a limit cardinal and the critical point of j is $\varkappa$ we know that $\varkappa$ is a cardinal in M.

Further, since the forcing is λ^+-saturated, it preserves stationary subsets of λ^+.

In the case where the cofinality of λ is greater than $\varkappa$, and $\varkappa$ is a regular limit, we choose distinct regular $\mu,\nu<\varkappa$ and force with the stationary set described in part (1) of the lemma. This contradicts the theorems of Shelah and Cummings.

If $\varkappa$ is the successor of $\varrho\geqslant\omega_1$ and the cofinality of $\lambda\geqslant\varkappa$, then we force below the stationary set defined in (2). Then all cardinals and cofinalities less than or equal to ϱ are preserved. In this case we see that in M (and hence in $V[G]$), $|\lambda|=\varrho$, but $\operatorname{cf}(\lambda)\neq\operatorname{cf}(\varrho)$. Again this contradicts the results of Shelah and Cummings.

This shows the Burke–Matsubara results in these cases.

3. $\varkappa=\omega_1$

In this section we introduce the notion of mutually stationarity and use it to show that the non-stationary ideal on $P_\varkappa(\lambda)$ is never saturated in the case where $\varkappa=\omega_1$ and $\lambda>\omega_1$. See §6 for more results about mutually stationary sets.

Following Burke and Matsubara, we first dispose of the case where λ is a regular cardinal bigger than ω_1.

We will use the following special case of a theorem of Gitik and Shelah [GS].

THEOREM 5 (Gitik–Shelah). *Suppose that λ is a regular cardinal. Then the non-stationary ideal on λ restricted to points of cofinality ω is not λ^+-saturated.*

Thus there is an antichain $\mathcal{A}$ of cardinality λ^+ consisting of stationary subsets of λ of points of cofinality ω. Standard arguments then show that for all $A\in\mathcal{A}$, $Y_A=\{x\in P_\varkappa(\lambda):\sup x\in A\}$ is stationary. Hence, $\{Y_A: A\in\mathcal{A}\}$ is an antichain in $P_\varkappa(\lambda)$ (modulo the non-stationary ideal) that has cardinality λ^+.

We now turn to the case where λ is singular. We begin with a general definition.

Definition 6. Let K be a collection of regular cardinals with supremum δ, and suppose that we have $S_\varkappa \subset \varkappa$ for each $\varkappa \in K$. Then the collection of sets $\{S_\varkappa : \varkappa \in K\}$ is mutually stationary if and only if for all algebras $\mathfrak{A}$ on δ there is an $N \prec \mathfrak{A}$ such that

$$\text{for all } \varkappa \in N \cap K, \quad \sup N \cap \varkappa \in S_\varkappa.$$

This definition has obvious equivalents; for example, we could equivalently require that for all algebras $\mathfrak{A}$ expanding any $H(\theta)$ where $\theta \geqslant \delta$, there is an $N \prec \mathfrak{A}$ satisfying the condition. Standard "proper forcing"-type tricks show that this is also equivalent to the existence of a single elementary substructure of $\langle H(\theta), \in, \{S_\varkappa : \varkappa \in K\}, \Delta\rangle$ (for some $\theta \gg 2^\delta$), satisfying the condition.

Note that being mutually stationary implies that every $S_\varkappa$ is stationary. Further if every $S_\varkappa$ is closed and unbounded, then the sequence is mutually stationary. The main theorem of this section is that every sequence of stationary sets of points of countable cofinality is mutually stationary.

THEOREM 7. *Let $\langle \varkappa_\alpha : \alpha \in \gamma\rangle$ be an increasing sequence of regular cardinals. Let $\langle S_\alpha : \alpha \in \gamma\rangle$ be a sequence of stationary sets such that $S_\alpha \subset \varkappa_\alpha$ and S_α consists of points of countable cofinality. If $\lambda = \sup_{\alpha<\gamma} \varkappa_\alpha$ and $\mathfrak{A}$ is an algebra on λ, then there is a countable $N \prec \mathfrak{A}$ such that for all $\alpha \in N$, $\sup N \cap \varkappa_\alpha \in S_\alpha$.*

COROLLARY 8. *Suppose that λ is a singular cardinal of cofinality μ. Then the non-stationary ideal on $P_\varkappa(\lambda)$ is not λ^μ-saturated.*

Proof of Corollary 8. Choose an increasing cofinal sequence $\langle \varkappa_\alpha : \alpha < \mu\rangle$ of regular cardinals in λ. For each α, divide the points of countable cofinality in $\varkappa_\alpha$ into $\varkappa_\alpha$ disjoint stationary sets, $\langle S^\alpha_\beta : \beta < \varkappa_\alpha\rangle$. For each function $f \in \prod_{\alpha<\mu} \varkappa_\alpha$, let $S_f = \{N \in P_{\omega_1}(\lambda) :$ for all $\alpha \in N, \sup N \cap \varkappa_\alpha \in S^\alpha_{f(\alpha)}\}$. By the theorem, each S_f is stationary in $P_{\omega_1}(\lambda)$. Further if $f \neq g$ then for any α with $f(\alpha) \neq g(\alpha)$, $\{N : N \in S_f \cap S_g$ and $\alpha \in N\}$ is empty. Hence the sets $\{S_f : f \in \prod_{\alpha<\mu} \varkappa_\alpha\}$ form an antichain in $P_{\omega_1}(\lambda)$ of cardinality λ^μ. □

Remark. In fact, if $\langle \varkappa_\alpha : \alpha < \gamma\rangle$ is any increasing sequence of cardinals cofinal in λ such that the non-stationary ideal on $\varkappa_\alpha$ restricted to points of countable cofinality is not λ_α-saturated, then the non-stationary ideal on $P_{\omega_1}(\lambda)$ is not $\prod_{\alpha<\gamma} \lambda_\alpha$-saturated.

Proof of Theorem 7. Fix an algebra $\mathfrak{A}$ on λ. Without loss of generality we can assume that $\mathfrak{A}$ codes all operations on λ definable in $\langle H(\lambda), \in, \Delta\rangle$ and has a predicate for $\{(\alpha, \varkappa_\alpha) : \alpha < \gamma\}$. Let $\mathfrak{T} \subset \lambda^{<\omega}$ be a tree. We will take $\mathfrak{T}$ so that we can label the nodes of $\mathfrak{T}$ with a function l from $\mathfrak{T}$ to $\{\varkappa_\alpha : \alpha < \gamma\}$ constructed so that:

(1) if $\sigma \in \mathfrak{T}$ and $l(\sigma) = \varkappa_\alpha$, then $\{\gamma : \sigma^\frown\gamma \in \mathfrak{T}\} \subset \varkappa_\alpha$ and has cardinality $\varkappa_\alpha$,

(2) if $\sigma \in \mathfrak{T}$ and $\varkappa_\alpha \in \operatorname{sk}^{\mathfrak{A}}(\sigma)$, then there are infinitely many $n \in \omega$ such that if $\tau \supset \sigma$, $\tau \in \mathfrak{T}$ has length n, then $l(\tau) = \varkappa_\alpha$.

Assume that $\mathfrak{T}$ is a tree with such a labelling. Let $\mathfrak{T}'$ be a subtree of $\mathfrak{T}$ with stem σ_0. Let $\lambda_1, ..., \lambda_n$ be a finite collection of $\varkappa_\alpha$'s such that each $\lambda_i \in \mathrm{sk}^{\mathfrak{A}}(\sigma_0)$. Then we say that $\mathfrak{T}'$ is an *acceptable* subtree (for $\{\lambda_1, ..., \lambda_n\}$) if and only if for all nodes $\sigma \in \mathfrak{T}'$, if $l(\sigma) \notin \{\lambda_1, ..., \lambda_n\}$ then $\{\gamma : \sigma^\frown\gamma \in \mathfrak{T}\} \subset l(\sigma)$ and has cardinality $l(\sigma)$, and if $l(\sigma) \in \{\lambda_1, ..., \lambda_n\}$ then there is a unique γ such that $\sigma^\frown\gamma \in \mathfrak{T}'$. If $\mathfrak{T}'$ is acceptable for $\{\lambda_1, ..., \lambda_n\}$ and $\varkappa \in \{\lambda_1, ..., \lambda_n\}$ we will say that $\mathfrak{T}'$ is *fixed* for $\varkappa$.

Our goal is to produce a decreasing sequence of acceptable subtrees $\mathfrak{T}_n$ such that the length of the stem of each $\mathfrak{T}_n$ is at least n, and so that:

(1) if $\varkappa_\alpha$ is in the Skolem hull of the stem of one of the $\mathfrak{T}_n$, there is an $m > n$ such that $\mathfrak{T}_m$ is fixed for $\varkappa_\alpha$,

(2) if $\mathfrak{T}_n$ is fixed for $\varkappa_\alpha$, there is a $\beta_\alpha \in S_\alpha$ such that for all branches b through $\mathfrak{T}_n$, $\sup(\mathrm{sk}^{\mathfrak{A}}(b) \cap \varkappa_\alpha) = \beta_\alpha$.

Clearly this suffices, as if b is the intersection of the $\mathfrak{T}_n$ we set $N = \mathrm{sk}^{\mathfrak{A}}(b)$. Then for all $\varkappa_\alpha \in N$ we have $N \cap \varkappa_\alpha = \beta_\alpha \in S_\alpha$.

What remains to prove is that there is such a sequence of $\mathfrak{T}_n$'s. We describe how to pass from $\mathfrak{T}_n$ to $\mathfrak{T}_{n+1}$ given a particular $\varkappa_\alpha$ in the Skolem hull of the stem of $\mathfrak{T}_n$, so that $\varkappa_\alpha$ is fixed for $\mathfrak{T}_{n+1}$, and $\mathfrak{T}_{n+1}$ satisfies (2). Easy "bookkeeping" completes the construction. Thus the following lemma is clearly enough.

LEMMA 9. *Given an acceptable tree $\mathfrak{T}$ for $\{\lambda_1, ..., \lambda_n\}$, and a $\varkappa_\alpha$ in the Skolem hull of the stem of $\mathfrak{T}$, there is a $\beta_\alpha \in S_\alpha$ and a subtree $\mathfrak{T}'$ of $\mathfrak{T}$ that is acceptable for $\{\lambda_1, ..., \lambda_n, \varkappa_\alpha\}$ such that for all branches b through $\mathfrak{T}'$,*

$$\sup(\mathrm{sk}^{\mathfrak{A}}(b) \cap \varkappa_\alpha) = \beta_\alpha.$$

Proof. To show this, for each ordinal $\delta \in \varkappa_\alpha$, we define a game $\mathfrak{G}_\delta$ played on $\mathfrak{T}$. The two players in the game, G (good) and B (bad), will alternate plays determining a branch through the tree $\mathfrak{T}$. At a stage of the game where a node $\sigma \in \mathfrak{T}$ has been determined:

(1) If $l(\sigma) \in \{\lambda_1, ..., \lambda_n\}$ then there is a unique γ such that $\sigma^\frown\gamma \in \mathfrak{T}$. The rules require that B must play this γ.

(2) If $l(\sigma) < \varkappa_\alpha$, B plays a γ so that $\sigma^\frown\gamma \in \mathfrak{T}$.

(3) If $l(\sigma) > \varkappa_\alpha$ (and not one of the λ_i's), then B chooses a subset D of $l(\sigma)$ of cardinality less than $l(\sigma)$, and G chooses an element of $\{\gamma : \sigma^\frown\gamma \in \mathfrak{T}\} \setminus D$.

(4) If $l(\sigma) = \varkappa_\alpha$, then B chooses an ordinal $\beta < \delta$, and G chooses a $\gamma > \beta$ so that $\sigma^\frown\gamma \in \mathfrak{T}$.

In this way the players determine a branch through the tree $\mathfrak{T}$. If either player is unable to play at any stage then that player loses, and if an infinite play of the game determines a branch b, then G wins provided that $\mathrm{sk}^{\mathfrak{A}}(b) \cap \varkappa_\alpha \leqslant \delta$. Note that this game is an open game for B, and hence determined.

