Random coverings in several dimensions

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1. Introduction and main results

A class of covering problems can be formulated as follows: Let K be a fixed "big set" and let $B_1, B_2, \ldots$ be a sequence of independent identically distributed random "small sets". We let N be the number of small sets required to cover K completely, i.e. $N = \inf\{n : \bigcup_1^n B_i \supset K\}$, and ask for various properties of the random variable N.

For example, K may be the unit circle and B_i uniformly distributed arcs of a fixed length a; see Solomon [15], Chapter 4 for a discussion and references. More generally, the lengths of the arcs may be random; cf. Siegel [13], Siegel and Holst [14], Jewell and Romano [9] and Janson [7].

One obvious generalization of this problem to two (and higher) dimensions is to let the big set be the surface of a sphere and the small sets be uniformly distributed spherical caps (with fixed or random radii); another generalization is to let the big set be a torus or a cube and the small sets be translates of some given set(s). Some results (different from ours) for the case of caps of fixed radius on a sphere have been obtained by Moran and Fazekas de St. Groth [11], Gilbert [6] and Peter [12]. Flatto and Newman [5] studied the more general problem of small geodesic balls on a compact Riemannian manifold, and obtained estimates for the distribution and the expectation of N.

The purpose of this paper is to derive, for all the situations described above, the asymptotic distribution of N as the small sets are uniformly shrunk. In fact, we will more generally give the asymptotic distribution of the number of small sets required to cover every point of the big set at least m times (where m is a fixed positive integer), although most details of the proofs will be given for $m=1$ only.

In the case of subsets of $\mathbf{R}^d$ or $\mathbf{T}^d$, we will for various technical reasons assume that the small sets are convex. (We conjecture that the results can be generalized to non-convex sets with nice boundaries.) Thus, we assume that A is a random convex set, that X is uniformly distributed on a set $V\subset\mathbf{R}^d$ and independent of A and that a is a positive scale factor and let the small sets be distributed as $aA+X$. As we will see in the final remark of Section 7, we should not take $V=K$ since in that case the boundary of K may be the last part to be covered. To avoid complications at the boundary, we assume on the contrary that $\bar{K}\subset V^\circ$, and thus $d(K, V^c)>0$. However, this implies that, at least for small a, many of the small sets miss K completely. We may choose not to count such sets and thus define

$$N' = \#\{i\leqslant N\colon B_i\cap K\neq\emptyset\}.$$

This eliminates the influence of the set V.

Another way to avoid boundary problems is to make everything periodic, i.e. take $K=V=\mathbf{T}^d$. In fact, the following theorem, and its proof, holds for sets $K\subset V\subset\mathbf{T}^d$ as well.

We let $|A|$ denote the Lebesgue measure of A and define $r(A)=\sup_{x\in A}|x|$. We assume that $\mathscr{E}|A|>0$.

THEOREM 1.1. *Suppose that K is a bounded subset of $\mathbf{R}^d$, $d\geqslant 1$, with $|\partial K|=0$, that $\bar{K}\subset V^\circ$ and that $|V|<\infty$. Suppose further that A is a random convex subset of $\mathbf{R}^d$ with $\mathscr{E}r(A)^{d+\varepsilon}<\infty$ for some $\varepsilon>0$, and that m is a positive integer.*

For $a>0$, let the small sets have the same distribution as $aA+X$, where X is uniformly distributed on V, and let $N_{a,m}$ be the number of small sets required to cover K m times.

Let $\alpha(A)$ be the constant given by (5.3) and Corollary 7.4 and let U have the extreme value distribution $\mathscr{P}(U\leqslant u)=\exp(-e^{-u})$. Then, as $a\to 0$,

$$\text{(i)}\ \frac{\mathscr{E}|aA|}{|V|}N_{a,m}-\log\frac{|K|}{\mathscr{E}|aA|}-(d+m-1)\log\log\frac{|K|}{\mathscr{E}|aA|}+\log(m-1)!-\log\alpha\xrightarrow{d} U. \tag{1.1}$$

(ii) *If*

$$|\{x\colon d(x,\partial K)<\varepsilon\}|=o(|\log\varepsilon|^{-1})\ \text{as}\ \varepsilon\to 0, \tag{1.2}$$

and $A\neq\emptyset$ a.s., and $N'_{a,m}$ is the number of the small sets that actually meet K, then

$$\frac{\mathscr{E}|aA|}{|K|}N'_{a,m}-\log\frac{|K|}{\mathscr{E}|aA|}-(d+m-1)\log\log\frac{|K|}{\mathscr{E}|aA|}+\log(m-1)!-\log\alpha\xrightarrow{d} U. \tag{1.3}$$

Note that the condition (1.2), although stronger than $|\partial K|=0$, is very weak and e.g. satisfied for all convex sets.

For the second version of our results, we let K be a C^2 compact Riemannian manifold, i.e. a compact C^2 manifold with a C^1 metric tensor. The metric tensor defines a metric on K, the geodesic distance, and a finite positive measure dv. We let the small sets be geodesic balls $B(x, r)=\{y$: there exists a curve of length less than r between x and $y\}$, and let the centres have the uniform distribution $v(K)^{-1}dv$.

THEOREM 1.2. *Suppose that K is a C^2 compact d-dimensional Riemannian manifold, $d\geq 1$. Suppose further that R is a positive random variable with $\mathscr{E}R^{d+\varepsilon}<\infty$ for some $\varepsilon>0$ and that m is a positive integer. For $a>0$, let $N_{a,m}$ be the number of independent random geodesic balls $B(X, aR)$, with X uniformly distributed on K (independently of R), that are needed to cover K m times.*

Let $b=v_d\mathscr{E}R^d/v(K)$, where $v_d=\pi^{d/2}/\Gamma(d/2+1)$ is the volume of the Euclidean unit sphere, and let α be the constant given by (9.24). *Then as $a\to 0$,*

$$ba^dN_{a,m}-\log(ba^d)^{-1}-(d+m-1)\log\log(ba^d)^{-1}+\log(m-1)!-\log\alpha\xrightarrow{d}U, \tag{1.4}$$

where U is as in Theorem 1.

In particular, this applies to the problem of covering the surface of a sphere in $\mathbf{R}^{d+1}$ by small spherical caps of fixed or random radii. If the (d-dimensional) area of the sphere is normalized to be one, ba^d in (1.4) may be replaced by (the expectation) of the area of the small caps.

Remarks. (1) For arcs of a non-random length on a circle, this was proved by Flatto [4]. (In this case $d=1$ and $\alpha=1$.)

(2) For arcs of random lengths on a circle and $m=1$, this was proved by a different method in Janson [7]. It was there shown that the moment condition $\mathscr{E}R^{1+\varepsilon}<\infty$ in this case can be weakened to $\mathscr{E}(RI(R>t))=o(1/\log t)$ as $t\to\infty$, but not to $O(1/\log t)$. Here and in the sequel $I(\dots)$ denotes the indicator function, i.e. I equals 1 when the condition inside the parenthesis holds, and 0 otherwise.

(3) There is also a zero-dimensional analogue, viz, the coupon collector's problem. Let the big set be a finite set with n elements and let the small sets consist of one element each (uniformly distributed). Then, for $m\geq 1$,

$$N/n-\log n-(m-1)\log\log n+\log(m-1)!\xrightarrow{d}U, \quad \text{as } n\to\infty,$$

Erdös and Renyi [2], which corresponds to (1.1) and (1.4) with $\alpha=1$.

(4) There exists a version of Theorem 1.2 (similar to Theorem 1.1) when K is a

relatively compact subset of a Riemannian manifold and X is uniformly distributed in a neighborhood V.

(5) The term

$$\frac{\mathscr{E}|aA|}{|V|} N_{a,m}$$

in (1.1) is the average number of sets covering any fixed point (ignoring complications at the boundary).

We note that the first order term in the asymptotic distributions is the logarithmic term. This term is independent of m, but the second order term (the $\log\log$) depends on m, and is furthermore the only term that explicitly depends on the dimension. The average volume of the small sets $\mathscr{E}|aA|$ enters in an obvious, normalizing way, but the shapes of the small sets and the variation of their volumes influence the asymptotic distribution only through the third order term $\log \alpha$. (If $d=1$, then α equals 1 and the asymptotic distribution is not influenced at all!) We will discuss this term in Section 9; for the moment we only note that a small value of α implies that the small sets cover efficiently compared to other sets of the same size. We refer the reader to the examples in Section 9 and the adjoining comments on the qualitative results that emerge.

The paper is organized as follows. Section 2 contains a preliminary discussion on Poisson processes. The basic idea of the proof is to reformulate the problem as a problem for Poisson processes. This is done in Section 3. Sections 4 and 5 contain further preliminaries. The core of the proofs of the theorems follows in Section 6, and the proofs are completed in Sections 7 and 8. The geometric constant $\alpha(A)$ is discussed in Section 9, where it also is computed for several examples.

Acknowledgements. I thank several of my colleagues for helpful discussions; in particular Sven-Erick Alm, Gunnar Brånvall, Allan Gut, and Lars Holst.

Part of this research was carried out during a visit to the University of Lund.

2. Poisson processes in general

Our proof will be based on properties of Poisson processes in general spaces. Thus, we begin with their definition, cf. e.g. Kallenberg [10].

Let $(\Omega, \mathscr{F})$ be a given measurable space. We let $\tilde{\Omega}$ be the set of extended integer-valued positive measures on $(\Omega, \mathscr{F})$, i.e. the measures Ξ such that $\Xi(A) \in \{0, 1, \ldots, \infty\}$ for every $A \in \mathscr{F}$. Thus $\Xi \to \Xi(A)$ is an extended integer valued function on $\tilde{\Omega}$ for each $A \in \mathscr{F}$, and we let $\tilde{\mathscr{F}}$ be the σ-field on $\tilde{\Omega}$ generated by these functions. Hence, if we provide $(\tilde{\Omega}, \tilde{\mathscr{F}})$ with a probability measure $\tilde{\nu}$, the mappings $\Xi \to \Xi(A)$, $A \in \mathscr{F}$, become random variables. Suppose that these random variables have the following properties, for some σ-finite measure ν on $(\Omega, \tilde{\mathscr{F}})$:

(i) if $A \in \mathscr{F}$, then $\Xi(A)$ has a Poisson distribution with expectation $\nu(A)$ (if $\nu(A)=\infty$, this is interpreted as $\Xi(A)=\infty$ a.s.);

(ii) if $A_1, \dots, A_n$ are disjoint, then $\Xi(A_1), \dots, \Xi(A_n)$ are independent.

