Regularity for stationary surfaces of constant mean curvature with free boundaries

by

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1. A partitioning problem. Boundary regularity of its solutions

Let $\mathscr{K}$ be a bounded convex body in $\mathbf{R}^3$ with boundary T. We consider the following *partitioning problem* $\mathscr{P}$:

Determine a rectifiable surface $\bar{S}$ of minimal or stationary area, with boundary Σ contained in T, which divides $\mathscr{K}$ into two parts $\mathscr{K}_1$ and $\mathscr{K}_2$ such that

$$\operatorname{meas} \mathscr{K}_1 = \sigma \operatorname{meas} \mathscr{K}, \quad \operatorname{meas} \mathscr{K}_2 = (1-\sigma) \operatorname{meas} \mathscr{K},$$

where σ denotes a preassigned constant with $0<\sigma<1$.

The existence theory for this problem is still in its infancy. Bokowski and Sperner jr. [2] have proved the existence of minimal partitionings of the ball, by employing tools from geometric measure theory, and in the same context Almgren [1] has proved existence and regularity almost everywhere of minimal partitionings of $\mathbf{R}^n$.

It seems intuitively clear that every solution of $\mathscr{P}$ must be a surface of constant mean curvature (in special situations: a minimal surface) perpendicular to the boundary T. This, in fact, will be confirmed here. The main aim of the present paper is the proof of boundary regularity for each solution of $\mathscr{P}$. In view of this emphasis, we shall already assume that the interior part S of $\bar{S}$ is a regular C^1-manifold which divides $\operatorname{int} \mathscr{K} - \mathscr{S}$ into two disjoint parts of preassigned measure. As a consequence, S has a conformal parameter representation on some Riemann surface B. Generally speaking, the surface S need not be of disc-type. It could, for instance, also have the topological type of an annulus, as is the case for a cylinder, a catenoid, or an arbitrary Delaunay surface.

Although our approach can easily be modified to handle this case or more general cases as well, we shall restrict ourselves here to solutions of $\mathcal{P}$ which can be parametrized over a disc.

In order to have a clear cut situation, we shall fix the assumptions which will be used during the proof of the subsequent boundary regularity theorem.

Assumption (A 1). (i) S has a conformal parameter representation $x\colon B\to\mathbf{R}^3$ of class $C^1(B,\mathbf{R}^3)$ on the unit disc B (that is, (2.7) holds).

(ii) S has finite area, so that its representation x is of class $H_2^1(B,\mathbf{R}^3)$. We assume that $\Sigma:=x(\partial B)$ is contained in T, where the boundary condition $\Sigma\subset T$ is to be interpreted in the sense of section 2 below.

(iii) The surface $S=x(B)$ omits a neighborhood of some point $p\in T$.

(iv) The map $x\colon B\to\mathbf{R}^3$ is an embedding of B into the interior of $\mathcal{K}$ such that

$$\operatorname{int}\mathcal{K}-S=\Omega_1\cup\Omega_2,\quad \Omega_1\cap\Omega_2=\emptyset,$$

where Ω_1 and Ω_2 are simply-connected regions and

$$\operatorname{meas}\Omega_1=\sigma\operatorname{meas}\mathcal{K},\quad \operatorname{meas}\Omega_2=(1-\sigma)\operatorname{meas}\mathcal{K}.$$

We then shall prove the following

THEOREM 1. *Let T be a regular surface of class C^3, or $C^{m,\alpha}(m\geq 3,\ 0<\alpha<1)$, or C^ω, respectively. Suppose that $S=\{x(w)\in\mathbf{R}^3\colon w\in B\}$ satisfies assumption* (A 1) *as well as the following variational property* (A 2):

S is stationary for the area functional (Dirichlet integral) in the class of all disc-type surfaces with boundary on T which partition $\mathcal{K}$ into two parts of measures $\sigma\operatorname{meas}\mathcal{K}$ and $(1-\sigma)\operatorname{meas}\mathcal{K}$ where $0<\sigma<1$.

Then $x(w)$ is a real-analytic surface of constant mean curvature on B. Moreover, $x(w)$ is of class $C^{1,\beta}(\bar{B},\mathbf{R}^3)$ for every $\beta\in(0,1)$, or of class $C^{m,\alpha}(\bar{B},\mathbf{R}^3)$, or real-analytic on $\bar{B}$, respectively, and $\bar{S}=S\cup\Sigma$ intersects T orthogonally in the points of $\Sigma=x(\partial B)$.

As it stands, assumption (A 2) seems to lack precision. This precision will be supplied in section 4 below.

It should be remarked that the convexity of the surface T is not essential for the proof of Theorem 1. Our main reason for using this assumption is to obtain a simple form of assumption (A 1).

We should also like to mention that, for area minimizing solutions of our isoperimetric problem, the assertions of Theorem 1 are consequences of results proved in [13]

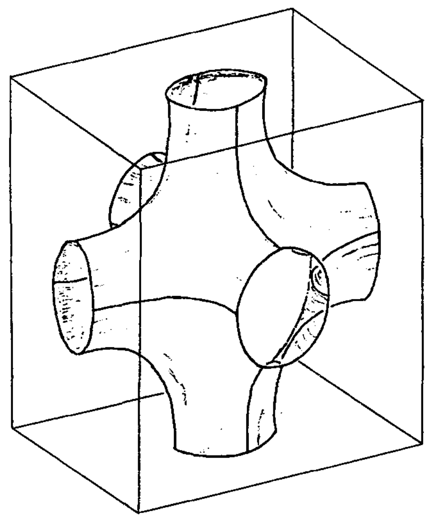

Fig. 1.

and [17]. Thus the novelty of our statement lies in the fact that we extend the consideration to stationary surfaces which may not necessarily be area minimizing. For instance, two of the three ellipses generated by the intersection of a three-axes ellipsoid $\mathscr{E}$ with its principal planes are only stationary solutions of the partitioning problem for $\mathscr{E}$ (with $\sigma=\frac{1}{2}$), whereas the third ellipse is area minimizing.

We note that, for a given value of the parameter σ, there may exist solutions of the partitioning problem $\mathscr{P}$ with different mean curvature. For an ellipsoid, this phenomenon has been discussed in [22]. To cite another example, if $\mathscr{K}$ is a solid cube, $\mathscr{P}$ is solved by three planar minimal surfaces as well as by hemispheres. Still other solutions are provided by spherical caps around the vertices and by cylindrical caps about the edges of the cube. If one would also admit surfaces of higher topological type as solutions of the partitioning problem, infinitely many solutions of different type may exist for the same value of the parameter σ; see [22]. A particular example is sketched in figure 1. It is part of a periodic minimal surface found by H. A. Schwarz.

The proof of Theorem 1 will be given in section 4. It rests on a regularity result for an auxiliary problem. This auxiliary problem, which is also of independent interest, is formulated in section 2. The pertinent Regularity Theorems 2 and 3 are stated and proved in section 3. In the proofs a technique is employed which has already been used by the authors to demonstrate the regularity of stationary minimal surfaces with free boundaries; see [9], and also [4] and [5].

We finally mention that Grüter and Jost recently have proved the existence of an embedded disc-type minimal surface S within an arbitrary closed and convex surface T of class C^4 that meets T at a right angle. [1] In an earlier paper [27], Struwe had shown the existence of a stationary minimal surface within T without reference to the question of embeddedness.

2. An auxiliary problem

We identify the two-dimensional Euclidean space $\mathbf{R}^2$ with $\mathbf{C}$ and write accordingly $w=(u,v)=u+iv$ for the points of $\mathbf{R}^2$. The open unit disc $B=\{w\in\mathbf{C}: |w|<1\}$ will be chosen as the parameter domain for the surfaces $x: B\to\mathbf{R}^3$, $x(w)=(x^1(w), x^2(w), x^3(w))$, which will be considered in the sequel.

The supporting surfaces T admissible for our discussion are two-dimensional submanifolds of $\mathbf{R}^3$ satisfying the following

Assumption (V). There are numbers $\varrho_0>0$, $K_0\geq 0$, and $K\geq 1$ such that the following holds:

For each point $x_0\in T$ there is a (full) neighborhood U of x_0 in $\mathbf{R}^3$ and a C^2-diffeomorphism $x=h(y)$ of $\mathbf{R}^3$ onto itself with the following two properties. Firstly, the inverse h^{-1} maps x_0 onto 0 and U onto the open ball $\{y: |y|<\varrho_0\}$ such that $T\cap U$ corresponds to the set $\{y: |y|<\varrho_0, y^3=0\}$ on the hyperplane $\{y^3=0\}$. Secondly, if we set

$$g_{ij}(y) = \delta_{kl}\, h^k_{y^i}(y)\, h^l_{y^j}(y), \tag{2.1}$$

then

$$K^{-1}|\xi|^2 \leq g_{ij}(y)\,\xi^i\xi^j \leq K|\xi|^2 \tag{2.2}$$

for all $\xi, y\in\mathbf{R}^3$, as well as

$$\left|\frac{\partial g_{ij}}{\partial y^k}\right|(y) \leq K_0 \tag{2.3}$$

for all $y\in\mathbf{R}^3$ and $i,j,k\in\{1,2,3\}$.

Every compact C^2-manifold in $\mathbf{R}^3$ satisfies assumption (V). Each submanifold of $\mathbf{R}^3$, compact or noncompact, for which assumption (V) holds, is a complete Riemannian manifold with respect to the induced metric of $\mathbf{R}^3$. For noncompact surfaces T,

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(1) M. Grüter and J. Jost, "On embedded minimal disks in convex bodies", and "Allard-type regularity results for varifolds with free boundaries". Preprints 1984.

assumption (V) imposes a certain uniformity condition on the metric $ds^2=g_{ij}(y)\,dy^i\,dy^j$ at infinity and is thus somewhat more stringent than the sole condition $T\in C^2$. Therefore, a C^2-submanifold of $\mathbf{R}^3$ satisfying (V) will be called a *strict C^2-surface* in $\mathbf{R}^3$.

We, moreover, note that every point $x\in\mathbf{R}^3$ possesses a foot f on T such that $|x-f|=\operatorname{dist}(x, T)$.

Next, we introduce the class $\mathscr{C}(T)$ of admissible surfaces $S=\{x(w)\colon w\in B\}$ with boundary on T by the stipulation

$$\mathscr{C}(T)=\{x(w)\colon x\in H_2^1(B,\mathbf{R}^3),\ x(\partial B)\subset T\}. \tag{2.4}$$

Here the inclusion "$x(\partial B)\subset T$" means that the $L^2(\partial B)$-trace of the Sobolev function x on ∂B maps $\mathscr{H}^1$-almost all points of ∂B into T.