CLAIM. *There is a closed unbounded set of $\delta<\varkappa_\alpha$ such that G has a winning strategy in the game $\mathfrak{G}_\delta$.*

Proof. Otherwise, let $S\subset\varkappa_\alpha$ be a stationary set of counterexamples. Since each game is determined, for each $\delta\in S$, we can fix a strategy $\mathcal{S}_\delta$ for B. Let θ be a regular cardinal greater than λ^{+5}, and $N\prec\langle H(\theta),\in,\Delta,\langle\mathcal{S}_\delta:\delta\in S\rangle,\mathfrak{A},\{\lambda_1,...,\lambda_n\},\mathfrak{T},...\rangle$ be such that $|N|<\varkappa_\alpha$, and $N\cap\varkappa_\alpha$ is an ordinal $\delta_0\in S$ (where as usual Δ is a well-ordering of $H(\theta)$).

We now derive a contradiction by exhibiting a play of the game according to $\mathcal{S}_{\delta_0}$ that produces a branch b with $\mathrm{sk}^{\mathfrak{A}}(b)\cap\varkappa_\alpha\leqslant\delta_0$.

Since B's plays are determined by the strategy $\mathcal{S}_{\delta_0}$, we need only describe what G does. Inductively we ensure that for all n, $b\restriction n\in N$ and is a legal play, with B following $\mathcal{S}_\delta$. Since N is closed under finite sequences, it suffices to show that each player plays ordinals that are elements of N.

If $l(\sigma)\in\{\lambda_1,...,\lambda_n\}$, then the unique γ such that $\sigma^\frown\gamma\in\mathfrak{T}$ lies in N.

At a stage where $l(\sigma)<\varkappa_\alpha$, B is supposed to extend some $\sigma\in\mathfrak{T}$ by playing an ordinal γ with $\sigma^\frown\gamma\in\mathfrak{T}$. Since $N\cap\varkappa_\alpha\in\varkappa_\alpha$, any ordinal $\gamma<l(\sigma)$ is an element of N.

Suppose that the play has constructed σ of length n, and that $l(\sigma)>\varkappa_\alpha$. By the induction hypothesis, $l(\sigma)\in N$. Since $\{\mathcal{S}_\delta:\delta\in S\}$ is in N and is a set of size $\varkappa_\alpha$, and the cardinality of $U=\bigcup\{\mathcal{S}_\delta(\sigma):\delta\in S\}$ is less than $l(\sigma)$, we know that $N\models l(\sigma)\setminus U\neq\varnothing$. At this stage G plays an element $\gamma\in N\cap(l(\sigma)\setminus U)$. Since $\gamma\in N$, $\sigma^\frown\gamma\in N$, and $\gamma\notin\mathcal{S}_\delta(\sigma)$.

To handle the last case, suppose now that the play has constructed σ of length n, and $l(\sigma)=\varkappa_\alpha$, and $\mathcal{S}_{\delta_0}$ tells B to play an ordinal $\beta<\delta_0$. Then G plays an arbitrary ordinal γ such that $\sigma^\frown\gamma\in\mathfrak{T}$ and $\beta<\gamma<\delta_0$. (Such an ordinal exists, since $N\models$ "the successors of σ are unbounded in $\varkappa_\alpha$".) Clearly, γ is in N, and hence $\sigma^\frown\gamma\in N$ and is a legal play.

Now suppose that b is the branch through $\mathfrak{T}$ produced in this way, and let $M=\mathrm{sk}^{\mathfrak{A}}(b)$. Then $M\prec N$, and so $\sup M\cap\varkappa_\alpha\leqslant\delta_0$.

This contradicts the fact that b is the result of a play of the game $\mathfrak{G}_{\delta_0}$ according to the strategy $\mathcal{S}_{\delta_0}$. This establishes the claim. $\square$

To finish the proof of the lemma (and hence the proof of Theorem 7) we describe the tree $\mathfrak{T}'$. Choose $\beta_\alpha\in S_\alpha$ such that G has a winning strategy in $\mathfrak{G}_{\beta_\alpha}$. Call this strategy $\mathcal{S}$. Fix a sequence $\langle\delta_m:m\in\omega\rangle$ increasing and cofinal in β_α. We define $\mathfrak{T}'$ by specifying, by induction on the length of $\sigma\in\mathfrak{T}$, whether σ is in $\mathfrak{T}'$. We will assume inductively that each $\sigma\in\mathfrak{T}'$ is the result of a partial play by G according to the strategy $\mathcal{S}$, and that if $l(\sigma)\notin\{\lambda_1,...,\lambda_n,\varkappa_\alpha\}$, then $\{\gamma:\sigma^\frown\gamma\in\mathfrak{T}'\}$ has cardinality $l(\sigma)$.

Suppose that we have put $\sigma\in\mathfrak{T}'$. We now break into cases, according to the rules of the game.

If $l(\sigma)\in\{\lambda_1,...,\lambda_n\}$, then σ has a unique successor in $\mathfrak{T}$, and this is B's only nonlosing play in the game. We put this successor into $\mathfrak{T}'$.

If $l(\sigma)<\varkappa_\alpha$, then we let the successors of σ in $\mathfrak{T}'$ be the same as the successors of σ in $\mathfrak{T}$. Note that in this case it is B's move in the game $\mathfrak{G}_{\beta_\alpha}$, and thus each successor of σ in $\mathfrak{T}$ is a legal play of the game according to the strategy $\mathcal{S}$.

If $l(\sigma)>\varkappa_\alpha$, we must define a collection of $\gamma<l(\sigma)$ of cardinality $l(\sigma)$ so that for each γ, $\sigma^\frown\gamma\in\mathfrak{T}$ and γ is the response by G to a move by B.

To do this, define by induction on $\nu<l(\sigma)$ ordinals $\gamma_\nu\in l(\sigma)$ so that γ_ν is the response by $\mathcal{S}$ to B playing $\{\gamma_{\nu'}:\nu'<\nu\}$ at the stage in the game where σ is played. Then we let the successors of σ in $\mathfrak{T}'$ be $\{\sigma^\frown\gamma_\nu:\nu\in l(\sigma)\}$.

The inductive hypothesis are clearly satisfied in this case.

The final case is where $l(\sigma)=\varkappa_\alpha$. In this case, since we want $\varkappa_\alpha$ to be fixed in $\mathfrak{T}'$, we must put a single successor of σ into $\mathfrak{T}'$. In the game, at this stage, B plays an ordinal $\beta<\varkappa_\alpha$. If the length of σ is m, let γ be G's response to B playing the ordinal δ_m. The only successor of σ in $\mathfrak{T}'$ is $\sigma^\frown\gamma$.

This defines a subtree $\mathfrak{T}'\subset\mathfrak{T}$. We must see that it satisfies the conclusions of Lemma 9. We verified inductively during the construction all of the conclusions except for the claim that if b is a branch through $\mathfrak{T}'$, then $\sup(\mathrm{sk}^{\mathfrak{A}}(b)\cap\varkappa_\alpha)=\beta_\alpha$. This is clear, however, since such a branch b is the result of a play of the game where G follows his strategy $\mathcal{S}$ in the game $\mathfrak{G}_{\beta_\alpha}$. Hence $\sup(\mathrm{sk}^{\mathfrak{A}}(b)\cap\varkappa_\alpha)\leqslant\beta_\alpha$. On the other hand, there are infinitely many m such that $l(b\restriction m)=\varkappa_\alpha$. For each such m, we have that the unique γ with $b\restriction m^\frown\gamma\in\mathfrak{T}'$ is bigger than or equal to δ_m. Since the δ_m are cofinal in β_α, b itself is cofinal in β_α, and hence the equality is verified.

This completes the proof of Theorem 7. □

4. The cofinality of λ is less than $\varkappa$

In this section we consider the case where the cofinality of λ is less than $\varkappa$, and $\varkappa\geqslant\omega_2$. We will show that the non-stationary ideal on $P_\varkappa(\lambda)$ is not even λ^{++}-saturated. These results extend the work of Burke and Matsubara, who showed in this case that the non-stationary ideal is not λ^+-saturated.

Our treatment of this case uses the following result of Shelah:

THEOREM 10 (Shelah [Sh2]). *Let λ be a singular cardinal, and suppose that $\mu<\lambda$ is a regular uncountable cardinal. Then there is a set $R\subset\lambda^+$ and a stationary set $A\subset\lambda^+$ consisting of ordinals of cofinality μ, such that whenever $N\prec\langle H(\theta),\in,\Delta,R\rangle$, if $\alpha=N\cap\lambda^+\in A$, then there is a cofinal sequence $C\subset\alpha$ of order type μ such that for all $\beta<\alpha$, $C\cap\beta\in N$.*

In Shelah's parlance, the set $A \in I[\lambda^+]$.

Our main use of this theorem is

LEMMA 11. *Let λ be a singular cardinal, and suppose that μ is a regular cardinal less than $\varkappa$. Then there is a stationary set $A \subset \lambda^+$ such that for all stationary $B \subset A$ and all expansions of $\mathfrak{A} = \langle H(\theta), \in, \Delta, R, \ldots \rangle$, there is an elementary substructure $N \prec \mathfrak{A}$ such that*

(1) $|N| < \varkappa$ *and* $N \cap \varkappa \in \varkappa$,

(2) $\sup(N \cap \lambda^+) \in B$,

(3) N *is internally approachable of length* μ.

Note that if $\mu = \omega$ then the result is immediate: A is the collection of all ordinals less than λ of countable cofinality. If $\mu > \omega$, this follows easily from Shelah's Theorem 10. The set $A \in I[\lambda]$ provided in the conclusion of Theorem 10 works for Lemma 11. Note that every stationary set $B \subset A$ is also in $I[\lambda^+]$. Further, if $B \in I[\lambda^+]$, then the set of elements N of $\mathrm{IA}(\mu)$, of cardinality less than $\varkappa$ with the property that the supremum of their intersection $N \cap \lambda^+$ lies in B, is stationary. (See [FM2] for a detailed proof of this fact. It is shown by finding an $\alpha \in B$ such that $\mathrm{sk}^{\mathfrak{A}}(\alpha) \cap \lambda^+ = \alpha$. If $C = \langle \alpha_i : i \in \mu \rangle$ is the witness to the approachability of α, then letting $N_j = \mathrm{sk}^{\mathfrak{A}}(\langle \alpha_i : i < j \rangle \cup \langle N_i : i < j \rangle)$ essentially works. A very similar lemma is in [BMg].)

Fix a set A as in the conclusion of Shelah's Theorem 10. Choose an increasing sequence $\langle \lambda_j : j \in \mathrm{cf}(\lambda) \rangle$ of regular cardinals cofinal in λ, such that $\prod_{j \in \mathrm{cf}(\lambda)} \lambda_j / I$ (where I is the ideal of bounded sets in $\mathrm{cof}(\lambda)$) has true cofinality λ^+. Fix a continuous scale $\langle f_\alpha : \alpha \in \lambda^+ \rangle$.

As is typical in the PCF theory if $N \prec H(\theta)$ we will define $\chi_N(j) = \sup(N \cap \lambda_j)$, so $\chi_N \in \prod \lambda_j$.

The main tool of this section is the following result:

LEMMA 12. *If $N \prec \langle H(\theta), \in, \Delta, \langle f_\alpha : \alpha \in \lambda^+ \rangle, \ldots \rangle$ is an internally approachable structure of length $\mu \neq \mathrm{cf}(\lambda)$ with $|N| < \varkappa$, $\mathrm{cf}(\lambda) \in N \cap \varkappa \in \varkappa$ and $\sup(N \cap \lambda^+) = \alpha$, then*

$$\chi_N \equiv f_\alpha \mod(I).$$

Proof. Let $\langle N_i : i < \mu \rangle$ be a witness to the fact that N is internally approachable. Then, since $N_i \in N$, $\chi_{N_i} \in N$, and since $\mathrm{cf}(\lambda) \subset N$, $\chi_N(j) = \sup\{\chi_{N_i}(j) : i < \mu\}$. Moreover, if $i < i' < \mu$, $\chi_{N_i}(j) < \chi_{N_{i'}}(j)$ for all $j < \mathrm{cf}(\lambda)$.