We then say that $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\nu})$ describes a Poisson process in $(\Omega, \mathscr{F})$ with intensity ν. A simple construction shows that for every σ-finite measure ν on $(\Omega, \mathscr{F})$, there exists a unique measure $\tilde{\nu}$ on $(\tilde{\Omega}, \check{\mathscr{F}})$ that describes a Poisson process with intensity ν, cf. [10], pp. 7–9.

In the definition above we regard the Poisson process as a random measure. We will use Ξ to denote this random measure. However, Ξ equals $\Sigma\, \delta_{\xi_i}$ for some (finite or infinite) sequence $\xi_1, \xi_2, \dots$ of points in Ω. (In fact, the Poisson process is constructed as such a sum in [10].) Consequently, we may identify Ξ with the set $\{\xi_i\}$ and regard the Poisson process as a random (countable) subset of Ω. We will often prefer this point of view and e.g. write $\xi \in \Xi$ for $\Xi(\{\xi\})>0$ and $\Xi \cup \{\xi\}$ for $\Xi+\delta_\xi$.

There are two minor technical problems in this identification: if $\mathscr{F}$ does not separate points in Ω, then $\{\xi_i\}$ is not unique, and, secondly, if ν has point masses (atoms), then there may be repetitions in $\xi_1, \xi_2, \dots$ so that some points in $\{\xi_i\}$ have to be counted more than once. We disregard these rather harmless complications, which in any case do not arise in our applications.

If f is a positive measurable function on $\Omega^n \times \tilde{\Omega}$ ($n \geq 0$), we may for each Ξ form the sum $\Sigma\,\{f(\xi_i, \dots, \xi_n, \Xi)\colon \xi_1, \dots, \xi_n$ are distinct elements of $\Xi\}$.

We denote this sum by $\Sigma'_\Xi f(\xi_1, \dots, \xi_n, \Xi)$. The (proof of the) following lemma shows that this sum is a measurable function of the Poisson process, and provides a formula for its expectation.

LEMMA 2.1. *If f is a positive measurable function on $\Omega^n \times \tilde{\Omega}$, then*

$$\mathscr{E}\sum_{\Xi}{}' f(\xi_1, \dots, \xi_n, \Xi) = \int \dots \int \mathscr{E} f(\omega_1, \dots, \omega_n, \Xi \cup \{\omega_i\}_1^n)\, d\nu(\omega_1) \dots d\nu(\omega_n). \tag{2.1}$$

Proof. By monotone convergence and linearity, it suffices to prove this formula when f is the indicator function of a measurable subset of $A^n \times \tilde{\Omega}$, with $\nu(A)<\infty$. By the monotone class theorem, we may restrict ourselves to f of the type $I(\xi_i \in A_i,\ i=1 \dots n$, and $\Xi(B_j) \in C_j, j=1 \dots m)$, where $m \geq 0$, $A_i, B_j \in \mathscr{F}$, $\nu(A_i)<\infty$, $\nu(B_j)<\infty$ and $C_j \subset Z$. Subdividing the sets A_i, B_j and C_j we may further assume that $B_1, \dots, B_m$ are disjoint, each A_i

equals some B_{j_i}, and $C_j = \{b_j\}$. Thus

$$f = I(\xi_i \in B_{j_i},\ i = 1 \dots n, \text{ and } \Xi(B_j) = b_j,\ i = 1 \dots m).$$

Put $n_j = \#\{i\colon A_i = B_j\}$, $j = 1 \dots m$. Then, with f as above,

$$\sum_{\Xi}{}' f(\xi_1, \dots, \xi_n, \Xi) = \prod_1^m (b_j)_{n_j} \cdot I(\Xi(B_j) = b_j, j = 1 \dots m).$$

Thus the left-hand side of (2.1) equals

$$\prod_1^m (b_j)_{n_j} \prod_1^m \mathcal{P}(\Xi(B_j) = b_j) = \prod_1^m \nu(B_j)^{b_j}((b_j - n_j)!)^{-1} e^{-\nu(B_j)}$$
$$= \prod_1^m \nu(B_j)^{n_j} \cdot \mathcal{P}(\Xi(B_j) + n_j = b_j, j = 1 \dots m),$$

which equals the right-hand side. Q.E.D.

Before we can formulate the second general property of Poisson processes, we need a definition.

Definition. A real-valued function f of the Poisson process is increasing if $f(\Xi) \geq f(\Xi')$ for every two realizations Ξ and Ξ' such that $\Xi - \Xi'$ is a positive measure (i.e. $\Xi \supset \Xi'$ regarded as sets). An event $E \in \bar{\mathcal{F}}$ is increasing if its indicator function is increasing.

In our applications Ξ will be a random set of sets and we will study the event that a certain set is covered by Ξ. This is obviously an example of an increasing event.

The importance of this property lies in the following correlation inequality.

LEMMA 2.2. *If f and g are two increasing non-negative measurable functions of a Poisson process Ξ, then*

$$\mathcal{E}(f(\Xi)\, g(\Xi)) \geq \mathcal{E}f(\Xi)\, \mathcal{E}g(\Xi). \tag{2.2}$$

In particular, if E_1 and E_2 are two increasing events,

$$\mathcal{P}(E_1 \text{ and } E_2) \geq \mathcal{P}(E_1)\, \mathcal{P}(E_2). \tag{2.3}$$

For a proof of this lemma and its relation to the FKG-inequality, see [8], Lemma 2.1.

3. Poisson processes in particular

We return to the situation of Theorem 1.1. We will in the sequel assume that the random convex set A is defined on a probability space $(\Omega_A, \mathcal{F}_A, \mu)$ such that the event $x \in A$ is measurable for every fixed $x \in R^d$.

We will in this section use the notation μ_a for the distribution of aA. (For notational convenience we assume that μ_a is defined on the same space Ω_A of convex sets.) Thus the small sets are defined as $aA+X$, where (aA, X) has the distribution $\mu_a \times |V|^{-1} dx$.

Let u be a fixed real number and let

$$\lambda(a) = \frac{1}{\mathscr{E}|aA|}\left(\log \frac{|K|}{\mathscr{E}|aA|} + (d+m-1) \log\log \frac{|K|}{\mathscr{E}|aA|} - \log(m-1)! + \log \alpha + u\right). \tag{3.1}$$

Then (1.1) may be written as

$$\mathscr{P}\left(\frac{\mathscr{E}|aA|}{|V|} N_{a,m} \leq \mathscr{E}|aA|\lambda(a)\right) \to e^{-e^{-u}} \quad \text{as } a \to 0 \tag{3.2}$$

or, since $\mathscr{P}(N_{a,m} \leq n) = \mathscr{P}(n$ sets cover m times)

$$\mathscr{P}(|V|\lambda(a) \text{ small sets cover } K\ m \text{ times}) \to \exp(-e^{-u}). \tag{3.3}$$

It is easy to see that we here may replace the fixed number $|V|\lambda(a)$ of small sets by a random number M with $M \sim \mathscr{P}o(|V|\lambda(a))$, i.e. M has a Poisson distribution with $\mathscr{E}M = |V|\lambda(a)$ (and, of course, M independent of everything else). (See e.g. [7], p. 70, where this is done in detail.) Consequently, (1.1) is equivalent to

$$\mathscr{P}(M \text{ small sets cover } K\ m \text{ times}) \to \exp(-e^{-u}). \tag{3.4}$$

However, let $\Xi^V_{\lambda,a}$ be a Poisson process on $\Omega_A \times V$ with intensity $|V|\lambda\mu_a \times |V|^{-1} dx = \lambda\mu_a \times dx$. Then $\Xi^V_{\lambda,a}$ consists of M, $M \sim \mathscr{P}o(|V|\lambda)$, independent elements (aA_i, X_i), each having the distribution $\mu_a \times |V|^{-1} dx$, and the set $\{aA+x\colon (aA,x) \in \Xi^V_{\lambda,a}\}$, is nothing but the collection of M small sets. With a minor abuse of notation, we let $\Xi^V_{\lambda,a}$ denote this set also, and (3.4) is the same as

$$\mathscr{P}(\Xi^V_{\lambda(a),a} \text{ cover } K\ m \text{ times}) \to \exp(-e^{-u}). \tag{3.5}$$

Finally, we extend $\Xi^V_{\lambda,a}$ to a Poisson process $\Xi_{\lambda,a}$ on $\Omega_A \times \mathbf{R}^d$ with intensity $\lambda\mu_a \times dx$. Thus, $\Xi_{\lambda,a} = \Xi^V_{\lambda,a} \cup \Xi^{V^c}_{\lambda,a}$, where $\Xi^{V^c}_{\lambda,a}$ is a Poisson process with the same intensity on

$\Omega_A \times V^c$. The number of sets of $\Xi^{V^c}_{\lambda, a}$ that meet K is Poisson distributed with expectation

$$\int_{\Omega_A}\int_{V^c} I((aA+x)\cap K \neq \varnothing)\,\lambda\, d\mu\, dx = \lambda \int_{\Omega_A}\int_{V^c} I(x \in K-aA)\, dx\, d\mu \tag{3.6}$$
$$= \lambda \mathscr{E}|(K-aA)\cap V^c|.$$

If $r(aA)<d(K, V^c)$, $(K-aA)\cap V^c = \varnothing$, and otherwise

$$|K-aA| \leq Cr(K-aA)^d \leq C(r(K)+r(aA))^d \leq Cr(aA)^d \leq Cr(aA)^{d+\varepsilon}.$$

Hence

$$\lambda(a)\,\mathscr{E}|(K-aA)\cap V^c| \leq C\lambda(a)\,a^{d+\varepsilon}\mathscr{E}r(A)^{d+\varepsilon} = C\lambda(a)\,a^{d+\varepsilon} \to 0, \quad \text{as } a \to 0 \tag{3.7}$$

and $\mathscr{P}$(some set in $\Xi^{V^c}_{\lambda(a), a}$ meets K)$\to 0$ whence (3.5) and hence (1.1) is equivalent to

$$\mathscr{P}(\Xi_{\lambda(a), a} \text{ covers } K\ m \text{ times}) \to \exp(-e^{-u}). \tag{3.8}$$

This will be proved in Section 7.

