ON THE ROOTS OF THE RIEMANN ZETA-FUNCTION

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It is the purpose of this paper to give an account of numerical calculations relating to the behavior of the Riemann zeta-function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \qquad (s = \sigma + i t)$$

on the critical line $\sigma = {}^1/_2$, $t > 0$. These results confirm those made previously by Gram [1], Hutchinson [2], Titchmarsh [3] and Turing [4] and extend these to the first 10,000 zeros of $\zeta(s)$. All these zeros have real parts equal to one half and are simple. Thus the Riemann Hypothesis is true at least for $t \leq 9878.910$. This extension of our knowledge of $\zeta({}^1/_2 + i t)$ is made possible by the use of the electronic computer known as the SWAC while it was the property of the United States National Bureau of Standards. Actually only a few hours of machine time was needed and much more could be done along the same lines by this or any other really high speed computer.

A brief history of previous results and contemplated calculations may be given as follows. The work of J. P. Gram (in 1902–4) was largely for real s. However, he gave the first ten roots of $\zeta(s)$ to 6 decimals and five further ones with less accuracy. He is also to be credited with a valuable observation, now known as Gram's Law, which may be stated as follows. Let n be a positive integer and let τ_n be the real positive root of the equation

$$\pi^{-1}\,\mathrm{Im}\,(\log\Gamma({}^1/_4 + \pi i \tau)) - \tau \log \pi = n.$$

We call τ_n the nth Gram point and the interval

$$I_n : (\tau_n, \tau_{n+1})$$

the nth Gram interval. Gram's Law states that $\zeta({}^1/_2 + 2\pi i \tau)$ has a single root in I_n. This law implies the Riemann Hypothesis and the verification of the latter depends largely on the verification of the former. Gram's Law, however, is known to fail for an infinity of n and, in fact, fails for about 800 values of n below 10,000. The first Gram interval not containing a root of $\zeta(^1/_2+2\pi i\tau)$ is I_{125}. However, I_{126} contains two roots. Similarly I_{133} contains two roots while I_{134} contains none. These facts were discovered in the neighborhoods of $t=282$ and $t=295$ by Hutchinson in 1925. He also extended Gram's list of zeros so as to include the first 29 below $t=100$, using 3 decimal accuracy. This list has never been extended. Further work has been in the direction of merely isolating and counting rather than approximating the roots. However, the early roots have received some attention lately. van der Pol [5] in 1947 constructed an electronic analog device for exhibiting the first 73 roots (for $t\leq 210$). Recent unpublished work of Haselgrove in England is being devoted to the determination of the early roots to considerable accuracy by electronic digital computation.

Returning to the work of Hutchinson, he showed that the first 138 roots of $\zeta(s)=0$ all have $\sigma=^1/_2$. Using a method of Backlund [6] he showed that inside the rectangular region

$$0<\sigma<1, \qquad 0<t<300.468$$

there are exactly 138 roots, and having found, by sign changes, at least 138 real roots of $\zeta(^1/_2+2\pi i\tau)=0$ he thus verified the non-existence of roots not on the abscissa $\sigma=^1/_2$ for $t<300$. As a basis for his computation, Hutchinson used the Euler-Maclaurin asymptotic expansion as suggested by Backlund

$$\zeta(s)=\sum_{\nu=1}^{m-1}\nu^{-s}+{}^1/_2\,m^{-s}+\frac{m^{1-s}}{s-1}+\sum_{\nu=1}^{k}T_\nu+R(k,m) \tag{1}$$

where

$$T_\nu=\frac{B_{2\nu}}{(2\nu)!}m^{1-s-2\nu}\prod_{j=0}^{2\nu-2}(s+j)$$

and $B_2=^1/_6$, $B_4=-^1/_{30}$, ... are the Bernoulli numbers with appropriate estimates, for the remainder $R(k,m)$. At $t=300$ the corresponding $m=O(t)$, of the order of 100, was occasionally needed and, with only a desk calculator available, the computation became too laborious.