CLAIM. *There are cofinal subsets $X \subset \mu$ and $Y \subset \alpha \cap N$, and a $j_0 < \mathrm{cf}(\lambda)$ such that if i, i' are successive elements of X, there is a unique $\beta \in Y$ such that*

$$\textit{for all } j > j_0, \quad \chi_{N_i}(j) < f_\beta(j) < \chi_{N_{i'}}(j).$$

Proof. For each $\beta \in N \cap \lambda^+$ there is an i such that $f_\beta \in N_i$ and $f_\beta \subset N_i$. Hence there is a cofinal subset $X \subset \mu$ such that if i, i' are successive points in X, there is a $\beta \in N \cap \alpha$ such that $\chi_{N_i} \leqslant_I f_\beta \leqslant \chi_{N_{i'}}$, where the latter inequality holds for all $j < \mathrm{cf}(\lambda)$.

Choose, for each successive pair in X, such a β and let Y be the resulting set. Let β_i be the element of Y corresponding to i, i'. Then for each successive pair $i, i' \in X$, there is a $j_i < \mathrm{cf}(\lambda)$ such that for all $j > j_i$, $\chi_{N_i}(j) < f_{\beta_i}(j) < \chi_{N_{i'}}(j)$.

We break into cases.

(1) $\mu < \mathrm{cf}(\lambda)$. Then there is a $j_0 > \sup\{j_i : i \in X\}$. Clearly this j_0 works.

(2) $\mu > \mathrm{cf}(\lambda)$. In this case we can refine X and Y, replacing them with cofinal subsets where for all i, $j_i = j_0$ for a particular fixed j_0. Again this j_0 clearly works.

This completes the proof of the claim. □

To finish the proof of the lemma, we argue again by cases. If $\mu < \mathrm{cf}(\lambda)$, then the equivalence class of f_α is arrived at by taking supremums along any cofinal subset of α. Hence for sufficiently large $j > j_0$, $f_\alpha(j) = \sup_{\beta \in Y} f_\beta(j) = \sup_{i \in X} \chi_{N_i}(j) = \chi_N(j)$.

If $\mathrm{cf}(\lambda) < \mu$, then Y and j_0 witness that α satisfies one of the equivalent definitions of "good point" given in the introduction, and hence, again for sufficiently large $j > j_0$, $f_\alpha(j) = \sup_{\beta \in Y} f_\beta(j) = \sup_{i \in X} \chi_{N_i}(j) = \chi_N(j)$.

This concludes the proof of Lemma 12. □

THEOREM 13. *Suppose that $\varkappa \geqslant \omega_2$ is regular and $\lambda > \varkappa$ is a cardinal with* $\mathrm{cf}(\lambda) < \varkappa$. *Then the non-stationary ideal on $P_\varkappa(\lambda)$ is not λ^{++}-saturated.*

Proof. Since $\varkappa \geqslant \aleph_2$ we can choose a regular cardinal $\mu < \varkappa$ such that $\mu \neq \mathrm{cf}(\lambda)$. Let A be as in Lemma 11. By Shelah's Theorem 2, we know that there is an antichain $\mathcal{A}$ of cardinality λ^{++} in $P(A)$ modulo the non-stationary ideal. If we enumerate $\mathcal{A}$ as $\{B_\alpha : \alpha \in \lambda^{++}\}$, then each B_α has non-stationary intersection with the diagonal union of $\{B_\beta : \beta < \alpha\}$. By intersecting each B_α with a club set, we can assume that the intersection is, in fact, empty. Thus without loss of generality, we can assume that for all B_1 and B_2 in $\mathcal{A}$, $B_1 \cap B_2$ is bounded.

By Lemma 11, for all stationary $B \subset A$, $S_B = \{N \in P_\varkappa(H(\theta)) : \sup(N \cap \lambda^+) \in B$ and N is internally approachable of length $\mu\}$ is stationary in $P_\varkappa(\lambda^+)$. By Lemma 12, we know that for all internally approachable N, $\chi_N \equiv_I f_\alpha$ where $\alpha = \sup(N \cap \lambda^+)$.

For each $B \in \mathcal{A}$, let $T_B = \{N \cap \lambda : N \in S_B\}$. Then each T_B is the projection to λ of a stationary set, and hence is stationary. We will be done if we can show:

If B, C are distinct elements of $\mathcal{A}$, then $T_B \cap T_C$ is non-stationary.

Let $\gamma \in \lambda^+$ be large enough that $(B \cap C) \setminus \gamma$ is empty. Let $N \in T_B$ be such that for all j, $f_\gamma(j) \in N$. Then, $\chi_N = f_\alpha$ for some $\alpha \in B \setminus \gamma$. Hence, there is no $M \in S_C$, with $M \cap \lambda = N$.

To summarize, if N is in the closed unbounded set of elements of $P_\varkappa(\lambda)$ including all of the ordinals $f_\gamma(j)$ for $j<\text{cf}(\lambda)$, then N cannot be in both T_B and T_C. Thus the intersection is non-stationary.

This completes the proof of Theorem 13. □

5. $\varkappa$ is weakly inaccessible

In this section we consider the case where $\varkappa$ is weakly inaccessible and λ has cofinality $\varkappa$. The main result of this section is

THEOREM 14. *Suppose that $\varkappa$ is a weakly inaccessible cardinal and $\lambda>\varkappa$. Then the non-stationary ideal on $P_\varkappa(\lambda)$ is not λ^+-saturated.*

As we have shown non-saturation in the cases where the cofinality of λ differs from $\varkappa$, we have only to handle the case where $\text{cf}(\lambda)=\varkappa$.

Define a map $\pi: P_\varkappa(\lambda^+)\to P_\varkappa(\lambda)$ by $\pi(x)=x\cap\lambda$. Then π induces a map from $P(P_\varkappa(\lambda^+))$ to $P(P_\varkappa(\lambda))$, which we shall also call π. As remarked in the introduction, it is a standard fact that π takes stationary sets to stationary sets and the inverse image by π of a stationary set is stationary.

LEMMA 15. *Let $S\subset P_\varkappa(\lambda^+)$ be stationary. Suppose that the non-stationary ideal restricted to $\pi(S)$ is saturated. Then there is a closed unbounded set $C\subset P_\varkappa(\lambda^+)$ such that for all stationary $S'\subset\pi(C\cap S)$, the set $\{N\in S:\pi(N)\in S'\}$ is stationary.*

Proof. If $B\subset\pi(S)$ is stationary, we will say that B is *bad* if $\{N\in S: N\cap\lambda\in B\}$ is not stationary. Let $\mathcal{A}\subset P(P_\varkappa(\lambda))$ be a maximal antichain among the bad sets. By the saturation, we can assume that $\mathcal{A}=\{B_\alpha:\alpha\in\lambda\}$. Further we can assume that if $N\in B_\alpha$, $\alpha\in N$.

For each $\alpha<\lambda$, let $C_\alpha\subset P_\varkappa(\lambda^+)$ be closed and unbounded such that for all $N\in C_\alpha\cap S$, $N\cap\lambda\notin B_\alpha$. Let $C=\Delta C_\alpha$, and let $T\subset\pi(C\cap S)$ be stationary. If $\{N\in S: N\cap\lambda\in T\}$ is not stationary, then $T\cap B_\alpha$ is stationary for some $\alpha<\lambda$. Hence without loss of generality we may assume that $T\subset B_\alpha$. Let $N\in C\cap S$ with $N\cap\lambda\in T$. Then, since $T\subset B_\alpha$, $\alpha\in N$. Hence $N\in C_\alpha\cap S$, and so $N\cap\lambda\notin B_\alpha$. □

Just for interest we prove

PROPOSITION 16. *The converse of this lemma is also true: if $\varkappa$, λ and λ^+ have this projection property for all stationary sets, then the non-stationary ideal on $P_\varkappa(\lambda)$ is saturated. This shows,* a fortiori, *that there is always a stationary set on which this projection property fails.*

To see why the proposition is true, suppose that if $S\subset P_\varkappa(\lambda^+)$ is stationary then there is a club set $C\subset P_\varkappa(\lambda^+)$ such that for all stationary $T\subset\pi(C\cap S)$, we know that $\pi^{-1}(T)$ is stationary in $P_\varkappa(\lambda)$.

Towards a contradiction we assume that $\{B_\alpha:\alpha\in\lambda^+\}$ is an antichain in $P_\varkappa(\lambda)$. Let $B'_\alpha=\pi^{-1}(B_\alpha)$ and S be the diagonal union of the B'_α. Find a closed unbounded set $C\subset P_\varkappa(\lambda^+)$ as in the assumption.

Then $\{B_\alpha\}_{\alpha\in\lambda^+}$ is a maximal antichain below $\pi(C\cap S)$. For if $B\subset\pi(C\cap S)$ is stationary, and for all α, $B\cap B_\alpha$ is not stationary, let D_α be a closed unbounded set such that for all α, $B_\alpha\cap D_\alpha\cap B$ is empty. Let $D'_\alpha=\pi^{-1}(D_\alpha)$ and D' be the diagonal intersection of the D'_α's. Let $N\in\pi^{-1}(B)\cap C\cap D'\cap S$. Then for some $\alpha\in N$, $N\in B'_\alpha$. But then $N\cap\lambda\in D_\alpha\cap B\cap B_\alpha$, a contradiction.

Let $T=P_\varkappa(\lambda)\setminus\pi(C\cap S)$. Then clearly $\{T\}\cup\{B_\alpha:\alpha\in\lambda^+\}$ is a maximal antichain in $P_\varkappa(\lambda)$ modulo the non-stationary ideal.

Let $T'=\pi^{-1}(T)$. We show that T' together with the B'_α form a maximal antichain in $P_\varkappa(\lambda^+)$. Otherwise there is a U' that is stationary in $P_\varkappa(\lambda^+)$ such that $\pi(U')\subset\pi(C\cap S)$ and for all α, $U'\cap B'_\alpha$ is non-stationary. Choose D'_α such that for all α, $U'\cap B'_\alpha\cap D'_\alpha=\varnothing$. Let $D=\Delta D'_\alpha$ and $U^*=U'\cap D$. By assumption we can assume (by further intersecting U^* with a closed unbounded set if necessary) that for all stationary $R\subset\pi(U^*)$, $\pi^{-1}(R)\cap U^*$ is stationary. Since $\pi(U^*)\subset\pi(C\cap S)$ and is stationary, there is an α such that $B_\alpha\cap\pi(U^*)$ is stationary. Hence $\pi^{-1}(B_\alpha\cap\pi(U^*))\cap U^*$ is stationary. Let $N\in\pi^{-1}(B_\alpha\cap\pi(U^*))\cap U^*$ be such that $\alpha\in N$. Then $N\in U^*\cap D_\alpha\cap B'_\alpha$, a contradiction.

Let $\mathfrak{A}$ be an algebra on λ^+ such that any $N\prec\mathfrak{A}$ in $P_\varkappa(\lambda^+)$ is in the diagonal union of this antichain. Let $D\subset P_\varkappa(\lambda)$ be the closed and unbounded set of N such that $\mathrm{sk}^{\mathfrak{A}}(N)\cap\lambda=N$. Let $N\in D$ be arbitrary. Let $N'=\mathrm{sk}^{\mathfrak{A}}(N)$. Then N' is in the diagonal union of the $\{B'_\alpha\}_{\alpha\in\lambda^+}\cup\{T'\}$, and hence for some $\alpha\in\mathrm{sk}^{\mathfrak{A}}(N)$, $N\in B_\alpha\cup T$. Thus D is contained in the diagonal union of $\{B_\alpha:\alpha\in\mathrm{sk}^{\mathfrak{A}}(\lambda)\}\cup\{T\}$. Since $\mathrm{sk}^{\mathfrak{A}}(\lambda)$ has cardinality λ, we have a contradiction to the fact that the antichain had cardinality λ^+. □

We now fix, as usual, an increasing sequence of regular cardinals $\langle\lambda_i:i\in\varkappa\rangle$ such that the true cofinality of the reduced product of the λ_i's modulo the ideal of bounded sets on $\varkappa$ is λ^+. Fix a continuous scale $\langle f_\alpha:\alpha<\lambda^+\rangle$ in this reduced product.

LEMMA 17. *Let T be the collection of $M\in P_\varkappa(\lambda^+)$ such that*

(1) $\mathrm{cf}(M\cap\varkappa)=\omega_1$,

(2) *there is a sequence $\langle\delta_n:n\in\omega\rangle\subset M\cap\lambda^+$, and an $i_0\in M\cap\varkappa$, such that $\chi_M(i)=\sup\{f_{\delta_n}(i):n\in\omega\}$ for all $i\in\varkappa\cap M$, $i>i_0$.*

Then T is stationary.