Similarly, since the number of sets in $\Xi_{\lambda, a}$ that meet K is Poisson distributed with expectation $\lambda\mathscr{E}|K-aA|$, it follows from (3.8) that

$$\mathscr{P}\left(\frac{\mathscr{E}|aA|}{\mathscr{E}|K-aA|} N'_a \leq \mathscr{E}|aA|\lambda(a)\right) \to \exp(-e^{-u}), \tag{3.9}$$

which, since (1.2) implies that

$$\mathscr{E}|K-aA|-|K| = o(|\log a|^{-1}), \quad \text{as } a \to 0,$$

yields (1.3). We omit the details.

A similar argument is used in the proof of Theorem 1.2, see Section 8.

4. Convex sets

We will need some properties of convex sets in $\mathbf{R}^d$. We denote the surface measure, i.e. the $d-1$ dimensional Hausdorff measure, by ω. (We will no longer need ω to denote points in Ω.) If A is convex, then $\omega(\partial A) \leq Cr(A)^{d-1}$.

Let $D^\delta_v(x) = \{y\colon |y-x|<\delta \text{ and } \langle v, y-x\rangle < 0\}$ (a hemisphere).

Definitions. A unit vector n is a normal to A at x if $x\in\partial A$ and $A\subset\{y: \langle y-x, n\rangle\leq 0\}$.

A vector $v\neq 0$ is special for $A_1,\dots,A_n$ at x if $x\in\bigcap_1^n\partial A_i$ and $\bigcup_1^n A_i\supset D_v^\delta(x)$ for some $\delta>0$.

A convex set has at least one normal at every point of ∂A. Furthermore, the boundary is ω-a.e. differentiable and thus, if the interior A° is non-empty, the normal is unique a.e. on ∂A. The significance of the special vectors will become clear in Section 6.

For the remainder of this section we assume that $A_1, A_2, \dots$ are fixed bounded convex sets in $\mathbf{R}^d$. We will prove several lemmas showing that random translates of these sets a.s. intersect in nice ways. The lemmas are intuitively obvious and more or less trivial to prove when the sets are e.g. spheres or polyhedra. We will nevertheless give complete proofs, but the reader that wants to come quickly to the point can skip these.

LEMMA 4.1. (i) *For a.e.* $\{x_i\}_1^d$, *the intersection* $\bigcap_1^d\partial(A_i+x_i)$ *is a finite set.*
(ii) *For a.e.* $\{x_i\}_1^{d+1}$, *the intersection* $\bigcap_1^{d+1}\partial(A_i+x_i)$ *is empty.*

Proof. (i) It suffices to prove this for x_i in a fixed large cube Q. Let Q_1 be a larger cube with $d(Q, Q_1^c)\geq\sup r(A_i)+1$. We put, for $0<\varepsilon<1$, $A_i^\varepsilon=\{x: d(x,\partial A_i)<\varepsilon\}$ and note that $|A_i^\varepsilon|\leq C_i\varepsilon$ for some constants $C_i<\infty$. Then

$$\begin{aligned}\int_Q\cdots\int_Q\Big|\bigcap_1^d(A_i^\varepsilon+x_i)\Big|dx_1\dots dx_d&=\int_{Q_1}\int_Q\cdots\int_Q I(x\in A_i^\varepsilon+x_i, i=1\dots d)\,dx\,dx_1\dots dx_d\\&=\int_{Q_1}\prod_1^d|Q\cap(x-A_i^\varepsilon)|\,dx\leq|Q_1|\prod_1^d|A_i^\varepsilon|\leq C\varepsilon^d.\end{aligned}\tag{4.1}$$

By Fatou's lemma,

$$\int\cdots\int\liminf_{\varepsilon\to 0}\varepsilon^{-d}\Big|\bigcap_1^d(A_i^\varepsilon+x_i)\Big|\,dx_1\dots dx_d\leq C<\infty.\tag{4.2}$$

However, if $x\in\bigcap_1^d\partial(A_i+x_i)$, then $\bigcap_1^d(A_i^\varepsilon+x_i)$ contains the ball with radius ε centred at x. Hence

$$\#\bigcap_1^d\partial(A_i+x_i)\leq C\liminf_{\varepsilon\to 0}\varepsilon^{-d}\Big|\bigcap_1^d(A_i^\varepsilon+x_i)\Big|$$

and thus

$$\int \dots \int \# \bigcap_1^d \partial(A_i+x_i)\, dx_1 \dots dx_d < \infty. \tag{4.3}$$

(ii) is proved in the same way, or by using (i). Q.E.D.

Remark. A different proof shows by induction that $\bigcap_1^k \partial(A_i+x_i)$ a.e. has a finite $(d-k)$-dimensional Hausdorff measure.

LEMMA 4.2. *Let $k \geq 1$. For a.e. $\{x_i\}_1^k$ holds that if $x \in \bigcap_1^k \partial(A_i+x_i)$ and n_i are normals to A_i+x_i at x, then $\{n_i\}_1^k$ span a proper cone in $\mathbf{R}^d$.*

Proof. Let F be the closed set $\{\{x_i\}_1^k \in \mathbf{R}^{dk} : \bigcap_1^k (A_i+x_i) \neq \emptyset\}$. Suppose that $\{x_i\} \in F$ and that n_i are normals to A_i+x_i at x but $\Sigma c_i n_i = 0$ for some $c_i \geq 0$ not all zero. If $\{x_i+z_i\} \in F$, then there exists $y \in \bigcap_1^k (A_i+x_i+z_i)$. Consequently, $y-z_i \in \bar{A}_i+x_i$, whence $\langle n_i, x-(y-z_i)\rangle \geq 0, i, \dots, k$, and

$$\sum_1^k c_i \langle n_i, z_i\rangle = \sum_1^k c_i \langle n_i, z_i\rangle + \left\langle \sum_1^k c_i n_i, x-y \right\rangle = \sum_1^k c_i \langle n_i, x-y+z_i\rangle \geq 0. \tag{4.4}$$

This restricts $\{z_i\}$ to a half-space in $\mathbf{R}^{dk}$. Consequently, $\{x_i\}_1^k$ is not a point of density of F. Since a.e. point in F is a point of density, see e.g. [16], p. 12, this completes the proof. Q.E.D.

LEMMA 4.3. (i) *For a.e. $\{x_i\}_1^k$, if $x \in \bigcap_1^k \partial(A_i+x_i)$, then $x \in \partial \bigcup_1^k \overline{A_i+x_i}$.*

(ii) *For a.e. $\{x_i\}_1^n$, $(\bigcup_1^n (A_i+x_i))^\circ = \bigcup_1^n (A_i+x_i)^\circ$.*

Proof. (i) Let n_i be normals at x. By Lemma 4.2 we may assume that there exists some vector v with $\langle v, n_i\rangle > 0$, $i=1 \dots k$. Consequently, $x+tv \notin \bigcup_1^k \overline{A_i+x_i}$ when $t>0$, and $x \notin (\bigcup_1^k \overline{A_i+x_i})^\circ$.

(ii) Suppose that $x \in (\bigcup_1^n (A_i+x_i))^\circ \setminus \bigcup_1^n (A_i+x_i)^\circ$. If the sets are indexed such that $x \in \overline{A_i+x_i}$ for $i=1, \dots, k$ but not for $i>k$, then $x \in \bigcap_1^k \partial(A_i+x_i)$ and $x \in (\bigcup_1^k (A_i+x_i))^\circ$. The result follows from (i). Q.E.D.

We denote the closed cone $\{\Sigma_1^k c_i n_i : c_i \geq 0\}$ spanned by some vectors $n_1, \dots, n_k$ by Cone $(n_1, \dots, n_k)$.

LEMMA 4.4. *Let $k \geq 1$. For a.e. $\{x_i\}_1^k$ holds that if v is special for $\{A_i+x_i\}_1^k$ at x and n_i is a normal to A_i+x_i at x, $i=1 \dots k$, then $v \in \mathrm{Cone}(n_1, \dots, n_k)$.*

Proof. Suppose that $v \notin \mathrm{Cone}(n_1, \ldots, n_k)$. Thus, there exists a vector e such that $\langle v, e\rangle < 0$ and $\langle n_i, e\rangle \geq 0$, $i=1 \ldots k$. Since, by Lemma 4.2, we may assume that there exists a vector f with $\langle n_i, f\rangle > 0$, it follows that with $e' = e + \varepsilon f$, $\varepsilon > 0$ small, $\langle v, e'\rangle < 0$ and $\langle n_i, e'\rangle > 0$, $i=1 \ldots k$. Consequently, if $\delta > 0$, $x + \delta e' \notin \bigcup_1^k (A_i + x_i)$ and v is not special at x.
Q.E.D.

LEMMA 4.5. *Let $1 \leq k \leq d-1$. For a.e. $\{x_i\}_1^k$, the set $\{v$: $\exists x$ such that v is special for $\{A_i + x_i\}_1^k$ at $x\}$ has Lebesgue measure zero.*

Proof. Let V_x be the linear span of all special vectors for $\{A_i + x_i\}_1^k$ at x. By Lemma 4.4, we may assume that $\dim V_x \leq k < d$ for every x.

Let v be special at x and let $D_v^\delta(x) \subset \bigcup_1^k (A_i + x_i)$. Suppose that $|y - x| < \delta$ and that some vector w is special at y. Then $y \in \bigcap_1^k \overline{A_i + x_i}$. If $\langle v, y - x\rangle > 0$, then for some small $\varepsilon > 0$, $x - \varepsilon(y - x) \subset D_v^\delta(x) \subset (\bigcup_1^k (A_i + x_i))^\circ$, and it follows, since each $\overline{A_i + x_i}$ is convex, that a neighborhood of x is included in $\bigcup_1^k \overline{A_i + x_i}$. We exclude this by Lemma 4.3 (i) and conclude that $\langle y - x, v\rangle \leq 0$. Then, for some $\eta > 0$, $D_v^\eta(y) \subset D_v^\delta(x)$ and v is special at y.