Let $Q\in C^{1,\beta}(\mathbf{R}^3,\mathbf{R}^3)$, $0<\beta<1$, be a vector field on $\mathbf{R}^3$ satisfying $\sup_{\mathbf{R}^3}\{|Q|+|Q_x|\}<\infty$. For any subset Ω of B and for any function $x\in H_2^1(B,\mathbf{R}^3)$ we introduce the two functionals

$$D_\Omega[x]=\frac{1}{2}\int\!\!\int_\Omega |\nabla x|^2\,du\,dv \quad \text{("Dirichlet integral")} \tag{2.5}$$

and

$$V_\Omega^Q[x]=\int\!\!\int_\Omega Q(x)\cdot(x_u\wedge x_v)\,du\,dv. \tag{2.6}$$

If $\Omega=B$, the abbreviations $D_B[x]=D[x]$, $V_B^Q[x]=V^Q[x]$ are used.

In our applications, it will be known that the surface S omits an open $\mathbf{R}^3$-neighborhood $U(p)$ of some point $p\in T$. It is then sufficient that the vector field $Q(x)$ has the stated properties in $\mathbf{R}^3-U_1(p)$ where $U_1(p)$ is a suitable smaller neighborhood of p contained in $U(p)$.

A surface $x(w)$ of class $H_2^1(B,\mathbf{R}^3)$ is said to be parametrized conformally if

$$|x_u|^2=|x_v|^2,\quad x_u\cdot x_v=0 \quad \text{a.e. in } B. \tag{2.7}$$

It is further necessary to define the concept of an *admissible variation of a surface* $x\in\mathscr{C}(T)$. By this we mean a family of surfaces $x_t\in\mathscr{C}(T)$, $|t|<t_0$ for some number $t_0>0$, where $\{x_t\}_{|t|<t_0}$ is of one of the following two types:

Type 1. The surfaces x_t are of the form $x_t=x(\tau_t)$ where $\{\tau_t\}_{|t|<t_0}$ is a family of diffeomorphisms from $\bar{B}$ to itself such that τ_0 is the identity and that $\tau(w,t)=\tau_t(w)$ is of class C^1 on $\bar{B}\times(-t_0,t_0)$.

Type 2. The surfaces x_t are of the form

$$x_t(w) = x(w) + t\Psi(w, t) \tag{2.8}$$

where

$$D[\Psi(\cdot, t)] \leq C \quad \text{for } |t| < t_0 \tag{2.9'}$$

with a bound C independent of t, and

$$\Psi(w, t) \to \Phi(w) \quad \text{as } t \to 0 \quad \text{for a.a. } w \in B, \tag{2.9''}$$

for some $\Phi \in H_2^1(B, \mathbf{R}^3)$.

Let $\{x_t\}_{|t|<t_0}$ be an admissible variation of a surface $x \in \mathscr{C}(T)$. Then the vector field

$$\Phi(w) = \lim_{t \to 0} \frac{1}{t} \{x_t(w) - x(w)\} \tag{2.10}$$

exists for a.a. $w \in B$ and is tangential to T at $x(w)$ for $\mathscr{H}^1$-almost all $w \in \partial B$, and we obtain

$$\lim_{t \to 0} \frac{1}{t} \{D[x_t] - D[x]\} = \int\int_B \nabla x \cdot \nabla \Phi \, du \, dv \tag{2.11}$$

and

$$\begin{aligned} \lim_{t \to 0} \frac{1}{t} \{V^Q[x_t] - V^Q[x]\} = & \int\int_B Q(x) \cdot [x_u \wedge \Phi_v + \Phi_u \wedge x_v] \, du \, dv \\ & + \int\int_B [Q_x(x)\, \Phi] \cdot (x_u \wedge x_v) \, du \, dv. \end{aligned} \tag{2.12}$$

In the following sections, we shall consider stationary points x of the Dirichlet integral in the class $\mathscr{C}(T)$ which are subjected to the subsidiary condition $V^Q[x]$=constant. It will then turn out that such surfaces are also stationary points, in the class $\mathscr{C}(T)$, of the modified functional

$$D[x] + \mu V^Q[x] \tag{2.13}$$

for some real number μ.

Suppose now that x is a stationary point of (2.13) in $\mathscr{C}(T)$. Employing admissible variations of type 1, we may then prove that (2.7) holds (see [21], § 299). If, in addition, the surface $x(w)$ is of class $C^2(B, \mathbf{R}^3) \cap C^1(\bar{B}, \mathbf{R}^3)$, it follows by a familiar argument that

$$\Delta x = \mu \operatorname{div} Q(x)\, x_u \wedge x_v \quad \text{in } B \tag{2.14}$$

and

$$\mu Q(x) \cdot n(x) = \cos \alpha(x) \quad \text{on } \partial B. \tag{2.15}$$

Here $n(x)$ denotes the outward normal unit vector of T at $x \in T$, and $\alpha(x)$ is the angle in which the surface S (described by the representation $x=x(w)$, $w \in B$) meets T at the point x. In view of (2.7) and (2.14) it is clear that the mean curvature $\mathcal{H}$ of our surface S in the point $x(w)$ is given by

$$\mathcal{H}(w) = H(x(w)) \quad \text{for all } w \in B,$$

where the function $H \in C^{0,\beta}(\mathbf{R}^3, \mathbf{R})$ is related to $Q(x)$ through

$$H(x) = \frac{\mu}{2} \operatorname{div} Q(x). \tag{2.16}$$

We shall investigate the question whether such stationary points of $D[x]$, or of $D[x]+\mu V^Q[x]$, are indeed of class C^1 up to the boundary ∂B. It will turn out that such a regularity theorem can be proved if the surface $x(w)$ meets the supporting surface everywhere at a right angle. By virtue of (2.15), this will be assured if the vector field Q is tangential to T along the trace $\Sigma = \{x(w) : w \in \partial B\}$ of $x(w)$ on T. It will be more convenient to assume that Q is tangential to T in a whole neighborhood of Σ on T. We mention already here that the concrete vector field Q leading to the volume functional V^Q, appropriate for our partitioning problem, has this property.

3. Regularity for the stationary solutions of the auxiliary problem

Throughout this section, we assume that $x \in \mathscr{C}(T)$ is a (conformally parametrized) stationary point of the Dirichlet integral in the class $\mathscr{C}(T) \cap \{z : V^Q[z]=c\}$, that is,

$$\lim_{t \to 0} \frac{1}{t} \{D[x_t] - D[x]\} = 0 \tag{3.1}$$

for every admissible variation x_t of x in $\mathscr{C}(T)$ which satisfies the subsidiary condition

$$V^Q[x_t] = c. \tag{3.2}$$

Moreover, the following assumptions will be made:

(i) There is an open $\mathbf{R}^3$-neighborhood $U(p)$ of some point $p \in T$ such that $x(B) \subset \mathbf{R}^3 - U(p)$.

(ii) The supporting surface T is a strict C^2-surface.

(iii) For a suitable open neighborhood $U_1(p) \subset\subset U(p)$, Q is of class $C^1(\mathbf{R}^3 - U_1(p), \mathbf{R}^3)$ and satisfies the inequalities $0 < |\operatorname{div} Q| \leq H_0 < \infty$ on $\mathbf{R}^3 - U_1(p)$ for some constant $H_0 > 0$.

(iv) $Q|_T$ is a tangential vector field on $T - U_1(p)$.

(v) $x(w) \not\equiv$ constant in B.

Assumption (v) implies that $D[x] > 0$. Then, for each $w_0 \in B$, there is a number r_1 with $0 < r_1 < \frac{1}{2} \operatorname{dist}(w_0, \partial B)$ such that

$$D_{B_{2r_1}(w_0)}[x] < \tfrac{1}{2} D[x].$$

Thus, by taking (2.7) into account, we find an open disc $\Omega^* \subset\subset B - B_{2r_1}(w_0)$ for which

$$\iint_{\Omega^*} |x_u \wedge x_v| \, du \, dv = D_{\Omega^*}[x] > 0.$$

Without loss of generality, it can be assumed that

$$\iint_{\Omega^*} |J_1| \, du \, dv > 0, \tag{3.3}$$

where $x_u \wedge x_v = \{J_1, J_2, J_3\}$. We now choose $\psi = \lambda \zeta e_1$. Here $\lambda \neq 0$ is a real number, $e_1 = \{1, 0, 0\}$, and $\zeta \in C_c^\infty(\Omega^*, \mathbf{R}^3)$ denotes a nonconstant function to be determined later.

Let us set

$$v(t) = V_{\Omega^*}^Q[x + t\psi]. \tag{3.4}$$

An integration by parts using (2.12) leads to

$$\begin{aligned} v'(0) &= \iint_{\Omega^*} \operatorname{div} Q(x) \, \psi \cdot (x_u \wedge x_v) \, du \, dv \\ &= \lambda \iint_{\Omega^*} \operatorname{div} Q(x) \, \zeta J_1 \, du \, dv. \end{aligned} \tag{3.5}$$

In view of (3.3) and assumption (iii), we can find a function $\zeta \in C_c^\infty(\Omega^*, \mathbf{R}^3)$ for which $v'(0) \neq 0$. By an appropriate choice of λ, we can arrange that $v'(0) = 1$. The resulting vector $\psi = \lambda \zeta e_1 \in C_c^\infty(\Omega^*, \mathbf{R}^3)$ has compact support in Ω^*.

Set $\Omega = B_{r_1}(w_0)$, and let $\eta \in C_c^\infty(\Omega, \mathbf{R}^3)$ be an arbitrary test vector. For sufficiently small $|s|$, $|t|$ define

$$\begin{aligned}
\varphi_0 \quad &= D_B[x] \\
\varphi_1(s) &= D_\Omega[x+s\eta] - D_\Omega[x] \\
\varphi_2(t) &= D_{\Omega^*}[x+t\psi] - D_{\Omega^*}[x] \\
v_1(s) &= V_\Omega^Q[x+s\eta] - V_\Omega^Q[x] \\
v_2(t) &= V_{\Omega^*}^Q[x+t\psi] - V_{\Omega^*}^Q[x].
\end{aligned}$$

The functions $\varphi_1, \varphi_2, v_1, v_2$ are differentiable, vanish at zero, and $v_2'(0) = 1$. Moreover,

$$\begin{aligned}
\varphi(s,t) &:= \varphi_0 + \varphi_1(s) + \varphi_2(t) = D[x+s\eta+t\psi] \\
v(s,t) &:= c + v_1(s) + v_2(t) = V^Q[x+s\eta+t\psi].
\end{aligned}$$

Since the Dirichlet integral is stationary in the class $\mathscr{C}(T) \cap \{z\colon V^Q[z]=c\}$, we may apply the reasoning of the proof of Lemma 3 in [17] to the functions $\varphi(s,t)$ and $v(s,t)$, and we obtain the condition

$$\varphi_1'(0) + \mu v_1'(0) = 0, \quad \mu = -\varphi_2'(0).$$

This condition implies

$$\iint_{B_{r_1}(w_0)} \{\nabla x \cdot \nabla \eta + \mu \operatorname{div} Q(x)\, \eta \cdot (x_u \wedge x_v)\}\, du\, dv = 0 \tag{3.6}$$

for all $\eta \in C_c^\infty(B_{r_1}(w_0), \mathbf{R}^3)$. Thus, $x(w)$ is a weak H_2^1-solution of

$$\Delta x = 2H(x)\,[x_u \wedge x_v], \quad H(x) = \frac{\mu}{2} \operatorname{div} Q(x) \tag{3.7}$$

in $B_{r_1}(w_0)$ which satisfies the conformality relation (2.7). It follows from Grüter's thesis [7], [8] that $x(w)$ is of class $C^{1,\beta}$ on $B_{r_1}(w_0)$, for each $\beta \in (0,1)$. Since w_0 was an arbitrary point of B, we conclude that $x \in C^{1,\beta}(B, \mathbf{R}^3)$, for each $\beta \in (0,1)$. A familiar argument [10], [12] implies that the branch points $w \in B$ of x must be isolated in B. In view of assumption (iii) it is then seen that the Lagrange multiplier μ in (3.6) or (3.7) above does not depend on the choice of $w_0 \in B$.