Proof. As usual let θ be a very large regular cardinal and $H(\theta)$ be a sufficiently rich structure to include all the information about the situation at hand, such as the scale, the various cardinals mentioned, etc. It suffices to produce an elementary substructure of $H(\theta)$ in T.

Let $M \prec H(\theta)$ be such that $\varkappa \subset M$, M has cardinality $\varkappa$ and is internally approachable by a sequence $\langle N_k : k \in \omega \rangle$ of length ω. Then for all k, there is a $\delta_k \in N_{k+1}$ such that $\chi_{N_k} <^* f_{\delta_k}$. Choose an increasing sequence $\langle M_\alpha : \alpha < \varkappa \rangle$ such that $M_\alpha \cap \varkappa \in \varkappa$, $|M_\alpha| < \varkappa$, each $N_k \in M_0$, and $M_\alpha \subset M$. Then for some $\alpha \in \mathrm{cf}(\omega_1)$, $\langle M_\alpha, \langle f_{\delta_k} \rangle \rangle \prec \langle M, \langle f_{\delta_k} \rangle \rangle$.

We show that $M_\alpha \cap \lambda^+$ is in T. Note that since each $f_{\delta_k} \in M_\alpha$ and α has uncountable cofinality, there is an $i_0 \in M_\alpha \cap \varkappa$ such that for all i between i_0 and $\varkappa$, and all k, $\chi_{N_k}(i) < f_{\delta_k}(i) < \chi_{N_{k+1}}(i)$. Thus for all i between i_0 and $\varkappa$, $\chi_{M_\alpha}(i) = \sup\{\chi_{N_k}(i) : k \in \omega\} = \sup\{f_{\delta_k}(i) : k \in \omega\}$. □

For each $N \in T$ we fix an increasing sequence $\langle \alpha_i^N : i \in \omega_1 \rangle$ of ordinals cofinal in $N \cap \varkappa$. Let $G \subset P(P_\varkappa(\lambda))/\mathrm{NS}$ be generic with $\pi(T)$ in G. If we form the generic ultrapower of V by G to get $j : V \to M \subset V[G]$, where M is transitive, then $j``\lambda \in M$ and the sequence $\langle \alpha_i^{j``\lambda} : i \in \omega_1 \rangle$ is a cofinal increasing sequence in $\varkappa$. This sequence determines a subsequence of the cardinals $\langle \lambda_i : i \in \varkappa \rangle$, which we will denote by $\langle \lambda_i^* : i \in \omega_1 \rangle$. Since this is a subsequence of the λ_i's, it makes sense to view each f_α as an element f_α^* of $\prod_{i<\omega_1} \lambda_i^*$. Also, for a typical $N \in T$, the sequence $\langle \alpha_i^N : i \in \omega_1 \rangle$ determines a version of the sequence of λ_i^* relative to $N \cap \lambda$, and for $i > i_0$ each $N \cap \lambda_i^*$ has cofinality ω.

By Lemma 15, by intersecting T with a closed unbounded set if necessary, we can assume that for all stationary $S \subset \pi(T)$, $\pi^{-1}(S) \cap T$ is stationary.

LEMMA 18. *For every generic $G \subset P(P_\varkappa(\lambda))/\mathrm{NS}$ with $\pi(T) \in G$, $V[G] \models \langle f_\alpha^* : \alpha \in \lambda^+ \rangle$ is unbounded in $\prod_{i<\omega_1} \lambda_i^*$ (modulo the filter of countable sets).*

Proof. Suppose that the lemma fails, and take G generic with $\pi(T) \in G$ as a counterexample. Let $j : V \to M \subset V[G]$ be the generic elementary embedding, where M is transitive. By shrinking T if necessary, we can assume that $\pi(T) \Vdash \langle f_\alpha^* : \alpha < \lambda^+ \rangle$ is bounded in $\prod_{i \in \omega_1} \lambda_i^*$. Hence there is an $h \in \prod_{i<\omega_1} j``\lambda_i^*$ such that for all $\alpha < \lambda^+$ and all large enough $i < \omega_1$, $h(i) > f_{j(\alpha)}^*(i)$. Note that $h \in M$. Hence, by the saturation there is a $g : \pi(T) \to V$ such that for almost every $x \in \pi(T)$, $g(x) \in \prod_{i \in \omega_1} (\lambda_i^* \cap x)$, and such that for all $\alpha < \lambda^+$, $\pi(T) \Vdash$ for all sufficiently large $i < \omega_1$, $[g]_M(i) > f_{j(\alpha)}^*(i)$.

Let N be a typical element of T. Then $g(N \cap \lambda) \in \prod_{i \in \omega_1} (\lambda_i^* \cap N)$. In particular, for all $i < \omega_1$, $g(N \cap \lambda)(i) < \chi_N^*(i)$ (where $\chi_N^*(i) = \sup N \cap \lambda_i^*$). Since $N \in T$, there is a sequence $\langle \delta_n : n \in \omega \rangle$ such that for all large enough $i \in \omega_1$, $\chi_N^*(i) = \sup\{f_{\delta_n}^*(i) : n \in \omega\}$. Hence for all large $i \in \omega_1$, there is an n such that $g(N \cap \lambda)(i) < f_{\delta_n}^*(i)$. Hence there is an unbounded set of $i < \omega_1$, and an n, such that $f_{\delta_n}^*(i) > g(N \cap \lambda)(i)$.

Using a regressive function argument, we can find a fixed δ and a stationary $T' \subset T$ such that for all $N \in T'$, $g(N \cap \lambda)(i) < f^*_\delta(i)$ for unboundedly many $i \in \omega_1$.

Suppose now that $\pi(T') \in G$. Then in M, $j(g)(j``\lambda)(i) < f^*_{j(\delta)}(i)$ for unboundedly many $i < \omega_1$, a contradiction. $\square$

Using Lemmas 15–18 we can find a generic $G \subset P(P_\varkappa(\lambda))$ such that in $V[G]$:

(1) $\langle f^*_\alpha : \alpha < \lambda^+ \rangle$ is unbounded in $\prod_{i<\omega_1} \lambda^*_i$,

(2) $\mathrm{cf}(\varkappa) = \omega_1$ and for all i, $\mathrm{cf}(\lambda^*_i) = \omega$.

Working now in $V[G]$, we apply the "Shelah Trichotomy Theorem" [Sh2] to see that in $OR^{\omega_1}/\{\text{bounded sets}\}$, the sequence $\langle f^*_\alpha : \alpha < \lambda^+ \rangle$ either

(1) has an exact upper bound g in $OR^{\omega_1}/\{\text{bounded sets}\}$ such that for almost all $i, j \in \omega_1$, $\mathrm{cf}(g(i)) = \mathrm{cf}(g(j))$, or

(2) there are sets $A_i \subset \lambda^*_i$, with $|A_i| \leqslant \omega_1$, and an ultrafilter $D \subset P(\omega_1)$ such that for all $\alpha < \lambda^+$ there is a $\beta < \lambda^+$ and a $g \in \prod_{i<\omega_1} A_i$ such that $f^*_\alpha <_D g <_D f^*_\beta$, or

(3) there is a $g \in \prod_{i<\omega_1} \lambda^*_i$ such that the sequence of equivalence classes of $\{i : f^*_\alpha(i) \leqslant g(i)\}$ modulo the ideal of bounded sets in ω_1 is *not* eventually constant.

Note that by Lemma 18, if the sequence $\langle f^*_\alpha : \alpha < \lambda^+ \rangle$ has an exact upper bound then it must be given by the function $g(i) = \lambda^*_i$. Since each λ^*_i has cofinality ω, we can choose cofinal countable sets $A_i \subset \lambda^*_i$. Since g is an exact upper bound, for every function $h \in \prod_{i\in\omega_1} A_i$, there is a β such that $h <^* f^*_\beta$. Further, since the A_i are cofinal, for all $\beta < \lambda^+$, there is an $h \in \prod_{i\in\omega_1} A_i$ such that $f^*_\beta <^* h$. Hence if there is an exact upper bound, the sets $\{A_i\}$ are a witness to being in case (2), for any ultrafilter D.

We argue that both (2) and (3) above lead to a contradiction. In either case there is an ordinal $\delta \in \lambda^+$ of cofinality ω_2 (in both V and $V[G]$) such that either

(in (2)) for all $\alpha < \delta$ there is a $\beta < \delta$ and a $g \in \prod_{i<\omega_1} A_i$ such that $f^*_\alpha <_D g <_D f^*_\beta$,

or

(in (3)) the sequence of equivalence classes of $\{i : f^*_\alpha(i) \leqslant g(i)\}$ (modulo the ideal of bounded sets in ω_1) for $\alpha < \delta$ is *not* eventually constant.

Returning to V for the moment, choose a cofinal $X \subset \delta$ of order type ω_2. Then, since $\varkappa$ is regular in V, there is a $j < \varkappa$ such that for all $i > j$ and all $\alpha < \beta$ in X, $f_\alpha(i) < f_\beta(i)$. Thus, in $V[G]$, there is an $i_0 \in \omega_1$ such that for all $\alpha < \beta$, $\alpha, \beta \in X$, and all $i > i_0$, $f^*_\alpha(i) < f^*_\beta(i)$. Work now again in $V[G]$.

If (2) holds: Build a cofinal set $X' \subset X$ such that for all $\alpha < \beta$ in X', there is a $g \in \prod_{i\in\omega_1} A_i$ such that $f^*_\alpha <_D g <_D f^*_\beta$. If $\alpha < \alpha'$ are successive elements of X', choose $i_\alpha > i_0$ and a $g_\alpha \in \prod_{i\in\omega_1} A_i$ such that $f_\alpha(i_\alpha) < g_\alpha(i_\alpha) < f_{\alpha'}(i_\alpha)$. Then there is a $j > i_0$ such that for cofinally many $\alpha \in X'$ we have that $i_\alpha = j$, and hence we can assume that for all $\alpha \in X'$, $i_\alpha = j$. But then if $\alpha < \beta$ are arbitrary elements of X',

$$f^*_\alpha(j) < g_\alpha(j) < f^*_{\alpha'}(i) < f^*_\beta(j) < g_\beta(j).$$

But this is a contradiction, since it implies that A_j has cardinality at least ω_2.

If (3) holds: Choose $X' \subset X$ cofinal in δ so that if $\alpha, \beta \in X'$ are distinct, then modulo the bounded subsets of ω_1, $[\{i : f_\alpha(i) \leqslant g(i)\}] \neq [\{i : f_\beta(i) \leqslant g(i)\}]$. Since the f_α's are increasing with $\alpha \in X'$ at *every* $i > i_0$, the sets $\{i > i_0 : f_\alpha(i) \leqslant g(i)\}$ are strictly decreasing with $\alpha \in X'$. However, it is impossible to have a strictly decreasing sequence of subsets of ω_1 of length ω_2, a contradiction.

This finishes the proof of Theorem 14 by showing that if $\varkappa$ is inaccessible, there is always an antichain in $P(P_\varkappa(\lambda))/\mathrm{NS}$ of cardinality λ^+.

6. Some examples

In this section we give some examples to show that our results cannot be extended in certain directions, by showing that various saturation properties of the non-stationary ideal are consistent. Curiously, the examples are arrived at by forcing over L.

One of the main tools in these results is a "reverse covering theorem" due to Magidor:

THEOREM 19 (Magidor [M]). *Suppose that there is no inner model with an Erdös cardinal. Then for all regular $\varkappa \geqslant \omega_1$ and all $\lambda \geqslant \varkappa$ there is a closed unbounded set $C \subset P_\varkappa(\lambda)$ such that each $N \in C$ is a countable union of elements of L.*

An immediate corollary of this theorem is that if one does countably closed forcing over L then there is a closed unbounded subset of $P_\varkappa(\lambda)$ consisting of elements of L.