Now, let $e_1, \ldots, e_l$, all special at x, be a basis of V_x. The above argument applied to each e_j shows that there exists some $\delta > 0$ (which may be chosen the same for every j) such that if $|y - x| < \delta$ and $V_y \neq 0$, then $\{e_j\}_1^l \in V_y$ and thus $V_x \subset V_y$. If furthermore $\dim V_x = \dim V_y$ then necessarily $V_x = V_y$. Hence, for every $l = 1, \ldots, d-1$, the set $\{x$: $\dim V_x = l\}$ may be covered by balls where V_x is constant. Consequently, there is at most a countable number of different spaces V_x and the set of special vectors is included in the union of countably many hyperplanes. Q.E.D.

LEMMA 4.6. *For a.e. $\{x_i\}_1^d$, the set $\{v$: there exist two different points x, x' such that v is special for $\{A_i + x_i\}_1^d$ both at x and at $x'\}$ has Lebesgue measure zero.*

Proof. Suppose that v is special at x and at x' and put $y = x' - x$. Let n_i be a normal to $A_i + x_i$ at x. Since $x' \in \bigcap_1^d \overline{A_i + x_i}$, $\langle n_i, y\rangle \leq 0$ for $i=1 \ldots d$. We study two cases separately.

Case (i): $\langle n_i, y\rangle = 0$ for all i. Then $x + ty + \varepsilon n_i \notin A_i + x_i$, when $\varepsilon > 0$. Since $x + ty \in \overline{A_i + x_i}$ by convexity for $0 \leq t \leq 1$, $\{x + ty$: $0 \leq t \leq 1\} \in \bigcap_1^d \partial(A_i + x_i)$. This is covered by Lemma 4.1 (i).

Case (ii): $\langle n_i, y\rangle < 0$ for some i, say $i=1$. Let $D_v^\delta(x) \subset \bigcup(A_i + x_i)$. Hence, if $\varepsilon > 0$ is small, $D_v^{\delta/2}(x - \varepsilon y) \subset D_v^\delta(x) \subset \bigcup_1^d (A_i + x_i)$. However, $x - \varepsilon y \notin \overline{A_1 + x_1}$, and

$x-\varepsilon y \notin (A_i+x_i)^\circ$, $i=2\dots d$, since otherwise $x\in(A_i+x_i)^\circ$ by convexity. It follows that v is special for a subset of $\{A_i+x_i\}_2^d$ at $x-\varepsilon y$. Thus, this case is covered by Lemma 4.5. Q.E.D.

LEMMA 4.7. *Let $l\geq 1$. For a.e. $\{x_i\}_1^{d+l}$, the set*

$$\{v\colon \exists x\in \bigcap_1^d \partial(A_i+x_i),\ y\in \bigcap_{l+1}^{d+l} \partial(A_i+x_i) \text{ such that } \langle v, x-y\rangle = 0\}$$

has Lebesgue measure zero.

Proof. By Lemma 4.1 (i) and (ii) the sets $\bigcap_1^d \partial(A_i+x_i)$ and $\bigcap_{l+1}^{d+l} \partial(A_i+x_i)$ are, for a.e. $\{x_i\}_1^{d+l}$, finite and disjoint. Thus, the set in question is the union of a finite number of hyperplanes. Q.E.D.

5. Further preliminaries

Let A be a random bounded convex set in $\mathbf{R}^d$ as before and let $0<\lambda<\infty$ We take $a=1$ for the time being and let Ξ be the Poisson process $\Xi_{\lambda,1}$ defined in Section 3.

LEMMA 5.1. *$\mathcal{P}$(There exist $d+1$ different sets $B_1,\dots,B_{d+1}\in\Xi$ such that $\bigcap_1^{d+1}\partial B_i\neq\emptyset$)$=0$.*

Proof. Put $f(B_1,\dots,B_{d+1})=I(\bigcap_1^{d+1}\partial B_i\neq\emptyset)$ and denote the sought probability by $\mathcal{P}$. Then, by Lemmas 2.1 and 4.1 (ii),

$$\begin{aligned}\mathcal{P} &\leq \mathcal{E}\sum_{\Xi}{}' f(B_1,\dots,B_{d+1})\\ &= \lambda^{d+1}\int\dots\int\int\dots\int f(A_1+x_1,\dots,A_{d+1}+x_{d+1})\,dx_1\dots dx_{d+1}\,d\mu(A_1)\dots d\mu(A_{d+1})\\ &= 0\end{aligned}$$

Q.E.D.

Definition. A vector $v\neq 0$ is admissible if

(i) for $k=1,\dots,d-1$,

$\mathcal{P}(\exists B_1,\dots,B_k\in\Xi$ and x such that v is special for $\{B_i\}_1^k$ at $x)=0$;

(ii) $\mathcal{P}(\exists B_1,\dots,B_d$ and x,x' with $x\neq x'$ such that v is special for $\{B_i\}_1^d$ both at x and at $x')=0$;

(iii) $\mathcal{P}(\exists B_1,\dots,B_d,B'_1,\dots,B'_d\in\Xi$ with $\{B_i\}\neq\{B'_i\}$ and x,x' with $\langle v,x-x'\rangle=0$ such that v is special for $\{B_i\}_1^d$ at x and for $\{B'_i\}_1^d$ at $x')=0$.

LEMMA 5.2. *Almost every $v\in\mathbf{R}^d$ is admissible.*

Proof. We treat the three conditions one by one.

(i) We may assume that $B_1,\dots,B_k$ are different (otherwise we reduce k). We let

$$f_v(B_1,\dots,B_k)=I(\exists x \text{ such that } v \text{ is special for } \{B_i\} \text{ at } x)$$

and use Lemmas 2.1 and 4.5 and Fubini's theorem to conclude

$$\int \mathscr{E}\sum_{\Xi}{}' f_v(B_1,\dots,B_k)\,dv=0.$$

(ii) The possibility that two B_i coincide is covered by (i). The case that $B_1,\dots,B_d$ are different follows from Lemmas 2.1 and 4.6 by the same argument.

(ii) Similar, using Lemma 4.7. Q.E.D.

Note that the property that v is admissible does not depend on λ. It is obviously invariant for a change of scale $A\to aA$ and a normalization $v\to v/|v|$.

We will perform an important change of variables in the next section and we compute the Jacobian here. Cf. Federer [3], Chapter 3.2 for changes of variables by Lipschitz mappings.

LEMMA 5.3. *Let $A_1,\dots,A_d$ be convex subsets of $\mathbf{R}^d$. Then Γ: $\mathbf{R}^d\times\partial A_1\times\dots\times\partial A_d\to(\mathbf{R}^d)^d$ defined by $\Gamma(x,y_1,\dots,y_d)=(x-y_1,\dots,x-y_d)$ has a.e. a Jacobian that equals $|\mathrm{Det}\,(n_i(y_i))_1^d|$, where $n_i(y_i)$ is the normal to A_i at y_i.*

Proof. Γ is a Lipschitz mapping and thus a.e. differentiable. Since the Jacobian is a function of the first order derivatives, its value at $(x,y_1,\dots,y_d)$ remains unchanged if $\partial A_1,\dots,\partial A_d$ are replaced by their respective tangent hyperplanes $\{y_i+H_i\}$, where $H_i=n_i(y_i)^\perp$. However, this linearized mapping $\Gamma^*:\mathbf{R}^d\times H_1\times\dots\times H_d\to(\mathbf{R}^d)^d$ with $\Gamma^*(x,z_1,\dots,z_d)=\{x-y_i-z_i\}_1^d$ (y_i are now held fixed!), equals the composition $\Gamma_3\circ(\Gamma_2\times I^d)\circ\Gamma$, where, with $n_i=n_i(y_i)$,

$$\Gamma_1\colon \mathbf{R}^d\times\Pi H_i\to\mathbf{R}^d\times\Pi H_i,\quad \Gamma_1(x,\{z_i\})=(x,\{x-\langle x,n_i\rangle n_i-z_i\}),$$
$$\Gamma_2\colon \mathbf{R}^d\to\mathbf{R}^d,\quad \Gamma_2(x)=\{\langle x,n_i\rangle\}_1^d,\ \text{and}$$
$$\Gamma_3\colon \mathbf{R}^d\times\Pi H_i\to(\mathbf{R}^d)^d,\quad \Gamma_3((a_i)_1^d,(z_i)_1^d)=\{a_in_i+z_i\}_1^d.$$

It is obvious that Γ_1 and Γ_3 are measure-preserving. Hence

$$|\mathrm{Det}\,\Gamma^*| = |\mathrm{Det}\,\Gamma_2| = |\mathrm{Det}\,(n_i)|. \qquad \text{Q.E.D.}$$

This lemma motivates the following,

Definition. Given d convex sets $A_1, \dots, A_d$, we let $\bar{\omega}$ denote the measure

$$|\mathrm{Det}\,(n_i(y_i))_1^d|\,d\omega(y_1)\dots d\omega(y_d) \quad \text{on} \quad \partial A_1 \times \dots \times \partial A_d,$$

where $n_i(y_i)$ is the normal to A_i at y_i. (Thus, $d\bar{\omega} = |d\omega \wedge \dots \wedge d\omega|$.)

We let, as in Section 4, $\mathrm{Cone}\,(n_1, \dots, n_k)$ denote the closed cone spanned by $n_1, \dots, n_k$ and let $\mathrm{Cone}^\circ\,(n_1, \dots, n_k)$ denote its interior.