Finally, if $H \in C^{m,\alpha}(\mathbf{R}^3, \mathbf{R}^3)$ or $H \in C^{m,\alpha}(\mathbf{R}^3 - U_1(p), \mathbf{R}^3)$, where $m = 0, 1, 2, \dots$ $0 < \alpha < 1$, the linear regularity theory yields that $x \in C^{m+2,\alpha}(B, \mathbf{R}^3)$.

We summarize our results in the

PROPOSITION 1. *The surface $x(w)$ is of class $C^{1,\beta}$ on B for every $\beta \in (0, 1)$. There is a real number μ such that x satisfies*

$$\int\int_B \{\nabla x \cdot \nabla \eta + 2H(x)\, \eta \cdot (x_u \wedge x_v)\}\, du\, dv = 0 \tag{3.8}$$

for all $\eta \in \mathring{H}_2^1 \cap L^\infty(B, \mathbf{R}^3)$ where

$$H(x) = \frac{\mu}{2} \operatorname{div} Q(x).$$

If $H \in C^{m,\alpha}(\mathbf{R}^3 - U_1(p), \mathbf{R}^3)$, where $m = 0, 1, 2, \dots$ and $0 < \alpha < 1$, then $x \in C^{m+2,\alpha}(B, \mathbf{R}^3)$, and

$$\Delta x = 2H(x)\, x_u \wedge x_v \quad \text{in } B. \tag{3.9}$$

It is our main goal to prove the

PROPOSITION 2. *The surface $x(w)$ is of class $C^0(\bar{B}, \mathbf{R}^3)$.*

In order to present a concise exposition of proof, we shall follow the approach of [9] as closely as possible. Obviously, it suffices to show that $x \in C^0(B \cup C, \mathbf{R}^3)$ for each proper connected open subarc C of ∂B. Given such an arc, there is a conformal mapping τ of the semi-disc

$$B^+ = \{w = u + iv\colon |w| < 1,\ v > 0\}$$

onto the disc B which can be extended to a homeomorphism of $\bar{B}^+$ onto $\bar{B}$ such that the diameter

$$I = \{w = u + iv\colon |w| < 1,\ v = 0\}$$

is mapped onto C. It is then only necessary to prove that $z := x \circ \tau \in C^0(B \cup I, \mathbf{R}^3)$. We observe that the equations (2.7) and (3.9) as well as the functionals D and V^Q are conformally invariant. Thus $z\colon B^+ \to \mathbf{R}^3$ can be treated in the same way as $x\colon B \to \mathbf{R}^3$. For this reason, we go back to our old notation and replace B^+, z by B, x, that is, we assume that

$$x\colon B = \{w\colon |w| < 1,\ v > 0\} \to \mathbf{R}^3$$

is stationary for the Dirichlet integral $D[y]$ in the class $\mathscr{C}(T)\cap\{y: V^{Q}(y)=c\}$, where

$$\mathscr{C}(T)=\{y\in H_2^1(B,\mathbf{R}^3): y(I)\subset T\};$$

see section 2. The assertions of Proposition 1 are assured, and we have to prove

PROPOSITION 2'. *The (reparametrized) surface $x(w)$ is of class $C^0(B\cup I,\mathbf{R}^3)$.*

This new notation enables us to a large extent to refer to the formulas of [9].

We begin with three lemmas, the proof of which will be omitted here. Lemmas 2 and 3 agree with the corresponding lemmas in [9]. Lemma 1 can be proved in the same way as Lemma 1 in [9] once one has convinced oneself that conformally parametrized weak solutions of (3.9) of class $C^1(B,\mathbf{R}^3)$ have the same asymptotic expansions at branch points as minimal surfaces (see [10] and [12]).

LEMMA 1. *For each open subset Ω of B and for every point $w^*\in\Omega$, we have*

$$\limsup_{\sigma\to 0}\frac{1}{\sigma^2}\iint_{\Omega\cap K_\sigma(x^*)}|\nabla x|^2\,du\,dv\geq 2\pi.$$

Here $x^*=x(w^*)$ and

$$K_\sigma(x^*)=\{w\in B: |x(w)-x^*|<\sigma\}. \tag{3.10}$$

In the following, $S_r(w_0)$ and $C_r(w_0)$ denote the sets

$$S_r(w_0)=\{w: |w-w_0|<r,\ v>0\}\quad\text{and}\quad C_r(w_0)=\{w: |w-w_0|=r,\ v>0\}.$$

LEMMA 2. *For each $z\in C^1(B,\mathbf{R}^3)$, for every $w_0\in I$, and for each $R_0\in(0,1-|w_0|)$, there is a number $r\in[R_0/2,R_0]$ such that*

$$\operatorname{osc}_{C_r(w_0)}z\leq\sqrt{\frac{2\pi}{\log 2}}\left\{\frac{1}{2}\iint_{S_{R_0}(w_0)}|\nabla z|^2\,du\,dv\right\}^{1/2}.$$

LEMMA 3. *Let $w_0\in I$, $r\in(0,1-|w_0|)$ and $z\in C^1(B,\mathbf{R}^3)$. Assume that, for some positive numbers α_1 and α_2,*

$$\operatorname{osc}_{C_r(w_0)}z\leq\alpha_1$$

and

$$\sup_{w^*\in S_r(w_0)}\ \inf_{w\in C_r(w_0)}|z(w)-z(w^*)|\leq\alpha_2.$$

Then

$$\operatorname{osc}_{S_r(w_0)} z \leq 2a_1 + 2a_2.$$

The crucial estimate for the proof of proposition 2′ is contained in the following

LEMMA 4. *Let ϱ_0, K_0, K be the constants appearing in assumption* (V). *Then there exist positive numbers ϱ_1 and K_1, depending on these constants and on μ and H_0, such that the following holds: If*

$$w_0 \in I, \quad w^* \in S_r(w_0), \quad 0 < 2r < 1 - |w_0|, \quad 0 < R < \varrho_1,$$

and if

$$\inf_{w \in C_r(w_0)} |x(w) - x(w^*)| > R,$$

then

$$R^2 \leq K_1^2 \iint_{S_r(w_0)} |\nabla x|^2 \, du \, dv.$$

We turn to the proof of Lemma 4. Let $w_0 \in I$, $w^* \in S_r(w_0)$ and $0 < R < \varrho_1 := \varrho_0 \sqrt{K}$, and set $x^* = x(w^*)$, $\delta = \delta(x^*) = \operatorname{dist}(x^*, T)$.

Case I. $\delta(x^*) > 0$. Here we choose some function $\lambda(s) \in C^1(\mathbf{R}, \mathbf{R})$ satisfying the conditions $\lambda'(s) > 0$ and $\lambda(s) = 0$ for $s \leq 0$. Let $R^* = \min(\delta(x^*), d^2 R)$, where $d = 1/(2K) \leq \frac{1}{2}$. For $0 < \varrho < R^*$ we define the function

$$\Phi(\varrho) = \frac{1}{2} \iint_{S_r(w_0)} \lambda(\varrho - |x - x^*|) \, |\nabla x|^2 \, du \, dv.$$

It follows as in [9] that the function

$$\eta(w) = \begin{cases} \lambda(\varrho - |x(w) - x^*|)\,(x(w) - x^*), & w \in S_r(w_0) \\ 0, & w \in B - S_r(w_0) \end{cases}$$

is of class $H_2^1(B, \mathbf{R}^3)$ and that $x(w, t) = x(w) + t\eta(w)$ represents an admissible variation of our surface $x(w) \in \mathscr{C}(T)$, permissible to be substituted in (3.8), to yield

$$\iint_{S_r(w_0)} \{\nabla x \cdot \nabla \eta + 2H(x)\, \eta \cdot (x_u \wedge x_v)\} \, du \, dv = 0.$$

Set $h_0 = \frac{1}{2}|\mu|\, H_0$ so that $|H(x)| \leq h_0$. Then a computation similar to those in [8] or [9] leads to the inequality

$$\frac{1}{\varrho^2}\Phi(\varrho) \leq (\varrho')^{-2} e^{2h_0\varrho'} \Phi(\varrho') \quad \text{for } 0<\varrho\leq\varrho'<R^*.$$

We now refine the choice of $\lambda(s)$ by imposing the further conditions $0 \leq \lambda(s) \leq 1$ and $\lambda(s)=1$ for $s \geq \varepsilon$, for some $\varepsilon>0$. Then

$$\frac{1}{\varrho^2}\iint_{S_r(w_0)\cap K_{\varrho-\varepsilon}(x^*)} |\nabla x|^2\,du\,dv \leq (\varrho')^{-2} e^{2h_0\varrho'} \iint_{S_r(w_0)\cap K_{\varrho'}(x^*)} |\nabla x|^2\,du\,dv.$$

The sets $K_\sigma(x^*)$ are defined in (3.10). By letting first $\varepsilon \to +0$ and then $\varrho' \to R^*$, we find

$$\frac{1}{\varrho^2}\iint_{S_r(w_0)\cap K_{\varrho}(x^*)} |\nabla x|^2\,du\,dv \leq (R^*)^{-2} e^{2h_0R^*} \iint_{S_r(w_0)\cap K_{R^*}(x^*)} |\nabla x|^2\,du\,dv.$$

If we now let ϱ go to zero, we infer from Lemma 1 that

$$2\pi \leq (R^*)^{-2} e^{2h_0R^*} \iint_{S_r(w_0)\cap K_{R^*}(x^*)} |\nabla x|^2\,du\,dv. \tag{3.11}$$

It is now necessary to distinguish two possibilities.