THEOREM 20. *Let $\varkappa \leqslant \lambda$ be cardinals in L with $\varkappa \geqslant \omega_2$ regular. Let $\mu = \lambda^{<\varkappa}$. If $\gamma \geqslant \mu^{++}$ and $G \subset \mathrm{Add}(\omega_1, \gamma)$ is generic over L, then in $L[G]$ the non-stationary ideal on $P_\varkappa(\lambda)$ is μ^{++}-saturated.*

COROLLARY 21. *More explicitly (working in L):*

(1) *If $\varkappa \geqslant \omega_2$ and $\mathrm{cf}(\lambda) < \varkappa$, we have $\lambda^{<\varkappa} = \lambda^+$. Hence adding λ^{+3} generic subsets to ω_1 makes the non-stationary ideal on $P_\varkappa(\lambda)$ λ^{+3}-saturated. Since $(\lambda^{<\varkappa})^{L[G]} = \lambda^{+3}$, the non-stationary ideal on $P_\varkappa(\lambda)$ is $\lambda^{<\varkappa}$-saturated.*

(2) *If $\mathrm{cf}(\lambda) \geqslant \varkappa \geqslant \omega_2$ then $\lambda^{<\varkappa} = \lambda$, and adding λ^{+2} generic subsets of ω_1 makes the non-stationary ideal on $P_\varkappa(\lambda)$ be $\lambda^{+2} = \lambda^{<\varkappa}$-saturated.*

Proof. Let $G \subset \mathrm{Add}(\omega_1, \gamma)$ be generic over L. Then by Theorem 19 we know that in $L[G]$ there is a closed unbounded subset of $P_\varkappa(\lambda)$ that lies in L. This set has cardinality μ.

Towards a contradiction, suppose that $\langle A_\alpha : \alpha < \mu^{+2} \rangle$ is an antichain in $P_\varkappa(\lambda)$ modulo the non-stationary ideal. We can assume that $A_\alpha \subset L$.

For each pair $\alpha,\beta\in\mu^{+2}$, choose a term $\dot{C}_{\alpha,\beta}$ for a closed unbounded set such that the empty condition forces that

$$A_\alpha\cap A_\beta\cap\dot{C}_{\alpha,\beta}=\varnothing.$$

We can assume that the empty condition forces $\dot{C}_{\alpha,\beta}\subset L$.

Now, as usual, let θ be a large regular cardinal.

If $M\prec\mathfrak{A}=\langle H(\theta),\in,\Delta,\langle\dot{C}_{\alpha,\beta}:\alpha,\beta<\mu^{+2}\rangle,\{\gamma,\lambda,\varkappa\}\rangle$ is an elementary substructure of size $<\varkappa$ such that $M\cap\varkappa$ is an ordinal, $\alpha,\beta\in M$ and p is an M-generic condition for $\mathrm{Add}(\omega_1,\gamma)$, then $p\Vdash M\cap\lambda\in C_{\alpha,\beta}$. Moreover, since $\varkappa\geqslant\omega_2$ the empty condition is generic over M.

In $L[G]$, for each $\alpha,\beta<\mu^{+2}$, $A_\alpha\cap\dot{C}_{\alpha,\beta}$ is a stationary subset of L. Hence, working in L, for each pair $\alpha<\beta\in\mu^{+2}$ we can choose $M_{\alpha,\beta}\prec\mathfrak{A}$ such that

(1) $\alpha,\beta\in M_{\alpha,\beta}$,

(2) if $N=M_{\alpha,\beta}\cap\lambda$ then there is an $M_{\alpha,\beta}$-generic condition $p^N_{\alpha,\beta}$ such that

$$p^N_{\alpha,\beta}\Vdash N\in A_\alpha.$$

Still in L, since $|P_\varkappa(\lambda)|=\mu$, we can apply the Erdös–Rado Theorem to conclude that there is a set $X\subset\mu^{+2}$ of cardinality μ^+ and an $N\in P_\varkappa(\lambda)$ such that for all $\alpha<\beta\in X$, $M_{\alpha,\beta}\cap\lambda=N$. We claim that for all $\alpha<\beta<\gamma<\delta\in X$, $p_{\alpha,\beta}$ is incompatible with $p_{\gamma,\delta}$. This yields a contradiction to the ω_2-c.c. of $\mathrm{Add}(\omega_1,\gamma)$.

Let $\alpha<\beta\in X$. Then $M_{\alpha,\beta}\cap\lambda=N$. Since the empty condition forces that $M_{\alpha,\beta}\cap\lambda\in C_{\alpha,\beta}$ we see that for all $\alpha<\beta\in X$, the empty condition forces that $N\in C_{\alpha,\beta}$. Since each $p_{\alpha,\beta}\Vdash N\in A_\alpha$ we see that if $q\leqslant p_{\alpha,\beta},p_{\gamma,\delta}$, then $q\Vdash N\in A_\alpha\cap A_\gamma\cap C_{\alpha,\gamma}$, a contradiction. □

7. Mutual stationarity

In this section we give some results describing our meager knowledge of mutual stationarity.

7.1. Definitions and basic facts

Recall:

Definition 22. Let K be a collection of regular cardinals with supremum δ, and suppose that we have $S_\varkappa\subset\varkappa$ for each $\varkappa\in K$. Then the collection of sets $\{S_\varkappa:\varkappa\in K\}$ is mutually stationary if and only if for all algebras $\mathfrak{A}$ on δ there is an $N\prec\mathfrak{A}$ such that

$$\text{for all }\varkappa\in N\cap K,\quad \sup N\cap\varkappa\in S_\varkappa.$$

We begin with a few easy observations about mutual stationarity. These ideas are implicit in Baumgartner's paper [Ba]:

(1) Any subset of a sequence of mutually stationary sets is mutually stationary.

(2) If $N \prec \langle H(\theta), \in, \Delta, ... \rangle$ (where $\theta > \delta$ is regular) and $\nu \in N$, then for all regular $\mu \in N \setminus (\nu+1)$,

$$\sup N \cap \mu = \sup(\mathrm{sk}^{\mathfrak{A}}(N \cup \nu) \cap \mu).$$

(3) Suppose that $\{S_\varkappa : \varkappa \in K\}$ is mutually stationary, $K \cap (\nu+1) = \varnothing$, and for all $\varkappa \in K$, $S_\varkappa \subset \mathrm{cf}(\leqslant \nu)$. Then there is an N that witnesses mutual stationarity with $|N| = \nu$.

This last remark can be seen as follows: Let $\mathfrak{A}$ be an expansion of $H(\theta)$. By the second remark, we can assume that there is a witness $M \prec \mathfrak{A}$ to mutual stationarity such that $\nu+1 \subset M$. Choose $F: (K \cap M) \times \nu \to M$ such that for all $\varkappa \in M \cap K$, $F(\varkappa, \cdot): \nu \to \varkappa$ cofinally in $\sup(M \cap \varkappa)$. Then if we let N be the Skolem hull of the expansion of M determined by F, ν and the functions in $\mathfrak{A}$, we find that for all $\varkappa \in N \cap K$, $\sup(N \cap \varkappa) = \sup(M \cap \varkappa)$.

Using this it is easy to prove the following lemma (essentially due, in different form, to Baumgartner [Ba]):

LEMMA 23. *Let ν be a regular cardinal less than the least element of K. If $\{S_\varkappa : \varkappa \in K\}$ is mutually stationary, and for all $\varkappa$, $S_\varkappa \subset \mathrm{cf}(\leqslant \nu)$, then for all $\lambda_1, ..., \lambda_n$ greater than ν and not in K, and all sequences of stationary sets $S_{\lambda_i} \subset \lambda_i \cap \mathrm{cf}(\leqslant \nu)$, the sequence $\{S_\varkappa : \varkappa \in K\} \cup \{S_{\lambda_1}, ..., S_{\lambda_n}\}$ is mutually stationary.*

Proof. Using induction it clearly suffices to prove this in the case where $n=1$. So suppose that λ is a regular cardinal greater than ν that is not in K, and $S \subset \lambda \cap \mathrm{cf}(\leqslant \nu)$.

Let $\mathfrak{A}$ be an algebra expanding $H(\theta)$ for some large θ. Suppose that $K = A \cup B$ where for all $\varkappa \in A$, $\varkappa > \lambda$, and for all $\varkappa \in B$, $\varkappa < \lambda$. Then $\{S_\varkappa : \varkappa \in A\}$ is mutually stationary, so by the remark, we can find an $M \prec \mathfrak{A}$ such that for all $\varkappa \in M \cap A$, $\sup(M \cap \varkappa) \in S_\varkappa$, $\lambda+1 \subset M$ and $|M| = \lambda$. Choose a function $F: (K \cap M) \times \nu \to M$ such that for all $\varkappa \in A \cap M$, $F(\varkappa, \cdot): \nu \to \varkappa \cap M_\alpha$ cofinally.

Let $\{M_\alpha : \alpha < \lambda\}$ be a continuous chain of elementary substructures of $\langle M, F \rangle$ such that $\lambda \in M_0$, $|M_\alpha| < \lambda$ and $M_\alpha \cap \lambda \in \lambda$. Then there is an α such that $M_\alpha \cap \lambda = \alpha \in S_\lambda$. Let $G: \nu \to \alpha$ cofinally.

Then M_α and the functions F, G determine an algebra $\mathfrak{B}$ on α such that if $N_0 \prec \mathfrak{B}$ then the Skolem hull of N_0 under the functions of $\mathfrak{A}$ and F and G determines an elementary substructure $N \prec \langle M_\alpha, F, G \rangle$ such that $N \cap \alpha = N_0$.

Since the sequence $\langle S_\varkappa : \varkappa \in K \cap \alpha \rangle$ is mutually stationary, there is an $N_0 \prec \mathfrak{B}$, $\nu \subset N_0$, such that for all $\varkappa \in N_0 \cap K$, $\sup(N_0 \cap \varkappa) \in S_\varkappa$. If we let N be the Skolem hull of N_0 under the functions of $\mathfrak{A}$ and F and G, then:

(1) for all $\varkappa\in K\cap\alpha$, $\sup(N\cap\varkappa)=\sup(N_0\cap\varkappa)\in S_\varkappa$,

(2) $\sup N\cap\lambda=\alpha\in S_\lambda$,

(3) for all $\varkappa\in N\cap K$, $\sup(N\cap\varkappa)=\sup(M\cap\varkappa)\in S_\varkappa$.

Hence N is the witness to the mutual stationarity of $\{S_\varkappa:\varkappa\in K\cup\{\lambda\}\}$. $\square$

Thus with minor restrictions, the notion of mutual stationarity is invariant under finite variations in the sequence.

There are several situations where the notion of mutual stationarity is trivial. In §3, we showed that if all of the sets $S_\varkappa$ consist of points of countable cofinality then the sequence is mutually stationary. For orthogonal reasons, if all of the cardinals $\varkappa\in K$ are measurable and $|K|$ is not too big, then indiscernible arguments yield the mutual stationarity of any sequence of stationary sets.

Thus we are mostly interested in the case where the cardinals $\varkappa$ are relatively small, and the stationary sets consist of points of uncountable cofinality.

Liu [L], answering a question of Baumgartner, showed that there are sequences $S^n\subset\aleph_n\cap\mathrm{cf}(>\omega)$ that are mutually stationary and such that the cofinalities of the point in S^n are not eventually constant. Shelah and Liu extended these results in [LS].

In what follows we generally restrict ourselves to sequences $\{S_\varkappa\}$ consisting of points of the same cofinality. For simplicity we focus on the case of a sequence of sets $S^n\subset\aleph_n$, though most of our results generalize in an obvious way.

Remark. The statement that there is a mutually stationary sequence of sets $S^n\subset\omega_n$ such that $S^n\subset\mathrm{cf}(\geqslant\omega_{k_n})$ and $\{k_n:n\in\omega\}$ is unbounded in ω implies that $\aleph_\omega$ is Jonsson. Silver has shown a strong converse to this fact. (Silver showed that if $\aleph_\omega$ is Jonsson, and $2^{\aleph_0}<\aleph_\omega$, then there is a sequence k_n tending to ∞ such that $S^n=\{\alpha\in\omega_n:\mathrm{cf}(\alpha)=k_n\}$ is mutually stationary. See [KM].)

For this reason for the rest of this section we will pay attention to the case where $\langle S^n\rangle$ concentrate on ordinals of bounded cofinality.

At this time we do not know if it is consistent that there is a model of set theory in which every sequence of stationary subsets of the $\aleph_n$'s of a fixed cofinality is mutually stationary. However, we do know that, unlike the case of countable cofinality, this is not a theorem of ZFC.