To meet this difficulty, Titchmarsh, in 1935, employed another asymptotic expansion of Riemann and Siegel in which only $[t^{1/2}]$ terms are required. This formula may be given as follows.

Let

$$f(\tau)=e^{i\theta}\zeta(^1/_2+2\pi i\tau)$$

where

$$\theta = \theta(\tau) = -\pi\tau \log \pi + \operatorname{Im}(\log \Gamma(1/_4 + \pi\tau))$$
$$= \pi(\tau \log \tau - \tau - 1/_8 + (192\pi^2 \tau)^{-1} + \cdots),$$

then $f(\tau)$ is real and is given by

$$f(\tau) = 2 \sum_{\nu=1}^{m} \nu^{-1/2} \cos 2\pi(\varkappa - \tau \log \nu) + g(\tau) + R \qquad (2)$$

where

$$m = [\tau^{1/2}]$$
$$\varkappa = \theta/2\pi$$
$$g(\tau) = (-1)^{m-1} \tau^{-1/4} h(\xi)$$
$$h(\xi) = (\sec 2\pi\xi) \cos 2\pi\phi$$
$$\xi = \tau^{1/2} - m$$
$$\phi = \xi - \xi^2 + 1/_{16}.$$

Titchmarsh gave a elaborate estimate for $|R|$ approximately equal to $\tau^{-3/4}$ for τ large enough to be of interest. Under the supervision of L. J. Comrie, a study was made in 1936 of the first 1041 roots in which automatic computing, in the form of punched card equipment, was first applied to the problem. Titchmarsh found that all roots had $\sigma = 1/_2$ and so the Riemann Hypothesis was verified up to $t = 1468$. There were 43 failures of Gram's Law.

Plans to extend the work of Titchmarsh by use of a differential analyzer were made in 1939 by the late A. M. Turing. These were interrupted by the war and later rendered obsolete by the advent of the electronic digital computers.

In 1947 the writer programmed an extension of the work of Titchmarsh for the ENIAC, the only electronic computer then in operation. However, before the program could be run, the ENIAC was drastically modified thus rendering it useless for the problem.

In June 1950, Turing used the Manchester University Mark 1 electronic digital computer to examine the zeta-function for $24{,}937.96 < t < 25{,}735.93$ (that is for $63 < \sqrt{\tau} < 64$) and found in this region of the critical strip that there are about 1070 simple zeros all with $\sigma = 1/_2$. In another short run the validity of the Riemann Hypothesis was verified between Titchmarsh's upper limit of $t = 1468$ and $t = 1540$. Only some twenty hours of machine time was used. Unfortunately no further time was made available and these incomplete results were published in 1953.

In 1949 the writer had suggested the zeta-function problem to J. B. Rosser, then Director of the National Bureau of Standards Institute for Numerical Analysis, and it was eventually accepted as a low priority project for the SWAC some two years before the completion of the machine. The coding and much of the actual operation has since been done by Mrs. Ruth Horgan. The initial part of the project consisted in determining the first 5000 Gram points τ_n. This was done by solving for τ the equation

$$\tau \log \tau - \tau = n + {}^1/_8$$

for $n = 1\ (1)\ 5000$ using the iterative subroutine

$$\tau^{(k+1)} = (n + {}^1/_8)/(\log \tau^{(k)} - 1), \qquad (\tau^{(0)} = \tau_{n-1}).$$

Next the function $f(\tau_n)$ was evaluated without remainder, that is

$$f_1(\tau_n) = 2 \sum_{\nu=1}^{m} \nu^{-1/2} \cos 2\pi \left(\frac{n}{2} - \tau \log \nu\right) + g(\tau_n)$$

was evaluated using a Chebyshev approximation for the cosine function and one of two step functions for g. When Gram's Law holds