Case I(a). $d^2R<\delta(x^*)$, i.e., $R^*=d^2R$. Then

$$R^2 \leq \frac{e^{2h_0Rd^2}}{2\pi d^4}\iint_{S_r(w_0)} |\nabla x|^2\,du\,dv. \tag{3.12}$$

This is the assertion of Lemma 4, with the constant

$$K_1 = \frac{\sqrt{8}\,K^2}{\sqrt{\pi}} e^{h_0\varrho_1/(2K)^2}.$$

Case I(b). $d^2R>\delta(x^*)$, i.e., $R^*=\delta(x^*)$. Then we obtain from (3.11) the estimate

$$2\pi \leq \frac{1}{\delta(x^*)^2} e^{2h_0\delta(x^*)} \iint_{S_r(w_0)\cap K_{\delta(x^*)}(x^*)} |\nabla x|^2\,du\,dv. \tag{3.13}$$

There exists a point $f \in T$ such that

$$|f-x^*| = \delta(x^*) < d^2R \leq \tfrac{1}{4}R.$$

We shall choose f as center of a new system of coordinates as described in assumption (V), with the defining diffeomorphism h and with the coefficients $g_{ij}(y)$ of the fundamental metric tensor introduced in (2.1). We set $y(w)=h^{-1}\circ x(w)$ and define the "norm" $\|y(w)\|$ by

$$\|y(w)\|^2 = g_{ij}(y(w))\, y^i(w)\, y^j(w).$$

For $\delta(x^*)/d<\varrho<dR$, we consider the function

$$\eta(w) = \begin{cases} \lambda(\varrho-\|y(w)\|)\, y(w), & w\in S_r(w_0) \\ 0, & w\in B-S_r(w_0) \end{cases}$$

which can be shown to be of class $H_2^1(B, \mathbf{R}^3)$ as in [9], pp. 398–399. Moreover, the one-parameter family

$$x_t(w) = h(y(w)+t\eta(w)) = x(w)+t\xi(w,t), \quad |t|<\tfrac{1}{2}$$

with

$$\xi(w,t) = A(w,t)\,\eta(w),$$

$$A(w,t) = \int_0^1 \frac{\partial h}{\partial y}(y(w)+\tau t\eta(w))\, d\tau$$

is seen to form an admissible variation of $x(w)$. The proof of this fact can be carried out as in [9], pp. 399–400.

Next, we choose some open disc $\Omega^*\subset\subset B-S_{r_1}(w_0)$, $r_1:=\frac{1}{2}(1-|w_0|)$. Since the branch points of $x(w)$ are isolated in B, we have

$$\iint_{\Omega^*} |x_u\wedge x_v|\, du\, dv > 0.$$

For sufficiently small $|s|$ and $|t|$ we introduce two real valued functions

$$\varphi(s,t) = \varphi_0+\varphi_1(s)+\varphi_2(t)$$

and

$$v(s,t) = c+v_1(s)+v_2(t)$$

where

$$\varphi_0 \quad = D_B[x]$$

$$\varphi_1(s) = D_{S_r(w_0)}[h(y+s\eta)]-D_{S_r(w_0)}[x]$$

$$\varphi_2(t) = D_{\Omega^*}[x+t\psi]-D_{\Omega^*}[x]$$

$$v_1(s) = V^Q_{S_r(w_0)}[h(y+s\eta)] - V^Q_{S_r(w_0)}[x]$$

$$v_2(t) = V^Q_{\Omega^*}[x+t\psi] - V^Q_{\Omega^*}[x]$$

where, as at the beginning of section 3, $\psi \in C_c^\infty(\Omega^*, \mathbf{R}^3)$ is chosen in such a way that $v_2'(0)=1$. The functions $\varphi_1, \varphi_2, v_1, v_2$ are differentiable, vanish at zero, and we see that

$$\varphi(s,t) = D[h(y+s\eta)+t\psi],$$

$$v(s,t) = V^Q[h(y+s\eta)+t\psi].$$

As in the beginning of section 3, we find

$$\varphi_1'(0) + \bar{\mu} v_1'(0) = 0, \quad \bar{\mu} = -\varphi_2'(0). \tag{3.14}$$

A computation shows that

$$\varphi_1'(0) = \int\int_{S_r(w_0)} \left\{ g_{ij}(y)\, \nabla y^i \nabla \eta^j + \frac{1}{2} \frac{\partial g_{ij}(y)}{\partial y^k} \nabla y^i \nabla y^j \eta^k \right\} du\, dv$$

and

$$v_1'(0) = \int\int_{S_r(w_0)} \left\{ \tilde{Q}(y) \cdot [(y_u \wedge \eta_v) + (\eta_u \wedge y_v)] + \eta^i \frac{\partial \tilde{Q}(y)}{\partial y^i} \cdot (y_u \wedge y_v) \right\} du\, dv \tag{3.15}$$

where we have set

$$\tilde{Q}(y) = \left\{ Q(h(y)) \frac{\partial h(y)}{\partial y^2} \wedge \frac{\partial h(y)}{\partial y^3}, Q(h(y)) \frac{\partial h(y)}{\partial y^3} \wedge \frac{\partial h(y)}{\partial y^1}, Q(h(y)) \frac{\partial h(y)}{\partial y^1} \wedge \frac{\partial h(y)}{\partial y^2} \right\}$$

and, of course, $h(y(w)) = x(w)$.

By virtue of Proposition 1, we conclude that the number $\bar{\mu}$ is the same constant as in Proposition 1 and hence independent of w_0 and r.

We know from [9] that $\eta(w)=0$ for $w \in C_r(w_0)$. Moreover, $y^3=0$ and $\eta^3=0$ on $I_r(w_0) = \{w : |w-w_0|<r,\ v=0\}$. Since $\partial h/\partial y^1 \wedge \partial h/\partial y^2$ is a normal vector field to T in $T \cap U$ and $Q(x)$ is a tangential vector field to T, it is also true that $\tilde{Q}^3=0$ on $\{y \in \mathbf{R}^3 : |y|<\varrho_0,\ y^3=0\} = h^{-1}(T\cap U)$. In view of this, an integration by parts leads from (3.15) to

$$v_1'(0) = \int\int_{S_r(w_0)} \operatorname{div} \tilde{Q}(y)\, \eta \cdot (y_u \wedge y_v)\, du\, dv. \tag{3.16}$$

This integration must be carried out with care. First, one approximates y in $H_2^1(S_r(w_0), \mathbf{R}^3)$ by regular mappings y_n vanishing on $C_r(w_0)$ and satisfying $y_n^3=0$ on

$I_r(w_0)$. Then one replaces y by y_n in (3.15), while η is left unaltered, performs an integration by parts and lets n go to infinity.

We now have arrived at the variational equation

$$\int\int_{S_r(w_0)} \left\{ g_{ij}(y)\, \nabla y^i \cdot \nabla \eta^j + \frac{1}{2} \frac{\partial g_{ij}(y)}{\partial y^k} \nabla y^i \cdot \nabla y^j \eta^k + \mu \operatorname{div} \tilde{Q}(y)\, (y_u \wedge y_v) \cdot \eta \right\} du\, dv = 0.$$

Set

$$g(y) = \det (g_{ij}(y)).$$

Then we see that

$$\operatorname{div} \tilde{Q} = (\operatorname{div} Q) \circ h \det \left(\frac{\partial h}{\partial y} \right) = (\operatorname{div} Q) \circ h \sqrt{g}\,,$$

whence by (3.7)

$$\mu \operatorname{div} \tilde{Q}(y) = 2H(h(y)) \sqrt{g(y)}\,,$$

and further

$$\int\int_{S_r(w_0)} \left\{ g_{ij}(y)\, \nabla y^i \cdot \nabla \eta^j + \frac{1}{2} \frac{\partial g_{ij}(y)}{\partial y^k} \nabla y^i \cdot \nabla y^j \eta^k \right.$$
$$\left. + 2H(h(y)) \sqrt{g(y)}\, (y_u \wedge y_v) \cdot \eta \right\} du\, dv = 0,$$

where

$$|H(h(y))| \sqrt{g(y)} \leq h_0 K^{3/2} = h_1.$$

Let us introduce the function

$$\Psi(\varrho) = \int\int_{S_r(w_0)} \lambda(\varrho - \|y\|)\, g_{ij}(y)\, \nabla y^i \cdot \nabla y^j\, du\, dv$$

and also note that the conformality relations (2.7) now take the form

$$g_{ij}(y)\, y_u^i y_u^j = g_{ij}(y)\, y_v^i y_v^j, \quad g_{ij}(y)\, y_u^i y_v^j = 0 \quad \text{in } B.$$

By the same computations as in [9], pp. 400–402, we see that the differential inequality

$$\frac{d}{d\varrho}\left[\frac{1}{\varrho^2} \Psi(\varrho) \right] + M_1 \frac{1}{\varrho^2} \Psi(\varrho) + M_2 \frac{d}{d\varrho} \left[\frac{1}{\varrho} \Psi(\varrho) \right] \geq 0 \qquad (3.18)$$

holds for $\delta(x^*)/d<\varrho<dR$, where we have set

$$M_1=(2K_0+2h_1)K^{3/2} \quad \text{and} \quad M_2=K_0K^{3/2}.$$

As in [9], pp. 402–403, this implies

$$\frac{1}{\delta^2(x^*)}\int\int_{S_r(w_0)\cap K_{2\delta(x^*)}(f)}|\nabla x|^2\,du\,dv\leqslant\frac{C(R)}{d^4}\,\frac{1}{R^2}\int\int_{S_r(w_0)}|\nabla x|^2\,du\,dv, \tag{3.19}$$

where

$$C(R)=(1+dRM_2)\,e^{dRM_1}.$$

Considering that $K_{\delta(x^*)}(x^*)\subset K_{2\delta(x^*)}(f)$, a combination of (3.13) and (3.19) gives

$$R^2\leqslant\frac{C(R)}{2\pi d^4}\,e^{2h_0d^2R}\int\int_{S_r(w_0)}|\nabla x|^2\,du\,dv. \tag{3.20}$$

Regarding the constant on the right hand side, recall that we had restricted R to the interval $0<R<\varrho_0\sqrt{K}=\varrho_1$.