7.2. Mutual stationarity in L

THEOREM 24. *Assume that $V=L$. For all $k\geqslant1$ there is a sequence of stationary sets $\langle S_n:n\in\omega\rangle$ such that $S_n\subset\omega_n\cap\mathrm{cf}(\omega_k)$ that is not mutually stationary.*

The proof of this theorem uses some elementary fine structure technique, as well as some of the techniques of Jensen's Covering Lemma. The theorem is actually much more general. We give a proof for the ω_n's for concreteness. It is easily seen to generalize to other sequences of cardinals.

For the rest of this discussion, assume that $V=L$. For each singular ordinal α, we can associate the least ordinal γ such that there is a $\delta<\alpha$, a finite set $\vec{p}\subset\gamma$ and an $n\in\omega$ such that $\mathrm{Hull}_n^{J_\gamma}(\delta\cup\vec{p})$ is cofinal in α (where $\mathrm{Hull}_n^{J_\gamma}$ denotes the Σ_n-Skolem hull in J_γ).

If α is singular, then such $\gamma,\delta,\vec{p},n$ exist, and we can associate to α the lexicographically least tuple $(\gamma,n,\delta,\vec{p})=(\gamma(\alpha),n(\alpha),\delta(\alpha),\vec{p}(\alpha))$ with $\mathrm{Hull}_n^{J_\gamma}(\delta\cup\vec{p})$ cofinal in α.

Fix natural numbers $m>k\geqslant 1$ and $n\geqslant 1$. Define

$$S_m^n=\{\alpha\in\omega_m\cap\mathrm{cf}(\omega_k):n(\alpha)=n\}.$$

LEMMA 25. *If $k,m,n\in\omega\setminus\{0\}$ and $m>k$ then S_m^n is stationary. Moreover, for fixed n,k the sequence $\langle S_m^n:n\in\omega\rangle$ is mutually stationary.*

Proof. Fix $m>k$ and $n\geqslant 1$. Suppose that S_m^n is not stationary. Let C, a closed unbounded subset of ω_m, be a witness.

Let $\varkappa=\omega_{m+1}$. Recall that an ordinal $\beta<\varkappa$ is $(n-1)$-stable if and only if $J_\beta\prec_{\Sigma_{n-1}}J_\varkappa$. Standard arguments show that the set of $(n-1)$-stable ordinals form a closed unbounded set in $\varkappa$.

Let γ be the ω_kth $(n-1)$-stable ordinal, above the stage where C is constructed.

LEMMA 26. *γ is Σ_{n-1}-admissible, but not Σ_n-admissible.*

The first assertion follows from the fact that the union of a Σ_{n-1}-elementary chain is a Σ_{n-1}-elementary extension of every structure on the chain.

To see the latter, we go by contradiction. There is a function $H:J_\gamma\times\omega\to J_\gamma$ that is Σ_{n-1}, and $\{\beta:C\in J_\beta$ and J_β is closed under $H\}$ is exactly the set of $(n-1)$-stable ordinals less than γ that construct C. (H is a function coding Σ_{n-1}-Skolem functions.) Since this set has order type ω_k and is definable in a Δ_n-way, we have a contradiction. The lemma follows. □

Let $N=\mathrm{Hull}_n^{J_\gamma}(\omega_k\cup\{C\})$. The proof of the previous lemma shows that N is cofinal in J_γ.

We show that $\sup(N\cap\omega_m)\in S_m^n\cap C$. Let $N'=\mathrm{Hull}_{n-1}^{J_\gamma}(N\cup\omega_{m-1})$. Then standard arguments verify:

- $\alpha=\sup(N\cap\omega_m)$ is an ordinal in C.
- N is a Σ_{n-1}-elementary substructure of N'.

- If τ is a Σ_{n-1}-term in N for a function from ω_{m-1} to ω_m, then τ is bounded in α. In particular, $\sup(N\cap\omega_m)=N'\cap\omega_m$.
- Similarly $\sup N=\sup N'$. Thus $N\prec_{\Sigma_n}N'$.

Let $\overline{N}$ be the transitive collapse of N'. Then:

- $\overline{N}=J_\nu$ for some ν.
- $J_\nu=\mathrm{Hull}_n^{J_\nu}(\omega_{m-1}\cup\{\overline{C}\})$ where $\overline{C}$ is the image of $\{C\}$ under the collapse map.
- $\nu=\nu(\alpha)$.
- $n=n(\alpha)$.

Thus, if α has cofinality ω_k, $\alpha\in C\cap S^n_m$.

What remains is to show that the cofinality of α is ω_k. We show that there is a closed unbounded set $D\subset\omega_k$ such that if $\xi<\eta$ are both in D, then $\sup(\mathrm{Hull}_n^{J_\gamma}(\xi\cup\{C\})\cap\omega_m)<\sup(\mathrm{Hull}_n^{J_\gamma}(\eta\cup\{C\})\cap\omega_m)$. This suffices to see that $\mathrm{cf}(\alpha)=\omega_k$.

For $\xi<\omega_k$, let β_ξ satisfy $\pi_\xi\colon J_{\beta_\xi}\cong\mathrm{Hull}_n^{J_\gamma}(\xi\cup\{C\})$ and let $\gamma_\xi=\sup(\mathrm{Hull}_n^{J_\gamma}(\xi\cup\{C\}))$. Then the sequence of γ_ξ is continuous, increasing and unbounded in γ. Without loss of generality we can assume that for $\xi\in D$, J_{γ_ξ} is a Σ_{n-1}-elementary substructure of J_γ. Hence the map π_ξ, viewed as a map from J_{β_ξ} to J_{γ_ξ} is Σ_{n-1}-elementary and cofinal. Hence the map is Σ_n-elementary, and thus $\mathrm{Hull}^{J_\gamma}(\xi\cup\{C\})\prec_{\Sigma_n}J_{\gamma_\xi}$. In particular, $\mathrm{Hull}_n^{J_\gamma}(\xi\cup\{C\})=\mathrm{Hull}_n^{J_{\gamma_\xi}}(\xi\cup\{C\})$.

Let $\xi<\eta$ be elements of D. Then $\gamma_\xi\in\mathrm{Hull}_n^{J_\gamma}(\eta\cup\{C\})$. Hence $\mathrm{Hull}_n^{J_\gamma}(\xi\cup\{C\})\in\mathrm{Hull}_n^{J_\gamma}(\eta\cup\{C\})$, and thus $\sup(\mathrm{Hull}_n^{J_\gamma}(\xi\cup\{C\})\cap\omega_m)\in\mathrm{Hull}_n^{J_\gamma}(\eta\cup\{C\})$, as desired.

To see the second claim (which is actually stronger) suppose that it is false, and let $\mathfrak{A}$ be a Skolemized algebra on $\aleph_\omega$ witnessing the non-mutual stationarity of $\langle S^n_m:m\in\omega\rangle$. We let $\varkappa$ be $\aleph_{\omega+1}$ and γ be the ω_kth $(n-1)$-stable ordinal in $\varkappa$ above the stage where $\mathfrak{A}$ is constructed. Then as in the first part of the lemma, γ is Σ_{n-1}-admissible, but not Σ_n-admissible. Moreover, the sequence of stable ordinals below γ that construct $\mathfrak{A}$ is definable in a Δ_n-way.

Let $N=\mathrm{Hull}_n^{J_\gamma}(\omega_k\cup\{\mathfrak{A}\})$. Then $(N\cap\aleph_\omega)\prec\mathfrak{A}$.

Let $\alpha_m=\sup N\cap\omega_m$. We claim that $\alpha_m\in S^n_m$, yielding a contradiction to $\mathfrak{A}$ being a counterexample to the mutual stationarity of the sequence $\langle S^n_m:m>k\rangle$. From here the proof goes as in the previous case, noting that the role of m in the previous case was only to fix γ.

This finishes the proof of the lemma.

THEOREM 27. *Fix $k\geqslant1$. Suppose that $h\colon\omega\to\omega$. Then the sequence $\langle S_m^{h(m)}:m>k\rangle$ is mutually stationary if and only if the function h is eventually constant.*

Proof. Suppose that h is eventually constant. Since mutual stationarity is preserved under finite variations we can assume that h is constant everywhere. Hence this direction of the theorem follows from the previous lemma.

For the difficult direction suppose that $\langle S_m^{h(m)} : m > k \rangle$ is mutually stationary. Let $\varkappa$ be a large regular cardinal, and let $N \prec J_\varkappa$ be a witness to the mutual stationarity for the algebra $\langle J_\varkappa, \in \rangle$. By earlier arguments we can assume that $\omega_k \subset N$. Let $\overline{N} \cong J_\beta$ be the transitive collapse, and $j: J_\beta \to J_\varkappa$ be the inverse of the collapsing map.

Let $\eta_l = \omega_l^{J_\beta}$. Then for all $l \geqslant k$, $\mathrm{cf}(\eta_l) = \omega_k$ and $\omega_k \subset J_\beta$.

Let γ be the least ordinal such that for some l there are $n, \delta, \vec{p}$ (with $\delta < \eta_l$ and $\vec{p} \subset \gamma$) such that $\mathrm{Hull}_n^{J_\gamma}(\delta \cup \vec{p})$ is cofinal in η_l. The following claim is standard.

CLAIM. *For all* $l' \geqslant l$, $\mathrm{Hull}_n^{J_\gamma}(\delta \cup \vec{p})$ *is cofinal in* $\eta_{l'}$. *Moreover, for all* $r \in \omega$, $H(\omega_r)^{J_\gamma} = H(\omega_r)^{J_\beta}$ *(where* $H(\omega_r)$ *is the collection of sets of hereditary cardinality less than* ω_r*).*

Let n_0 be the least integer such that for some η_l, $\mathrm{Hull}_{n_0}^{J_\gamma}(\delta \cup \vec{p})$ is cofinal in some η_l. We show that $h(m) = n_0$ for all $m \geqslant l$.

Fix an $m \geqslant l$, and let $N_m = \mathrm{Hull}_\omega^{J_\varkappa}(N \cup \omega_{m-1})$. Then the following claims are standard:

(1) If $\alpha_m = \sup(N \cap \omega_m)$ then $\alpha_m = N_m \cap \omega_m$.
(2) $\sup(N_m \cap \omega_{m'}) = \sup(N \cap \omega_{m'})$ for all $m' \geqslant m$.
(3) $\sup(N) = \sup(N_m)$.
(4) $j: N \to N_m$ is elementary.

Following the proof of Jensen's Covering Lemma, we can form a directed system $\mathcal{S}$ of Σ_{n_0-1}-Skolem hulls of sets of the form $\delta' \cup \vec{p}'$ (for $\delta' < \eta$ and $\vec{p}' \in [\gamma]^{<\omega}$) in J_γ, and sending them by j to a directed system $\mathcal{S}^j$ of sets with Σ_{n_0-1}-embeddings between them. The elements of $\mathcal{S}^j$ and the embeddings lie in N_m.

We will argue later that $\varinjlim \mathcal{S}^j$ is well-founded and hence isomorphic to some J_{γ_m}. Assuming this to be true, there is a cofinal Σ_{n_0-1}-embedding $\tilde{j}: J_\gamma \to J_{\gamma_m}$. Since $\tilde{j}$ is cofinal, it is Σ_{n_0}-elementary.

In particular, this implies that

(1) $\mathrm{Hull}_n^{J_{\gamma_m}}(j(\delta) \cup j(\vec{p}))$ is unbounded in α_m,
(2) for all $\delta' < \alpha_m$, $\vec{p} \in [\gamma_m]^{<\omega}$ we know that $\mathrm{Hull}_{n-1}^{J_{\gamma_m}}(\delta' \cup \vec{p}) \in J_{\gamma_m}$,
(3) for all $\gamma' < \gamma_m$, $\delta' < \alpha_m$, $\vec{p} \in [\gamma_m]^{<\omega}$, we know that $\mathrm{Hull}_\omega^{J_{\gamma'}}(\delta' \cup \vec{p}) \in J_{\gamma_m}$.

Thus, we know that $n(\alpha_m) = n_0$, as desired.

What remains is to outline the Covering Lemma argument.