$$\operatorname{sgn} f(\tau_n) = (-1)^n. \tag{3}$$

Whenever $|f_1(\tau_n)|$ was below the estimate for $|R|$ in (2), so that the sign of $f(\tau_n)$ was in doubt, a special indication of this fact was put out by the machine. Another coded output was made when $|f_1(\tau_n)|$ was large enough to prove that (3) did not hold. Such failures (or possible failures) of Gram's Law were examined more closely by a further computation subdividing each of the Gram intervals I_{n-1} and I_n into 8 subintervals (equally spaced in the scale of n) with the expectation of finding sufficiently large values of $|f_1(\tau)|$ to indicate clearly the behavior of $f(\tau)$ in the combined interval (τ_{n-1}, τ_{n+1}). A study of these exceptional cases revealed about 360 failures in Gram's Law. In all but one interval the absence of a root was exactly compensated for by the presence of an extra root in an adjoining interval. In the case of I_{4763} the function $f_1(\tau)$ rises to a very low maximum less than the estimated $|R|$ and so (2) was inadequate to determine whether I_{4763} has two real roots or no real roots of $f(\tau)$.

The discovery of this interesting phenomenon led to two further programs: (a) an immediate analysis by the previous method of the next 5000 Grams points and (b) a new program based on (1), with a much smaller remainder than (2), for the more minute investigation of the zeta-function in a selected neighborhood.

Program (a) disclosed another doubtful Gram interval I_{6707} and about 450 additional failures of Gram's Law.

Program (b) disclosed that, in conformity with the Riemann Hypothesis, $f(\tau)$ has two real roots in each of I_{4673} and I_{6707} and no root in I_{4674} and I_{6708}.

An account of the failures of Gram's Law among the first 10,000 intervals can be given briefly as follows. All failures produce one Gram interval without a root and it is convenient to assign the failure to this interval as its location. All but 14 failures are of the simple type in which the "missing" root is found in one of the two adjacent intervals. Very often this displacement is a minor one and in 18 cases it was impossible to decide, without further calculation, whether such a displacement had actually occurred since $f(\tau_n)$ was too small in absolute value to determine its sign. The 14 failures mentioned above are such that the missing root is displaced into the next interval beyond the adjacent one. Thus for $n = 3358$, 3777, 4541, 5105, 6413, 6536, 6810, 7002, 7544 and 9609, there are two roots in I_{n-2}, one in I_{n-1} and none in I_n. There are two roots in I_{n+2}, one in I_{n+1} and none in I_n for $n = 4921$, 5491, 5816 and 5936.

The failures of Gram's Law become more and more frequent as n increases. The following table gives the number $E(n)$ of failures among the first n Gram intervals for $n = 1000\,(1000)\,10{,}000$. Numbers in parentheses refer to cases in which there is some doubt, as mentioned above.

n	$E(n)$	$\Delta E(n)$
1000	40 + (4)	72 + (7)
2000	112 + (11)	76
3000	188 + (11)	82 + (3)
4000	270 + (14)	86 + (1)
5000	356 + (15)	87 + (1)
6000	443 + (16)	93
7000	536 + (16)	87 + (2)
8000	623 + (18)	91
9000	714 + (18)	100
10000	814 + (18)	

In 419 failures the shift of the missing root was to the right; in 395 cases to the left.

Although in about 546 cases failures are more or less isolated, some 268 occur in conjunction with one or more other failures. For example, there are three roots in I_n for $n = 2145$, 4085, 4509, 6704, 9100 and possibly 7334, the two adjacent intervals I_{n-1} and I_{n+1} being devoid of roots. There are failures in I_{8997}, I_{8999}, I_{9002}, the numbers of roots in I_{8995} to I_{9004} being