(3.12) and (3.20) contain the desired inequalities for the case $\delta(x^*)>0$. We must still consider the

Case II. $\delta(x^*)=0$. Then $x^*=f$. As in the previous case I(b), we can derive the differential inequality (3.18) for all $\varrho\in(0,dR)$. We integrate this inequality between the limits ϱ and dR and find

$$\frac{1}{\varrho^2}\int\int_{S_r(w_0)\cap K_{\varrho/K}(x^*)}|\nabla x|^2\,du\,dv\leqslant\frac{C(R)}{d^2R^2}\int\int_{S_r(w_0)}|\nabla x|^2\,du\,dv.$$

By employing once more Lemma 1, we conclude that

$$R^2\leqslant\frac{C(R)}{8\pi d^4}\int\int_{S_r(w_0)}|\nabla x|^2\,du\,dv. \tag{3.21}$$

If we set

$$K_1=4K^2e^{h_0\varrho_1/(4K^2)}\sqrt{\frac{C(\varrho_1)}{2\pi}}\,,$$

we can combine the inequalities (3.12), (3.20) and (3.21) to give the assertion of Lemma 4. Lemma 4 is thus proved.

Proposition 2′, that is, the continuity of $x(w)$ on $B \cup I$, is a direct consequence of Lemmas 2, 3 and 4. To see this, choose an arbitrary point $w_0 \in I$ and an arbitrary number $R \in (0, \varrho_1)$. Then there will exist a number r_0 with $0<2r_0<1-|w_0|$ such that

$$K_1^2 \iint_{S_{r_0}(w_0)} |\nabla x|^2 \, du \, dv < R^2.$$

Lemma 4 implies that

$$\sup_{w^* \in S_r(w_0)} \inf_{w \in C_r(w_0)} |x(w)-x(w^*)| < R$$

for all $r \in (0, r_0)$. On the other hand, Lemma 2 guarantees the existence of a number $r \in [r_0/2, r_0]$ for which

$$\operatorname{osc}_{C_r(w_0)} x \leq K_2 \left\{ \iint_{S_{r_0}(w_0)} |\nabla x|^2 \, du \, dv \right\}^{1/2} \leq \frac{K_2}{K_1} R, \quad \text{where } K_2 = \sqrt{\frac{\pi}{\log 2}}.$$

An application of Lemma 3 shows that

$$\operatorname{osc}_{S_{r_0/2}(w_0)} x \leq \operatorname{osc}_{S_r(w_0)} x \leq 2\left(1+\frac{K_2}{K_1}\right) R$$

and therefore

$$\lim_{r \to 0} \operatorname{osc}_{S_r(w_0)} x = 0.$$

The desired continuity of $x(w)$ is a consequence of this relation.

The continuity of $x(w)$ is the starting point for the proof of the higher regularity. As a first step toward this goal, we shall now prove

PROPOSITION 3′. *The (reparametrized) surface* $x(w)$ *is of class* $C^{0,\alpha}(B \cup I, \mathbf{R}^3)$ *for some* $\alpha \in (0, 1)$.

Let $w_0 \in I$, and choose $f=x(w_0)$ as center of a new system of coordinates as described in assumption (V), with the defining diffeomorphism h. Set $y(w)=h^{-1} \circ x(w)$. Then $y \in C^0(B \cup I, \mathbf{R}^3)$, $y(w_0)=0$ and $y^3(w)=0$ for $w \in I_{R_0}(w_0)$ if $R_0 \in (0, 1-|w_0|)$ is chosen so small that $|y(w)| \leq \varrho_0$ for $w \in S_{R_0}(w_0)$. As in the derivation of (3.17), it can be shown that

$$\iint_{S_r(w_0)}\left\{g_{ij}(y)\,\nabla y^i\cdot\nabla\varphi^j+\frac{1}{2}\,\frac{\partial g_{ij}(y)}{\partial y^k}\nabla y^i\cdot\nabla y^j\varphi^k\right.$$
$$\left.+2H(h(y))\sqrt{g(y)}\,(y_u\wedge y_v)\cdot\varphi\right\}du\,dv=0 \tag{3.22}$$

for each test vector $\varphi=\{\varphi^1,\varphi^2,\varphi^3\}\in H_2^1\cap L^\infty(S_r(w_0),\mathbf{R}^3)$ with $\varphi=0$ on $C_r(w_0)$ and $\varphi^3=0$ on $I_r(w_0)$, where $0<r<R_0$. Let $R\in(0,R_0/2)$, and choose a function $\zeta\in C_c^\infty(B_{2R}(w_0),\mathbf{R})$ with $0\leq\zeta\leq 1$, $\zeta\equiv 1$ on $B_R(w_0)$ and $|\nabla\zeta|\leq 2/R$. For $r=2R$, φ be the vector with the components

$$\varphi^i=(y^i-\omega^i)\,|\zeta|^2,\quad i=1,2,3,$$

where

$$\omega^j=\fint_{S_{2R}(w_0)-S_R(w_0)} y^j\,du\,dv\quad \text{for } j=1,2,\ \omega^3=0,$$

and the symbol $\fint$ indicates the mean value. We note that φ is admissible for (3.22). Therefore,

$$\iint_{S_{2R}(w_0)}\left\{g_{ij}(y)\,\nabla y^i\cdot\nabla y^j+\left[\frac{1}{2}\,\frac{\partial g_{ij}(y)}{\partial y^k}\nabla y^i\cdot\nabla y^j\right.\right.$$
$$\left.\left.+2H(h(y))\sqrt{g(y)}\,(y_u\wedge y_v)^k\right](y^k-\omega^k)\right\}\zeta^2\,du\,dv$$
$$=-\iint_{S_{2R}(w_0)}2\zeta g_{ij}(y)\,(y^i-\omega^i)\,\nabla y^j\cdot\nabla\zeta\,du\,dv$$
$$\leq\delta\iint_{S_{2R}(w_0)}g_{ij}(y)\,\nabla y^i\cdot\nabla y^j\zeta^2\,du\,dv$$
$$+\frac{4}{\delta R^2}\iint_{S_{2R}(w_0)-S_R(w_0)}g_{ij}(y)\,(y^i-\omega^i)\,(y^j-\omega^j)\,du\,dv$$

for every $\delta>0$. Since $y(w_0)=0$ and $y\in C^0(B\cup I,\mathbf{R}^3)$, we can make $|y-\omega|^2$ on $S_{2R}(w_0)$ as small as desired by selecting a sufficiently small $R>0$. Thus, in view of (iii) and of assumption (V), we can find a number $R_1\in(0,R_0/2)$ and a constant K_3 independent of w_0 and R such that the above integral inequality can be transformed into the inequality

$$\iint_{S_R(w_0)}|\nabla y|^2\,du\,dv\leq\frac{K_3}{R^2}\iint_{S_{2R}(w_0)-S_R(w_0)}|y-\omega|^2\,du\,dv$$

valid for all $R \in (0, R_1]$. In view of Poincaré's inequality, the right hand side of this inequality can be bounded by a term of the form

$$K_4 \iint_{S_{2R}(w_0) - S_R(w_0)} |\nabla y|^2 \, du \, dv$$

where K_4 denotes another constant independent of w_0 and R. With the help of the "hole-filling" technique of [18], [31], we can now conclude the existence of numbers $\alpha \in (0,1)$ and $K_5 > 0$, $K_6 > 0$, independent of w_0 and R, such that

$$\iint_{S_R(w_0)} |\nabla y|^2 \, du \, dv \leq K_5 \left(\frac{R}{R_0}\right)^{2\alpha} D[y]$$

and, as well,

$$\iint_{S_R(w_0)} |\nabla x|^2 \, du \, dv \leq K_6 \left(\frac{R}{R_0}\right)^{2\alpha} D[x]$$

for all $w_0 \in I$, $R \in (0, 1 - |w_0|)$.

By combining the last inequality with appropriate interior estimates, which can be derived in a similar way, a well known device (see e.g. [14], pp. 259–260) yields that for all $r > 0$, for all $d \in (0,1)$ and for all $w_0 \in Z_d$, where $Z_d = \{w : w \in B, |w| < 1 - d\}$, the following estimate holds

$$\iint_{B \cap B_r(w_0)} |\nabla x|^2 \, du \, dv \leq K_7 \left(\frac{r}{d}\right)^{2\alpha} D[x].$$

Here K_7 is a constant similar to the previous constants. This implies $x \in C^{0,\alpha}(B \cup I, \mathbf{R}^3)$.

Let us return to our original notations and assumptions from the beginning of this section. Then Proposition 3′ leads to the following result:

PROPOSITION 3. *The surface $x(w)$ is of class $C^{0,\alpha}(\bar{B}, \mathbf{R}^3)$, for some $\alpha \in (0,1)$.*

The higher regularity of our surface is expressed by the

PROPOSITION 4. *Suppose, that the supporting surface T also belongs to class C^3. Then the surface $x(w)$ is of class $C^{1,\beta}(\bar{B}, \mathbf{R}^3)$ for every $\beta \in (0,1)$, and it intersects T orthogonally in its trace curve $\Sigma = \{x = x(w); w \in \partial B\}$.*

Proof. Let us return to the proof of proposition 3′ and, in particular, to equation (3.22). If $(g^{ij}(y))$ denotes the inverse of the matrix $(g_{ij}(y))$, and if $\psi = \{\psi^1, \psi^2, \psi^3\} \in$

$C_c^\infty(S_r(w_0), \mathbf{R}^3)$, then the vector $\varphi=\{\varphi^1, \varphi^2, \varphi^3\}$ with components $\varphi^i = g^{ij}(y)\,\psi^j$ is admissible for (3.22). Substitution into (3.22) yields

$$\iint_{S_r(w_0)} \{\nabla y^i \cdot \nabla \psi^i - \Gamma_{ij}^k(y)\,\nabla y^i \cdot \nabla y^j \psi^k + 2H(h(y))\sqrt{g(y)}\,g^{ij}(y)\,(y_u \wedge y_v)^i \psi^j\}\,du\,dv = 0. \tag{3.23}$$

Here Γ_{ij}^k are the Christoffel symbols of the second kind with respect to the metric (g_{ij}), and $(y_u \wedge y_v)^i$ denotes the ith component of the vector product $y_u \wedge y_v$. An application of the fundamental lemma of the calculus of variations shows that

$$\Delta y^k + \Gamma_{ij}^k(y)\,\nabla y^i \cdot \nabla y^j = 2H(h(y))\sqrt{g(y)}\,g^{jk}(y)\,(y_u \wedge y_v)^j, \quad k=1,2,3, \quad \text{in } S_r(w_0). \tag{3.24}$$

Since T is assumed to be of class C^3, the local coordinates y can be chosen in such a way that, in addition to the condition of assumption (V), also

$$g_{ij}(y) = 0 \quad \text{for } i \neq j,\ |y| < \varrho_0,\ y^3 = 0. \tag{3.25}$$

(Details of the necessary construction can be found in [14], pp. 265–266.) Then also every test vector $\psi = \{\psi^1, \psi^2, \psi^3\}$ with

$$\psi^1,\ \psi^2 \in C_c^\infty(B_r(w_0), \mathbf{R}), \quad \psi^3 \in C_c^\infty(S_r(w_0), \mathbf{R}), \quad 0 < r < R_0$$

is admissible in (3.23). In view of (3.24), we therefore obtain the condition

$$\lim_{\delta \to +0} \int_{\Gamma_r^\delta(w_0)} [y_v^1 \psi^1 + y_v^2 \psi^2]\,du = 0, \tag{3.26}$$

where

$$\Gamma_r^\delta(w_0) = \{w\colon w \in S_r(w_0),\ v = \delta\}, \quad \delta > 0.$$

We extend $y(w)$ from $S_r(w_0)$ to $B_r(w_0)$ by setting

$$y^1(w) = y^1(\bar{w}), \quad y^2(w) = y^2(\bar{w}), \quad y^3(w) = -y^3(\bar{w}), \quad \text{if } v < 0.$$

The extended vector $y(w)$ is of class $H_2^1 \cap C^{0,\alpha}(B_r(w_0), \mathbf{R}^3)$, and (3.23), (3.26) imply that

$$\iint_{B_r(w_0)} [\nabla y \cdot \nabla \psi + f(w, y, \nabla y) \cdot \psi]\,du\,dv = 0$$

for all $\psi \in \mathring{H}_2^1 \cap L^\infty(B_r(w_0), \mathbf{R}^3)$, $0<r<R_0$. The vector function $f(w, y, \nabla y)$ can be computed from (3.23); all we need to know here is the inequality

$$|f(w, y(w), \nabla y(w))| \leq a|\nabla y(w)|^2, \quad w \in B_r(w_0),$$

for a suitable constant $a \geq 0$.