Define a directed system of transitive structures indexed by pairs $(\delta, \vec{p})$ such that $\delta < \eta_m$ and $\vec{p} \in [\gamma_m]^{<\omega}$. To such an index $i = (\delta, \vec{p})$, we associate the transitive collapse N_i of $\mathrm{Hull}_{n_0-1}^{J_\gamma}(\delta \cup \vec{p})$. Note that $N_i \in H(\eta_m)^{J_\gamma}$.

If $i = (\delta, \vec{p})$ and $i' = (\delta', \vec{p}')$ are indices, set $i <^* i'$ if and only if $\delta < \delta'$ and $\vec{p} \subset \vec{p}'$. For

$i<^* i'$, $\mathrm{Hull}^{J_\gamma}_{n_0-1}(\delta\cup\vec{p})\prec_{\Sigma_{n_0-1}}\mathrm{Hull}^{J_\gamma}_{n_0-1}(\delta'\cup\vec{p}\,')$. This induces a natural Σ_{n_0-1}-embedding $f_{i,i'}$ from N_i into $N_{i'}$, induced by taking collapses.

Thus we get a directed system $\mathcal{S}=\{N_i, f_{i,i'}: i<^* i'\}\subset H(\eta_m)^{J_\gamma}=H(\eta_m)^{J_\beta}$. By applying j pointwise to this system we get a new directed system $\mathcal{S}^j=\{j(N_i), j(f_{i,i'}): i<^* i'\}$. We need to see that the direct limit of $\mathcal{S}^j$ is well-founded.

If this fails there is a countable set of indices I such that $\varinjlim\{j(N_i), j(f_{i,i'}):$ $i<^* i'$ and $i,i'\in I\}$ is ill-founded. Since η_m has uncountable cofinality there is a $\nu<\eta_m$ such that for all $i\in I$, if $i=(\delta,\vec{p})$, then $\delta\subset\nu$. Further there is a countable C such that if $i=(\delta,\vec{p})\in I$, then $\vec{p}\in C$. Let J_ϱ be the transitive collapse of $\mathrm{Hull}^{J_\gamma}_{n_0-1}(\nu\cup C)$. Since η_m has uncountable cofinality, $\varrho<\eta_m$.

For all $i\in I$, there is a natural Σ_{n_0-1}-embedding g_i from N_i to J_ϱ such that if $i<^* i'$, then $g_{i'}\circ f_{i,i'}=g_i$. Moreover, $g_i\in J_\gamma$. Applying j, we get natural embeddings $j(g_i)$ from $j(N_i)$ to $J_{j(\varrho)}$ that commute with $j(f_{i,i'})$. Hence there is an embedding $\varinjlim j(g_i)=g$ from the direct limit of $\{j(N_i): i\in I\}$ into $J_{j(\varrho)}$, contradicting ill-foundedness.

7.3. Splitting mutually stationary sequences

We now investigate the splitting properties of mutually stationary sets, about which we know very little. Again, we work on the $\aleph_n$'s for clarity, although the results clearly generalize.

We will adopt the following notation for simplicity: If N is a set, we will define the *characteristic function* of N to be $\chi_N(n)=\sup(N\cap\omega_n)$. Thus the statement that $S^n\subset\omega_n$ $(n\in\omega)$ is mutually stationary can be formulated: for all algebras $\mathfrak{A}$ on $\aleph_\omega$, there is an $N\prec\mathfrak{A}$ such that for all n, $\chi_N(n)\in S^n$.

THEOREM 28. *Assume that* $2^{\aleph_0}=\aleph_k<\aleph_\omega$. *Suppose that* $S^n\subset\aleph_n$ $(n>k)$ *is a mutually stationary sequence of sets. If we split each* S^n *into two sets* S^n_0 *and* S^n_1, *then there is a function* $f:\omega\to 2$ *such that the sequence* $S^n_{f(n)}$ *is mutually stationary.*

Proof. Suppose that the theorem is false. Then for each $f:\omega\to 2$, there is an algebra $\mathfrak{A}_f$ on $\aleph_\omega$ such that if $N\prec\mathfrak{A}_f$ then there are arbitrarily large n such that $\chi_N(n)\notin S^n_{f(n)}$. Since $2^{\aleph_0}<\aleph_\omega$, there is an algebra $\mathfrak{A}$ on $\aleph_\omega$ such that if $2^{\aleph_0}\subset N\prec\mathfrak{A}$ then for all f, $N\prec\mathfrak{A}_f$.

Since the original sequence of S^n is mutually stationary we can find an N such that $N\prec\mathfrak{A}$, $2^{\aleph_0}\subset N$ and for all $\aleph_n>2^{\aleph_0}$, $\chi_N(n)\in S^n$. Then this N determines an $f:\omega\to 2$ such that for all $\aleph_n>2^{\aleph_0}$, $\chi_N(n)\in S^n_{f(n)}$.

But $N\prec\mathfrak{A}_f$, a contradiction. □

We note that this result, in conjunction with Lemma 23, gives similar results for sequences $\langle S_n: n>0\rangle$ of ordinals of the appropriate cofinality.

Definition 29. Let $S^n \subset \aleph_n$ be a mutually stationary sequence of sets. We say that the sequence *weakly splits* if there is a partition of each S^n into S_0^n and S_1^n, and functions $f, g: \omega \to 2$ such that for infinitely many m, $f(m) \neq g(m)$, and both $\{S_{f(n)}^n : n \in \omega\}$ and $\{S_{g(n)}^n : n \in \omega\}$ are mutually stationary. Equivalently there is an infinite set $A \subset \omega$ and a sequence of partitions $S^n = S_0^n \cup S_1^n$ such that for $i=0$ or 1, the sequence T^n defined by taking $T^n = S_i^n$ for $n \in A$, and $T^n = S^n$ for $n \notin A$, is mutually stationary.

The sequence *splits* if and only if there is a partition of each S^n into S_0^n and S_1^n, and a function $f: \omega \to 2$ such that both $\{S_{f(n)}^n : n \in \omega\}$ and $\{S_{1-f(n)}^n : n \in \omega\}$ are stationary. Of course, in this case, we can always find S_i^n so that f is constantly 0.

We do not know whether every mutually stationary sequence of subsets of the $\aleph_n$'s can even be weakly split. However, we can show that if the continuum is less than $\aleph_\omega$, and there is a mutually stationary sequence $S_n \subset \omega_n$ $(n \in \omega)$ that does not weakly split, then there is an indecomposable ultrafilter on $\aleph_\omega$. This implies that every sequence of mutually stationary sets can be split in the constructible universe L.

Definition 30. Suppose that $A \subset \omega$, and that $\vec{U} = \langle U^n : n \in A \rangle$ and $\vec{T} = \langle T^n : n \in A \rangle$ are sequences of mutually stationary sets. We say that $\vec{U} \subset \vec{T}$ if and only if for all $n \in A$, $U^n \subset T^n$. If $\mathcal{F}$ is a collection of mutually stationary sequences of sets $\vec{T} = \langle T^n : n > k \rangle$, then we say that $\mathcal{F}$ is a *filter* provided that $\mathcal{F}$ is closed upwards under $\subset$, and whenever $\vec{T}$, $\vec{U}$ are elements of $\mathcal{F}$, there is a $\vec{V} \in \mathcal{F}$ such that for all large enough $n \in \omega$, $V^n \subset U^n \cap T^n$.

LEMMA 31. *Suppose that $2^{\aleph_0} < \aleph_\omega$ and $\vec{S} = \langle S^n : n > k \rangle$ is a mutually stationary sequence of sets that does not weakly split with $S^n \subset \omega_n \cap \mathrm{cof}(\leqslant \omega_k)$. Let $\mathcal{F}$ be the collection of sequences $\vec{T} \subset \vec{S}$ that are mutually stationary. Then $\mathcal{F}$ is a filter that is closed under $\subset$-descending sequences of uncountable cofinality.*

Proof. We first show that if $\vec{T}, \vec{U} \in \mathcal{F}$ then for some $k' \geqslant k$, $\langle U^n \cap T^n : n > k' \rangle$ is mutually stationary.

Since $\vec{S}$ does not weakly split, for each infinite $A \subset \omega$ we can find a structure $\mathfrak{A}_A$ such that for all $N \prec \mathfrak{A}_A$ with $\chi_N(n) \in S^n$ for all $n > k$, there are infinitely many $n \in A$ such that $\chi_N(n) \in T^n$. Putting all of the $\mathfrak{A}_A$ together into one structure we can find a structure $\mathfrak{A}_0$ such that if $N \prec \mathfrak{A}_0$, $2^{\aleph_0} \subset N$ and for all large n, $\chi_N(n) \in S^n$, then for all but finitely many n, $\chi_N(n) \in T^n$.

Similarly we can find an $\mathfrak{A}_1$ such that for all $N \prec \mathfrak{A}_1$ with $2^{\aleph_0} \subset \mathfrak{A}_1$ and for all large enough n, $\chi_N(n) \in S^n$, then for all but finitely many n, $\chi_N(n) \in U^n$.

Putting $\mathfrak{A}_0$ and $\mathfrak{A}_1$ together we can find an algebra $\mathfrak{A}$ such that for all $N \prec \mathfrak{A}$, if $\chi_N(n) \in S^n$ for all large enough n and $2^{\aleph_0} \subset N$, then $\chi_N(n) \in T^n \cap U^n$ for all but finitely many n.

If $\mathfrak{B}$ is any structure expanding $\mathfrak{A}$, by the mutual stationarity of $\vec{S}$ there is an $N \prec \mathfrak{B}$ such that $\chi_N(n) \in S^n$ for all large n. Since $\mathfrak{B}$ is an expansion of $\mathfrak{A}$, we know that for all large enough n, $\chi_N(n) \in U^n \cap T^n$. By the argument of Lemma 23, we see that the sequence $\langle U^n \cap T^n : k \leqslant n \in \omega \rangle$ is mutually stationary.

Now suppose that $\langle \vec{T}_\alpha : \alpha < \varkappa \rangle$ is a $\subset$-descending sequence of elements of $\mathcal{F}$ for some cardinal $\varkappa$, $\omega < \varkappa < \aleph_\omega$. Suppose that $\vec{T}_\alpha = \langle T^n_\alpha : n > k \rangle$. Define $U^n = \bigcap \{T^n_\alpha : \alpha < \varkappa\}$. It suffices to show that for some L, the sequence $\vec{V}$ defined by setting $V^n = U^n$ for $n > L$, and $V^n = S^n$ for $k < n \leqslant L$, is mutually stationary.

For each α there is a structure $\mathfrak{A}_\alpha$ such that if $N \prec \mathfrak{A}_\alpha$ and $\chi_N(n) \in S^n$ for all $n > k$, there is an l such that for all $n > l$, $\chi_N(n) \in T^n_\alpha$. Let $\mathfrak{A}$ be an algebra such that if $N \prec \mathfrak{A}$ and $\varkappa \subset N$ then for all $\alpha < \varkappa$, $N \prec \mathfrak{A}_\alpha$. We can also assume that if some U^n is non-stationary, then for any $N \prec \mathfrak{A}$, $\chi_N(n) \notin U^n$.

Let $\mathfrak{B}$ expand $\mathfrak{A}$ and suppose that $N \prec \mathfrak{B}$, $\varkappa \subset N$ and $\chi_N(n) \in S^n$ for all $n > k$, $\omega_n > \varkappa$. For each $\alpha < \varkappa$, there is an $l_\alpha \in \omega$ such that for all $n > l_\alpha$, $\chi_N(n) \in T^n_\alpha$. Since $\varkappa$ is regular, there is an L such that for cofinally many α, $l_\alpha = L$.

Since the sequence of $\vec{T}_\alpha$'s is decreasing, this implies that for all $\alpha < \varkappa$ and all $l > L$, $\chi_N(l) \in T^n_\alpha$.

For each $\mathfrak{B}$, let $L(\mathfrak{B})$ be the least L such that there is an $N \prec \mathfrak{B}$ such that for all $l > L$, $\chi_N(l) \in U^l$, and for all $n > k$, $\chi_N(n) \in S^n$. Then $L(\mathfrak{B}) \leqslant L(\mathfrak{B}')$ if $\mathfrak{B}'$ is an expansion of $\mathfrak{B}$. Hence there is an $L \in \omega$ such that for all $\mathfrak{B}$, $L(\mathfrak{B}) \leqslant L$.