We define, for $v \in \mathbf{R}^d$, if the interiors $A_1^\circ, \dots, A_d^\circ$ are nonempty,

$$\beta(A_1, \dots, A_d, v) = \int_{\partial A_1} \dots \int_{\partial A_d} I(v \in \mathrm{Cone}\,(n_1(y_1), \dots, n_d(y_d)))\, d\bar{\omega} \tag{5.1}$$

and

$$\beta_0(A_1, \dots, A_d, v) = \int_{\partial A_1} \dots \int_{\partial A_d} I(v \in \mathrm{Cone}^\circ\,(n_1(y_1), \dots, n_d(y_d)))\, d\bar{\omega}, \tag{5.2}$$

and $\beta(A_1, \dots, A_d, v) = \beta_0(A_1, \dots, A_d, v) = 0$ if some $A_i^\circ = \varnothing$, and further, if A denotes our random convex set,

$$\alpha(A, v) = (d!)^{-1}(\mathscr{E}|A|)^{-(d-1)}\mathscr{E}\beta(A_1, \dots, A_d, v), \tag{5.3}$$

$$\alpha_0(A, v) = (d!)^{-1}(\mathscr{E}|A|)^{-(d-1)}\mathscr{E}\beta_0(A_1, \dots, A_d, v), \tag{5.4}$$

where $A_1, \dots, A_d$ are independent random sets with the same distribution as A. (We write the first argument of α and α_0 as A although they actually are functionals of the distribution μ of A.) Note that the factors $\mathscr{E}|A|$ in α and α_0 make them homogeneous; $\alpha(aA, v) = \alpha(A, v)$. Also, $\beta_0 \leq \beta \leq \Pi_1^d\ \omega(\partial A_i)$ and thus

$$\alpha_0 \leq \alpha \leq (\mathscr{E}\omega(\partial A))^d (\mathscr{E}|A|)^{1-d} < \infty \quad \text{if} \quad \mathscr{E}r(A)^d < \infty.$$

LEMMA 5.4. $\beta(A_1, \dots, A_d, v) = \beta_0(A_1, \dots, A_d, v)$ *a.e.* (5.5)

and

$$\alpha(A, v) = \alpha_0(A, v) \quad a.e. \tag{5.6}$$

Proof. If B is a ball in $\mathbf{R}^d$,

$$|B\cap \operatorname{Cone}(n_1(y_1),\dots,n_d(y_d))| = |B\cap \operatorname{Cone}^\circ(n_1(y_1),\dots,n_d(y_d))|.$$

Hence, by the Fubini theorem

$$\int_B \beta(A_1,\dots,A_d,v)\,dv = \int_B \beta_0(A_1,\dots,A_d,v)\,dv,$$

whence, since B is arbitrary, $\beta=\beta_0$ for a.e. v. Similarly, $\alpha=\alpha_0$ for a.e. v. Q.E.D.

We will later (Corollary 7.4) show that in fact $\alpha(A,v)$ equals a constant $\alpha(A)$ for a.e. v (this is the constant appearing in Theorem 1.1), and similarly that $\beta(A_1,\dots,A_d,v)$ is a.e. independent of v. We do not know any direct proof of this fact of integral geometry.

We will compute α for some special cases in Section 9.

We also will need modified versions of these functionals. Let $\Xi'=\{B\colon B\in\Xi$ and $0\notin B\}$ be our usual Poisson process with all sets containing 0 removed, and put, with $D_v^\delta=D_v^\delta(0)=\{x\colon |x|<\delta,\ \langle x,v\rangle<0\}$,

$$\beta_+(A_1,\dots,A_d,v,\lambda,\delta)=\beta(A_1,\dots,A_d,v)/\mathscr{P}(\Xi \text{ covers } D_v^\delta), \tag{5.7}$$

$$\begin{aligned}\beta_-(A_1,\dots,A_d,v,\lambda,\delta)=\int_{\partial A_1}\cdots\int_{\partial A_d} & I(v\in\operatorname{Cone}^\circ(n_1(y_1),\dots,n_d(y_d)))\\ &\times\mathscr{P}(\Xi'\cup\{A_i-y_i\}_1^d \text{ cover } D_v^\delta)\,d\bar{\omega}\end{aligned} \tag{5.8}$$

if $A_1^\circ,\dots,A_d^\circ\neq\varnothing$, and $\beta_+=\beta_-=0$ otherwise, and, parallelling the definitions above,

$$\alpha_+(A,v,\lambda,\delta)=(d!)^{-1}(\mathscr{E}|A|)^{1-d}\mathscr{E}\beta_+=\alpha/\mathscr{P}\,(\Xi \text{ covers } D_v^\delta) \tag{5.9}$$

$$\alpha_-(A,v,\lambda,\delta)=(d!)^{-1}(\mathscr{E}|A|)^{1-d}\mathscr{E}\beta_-. \tag{5.10}$$

Finally we define

$$\gamma(A,\lambda)=\lambda^d(\mathscr{E}|A|)^{d-1}e^{-\lambda\mathscr{E}|A|}. \tag{5.11}$$

6. Covering a cylinder

It will be convenient to do the central calculations on the infinite cylinder $\mathbf{R}\times\mathbf{T}^{d-1}$.

We assume that $r(A)$ is bounded above; $\mathscr{P}(2r(A)\geq\delta)=0$ for some $\delta<1/2$, and let in this section Ξ denote a Poisson process on $\Omega_A\times\mathbf{R}\times\mathbf{T}^{d-1}$ with intensity $\lambda d\mu\times dx$

constructed as the process of random sets in $\mathbf{R}^d$ studied in the preceding section. Locally, the two processes are the same, and the earlier lemmas hold for the process on the cylinder as well.

For simplicity, we will assume that $m=1$ and leave the modifications for $m>1$ until the end of Section 7.

Let, for $t\geq 0, C_t$ be the cylinder $[0,t)\times\mathbf{T}^{d-1}=\{(s,x')\in\mathbf{R}\times\mathbf{T}^{d-1}:0\leq s<t\}$ and define $\tau=\tau(\Xi)=\inf\{s\geq 0: (s,x')\notin\bigcup\Xi\}$. Hence, $\tau\geq t$ iff C_t is covered by Ξ. The main idea is, loosely speaking, to show that τ is approximately exponentially distributed with parameter $\gamma\alpha$.

Let e denote the vector $(1,0,\dots,0)$.

LEMMA 6.1. *Suppose that e is admissible and that $2r(A)<\delta<1/2$ a.s. Then the distribution of τ is, apart from a point mass at 0, absolutely continuous with a density function $\varphi(t)$ satisfying*

$$\gamma(A,\lambda)\,\alpha_-(A,e,\lambda,\delta)\,\mathcal{P}(\tau\geq t)\leq\varphi(t)\leq\gamma(A,\lambda)\,\alpha_+(A,e,\lambda,\delta)\,\mathcal{P}(\tau\geq t),\quad t>0. \tag{6.1}$$

Proof. Suppose that we have proved this with A replaced by the random open set A° (note that $\alpha_\pm$ and γ are the same for A° as for A), and let Ξ° be the corresponding Poisson process. Using Lemma 4.3(ii) it follows that a.s. $\bigcup\Xi^\circ=(\bigcup\Xi)^\circ$ and thus Ξ° covers $C_t^\circ\Leftrightarrow\Xi$ covers C_t°. However, for $\varepsilon>0$, $\mathcal{P}(C_t^\circ$ is covered$)\leq\mathcal{P}(C_{t-\varepsilon}$ is covered$)=\mathcal{P}(\tau\geq t-\varepsilon)$, whence $\mathcal{P}(C_t^\circ$ is covered$)=\mathcal{P}(\tau\geq t)=\mathcal{P}(C_t$ is covered$)$, and thus the result for A follows from the result for A°.

Hence, we may without loss of generality assume that A is a random bounded open convex set. Then, since $\mathbf{T}^{d-1}$ is compact, $\tau=t$ iff C_t is covered by Ξ, but there exists an uncovered point $x=(t,x')$. Let $B_1\dots B_k$ be the sets $B\in\Xi$ whose closure contain x. Thus $x\in\bigcap_1^k\partial B_i$ and, if $t>0$, e is special for $B_1,\dots,B_k$ at x. We may ignore the possibility that $k>d$ by Lemma 5.1 and the possibility that $k<d$ by the assumption that e be admissible. Hence $k=d$ and thus a.s. (with a common exceptional null set for all $t>0$)

$$\tau=t\Leftrightarrow\exists x=(t,x')\text{ and sets }B_1,\dots,B_d\in\Xi\text{ such that }x\in\bigcap_1^d\partial B_i,$$
$$e\text{ is special for }B_1,\dots,B_d\text{ at }x,\ x\notin\bigcup\Xi\text{ and }\Xi\text{ covers }C_t.$$

Since e is admissible, the point x and the sequence $B_1,\dots,B_d$ are a.s. unique (up to the ordering of $\{B_i\}$).

Let g be a positive measurable function on $[0,\infty)$ with $g(0)=0$ and define

$$\psi(B_1,\dots,B_d,\Xi)=\begin{cases} g(t)\cdot I(\Xi \text{ covers } C_t \text{ and } x\notin\bigcup\Xi) & \text{when there exists} \\ \quad \text{a point } x=(t,x')\in\bigcap_1^d \partial B_i \text{ such that } e \text{ is special} \\ \quad \text{for } B_1,\dots,B_d \text{ at } x, \text{ and } t>0. \\ 0 \quad \text{otherwise.} \end{cases} \tag{6.2}$$

Then, by the argument above,

$$\sum_{\Xi}{}' \psi(B_1,\dots,B_d,\Xi)=d!g(\tau) \quad \text{a.s.} \tag{6.3}$$

($d!$ terms, differing in the order of $B_1,\dots,B_d$, equal $g(\tau)$ while the others are zero). Hence, by Lemma 2.1,

$$d!Eg(\tau)=\int\dots\int \mathscr{E}\psi(A_1+x_1,\dots,A_d+x_d,\Xi\cup\{A_i+x_i\}_1^d)\lambda^d\,dx_1\dots dx_d\,d\mu(A_1)\dots d\mu(A_d). \tag{6.4}$$