$$1,\ 2,\ 0,\ 1,\ 0,\ 2,\ 1,\ 0,\ 2,\ 1.$$

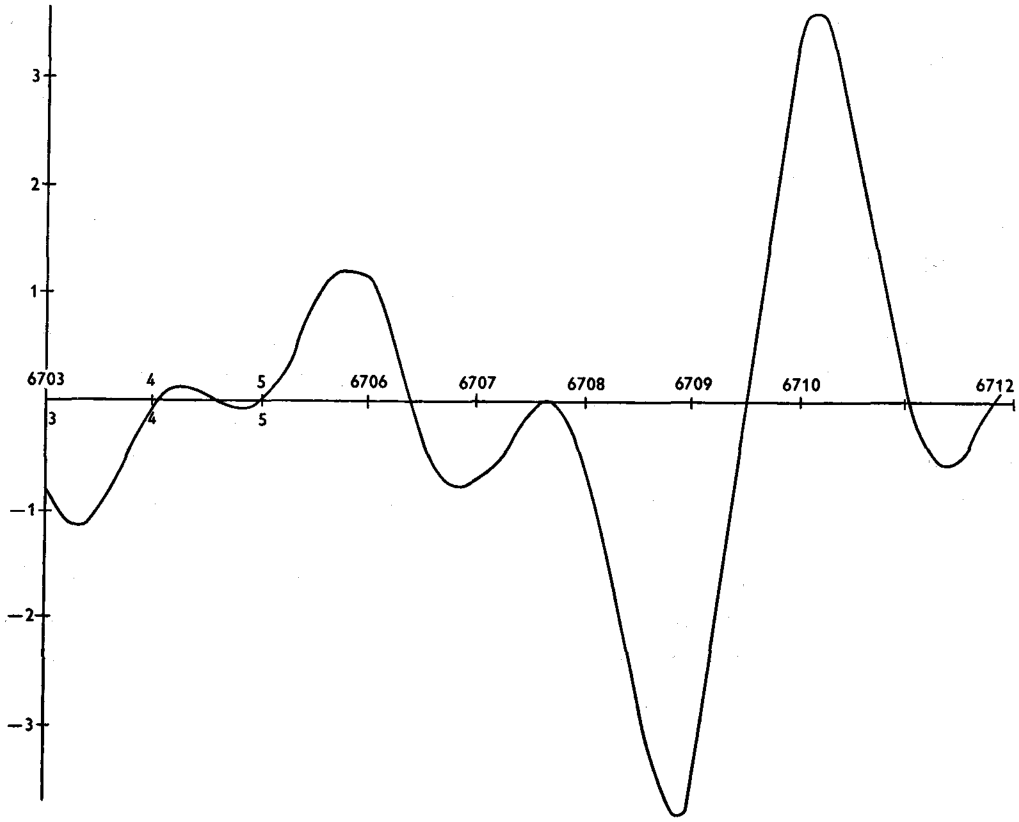

Fig. 1.

Eight similar phenomena occur in the neighborhood of I_n for $n = 4823$, 5747, 5935, 6154, 7003, 7545, 9483, 9581. There are 47 and possibly 49 cases in which failures occur in both I_n and I_{n+1} and 61 or 62 cases in which both I_n and I_{n+2} have failures.

The above anomalies in Gram's Law may be thought of as minor modulations in the frequency of the oscillations of $f(\tau)$. Perhaps of more interest to the problem of the Riemann Hypothesis is the question of the amplitude of these oscillations. What interests us especially are low maxima and high minima and the related matter of two nearly coincident roots.

For some 25 values of n, I_n has two roots differing by less than one fourth of I_n between which the extreme value of f has a particularly small absolute value. Five of these cases required a special fine mesh calculation to insure that two roots actually exist. These are the cases $n = 3775$, 4085, 4509, 4763 and 6707. In the first