Define $\vec{V}$ as above, using this L. Then for all $\mathfrak{B}$, there is an $N \prec \mathfrak{B}$ such that $\chi_N(n) \in V^n$, $n > k$. Hence $\vec{V}$ is mutually stationary and in $\mathcal{F}$. □

THEOREM 32. *Suppose that $2^{\aleph_0} < \aleph_\omega$, and that there is a mutually stationary sequence $S^n \subset \omega_n$ that does not weakly split. Then there is an ultrafilter on $\aleph_\omega$ that is $\varkappa$-indecomposable for all $\varkappa$ between $2^{\aleph_0}$ and $\aleph_\omega$.*

COROLLARY 33. *If there is a mutually stationary sequence that does not weakly split, then there is an inner model with a measurable cardinal (see* [D]*). In particular, if $V = L$, every sequence of mutually stationary sets weakly splits.*

Proof of Theorem 32. We construct an ultrafilter $\mathcal{V}$ on $\aleph_\omega$ that is $\varkappa$-descendingly complete for all $\varkappa$ between $2^{\aleph_0}$ and $\aleph_\omega$.

Let $\mathcal{U}$ be an ultrafilter on ω. For each set $X \subset \aleph_\omega$, associate the sequence $X_n = (X \cap S^n) \setminus \omega_{n-1}$. Define an ultrafilter $\mathcal{V}$ on $\aleph_\omega$ by putting $X \in \mathcal{V}$ if and only if there is a set $A \in \mathcal{U}$ such that the sequence

$$Y^n = \begin{cases} X^n & \text{if } n \in A, \\ S^n & \text{otherwise,} \end{cases}$$

is mutually stationary. We show that $\mathcal{V}$ is an ultrafilter.

We first note that Lemma 31 implies that $\mathcal{V}$ is a filter. For if X, X' are subsets of $\aleph_\omega$ that lie in $\mathcal{V}$, then we can take $A, B \in \mathcal{U}$ witnessing this. By Lemma 31, $A \cap B$ witnesses that $X \cap X' \in \mathcal{V}$.

Moreover, an application of Theorem 28 shows that if $X \subset \aleph_\omega$ then there is an $A \in \mathcal{U}$ such that $\bigcup\{X \cap S^n : n \in A\} \in \mathcal{V}$ or $\bigcup\{S^n \setminus X : n \in A\} \in \mathcal{V}$. Hence $\mathcal{V}$ is an ultrafilter.

Suppose that $\langle X_\alpha : \alpha < \varkappa\rangle$ is a decreasing sequence of elements of $\mathcal{V}$. Then we can choose sets $\langle A_\alpha : \alpha < \varkappa\rangle \subset \mathcal{U}$ such that A_α witnesses that $X_\alpha \in \mathcal{U}$. Since $\varkappa > 2^\omega$, there is a set $A \in \mathcal{U}$ such that for cofinally many $\alpha < \varkappa$, A is a witness that $X_\alpha \in U$. It follows that A is a witness for a tail of $\alpha < \varkappa$. Define Y^n_α using A as in the definition of $\mathcal{V}$.

Then the sequence $\langle \vec{Y}_\alpha : \alpha < \varkappa\rangle$ is a subset of the $\mathcal{F}$ defined in Lemma 31. Hence there is a sequence $\vec{Z} \subset \vec{Y}_\alpha$ for all α such that $\vec{Z} \in \mathcal{F}$. Define $X_\varkappa = \bigcup_{n \in A} Z_n$. Then $X_\varkappa \in \mathcal{V}$ and $X_\varkappa \subset X_\alpha$ for all $\alpha < \varkappa$. □

We can present a sufficient condition for splitting.

Definition 34. Let $N \prec H(\theta)$. Then N is *tight* if and only if $N \cap \prod \aleph_n$ is cofinal in $\prod (N \cap \aleph_n)$.

It is an easy exercise to show that if $N \prec H(\theta)$ is internally approachable of length ν where ν is an uncountable regular cardinal, then N is tight.

With the GCH one can give a criterion for tightness using Shelah's PCF theory. (There are more complicated variants of this criterion with the failure of the GCH.) We give this criterion next, but will not use it for the results at the end of this section.

Fix a continuous scale $\{f_\alpha : \alpha < \aleph_{\omega+1}\} \subset \prod \aleph_n$. Recall [Sh2] that a point α of uncountable cofinality is *good* if and only if there is a cofinal set $A \subset \alpha$ and an m such that for all $n > m$, $\{f_\beta(n) : \beta \in A\}$ is increasing. In this case there is an exact upper bound for the sequence $\{f_\beta : \beta \in \alpha\}$, and it is given by $f_\alpha(m) = \sup\{f_\beta(m) : \beta \in A\}$.

Shelah has shown that there is always a stationary set of good points in every uncountable cofinality (or see e.g. the section of this paper covering the case where $\mathrm{cf}(\lambda) < \varkappa$). Further, under the CH, every point α of cofinality at least ω_2 is good, and if very weak square principles hold, then almost every point of cofinality ω_1 is good [FM2].

We now remark that, for an $N \prec H(\theta)$ with the property that there is an uncountable ν such that for all n, $\mathrm{cf}(\sup(N \cap \aleph_n)) = \nu$, the following are equivalent:

(1) N is tight.

(2) If $\gamma = \sup(N \cap \aleph_{\omega+1})$, then γ is good, and $\chi_N \equiv f_\gamma$ almost everywhere.

Unfortunately it is not true in general that almost all N of the right cofinality are tight as the following example shows:

Example 35 (due to Zapletal). Suppose that the GCH holds, and let $\mathfrak{A}$ be an algebra on $H(\theta)$. Choose a good $\gamma<\aleph_{\omega+1}$ of cofinality (say) ω_1 such that $\mathrm{sk}^{\mathfrak{A}}(\gamma)\cap\aleph_{\omega+1}=\gamma$. Let $M=\mathrm{sk}^{\mathfrak{A}}(\gamma)$ and build an increasing chain of elementary substructures $\langle N_\alpha:\alpha<\omega_1\rangle$ of M such that $f_\gamma\subset N_0$, N_0 is cofinal in γ, and for $\alpha<\beta$ and all n, $\chi_{N_\alpha}(n)<\chi_{N_\beta}(n)$. If we let $N=\bigcup_{\alpha<\omega_1}N_\alpha$, then for all n, $\mathrm{cf}(\chi_N(n))=\omega_1$, $\sup(N\cap\aleph_{\omega+1})=\gamma$, but for all n, $\chi_N(n)>f_\gamma(n)$.

We now define a more restrictive version of mutual stationarity, which is more tractable than the original.

Definition 36. A sequence of sets $\{S_n:n\in\aleph_n\}$ is tightly stationary if and only if for all algebras $\mathfrak{A}=\langle H(\theta),\in,\Delta,\{S^n\},\ldots\rangle$ there is a tight $N\prec\mathfrak{A}$ such that for all n, $\chi_N(n)\in S^n$.

Tightly stationary sequences are invariant under similar finite variations as mutually stationary sequences.

For tightly stationary sequences of sets we can show both a version of Fodor's Theorem and a splitting theorem.

THEOREM 37 (Fodor's Lemma). *Suppose that $\langle S^n:k<n<\omega\rangle$ is a sequence of tightly stationary sets. If $f:\aleph_\omega\to\aleph_\omega$ is regressive (i.e. for all α, $f(\alpha)<\alpha$), then there is a function $g\in\prod_{n\in\omega}\aleph_n$ such that if $T^n_g=\{\alpha\in S^n:f(\alpha)<g(n)\}$, then the sequence $\langle T^n_g:k<n<\omega\rangle$ is tightly stationary.*

Again we remark that this lemma can be generalized. However, if the set K of cardinals we are considering has limit points inside K, we view the function f as a function on pairs.

Proof of Theorem 37. Suppose that the theorem is false. Then for each $g\in\prod_{n\in\omega}\aleph_n$, there is an algebra $\mathfrak{B}_g$ on $\aleph_\omega$ witnessing the failure of the tight stationarity of the sequence $\langle T^n_g:k<n<\omega\rangle$.

Let $\mathfrak{A}=\langle H(\theta),\in,\Delta,\langle S^n\rangle,\{f\},\langle\mathfrak{B}_g:g\in\prod_{n\in\omega}\aleph_n\rangle,\ldots\rangle$. Let N be a tight elementary substructure of $\mathfrak{A}$ such that for all $n>k$, $\chi_N(n)\in S^n$.

Consider the function $h\in\prod\aleph_n$ defined by $h(n)=f(\chi_N(n))$. (In other words, h enumerates the values of f on the sequence of ordinals $\langle\sup(N\cap\aleph_n)\rangle$.) Then for all $n>k$, $h(n)<\chi_N(n))$, and hence by tightness there is a $g\in N\cap\prod\aleph_n$ such that for all $n>k$, $g(n)>h(n)$. But then $N\cap\aleph_\omega\prec\mathfrak{B}_g$, N is tight, and for all $n>k$, $\chi_N(n)\in T^n_g$, a contradiction. $\square$

We can now show a splitting theorem:

THEOREM 38. *Let $\langle S^n : k<n<\omega\rangle$ be a tight sequence of stationary sets, and suppose that for all n, $S^n \subset \mathrm{cf}(\aleph_k)$. Then there is a partition of each S^n into $\{S^n_\beta : \beta<\aleph_k\}$ such that for all β the sequence $\langle S^n_\beta : k<n<\omega\rangle$ is tightly stationary.*

Proof. Let $\mu=\aleph_k$. For each $\alpha\in S^n$ choose a sequence $\langle\alpha_\delta : \delta<\mu\rangle$ increasing and cofinal in α. By Fodor's Lemma for tightly stationary sequences, for each δ, we can find a function $g_\delta\in\prod\aleph_n$ such that if we set

$$T^n_{g_\delta,\delta}=\{\alpha\in S^n : \alpha_\delta<g_\delta(n)\},$$

then the sequence of $T^n_{g_\delta,\delta}$'s is tightly stationary. Since $\prod_{n>k}\aleph_n$ is μ-closed (under everywhere domination), there is a single function $g\in\prod_{n>k}\aleph_n$ such that for all $\delta<\mu$, $T^n_{g,\delta}=\{\alpha\in S^n : \alpha_\delta<g(n)\}$ is tightly stationary.

Note that the $T^n_{g,\delta}$ are decreasing as δ is increasing. We claim that there are cofinally many pairs $\delta<\delta'<\mu$ such that the sequence $T^n_{g,\delta}\setminus T^n_{g,\delta'}$ is tightly stationary. This clearly suffices for the theorem.

If the claim failed, then for all sufficiently large $\delta<\delta'$ (say $\delta,\delta'>\delta_0$) there is an algebra $\mathfrak{B}_{\delta,\delta'}$ such that if $N\prec\mathfrak{B}_{\delta,\delta'}$ is tight and for all $n>k$, $\chi_N(n)\in T^n_{g,\delta}$, then for infinitely many n, $\chi_N(n)\in T^n_{g,\delta'}$. Again we can find an algebra $\mathfrak{B}$ on $\aleph_\omega$ such that if $N\prec\mathfrak{B}$ and $\mu\subset N$, then for all $\delta<\delta'<\mu$, $N\prec\mathfrak{B}_{\delta,\delta'}$.

By construction, the sequence T^n_{g,δ_0} is tightly stationary, so we can find stationarily many tight $N\prec\langle H(\theta),\in,\Delta,\mathfrak{B},\{g\},...\rangle$ such that for all $n>k$, $\chi_N(n)\in T^n_{g,\delta_0}$. Fix such an N. For $\delta_0<\delta$, let $X_\delta=\{n:\chi_N(n)\in T^n_{g,\delta}\}$. Since the sequence $T^n_{g,\delta_0}\setminus T^n_{g,\delta}$ is not tightly stationary, and $N\prec\mathfrak{B}$, for all δ, $|X_\delta|=\omega$. Then for $\delta<\delta'$, $X_{\delta'}\subset X_\delta\subset\omega$. Hence there is a δ_1 and a set $X\subset\omega$ such that for all $\delta>\delta_1$, $X_\delta=X$.

Let $n\in X$ and $\alpha=\chi_N(n)$. Then for all δ, $\alpha_\delta<g(n)$. Hence $\alpha\leqslant g(n)$. But $g\in N$, so $g(n)\in N$, a contradiction. □

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