We define

$$\Psi(A_1,\dots,A_d)=\int\dots\int \mathscr{E}\psi(A_1+x_1,\dots,A_d+x_d,\Xi\cup\{A_i+x_i\}_1^d)\,dx_1\dots dx_d \tag{6.5}$$

and (6.4) may be written

$$Eg(\tau)=(d!)^{-1}\lambda^d\mathscr{E}\,\Psi(A_1,\dots,A_d), \tag{6.6}$$

where $A_1,\dots,A_d$ are independent random sets with the distribution μ. By the definition of ψ, the integrand in (6.5) is zero unless there exist $x\in\mathbf{R}\times\mathbf{T}^{d-1}$ and $y_i\in\partial A_i$, $i=1,\dots,d$, such that $x=x_i+y_i$. Hence, we make the change of variables $x_i=x-y_i$. This is a Lipschitz mapping of $(\mathbf{R}\times\mathbf{T}^{d-1})\times\partial A_1\times\dots\times\partial A_d$ into $(\mathbf{R}\times\mathbf{T}^{d-1})^d$ and, by Lemma 5.3, its Jacobian equals $|\mathrm{Det}(n_i(y_i))_1^d|$ a.e. where $n_i(y_i)$ is the normal vector of A_i at y_i. Consequently, with $x=(t,x')$ (note that $x\notin A_i+x-y_i$) and $\bar\omega$ as defined in Section 5,

$$\begin{aligned}\Psi(A_1,\dots,A_d)=\int_0^\infty\int_{\mathbf{T}^{d-1}}\int_{\partial A_1}\dots\int_{\partial A_d} &\mathscr{E}(g(t)\,I(\Xi\cup\{A_i+x-y_i\}_1^d \text{ covers } C_t \text{ and } x\notin\bigcup\Xi) \\ &\times I(e \text{ is special for } \{A_i+x-y_i\}_1^d \text{ at } x))\,dt\,dx'\,d\bar\omega.\end{aligned} \tag{6.7}$$

We write

$$\begin{aligned}\Phi(x,A_1,\dots,A_d)=\int_{\partial A_1}\dots\int_{\partial A_d} &\mathscr{P}(\Xi\cup\{A_i+x-y_i\} \text{ covers } C_t \text{ and } x\notin\bigcup\Xi) \\ &\times I(e \text{ is special for } \{A_i-y_i\}_1^d \text{ at } 0)\,d\bar\omega.\end{aligned} \tag{6.8}$$

By symmetry, this function is independent of x' and we will write it as $\Phi(t, A_1, \dots, A_d)$. Thus

$$\begin{aligned}\Psi(A_1, \dots, A_d) &= \int_0^\infty \int_{\mathbf{T}^{d-1}} g(t)\,\Phi(t, A_1, \dots, A_d)\,dt\,dx' \\ &= \int_0^\infty g(t)\,\Phi(t, A_1, \dots, A_d)\,dt\end{aligned} \tag{6.9}$$

and, by (6.6),

$$\mathscr{E}g(\tau) = \frac{1}{d!}\lambda^d \int_0^\infty g(t)\,\mathscr{E}\Phi(t, A_1, \dots, A_d)\,dt. \tag{6.10}$$

Since g is arbitrary with $g(0)=0$, this formula shows that τ is absolutely continuous on $\tau>0$, with density function given by

$$\varphi(t) = \frac{1}{d!}\lambda^d \mathscr{E}\Phi(t, A_1, \dots, A_d). \tag{6.11}$$

In order to estimate ϕ we proceed as follows. We fix $x=(t, x')$, $A_1, \dots, A_d$ and $y_1, \dots, y_d$ such that e is special for $\{A_i - y_i\}$ at 0. Let Ξ'_x be the restriction of the Poisson process Ξ to $\{B\colon x \notin B\}$, i.e. Ξ with all sets covering x excluded. (Thus Ξ' in Section 5 equals Ξ'_0.) Note that the distribution of Ξ'_x equals the conditional distribution of Ξ given $x \notin \bigcup \Xi$. Furthermore, the number of sets $B \in \Xi$ that contain x is Poisson distributed with expectation $\int\int I(x \in A+y)\,\lambda\,d\mu(A)\,dy = \lambda\mathscr{E}|A|$, and thus $\mathscr{P}(x \notin \bigcup \Xi) = e^{-\lambda\mathscr{E}|A|}$. Hence

$$\begin{aligned}&\mathscr{P}\{\Xi \cup \{A_i + x - y_i\} \text{ covers } C_t \text{ and } x \notin \bigcup\Xi)\\ &\quad= \mathscr{P}(\Xi \cup \{A_i + x - y_i\} \text{ covers } C_t \mid x \notin \bigcup\Xi)\cdot \mathscr{P}(x \notin \bigcup\Xi)\\ &\quad= \mathscr{P}(\Xi'_x \cup \{A_i + x - y_i\} \text{ covers } C_t)\cdot e^{-\lambda\mathscr{E}|A|}.\end{aligned} \tag{6.12}$$

We define $D_e^\delta(x)$ and $D_e^\delta = D_e^\delta(0)$ as on $\mathbf{R}^d$ and let $D = D_e^\delta(x)$, $E = C_t \setminus D$. Let Ξ^* denote $\Xi'_x \cup \{A_i + x - y_i\}$. The correlation inequality (2.3) yields, the events obviously being increasing functions of Ξ'_x,

$$\mathscr{P}(\Xi^* \text{ covers } C_t) \geq \mathscr{P}(\Xi^* \text{ covers } E \cup D) \geq \mathscr{P}(\Xi^* \text{ covers } E)\,\mathscr{P}(\Xi^* \text{ covers } D). \tag{6.13}$$

However, since neither any set $A_i + x - y_i$ nor any $B \in \Xi$ with $x \in B$ meets E (because their diameters are less than δ), Ξ^* covers $E \Leftrightarrow \Xi'_x$ covers $E \Leftrightarrow \Xi$ covers E. Hence

$$\begin{aligned}\mathscr{P}(\Xi^* \text{ covers } C_t) &\geq \mathscr{P}(\Xi \text{ covers } E)\,\mathscr{P}(\Xi^* \text{ covers } D)\\ &\geq \mathscr{P}(\Xi \text{ covers } C_t)\,\mathscr{P}(\Xi^* \text{ covers } D).\end{aligned} \tag{6.14}$$

Similarly we obtain, reversing the roles of Ξ and Ξ^*,

$$\begin{aligned}\mathcal{P}(\Xi \text{ covers } C_t) &\geq \mathcal{P}(\Xi \text{ covers } E)\,\mathcal{P}(\Xi \text{ covers } D)\\ &= \mathcal{P}(\Xi^* \text{ covers } E)\,\mathcal{P}(\Xi \text{ covers } D)\\ &\geq \mathcal{P}(\Xi^* \text{ covers } C_t)\cdot\mathcal{P}(\Xi \text{ covers } D).\end{aligned} \tag{6.15}$$

These inequalities, translation invariance, and the fact that Ξ covers $C_t \Leftrightarrow \tau \geq t$, yield

$$\begin{aligned}\mathcal{P}(\tau \geq t)\cdot\mathcal{P}(\Xi' \cup \{A_i - y_i\} \text{ covers } D_e^\delta) &\leq \mathcal{P}(\Xi^* \text{ cover } C_t)\\ &\leq \mathcal{P}(\tau \geq t)/\mathcal{P}(\Xi \text{ covers } D_e^\delta).\end{aligned} \tag{6.16}$$

Furthermore, if e is special for $\{A_i - y_i\}_1^d$ at 0, we may assume using Lemma 4.4 that $e \in \mathrm{Cone}\,(n_i(y_i))_1^d$, and, conversely, if $e \in \mathrm{Cone}^\circ\,(n_i(y_i))_1^d$ and each ∂A_i is differentiable at y_i (which holds a.e.), then e is special. Hence, by (6.12), (6.16), (6.8)

$$\begin{aligned}\Phi(t, A_1, \ldots, A_d) &\leq \int_{\partial A_1} \cdots \int_{\partial A_d} \mathcal{P}(\tau \geq t)\,\mathcal{P}(\Xi \text{ covers } D_e^\delta)^{-1} e^{-\lambda\mathscr{E}|A|}\cdot I(e \in \mathrm{Cone}\,(n_i(y_i))_1^d)\,d\bar{\omega}\\ &= e^{-\lambda\mathscr{E}|A|}\beta_+(A_1, \ldots, A_d, e)\,\mathcal{P}(\tau \geq t)\end{aligned} \tag{6.17}$$

and, similarly,

$$\Phi(t, A_1, \ldots, A_d) \geq e^{-\lambda\mathscr{E}|A|}\beta_-(A_1, \ldots, A_d, e)\,\mathcal{P}(\tau \geq t). \tag{6.18}$$

(6.1) follows by (6.11), and the definitions of α_+, α_- and γ. Q.E.D.

LEMMA 6.2. *If e is admissible and $2r(A) < \delta < 1/2$ a.s., then*

$$\mathcal{P}(\tau \geq t) \leq e^{-\gamma\alpha_- t} \tag{6.19}$$

$$\mathcal{P}(\tau \geq t) \geq e^{-\gamma\alpha_+ t}\mathcal{P}(\tau \geq 0) \geq e^{-\gamma\alpha_+(t+\delta)}. \tag{6.20}$$

Proof. By (6.1),

$$\frac{d}{dt}(\mathcal{P}(\tau \geq t)\,e^{\gamma\alpha_- t}) = (-\varphi(t) + \gamma\alpha_-\,\mathcal{P}(\tau \geq t))\,e^{\gamma\alpha_- t} \leq 0, \quad t > 0.$$

Thus $\mathcal{P}(\tau \geq t)\,e^{\gamma\alpha_- t}$ is decreasing and, if $t > 0$,

$$\mathcal{P}(\tau \geq t)\,e^{\gamma\alpha_- t} \leq \lim_{s \to 0} \mathcal{P}(\tau \geq s)\,e^{\gamma\alpha_- s} = \mathcal{P}(\tau > 0) \leq 1.$$

Similarly $\mathcal{P}(\tau \geq t)\,e^{\gamma\alpha_+ t} \geq \mathcal{P}(\tau > 0)$.

Finally, note that if $\varepsilon > 0$, the two sets $[0, \varepsilon) \times \mathbf{T}^{d-1}$ and $[\delta + \varepsilon, \delta + 2\varepsilon) \times \mathbf{T}^{d-1}$ are covered with the same probability $\mathcal{P}(\tau\geq\varepsilon)$ independently of each other. Hence

$$\mathcal{P}(\tau\geq\varepsilon)^2\geq\mathcal{P}(\tau\geq\delta+2\varepsilon)\geq e^{-\gamma a_+(\delta+2\varepsilon)}\mathcal{P}(\tau>0).$$

We obtain, as $\varepsilon\to 0$, since $\mathcal{P}(\tau\geq\varepsilon)\to\mathcal{P}(\tau>0)>0$, $\mathcal{P}(\tau>0)\geq e^{-\gamma a_+\delta}$, which completes the proof. Q.E.D.