The assertion of regularity is now a consequence of a regularity theorem of F. Tomi [18]. Moreover, we have $y^3=0$ on $I_r(w_0)$, and (3.26) implies that $y_v^1=y_v^2=0$ on $I_r(w_0)$. It follows from this that the surface $y(w)$ is perpendicular to the plane $\{y^3=0\}$ along the curve $\{y(w)\colon w \in I_r(w_0)\}$. Going back to our original coordinates and remembering (3.26), we see that $x(w)$ intersects T orthogonally along Σ. Proposition 4 is thus proved.

The main result of the present section can now be formulated:

THEOREM 2. (a) *If, in addition to the previously stated assumptions, T is of class $C^{m,\alpha}$, where $m \geq 3$, $0<\alpha<1$, and Q is of class $C^{m-1,\alpha}$ in $\mathbf{R}^3 - U_1(p)$, then* $x(w) \in C^{m,\alpha}(\bar{B}, \mathbf{R}^3)$.

(b) *If T is real analytic, and if Q is real analytic on $\mathbf{R}^3 - U_1(p)$,* (2) *then $x(w)$ is real analytic on $\bar{B}$.*

In both cases, x intersects T orthogonally along its trace $\Sigma = \{x = x(w)\colon w \in \partial B\}$.

The proof follows from Propositions 1–4 by the techniques employed in [14]. For the applicability of the Agmon-Douglis-Nirenberg results, one has to check that the complementing condition with respect to (3.24) is satisfied for the boundary conditions of y; cf. [14], pp. 304–307. However, this is an immediate consequence of assumption (iv). Combining the above results with the reasoning of [14], we obtain

THEOREM 3. *If T is also of class C^3, then the solution surface $x(w)$ possesses only finitely many branch points in $\bar{B}$. Every branch point $w_0 \in \bar{B}$ is associated with a constant vector $b \in \mathbf{C}$, satisfying $b \neq 0$ and $b \cdot b = 0$, and an integer $\nu \geq 1$ such that*

$$x_w(w) := \tfrac{1}{2}[x_u(w) - ix_v(w)] = b(w-w_0)^\nu + o(|w-w_0|^\nu), \tag{3.27}$$

for $w \to w_0$. The tangent plane of $x(w)$ tends to a limiting position as $w \to w_0$. Finally, if $w_0 \in \partial B$, then the nonoriented tangent of the trace Σ moves continuously through w_0. The oriented tangent of Σ is continuous at branch points of even order ν, while for

(2) It clearly suffices to assume that $Q(x)$ is real analytic in a neighborhood of $x(\bar{B})$.

branch points of odd order, the tangent direction jumps by 180 *degrees, that is, the trace* Σ *has a cusp at* $x(w_0)$.

There are, of course, special situations for which more extensive information is available regarding the existence of branch points. For instance, it follows from the asymptotic expansion (3.27) that *if T is the boundary of a convex body, any solution surface of the partitioning problem formulated in section* 1 *is free of boundary branch points.*

Let us consider such a convex supporting surface $T=\partial\mathcal{K}$, and let $S=\{x(w): w\in\bar{B}\}$ be a solution of our partitioning problem. Denote by R_0 the radius of the largest sphere inscribed in $\mathcal{K}$, and assume that the principal curvatures $\varkappa_1, \varkappa_2$ of T satisfy the inequality $|\varkappa_1|, |\varkappa_2|\leq 1/R$ for all points of T.

Green's formula gives

$$\int_{\partial B} x_\varrho(e^{i\theta})\, d\theta = \int\int_B \Delta x\, du\, dv \tag{3.28}$$

and, for any test vector $y(w)\in H_2^1(B,\mathbf{R}^3)\cap L^\infty(B,\mathbf{R}^3)$,

$$\int_{\partial B} y\cdot x_\varrho(e^{i\theta})\, d\theta = \int\int_B y\cdot\Delta x\, du\, dv + \int\int_B \nabla x\cdot\nabla y\, du\, dv. \tag{3.29}$$

As before, let $n(x)$ be the outward normal unit vector of T at $x\in T$; also introduce the arc length parameter $s=s(\theta)$ on the trace curve Σ and set $\xi(s):=x(e^{i\theta(s)})$. Then $x_\varrho(e^{i\theta})=|x_\varrho(e^{i\theta})|\, n(x(e^{i\theta}))$ so that

$$\begin{aligned}\int_{\partial B} x_\varrho(e^{i\theta})\, d\theta &= \int_\Sigma n(\xi(s))\, ds,\\ \int_{\partial B} y\cdot x_\varrho(e^{i\theta})\, d\theta &= \int_\Sigma y\cdot n(\xi(s))\, ds.\end{aligned} \tag{3.30}$$

Since

$$2\int\int_B x_u\wedge x_v\, du\, dv = \int_{\partial B} x\wedge dx = \int_\Sigma \xi\wedge d\xi, \tag{3.31}$$

it follows from (3.28) and (3.30) that

$$\int_\Sigma \{n(\xi)\, ds - H\xi\wedge d\xi\} = 0 \tag{3.32}$$

and in particular

$$\int_{\Sigma} n(\xi(s))\, ds = 0 \tag{3.32'}$$

if $x=x(w)$ is a minimal surface.

Denote by

$$L = \int_{\partial B} |x_\theta(e^{i\theta})|\, d\theta = \int_{\Sigma} |d\xi|$$

the length of Σ, and let

$$\bar{\xi} := \int_0^L \xi(s)\, ds$$

be the barycenter of the parameter representation $\xi=\xi(s)$ of Σ with respect to the arc length parameter s. Now we employ an idea due to Croke and Weinstein [3]. First, we infer from Wirtinger's inequality that

$$\int_0^L |\xi-\bar{\xi}|^2\, ds \leq \left(\frac{L}{2\pi}\right)^2 \int_0^L |\xi'|^2\, ds = \frac{L^3}{4\pi^2}.$$

We can assume that the ball $\{z\in\mathbf{R}^3\colon |z|\leq R_0\}$ is contained in $\mathcal{K}$. Then the support function $p(z)=z\cdot n(z)$ satisfies $p(z)\geq R_0$ for all $z\in T$, whence

$$\begin{aligned} R_0 L - \int_0^L \bar{\xi}\cdot n(\xi)\, ds &\leq \int_0^L (\xi-\bar{\xi})\cdot n(\xi)\, ds \\ &\leq \sqrt{L}\cdot\left(\int_0^L |\xi-\bar{\xi}|^2\, ds\right)^{1/2} \leq \frac{L^2}{2\pi}. \end{aligned}$$

By virtue of (3.32), we arrive at

$$R_0 L \leq \frac{L^2}{2\pi} + H\int_0^L (\bar{\xi}, \xi, \xi')\, ds,$$

where $(a, b, c)=a\cdot(b\wedge c)$ for $a, b, c\in\mathbf{R}^3$. Since

$$(\bar{\xi}, \xi, \xi') = (\bar{\xi}, \xi-\bar{\xi}, \xi'),$$

we obtain

$$R_0 L \leq \frac{L^2}{2\pi} + |H|\int_0^L |\bar{\xi}|\, |\xi-\bar{\xi}|\, |\xi'|\, ds$$

$$\leq \frac{L^2}{2\pi} + |H|\,|\xi| \left(\int_0^L |\xi - \bar{\xi}|^2\, ds \right)^{1/2} \sqrt{L}$$

$$\leq (1 + |H|\,|\bar{\xi}|) \frac{L^2}{2\pi}.$$

An elementary consideration yields $|\bar{\xi}| \leq \operatorname{diam} \mathscr{K} - R_0$. Thus we conclude that the length $L = L(\Sigma)$ of Σ is estimated from below by

$$L(\Sigma) \geq 2\pi R_0 [1 + (\operatorname{diam} \mathscr{K} - R_0)\,|H|]^{-1}. \tag{3.33}$$

If $x = x(w)$ is a minimal surface, we have in particular

$$L(\Sigma) \geq 2\pi R_0, \tag{3.33'}$$

and also

$$\pi R_0^2 \leq D[x], \tag{3.33''}$$

since

$$R_0 L \leq \int_0^L \xi \cdot n(\xi)\, ds = \int_{\partial B} x \cdot x_\varrho\, d\theta = \int\int_B |\nabla x|^2\, du\, dv = 2D[x],$$

and thus, by the isoperimetric inequality,

$$R_0^2 L^2 \leq 4D[x]\, D[x] \leq 4D[x] \frac{1}{4\pi} L^2 = \frac{L^2}{\pi} D[x].$$

For an estimate of $L(\Sigma)$ from above, let us, as in [19], choose a vector field $\varphi(x)$ which coincides with the normal vector field $n(x)$ on T and which vanishes outside a suitable parallel strip of T. Then, for every $\varepsilon > 0$, one can construct $\varphi(x)$ in such a way that

$$|\varphi(x)| \leq 1, \quad |\operatorname{grad} \varphi(x)| \leq \frac{1+\varepsilon}{R}.$$

Using $y(w) = \varphi(x(w))$ in (3.29) and (3.30), and observing that

$$|\nabla x \cdot \nabla y| \leq |\operatorname{grad} \varphi|\, |\nabla x|^2 \leq \frac{2}{R}(1+\varepsilon) \cdot \tfrac{1}{2} |\nabla x|^2, \quad |x_u \wedge x_v| = \tfrac{1}{2} |\nabla x|^2,$$

we obtain the estimate

$$L(\Sigma) \leq 2\left(|H| + \frac{1}{R} \right) D[x] \tag{3.34}$$

as $\varepsilon \to 0$, since there are no branch points of $x(w)$ on ∂B. For $H=0$, this inequality was found in [16] and [19], and, for $H \neq 0$, the same estimate has been stated in [20].