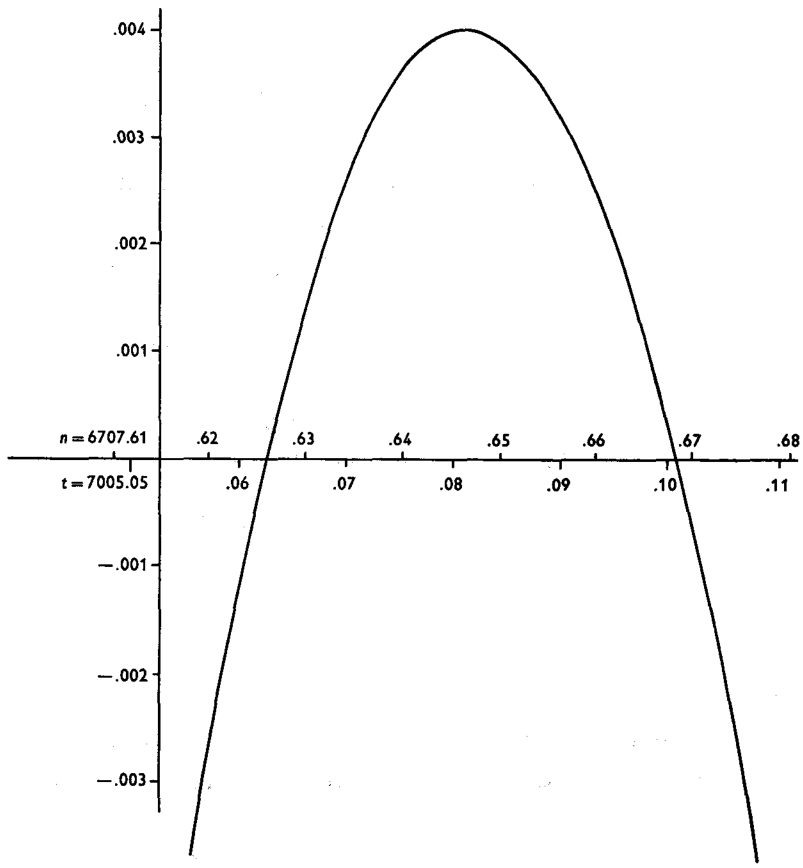

Fig. 2.

three cases extreme values of $f_1(\tau)$ were found sufficiently large in absolute value to exceed the estimated value of $|R|$ in (2). The last two cases remained uncertain and required the use of (1) as mentioned above. The number m of terms used was 1500 and 2000. Both real and imaginary parts of $\zeta(^1/_2 + 2\pi i \tau)$ were found at some 20 points in each case, the time required being about 30 seconds for each point. The results obtained were gratifyingly free from round-off errors. The behavior of $f(\tau)$ in the neighborhood of I_{6707} is illustrated in the accompanying two figures.

Figure 1 is a graph of $f(\tau_n)$ with n as a continuous independent variable between 6703 and 6711, showing three roots in I_{6704} and the low maximum in I_{6707}.

Figure 2 shows, greatly magnified, the critical region near $\tau = 1114.89$ ($n = 6707.6$) where $f(\tau)$ has its lowest maximum in the range of its first 10,000 roots. The actual maximum is only .0039675 and occurs at $\tau = 1114.89340$ ($n = 6707.64686$, $t = 7005.0819$). The two roots of $\zeta(s)$ in I_{6707} are

$$^1/_2 + i\ 7005.0629,$$
$$^1/_2 + i\ 7005.1006.$$

It is perhaps appropriate to make a few informal comments in conclusion about the significance of the above results in relation to the Riemann Hypothesis. These comments repeat to some extent those made already by Titchmarsh. A tendency for the behavior of the zeta-function to become more and more capricious as t increases is evident from the values thus far computed. Nearly one interval in ten now shows a departure from Gram's Law.

There are relations between the logarithms of the primes, especially the early ones, which, at least as far as $\tau = 1572$, prevent a negative maximum of a positive minimum of $f(\tau)$. Whether this conspiracy will ever break down is equivalent to the question of the truth or falsity of the Riemann Hypothesis. The low maximum of $f(\tau)$ at $\tau = 1114.89$ has a value approximately of the order of magnitude of the 25,000th term of the series Σn^{-s} so that there is a very delicate balance at this point. By inspecting the terms of $f(\tau)$ in I_{6707} one sees the first few rapidly rising terms being steadily beaten back by a conspiracy of slowly descending terms so that the resulting maximum is nearly negative, the contest being followed by a violent oscillation. This phenomenon, which has occurred before in a slightly less pronounced way, will occur infinitely often, by Kronecker's theorem, and investigations at suitable neighborhoods at immense distances up the critical strip may well prove to be the best future attack on the Riemann Hypothesis.

NOTE ADDED IN PROOF: Since the above was written a few more hours of machine time was used to examine the next 15000 roots.

The zeta-function's behavior becomes steadily "worse" but all roots have $\sigma = {}^1/_2$.

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