7. Covering a set in $\mathbf{R}^d$

In this section we return to $\mathbf{R}^d$. We first estimate the probability of covering a cube. We keep the notation of the preceding sections, in particular $e=(1,0,\ldots,0)$. In this section all cubes are closed and have sides parallel to the coordinate axes.

LEMMA 7.1. *Suppose that e is admissible and that $2r(A)<\delta$ a.s. If Q is a cube in $\mathbf{R}^d$ with side $s>\delta$, then*

$$\mathcal{P}(\Xi \text{ cover } Q)\leq e^{-\gamma(A,\lambda)a_-(A,e,\lambda,\delta)s^d} \tag{7.1}$$

$$\mathcal{P}(\Xi \text{ cover } Q)\geq e^{-\gamma(A,\lambda)a_+(A,e,\lambda,\delta)(s+\delta)^d} \tag{7.2}$$

Proof. The change of scale $s\to as$, $\delta\to a\delta$, $A\to aA$, $\lambda\to a^{-d}\lambda$ preserves $\mathcal{P}(\Xi$ covers $Q)$ and $a_\pm$ and changes γ into $\gamma(aA,a^{-d}\lambda)=a^{-d}\gamma(A,\lambda)$. Hence it is sufficient to prove the inequalities for a specific s.

For (7.1) we may thus assume that $s=1/2$. The closed cylinder $\bar{C}_1$ may be decomposed into 2^d cubes of side 1/2. Each of these is covered (by the process on $\mathbf{R}\times\mathbf{T}^{d-1}$) with the same probability as Q, and thus, by Lemma 2.2, $\mathcal{P}(\tau\geq 1)=\mathcal{P}(C_1$ is covered$)\geq\mathcal{P}(Q$ is covered$)^{2^d}$. Hence (7.1) follows from (6.19).

For (7.2) we assume that $s+\delta=1$. Then Q may be regarded as a subset of the cylinder $\bar{C}_{1-\delta}$ and, by (6.20),

$$\mathcal{P}(Q \text{ is covered})\geq\mathcal{P}(\bar{C}_{1-\delta} \text{ is covered})=\mathcal{P}(\tau>1-\delta)\geq e^{-\gamma a_+}. \qquad \text{Q.E.D.}$$

The next step is to approximate the set K by a union of cubes. (We are squaring the circle!)

Let $\mathcal{F}_s$ be the family of cubes $\{x\colon n_is\leq x_i\leq(n_i+1)s,\ i=1\ldots d\}$, where $n_1,\ldots,n_d$ are integers.

LEMMA 7.2. *Suppose that K is a bounded set in $\mathbf{R}^d$ and let $n_s=\#\{Q\in\mathcal{F}_s\colon Q\subset K\}$, $m_s=\#\{Q\in\mathcal{F}_s\colon Q\cap\partial K\neq\emptyset\}$. If e is admissible and $2r(A)<\delta<s$ a.s., then*

$$\mathcal{P}(\Xi \text{ covers } K) \le e^{-\gamma\alpha_-(s-\delta)^d n_s} \tag{7.3}$$

$$\mathcal{P}(\Xi \text{ covers } K) \ge e^{-\gamma\alpha_+(s+\delta)^d(n_s+m_s)} \tag{7.4}$$

Proof. Let $\{Q_i\}_1^{n_s}=\{Q\in\mathcal{F}_s\colon Q\subset K\}$, and $\{Q_i\}_{n_s+1}^{n_s+m_s}=\{Q\in\mathcal{F}_s\colon Q\cap\partial K\neq\emptyset\}$. Furthermore, let $\tilde{Q}_i$ be the cube with the same center as Q_i and with side $s-\delta$. Then, the events $\{\Xi \text{ covers } \tilde{Q}_i\}$, $i=1,\dots,n_s$ are independent and, using (7.1),

$$\begin{aligned}\mathcal{P}(\Xi \text{ covers } K) &\le \mathcal{P}(\Xi \text{ covers every } \tilde{Q}_i, i=1,\dots,n_s)\\ &=\prod_1^{n_s}\mathcal{P}(\Xi \text{ covers } \tilde{Q}_i)\le e^{-\gamma\alpha_-(s-\delta)^d n_s}.\end{aligned} \tag{7.5}$$

In the opposite direction we note that $K\subset\bigcup_1^{n_s+m_s}Q_i$. Hence, the correlation inequality (2.3) yields

$$\begin{aligned}\mathcal{P}(\Xi \text{ covers } K) &\ge \mathcal{P}(\Xi \text{ covers every } Q_i,\ i=1,\dots,n_s+m_s)\\ &\ge\prod_1^{n_s+m_s}\mathcal{P}(\Xi \text{ covers } Q_i).\end{aligned} \tag{7.6}$$

Lemma 7.1 now completes the proof. Q.E.D.

LEMMA 7.3. *Suppose that K is a bounded set in $\mathbf{R}^d$ such that $|\partial K|=0$ and that $\mathcal{E}r(A)^{d+\varepsilon}<\infty$ for some $\varepsilon>0$. Suppose further that v is admissible and that $\alpha(A,v)=\alpha_0(A,v)$. If $a\to 0$ and*

$$\mathcal{E}|aA|\lambda-\log\frac{|K|}{\mathcal{E}|aA|}-d\log\log\frac{|K|}{\mathcal{E}|aA|}-\log\alpha(A,v)\to u,\quad -\infty<u<\infty, \tag{7.7}$$

then

$$\mathcal{P}(\Xi_{\lambda,a} \text{ covers } K)\to\exp(-e^{-u}). \tag{7.8}$$

Note. The conditions on v hold a.e. by Lemmas 5.2 and 5.4.

Proof. We may assume that v is a unit vector; further we may assume that $v=e$. (Otherwise we rotate everything.)

We truncate the distribution of A by defining

$$A^{(R)}=\begin{cases}A & \text{when } r(A)<R\\ \emptyset & \text{when } r(A)\ge R.\end{cases} \tag{7.9}$$

Let $\Xi_{\lambda,a}^{(R)}$ denote the Poisson process defined as $\Xi_{\lambda,a}$ but based on $A^{(R)}$. Thus $\Xi_{\lambda,a}^{(R)}$ differs from $\Xi_{\lambda,a}$ only in that the sets $aA+x$ with $r(aA)\geqslant aR$ have been deleted.

We choose $R=a^{\eta-1}$ where $\eta>0$ is such that $1-\eta>d/(d+\varepsilon)$. (Thus $R\to\infty$ as $a\to 0$.) Then, the number of sets in $\Xi_{\lambda,a}\setminus\Xi_{\lambda,a}^{(R)}$ that meet K is a Poisson distributed random variable with expectation (a is tacitly assumed to be small enough)

$$\begin{aligned}
\int\int I(x\in aA-K)\cdot I(r(A)\geqslant R)\,\lambda\,dx\,d\mu(A) &= \lambda\mathscr{E}(|aA-K|I(r(A)\geqslant R)) \\
&\leqslant \lambda\mathscr{E}(C(ar(A)+r(K))^d I(r(A)\geqslant R)) \\
&\leqslant C(a^{-d}\log\frac{1}{a})\,\mathscr{E}((a^d r(A)^d+1)\,I(r(A)\geqslant R)) \qquad (7.10)\\
&\leqslant C\mathscr{E}((r(A)^d\log R+R^{d/(1-\eta)}\log R)\,I(r(A)\geqslant R)) \\
&\leqslant C\mathscr{E}(r(A)^{d/(1-\eta)}\log r(A)\cdot I(r(A)\geqslant R))\to 0 \quad \text{as } a\to 0.
\end{aligned}$$

Hence

$$\mathscr{P}(\Xi_{\lambda,a}\text{ covers }K)-\mathscr{P}(\Xi_{\lambda,a}^{(R)}\text{ covers }K)\to 0. \qquad (7.11)$$

For $\mathscr{P}(\Xi_{\lambda,a}^{(R)}$ covers $K)$ we use Lemma 7.2 (with A replaced by $aA^{(R)}$), taking $\delta=2aR$ and $s=\delta^{1/2}$. Thus, $s\to 0$ and $\delta/s\to 0$ as $a\to 0$.

With the notation of the proof of Lemma 7.2,

$$\bigcup_1^{n_s} Q_i\subset K\subset\bigcup_1^{n_s+m_s} Q_i$$

and thus

$$n_s s^d\leqslant|K|\leqslant(n_s+m_s)\,s^d. \qquad (7.12)$$

Furthermore,

$$\bigcup_{n_s+1}^{n_s+m_s} Q_i\subset\{x\colon d(x,\partial K)\leqslant\sqrt{d}\,s\},$$

and thus

$$m_s s^d\leqslant|\{x\colon d(x,\partial K)\leqslant\sqrt{d}\,s\}|\to|\partial K|=0\quad\text{as } s\to 0. \qquad (7.13)$$

Consequently,

$$m_s s^d\to 0\quad\text{and}\quad n_s s^d\to|K|,$$

whence

$$n_s(s-\delta)^d \to |K| \quad \text{and} \quad (n_s+m_s)(s+\delta)^d \to |K|. \tag{7.14}$$

Furthermore,

$$\begin{aligned}\gamma(aA,\lambda) &= (\lambda\mathscr{E}|aA|)^d(\mathscr{E}|aA|)^{-1}e^{-\lambda\mathscr{E}|aA|}\\ &= \left(\log\frac{|K|}{\mathscr{E}|aA|}\right)^d(1+o(1))(\mathscr{E}|aA|)^{-1}\\ &\quad\times\exp\left(-\log\frac{|K|}{\mathscr{E}|aA|}-d\log\log\frac{|K|}{\mathscr{E}|aA|}-\log\alpha(A,e)-u+o(1)\right)\\ &\to |K|^{-1}\alpha(A,e)^{-1}e^{-u}.\end{aligned} \tag{7.15}$$