If $H=0$, we may apply the isoperimetric inequality $4\pi D[x] \leq L^2$ to (3.34), whence we get

$$L(\Sigma) \geq 2\pi R. \tag{3.34'}$$

Since $R_0 \geq R$, this estimate is slightly worse than (3.33').

It seems generally to be a difficult task to determine the area $D[x]$ and the mean curvature H by the geometric properties of $\mathcal{K}$ and by the value of σ. As we have seen, there can exist stationary solutions of the σ-partitioning problem with $H=0$ and $H \neq 0$. Moreover, the plane minimal surface

$$x(w) = (\operatorname{Re} w^m, \operatorname{Im} w^m, 0)$$

$(m=1, 2, 3, \ldots)$ is stationary for the partitioning problem of the unit ball $\{z \in \mathbf{R}^3 : |z| \leq 1\}$ with $\sigma = 1/2$, and

$$D[x] = m\pi.$$

Thus it is impossible to bound the area in terms of $\mathcal{K}$ and σ if x is allowed to have interior branch points. One might, however, conjecture that the area of embedded solutions of the partitioning problem can be estimated from above.

A general stationary solution $S = \{x(w) \colon w \in \bar{B}\}$ of a partitioning problem for a convex body $\mathcal{K}$, as defined in section 2, has no boundary branch points and only finitely many interior branch points $w_j \in B$, $j=1, \ldots, l$, of order $m_j \geq 1$, and, by [11], we have

$$2\pi\left(1+\sum_{j=1}^{l} m_j\right) = \int_\Sigma \varkappa_g \, ds + \int_S K \, dA \tag{3.35}$$

where $\varkappa_g$ is the geodesic curvature of the boundary curve Σ of S with respect to S, ds its line element, and where K is the Gauß curvature of S and dA its area element.

If H denotes the (constant) mean curvature of S, then we have $K \leq H^2$. Since $|\varkappa_g| \leq \varkappa$, we then obtain

$$2\pi\left(1+\sum_{j=1}^{l} m_j\right) \leq \int_\Sigma \varkappa_g \, ds + H^2 D[x]. \tag{3.36}$$

Let us collect the results.

THEOREM 4. *Let $\mathcal{K}$ be a convex body with a smooth boundary T which contains a ball of radius R_0. Assume also that the principal curvatures $\varkappa_1$, $\varkappa_2$ of T satisfy $|\varkappa_1|$,*

$|\varkappa_2| \leq 1/R$. *Let* $S = \{x(w): w \in \bar{B}\}$ *be a stationary solution of a partitioning problem for* $\mathcal{K}$ *which has the free trace* $\Sigma = \{x(w): w \in \partial B\}$ *on* T *and the (constant) mean curvature* H. *Then the length* $L(\Sigma)$ *of* Σ *can be estimated by*

$$2\pi R_0[1 + (\operatorname{diam} \mathcal{K} - R_0)\,|H|]^{-1} \leq L(\Sigma) \leq 2(|H| + R^{-1})\,D[x].$$

Moreover, S has only finitely many branch points of order m_j, *and*

$$2\pi\left(1 + \sum_j m_j\right) \leq \int_\Sigma \varkappa\, ds + H^2 D[x],$$

where $\varkappa$ *denotes the curvature of the curve* Σ.

4. Proof of Theorem 1

Let us return to the notation of section 1, and suppose that the surface $S = \{x = x(w): w \in B\}$ satisfies assumption (A 1). It is assumed that an $\mathbf{R}^3$-neighborhood $U(p)$ of a specific point $p \in T$ is omitted by S, and we choose a smaller neighborhood $U_1(p) \subset\subset U(p)$. It is our aim to construct a vector function $Q(x) = (Q^1(x^1, x^2, x^3), Q^2(x^1, x^2, x^3), Q^3(x^1, x^2, x^3))$ which satisfies conditions (iii), (iv) of section 2, and for which

$$\operatorname{meas} \Omega_1^* = \iint_B Q(z) \cdot (z_u \wedge z_v)\, du\, dv \tag{4.1}$$

holds for every embedded regular surface $z(w) \in C^1(B, \mathbf{R}^3)$ which maps B into the interior of $\mathcal{K}$ dividing int $\mathcal{K} - z(B)$ into two open simply-connected parts Ω_1^* and Ω_2^*, and which is omitting $U(p)$.

Remark. We prefer to present a direct and explicit determination of the vector field $Q(x)$ which is tangential to T except in the neighborhood $U_1(p) \cap T$. For the reader who prefers a minimum of explicit calculations involved we shall indicate an abstract existence proof at the end of this section. There is a wide freedom in the choice of a suitable vector function, see [23]. As an example, we mention that for an ellipsoid $\mathcal{K}$: $(x^1/a)^2 + (x^2/b)^2 + (x^3/c)^2 < 1$, $Q(x)$ can take the interesting form

$$Q(x) = \frac{1}{3}\,\frac{2c - x^3}{(c - x^3)^2}\,\{-x^1 x^3, -x^2 x^3, c^2 - (x^3)^2\} \tag{4.2}$$

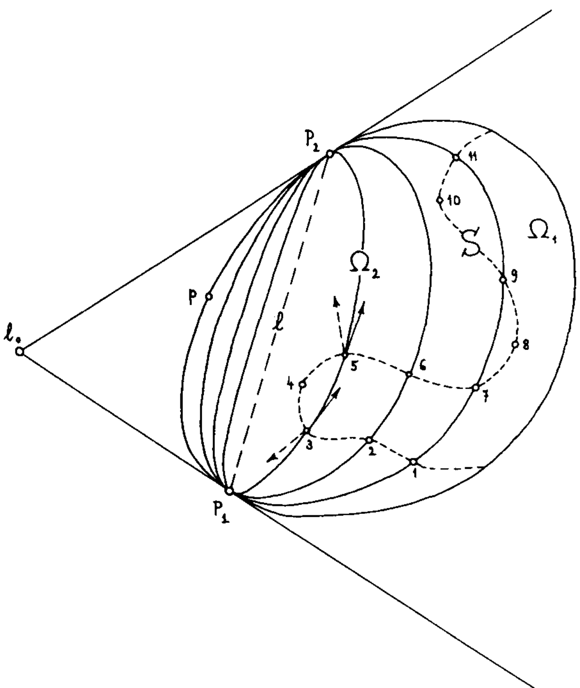

Fig. 2.

(or analogous expressions with one of the other axes preferred). (4.1) is applicable if p is the point $(0, 0, c)$. For the general case, we proceed as follows.

We select two points $p_1=(x_1^1, x_1^2, x_1^3)$ and $p_2=(x_2^1, x_2^2, x_2^3)$ on T such that the chord $l=p_1p_2$ is contained in $U_1(p)$ and that the planes tangent to T at these points intersect in a straight line l_0 outside of $\mathcal{K}$. (3) A simple foliation of $\mathcal{K}$ will be used considering $\mathcal{K}$ as the union of shells, much like an onion, by taking the line l as the stem and the point p_1 as the stemplate; see figure 2. In a suitably chosen coordinate system, the line l_0 becomes the x^2-axis, the body $\mathcal{K}$ lies in the halfspace $x^1>0$, and p_1 is situated "below" p_2 so that $x_1^3/x_1^1=-x_2^3/x_2^1$. Each plane in the pencil of planes with carrier line l_0 intersects $\mathcal{K}$ in a convex region, and we obtain the following representation for the points of $\mathcal{K}$:

$$\mathring{x}(a, \lambda, \varphi) = \begin{cases} \mathring{x}^1 = x^1(a) + \lambda R(a, \varphi) \cos\varphi \cos a \\ \mathring{x}^2 = x^2(a) + \lambda R(a, \varphi) \sin\varphi \\ \mathring{x}^3 = x^3(a) + \lambda R(a, \varphi) \cos\varphi \sin a \end{cases} \tag{4.3}$$

(3) It is assumed here that $\mathcal{K}$ is not flat in the neighborhood of p.

where $|\alpha|\leq\alpha_0$, $0\leq\lambda\leq 1$, $0\leq\varphi\leq 2\pi$ while $\{x=x(\alpha): |\alpha|\leq\alpha_0\}$ is a representation of l:

$$x^1(\alpha)=\frac{(x_2^1x_1^3-x_1^1x_2^3)\cos\alpha}{(x_2^1-x_1^1)\sin\alpha-(x_2^3-x_1^3)\cos\alpha}$$

$$x^2(\alpha)=\frac{(x_2^1x_1^2-x_1^1x_2^2)\sin\alpha+(x_2^2x_1^3-x_1^2x_2^3)\cos\alpha}{(x_2^1-x_1^1)\sin\alpha-(x_2^3-x_1^3)\cos\alpha}$$

$$x^3(\alpha)=\frac{(x_2^1x_1^3-x_1^1x_2^3)\sin\alpha}{(x_2^1-x_1^1)\sin\alpha-(x_2^3-x_1^3)\cos\alpha}.$$

Every point (x^1,x^2,x^3) in the interior of $\mathscr{K}$, not in $U_1(p)$ (and thus not on l), corresponds to a unique triple (α,λ,ψ):

$$\begin{aligned}\alpha&=\alpha(x^1,x^2,x^3)=\operatorname{arc\,tan}[x^3/x^1]\\ \varphi&=\varphi(x^1,x^2,x^3)=\operatorname{arc\,tan}\left[(x^2-x^2(\alpha))/\sqrt{(x^1-x^1(\alpha))^2+(x^3-x^3(\alpha))^2}\,\right]\\ \lambda&=\lambda(x^1,x^2,x^3)=R^{-1}(\alpha(x),\varphi(x))\sqrt{(x^1-x^1(\alpha))^2+(x^2-x^2(\alpha))^2+(x^3-x^3(\alpha))^2}.\end{aligned}\tag{4.4}$$

In the second and third formulas it is understood that $\tan\alpha=x^3/x^1$ is substituted in the expressions $x^1(\alpha)$, $x^2(\alpha)$, $x^3(\alpha)$. A computation shows that

$$\Delta(\alpha,\lambda,\varphi):=\mathring{x}_\alpha\cdot(\mathring{x}_\lambda\wedge\mathring{x}_\varphi)=\frac{\lambda R^2(\alpha,\varphi)\,x^1(\alpha)}{\cos\alpha}+\lambda^2R^3(\alpha,\varphi)\cos\varphi.\tag{4.5}$$

The various leaves of the foliation are the loci λ=constant; for $\lambda=1$ we obtain a representation of the boundary $\partial\mathscr{K}=T$.

Let now $z(w)$ be the surface under consideration and assume that the orientation is chosen so that the point p lies on the boundary of $\Omega_2^\ddagger$ and that the normal vector $z_u\wedge z_v$ points toward $\Omega_2^\ddagger$. We use the abbreviations

$$\bar\alpha(w)=\alpha(z^1(w),z^2(w),z^3(w))$$

$$\bar\lambda(w)=\lambda(z^1(w),z^2(w),z^3(w))$$

$$\bar\varphi(w)=\varphi(z^1(w),z^2(w),z^3(w)).$$

Then it turns out that

$$\frac{\partial(\bar\lambda,\bar\varphi)}{\partial(u,v)}=\Delta^{-1}(\bar\alpha,\bar\lambda,\bar\varphi)\,\mathring{x}_\alpha\cdot(z_u\wedge z_v).\tag{4.6}$$

On the surface $z(w)$ we consider the vector $\bar{\mathring{x}}_\alpha=\mathring{x}_\alpha(\bar\alpha(w),\bar\lambda(w),\bar\varphi(w))$ and we define the open subsets of B,

$$B^+ = \{w\colon w \in B,\ \bar{\lambda}(w) > 0,\ \bar{\mathring{x}}_\alpha \cdot (z_u \wedge z_v) > 0\}$$

$$B^- = \{w\colon w \in B,\ \bar{\lambda}(w) > 0,\ \bar{\mathring{x}}_\alpha \cdot (z_u \wedge z_v) < 0\}$$

and the set

$$B^0 = \{w\colon w \in B,\ \bar{\lambda}(w) > 0,\ \bar{\mathring{x}}_\alpha \cdot (z_u \wedge z_v) = 0\}.$$

The sets B^+ and B^- define those portions of $z(w)$ which are intersected transversally by the curves λ=constant, φ=constant of our foliation. In figure 2, the points $z_j = z(w_j)$ on $z(w)$ are images of points w_j in B^+ for $j=5,6,7,11$, in B^- for $j=1,2,3,9$, and in B^0 for $j=4,8,10$.

Consider a small neighborhood σ in B^+. Its image $z(\sigma)$ on the surface $z(w)$ is the base of a curved cone with vertex p_1 (in the negative direction from $z(w)$). The volume element is $(\mathring{x}_\alpha, \mathring{x}_\lambda, \mathring{x}_\varphi)\, d\alpha\, d\lambda\, d\varphi$; thus the volume of the cone comes to

$$\iint_{\hat{\sigma}} \left[\int_{-x_0}^{\hat{\alpha}(\lambda,\varphi)} (\mathring{x}_\alpha, \mathring{x}_\lambda, \mathring{x}_\varphi)\, d\alpha \right] d\lambda\, d\varphi.$$

Here $(\mathring{x}_\alpha, \mathring{x}_\lambda, \mathring{x}_\varphi) = \mathring{x}_\alpha \cdot (\mathring{x}_\lambda \wedge \mathring{x}_\varphi)$, $\hat{\sigma}$ is the image of σ in the (λ, φ)-plane and $\hat{\alpha}(\lambda, \varphi) = \bar{\alpha}(u, v)$. In view of (4.5), (4.6), by a change of variables, the above integral is equal to

$$\iint_{\sigma} \left[\frac{1}{\Delta(\bar{\alpha}, \bar{\lambda}, \bar{\varphi})} \int_{-\alpha_0}^{\bar{\alpha}(u,v)} \Delta(\beta, \bar{\lambda}, \bar{\varphi})\, d\beta \right] (\bar{\mathring{x}}_\alpha, z_u, z_v)\, du\, dv$$

or

$$\iint_{\sigma} Q(z) \cdot (z_u \wedge z_v)\, du\, dv,$$

if we define generally

$$Q(x) = \frac{1}{\Delta(\alpha, \lambda, \varphi)} \left[\int_{-\alpha_0}^{\alpha} \Delta(\beta, \lambda, \varphi)\, d\beta \right] \mathring{x}_\alpha(\alpha, \lambda, \varphi) \tag{4.7}$$

where on the right hand side the values $\alpha(x)$, $\lambda(x)$, $\varphi(x)$ must be substituted from (4.4). For the change from the parameters λ, φ to the parameters u, v, the inequality $\partial(\bar{\lambda}, \bar{\varphi})/\partial(u, v) > 0$ has been used.

Near its base, the cone lies in the domain Ω_1^*, but its generating lines may intersect the surface $z(w)$ repeatedly—alternatingly in the images of points in B^- and B^+. Since $\partial(\bar{\lambda}, \bar{\varphi})/\partial(u, v) < 0$ for $w \in B^-$, the contribution to the volume will be negative for a base $z(\sigma)$, $\sigma \subset B^-$. Therefore, ultimately only those parts of the original cone which lie in Ω_1^*

will be recorded. Formula (4.1) is obtained by an extension of the integration over all of B. (Employing an approximation argument, if necessary, it may be assumed that the set B^0 has measure zero.)

The vector function $Q(x)$ of (4.7) has the following properties:

(1) $Q(x)$ is continuously differentiable in $\mathcal{K}-U_1(p)$; the precise regularity of $Q(x)$ depends on that of T.

(2) $\operatorname{div} Q(x)=1$ in $\mathcal{K}-U_1(p)$.

(3) For $x\in T-U_1(p)$, $Q(x)$ is a vector tangent to T.

Property (1) is clear. Property (3) is a consequence of the fact that the curves $\lambda=1$, φ=constant lie on the boundary T and can be parametrized with the help of the variable α. As for property (2), let us set

$$q(\alpha,\lambda,\varphi)=\frac{1}{\Delta(\alpha,\lambda,\varphi)}\int_{-\alpha_0}^{\alpha}\Delta(\beta,\lambda,\varphi)\,d\beta$$

and observe that

$$\operatorname{grad}\alpha(x)=\frac{1}{\Delta}(\mathring{x}_\lambda\wedge\mathring{x}_\varphi)$$

$$\operatorname{grad}\lambda(x)=\frac{1}{\Delta}(\mathring{x}_\varphi\wedge\mathring{x}_\alpha)$$

$$\operatorname{grad}\varphi(x)=\frac{1}{\Delta}(\mathring{x}_\alpha\wedge\mathring{x}_\lambda).$$

Then

$$\mathring{x}_\alpha\cdot\operatorname{grad}\alpha=1,\quad \mathring{x}_\alpha\cdot\operatorname{grad}\lambda=0,\quad \mathring{x}_\alpha\cdot\operatorname{grad}\varphi=0$$

$$\mathring{x}_{\alpha\alpha}\cdot\operatorname{grad}\alpha+\mathring{x}_{\alpha\lambda}\cdot\operatorname{grad}\lambda+\mathring{x}_{\alpha\varphi}\cdot\operatorname{grad}\varphi=\frac{\Delta_\alpha}{\Delta}$$

and

$$q_\alpha=1-\frac{\Delta_\alpha}{\Delta}q.$$

It follows that

$$\begin{aligned}\operatorname{div} Q(x)&=\frac{\partial Q^i}{\partial x^i}=\frac{\partial Q}{\partial\alpha}\cdot\operatorname{grad}\alpha+\frac{\partial Q}{\partial\lambda}\cdot\operatorname{grad}\lambda+\frac{\partial Q}{\partial\varphi}\cdot\operatorname{grad}\varphi\\&=(q_\alpha\operatorname{grad}\alpha+q_\lambda\operatorname{grad}\lambda+q_\varphi\operatorname{grad}\varphi)\cdot\mathring{x}_\alpha\\&\quad+q(\mathring{x}_{\alpha\alpha}\cdot\operatorname{grad}\alpha+\mathring{x}_{\alpha\lambda}\cdot\operatorname{grad}\lambda+\mathring{x}_{\alpha\varphi}\cdot\operatorname{grad}\varphi)\\&=1-\frac{\Delta_\alpha}{\Delta}q+\frac{\Delta_\alpha}{\Delta}q=1.\end{aligned}$$

Our vector function $Q(x)$ of (4.7) and the surface T satisfy the conditions (i)–(iv) stated at the beginning of section 3, and the volume formula (4.1) applies to the representation $x: B \to \mathbf{R}^3$ of S. Let now $\mathscr{C}(T)$ be the class of surfaces defined in section 2, and set

$$\mathscr{C}^* := \left\{ z(w) \in \mathscr{C}(T) : \int\int_B Q(z) \cdot (z_u \wedge z_v)\, du\, dv = \sigma \operatorname{meas} \mathscr{K} \right\}.$$

We shall now give the precise formulation of

Assumption (A2). $x(w)$ is stationary for the Dirichlet integral within the class $\mathscr{C}^*$.

By virtue of our construction, the assertion of Theorem 1 follows from the Theorems 2 and 3. As had been already mentioned, $x(w)$ cannot have branch points on ∂B.

Let us finally give an abstract reason for the existence of a vector field Q as in assumption (A2). We can try to find a solution of

$$\operatorname{div} Q = 1 \text{ in } \mathscr{K}, \quad Q_n = 0 \text{ on } T - U_1(p), \quad (\text{where } Q_n = n \cdot Q), \tag{4.8}$$

which is of the special form

$$Q(x) = \operatorname{grad} f(x).$$

For this purpose, we determine the scalar function $f(x)$ as solution of a boundary value problem

$$\begin{aligned} \Delta f = 1 \quad & \text{in } \mathscr{K} \\ \frac{\partial f}{\partial n} + \gamma f = 0 \quad & \text{on } T = \partial \mathscr{K} \end{aligned} \tag{4.9}$$

where γ is some sufficiently regular function on T which satisfies $\gamma \geq 0$, $\gamma \not\equiv 0$, and $\gamma \equiv 0$ on $T - U_1(p)$. The boundary condition in (4.9), which was suggested by Fanghua Lin, has been selected to make sure that a unique regular solution $f(x)$ exists; see, for instance [6], especially p. 124.

This approach has the advantage that it can also be applied to nonconvex sets $\mathscr{K}$. If $x(w)$ is a solution of the partitioning problem for $\mathscr{K}$, we infer from (4.8) that

$$\operatorname{meas} \Omega_1^* = \int\int\int_{\Omega_1^*} dx^1\, dx^2\, dx^3 = \int\int\int_{\Omega_1^*} \operatorname{div} Q\, dx^1\, dx^2\, dx^3$$

$$= \int\int_{\partial\Omega_1^*} Q_n \, dS = \int\int_B Q(x) \cdot x_u \wedge x_v \, du \, dv.$$

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