By (7.10),

$$\lambda(\mathscr{E}|aA|-\mathscr{E}|aA^{(R)}|) = \lambda\mathscr{E}(|aA|I(r(A)\geqslant R))\to 0$$

and thus

$$\gamma(aA^{(R)},\lambda)/\gamma(aA,\gamma)\to 1$$

and

$$\gamma(aA^{(R)},\lambda)\to|K|^{-1}a(A,e)^{-1}e^{-u}. \tag{7.16}$$

The only step remaining before we can deduce (7.8) from (7.3) and (7.4) is to show that $a_\pm(aA^{(R)},e,\lambda,2aR)$ converge to $a(A,e)$. We fix r such that $\mathscr{E}A^{(r)}/\mathscr{E}A\geqslant 1-\eta/2$. Then, since D_e^{2R} is included in a cube of side $4R$, Lemma 7.1 (with $\delta=2r$) yields, for $R>r$,

$$\mathscr{P}\left(\Xi^{(r)}_{a^d\lambda} \text{ covers } D_e^{2R}\right)\geqslant \exp\left(-\gamma(A^{(r)},a^d\lambda)\,\alpha_+(A^{(r)},e,a^d\lambda,2r)(4R+2r)^d\right). \tag{7.17}$$

By the definition of γ and (7.15),

$$\begin{aligned}\gamma(A^{(r)},a^d\lambda)R^d &= R^d(a^d\lambda)^d(\mathscr{E}A^{(r)})^{d-1}e^{-a^d\lambda\mathscr{E}|A^{(r)}|}\\ &\leqslant \gamma(A,a^d\lambda)^{1-\eta/2}(a^d\lambda)^{d\eta/2}(\mathscr{E}A^{(r)})^{(d-1)\eta/2}R^d\\ &= C(a^d\gamma(aA,\lambda))^{1-\eta/2}(a^d\lambda)^{d\eta/2}R^d\\ &\leqslant Ca^{d(1-\eta/2)}\left(\log\frac{1}{a}\right)^{d\eta/2}R^d = C\left(a\log\frac{1}{a}\right)^{d\eta/2}\to 0.\end{aligned} \tag{7.18}$$

Furthermore, $\mathscr{P}(\Xi^{(r)}_{a^d\lambda}$ covers $D_e^{2r})$ increases as $a\searrow 0$, whence $\alpha_+(A^{(r)},e,a^d\lambda,2r)\leqslant Ca(A^{(r)},e)<\infty$. Consequently, the exponent in (7.17) tends to 0 and

$$\mathcal{P}\left(\Xi_{a^d\lambda}^{(R)} \text{ covers } D_e^{2R}\right) \geq \mathcal{P}\left(\Xi_{a^d\lambda}^{(r)} \text{ covers } D_e^{2R}\right) \to 1. \tag{7.19}$$

Thus

$$\begin{aligned} a_+(aA^{(R)}, e, \lambda, 2aR) &= a_+(A^{(R)}, e, a^d\lambda, 2R) \\ &= \alpha(A^{(R)}, e) \cdot \mathcal{P}\left(\Xi_{a^d\lambda}^{(R)} \text{ covers } D_e^{2R}\right)^{-1} \to \alpha(A, e). \end{aligned} \tag{7.20}$$

In order to show that α_- converges we fix $A_1, \dots, A_d$ and $y_1, \dots, y_d$ such that e is special for $\{A_i - y_i\}_1^d$ at 0. Thus $\{A_i - y_i\}$ covers $D_e^\varkappa$ for some $\varkappa > 0$. As $a^d\lambda$ increases $\Xi_{a^d\lambda}^{(r)\prime}$ contains more and more sets and it is obvious that $D_e^{2r} \setminus D_e^\varkappa$ eventually becomes covered, i.e.

$$\mathcal{P}\left(\Xi_{a^d\lambda}^{(r)\prime} \text{ covers } D_e^{2r} \setminus D_e^\varkappa\right) \to 1. \tag{7.21}$$

Since $\Xi_{a^d\lambda}^{(r)\prime}$ and $\Xi_{a^d\lambda}^{(r)}$ coincide on $D_e^{2R} \setminus D_e^{2r}$, it follows from (7.19) that

$$\mathcal{P}\left(\Xi_{a^d\lambda}^{(r)\prime} \text{ covers } D_e^{2R} \setminus D_e^{2r}\right) \to 1. \tag{7.22}$$

Together, these estimates yield

$$\mathcal{P}\left(\Xi_{a^d\lambda}^{(R)\prime} \cup \{A_i - y_i\}_1^d \text{ covers } D_e^{2R}\right) \geq \mathcal{P}\left(\Xi_{a^d\lambda}^{(r)\prime} \cup \{A_i - y_i\} \text{ covers } D_e^{2R}\right) \to 1. \tag{7.23}$$

By dominated convergence, it now is clear from the definitions (5.8) and (5.2) that

$$\beta_-(A_1, \dots, A_d, e, a^d\lambda, 2R) \to B_0(A_1, \dots, A_d, e) \tag{7.24}$$

and

$$a_-(aA^{(R)}, e, \lambda, 2aR) = a_-(A^{(R)}, e, a^d\lambda, 2R) \to a_0(A, e). \tag{7.25}$$

Q.E.D.

COROLLARY 7.4. *$\alpha(A, v)$ is a.e. independent of v, i.e. there exists a constant $\alpha(A)$ with $\alpha(A, v) = \alpha(A)$ a.e. Similarly, $\beta(A_1, \dots, A_d, v) = \beta(A_1, \dots, A_d)$ a.e.*

Proof. If Lemma 7.3 applies to both v and w (for some K), then obviously $\alpha(A, v) = \alpha(A, w)$. This proves the statement for α by Lemmas 5.2 and 5.4. The result for β follows easily from this if we, for fixed $A_1, \dots, A_d$, consider the random set A that equals A_i with probability p_i ($\Sigma_1^d p_i = 1$) and vary $\{p_i\}$. Q.E.D.

Lemma 7.3 proves (3.8) and thus Theorem 1.1 for $m=1$.

For $m>1$, essentially the same argument works. The main modifications are as follows: We define τ_m as $\inf\{s\geq 0: (s,x')\in \mathbf{R}\times\mathbf{T}^{d-1}$ is not covered m times by $\Xi\}$ and find as before that (a.s.) $\tau_m=t>0 \Leftrightarrow$ there exists $x=(t,x')$ which belongs to the boundary of exactly d sets $B_1,\dots,B_d$ and to the interior of exactly $m-1$ sets $B_{d+1},\dots,B_{d+m-1}$ such that e is special for $B_1,\dots,B_d$ at x. We define $\psi_m(B_1,\dots,B_{d+m-1},\Xi)$ as ψ but with the extra conditions that Ξ covers C_t m times and $x\in B_{d+1},\dots,B_{d+m-1}$. This gives

$$\sum_{\Xi}{}' \psi_m(B_1,\dots,B_{d+m-1},\Xi)=d!(m-1)!g(\tau).$$

We use Lemma 2.1 as before and make the same change of variables $x_i=x-y_i$, $y_i\in\partial A_i$, $i=1,\dots,d$ as before and put $x_i=x-y_i$ for $i=d+1,\dots,d+m-1$. This yields that τ_m has a density function

$$\varphi_m(t)=((m-1)!d!)^{-1}\lambda^{d+m-1}\mathscr{E}\Phi_m(t,A_1,\dots,A_{d+m-1})$$

with

$$\Phi_m(x,A_1,\dots,A_{d+m-1})=\int_{\partial A_1}\cdots\int_{\partial A_d}\int_{A_{d+1}}\cdots\int_{A_{d+m-1}}\mathscr{P}(\Xi\cup\{A_i+x-y_i\}_1^{d+m-1}$$

$$\text{covers } C_t\ m \text{ times and if } B\in\Xi \text{ then } x\notin B)\cdot I(e \text{ is special for}$$

$$\{A_i-y_i\}_1^d \text{ at } 0)\,d\bar{w}\,dy_{d+1}\cdots dy_{d+m-1}.$$

We define $\alpha_{m+}=\alpha/\mathscr{P}(\Xi$ covers D_v^δ m times),

$$\alpha_{m-}=\frac{1}{d!}(\mathscr{E}|A|)^{-(d+m-2)}\mathscr{E}\int_{\partial A_1}\cdots\int_{\partial A_d}\int_{A_{d+1}}\cdots\int_{A_{d+m-1}}\mathscr{P}(\Xi'\cup\{A_i-y_i\}_1^{d+m-1}$$

$$\text{cover } D_v^\delta\, m \text{ times})\cdot I(e\in\mathrm{Cone}^\circ(n_1(y_1)\dots n_d(y_d)))\,d\bar{w}\,dy_{d+1}\cdots dy_{d+m-1}$$

and

$$\gamma_m=\frac{1}{(m-1)!}\lambda^{d+m-1}(\mathscr{E}|A|)^{d+m-2}e^{-\lambda\mathscr{E}|A|}$$

and proceed as above.

This completes the proof of Theorem 1.1.

Remark. We close this section with an example of the misbehavior when the condition $\bar{K}\subset V^\circ$ of Theorem 1.1 is not satisfied.

Let $V=K$ be the unit cube $[0,1]^d$ and let A be the cube $[-1/2,1/2]^d$. Thus we try to cover a cube with small cubes of side a and centres uniformly distributed inside the big cube. We take $m=1$. By Example 1 of Section 9, $\alpha=1$.

First, we assume that $d=2$. Suppose that $a\to 0$ and $a^2\lambda-\log a^{-2}-2\log\log a^{-2}\to u$ (7.7). It follows from Lemma 7.3 that if $K_a=[a/2,1-a/2]^2$, $\mathcal{P}(\Xi^V_{\lambda,a}$ covers $K_a)\to\exp(-e^{-u})$ since $\Xi^V_{\lambda,a}$ and $\Xi_{\lambda,a}$coincide on K_a. However, on the boundary the intensity of $(\Xi^V_{\lambda,a}$ is smaller. For example, on $K^{(1)}_a=\{0\}\times[a/2,1-a/2]$, $\Xi^V_{\lambda,a}$ is a Poisson process of intervals of length a with intensity $\lambda_1=a\lambda/2$. Since $a\lambda_1-\log a^{-1}-\log\log a^{-1}\to u/2+\log 2$, Lemma 7.3 yields

$$\mathcal{P}(\Xi^V_{\lambda,a}\text{ covers }K^{(1)}_a)\to\exp(-\tfrac{1}{2}e^{-u/2}).$$

Hence (7.8) fails for $\Xi^V_{\lambda,a}$ and (1.1) fails for this V. In fact, it is not difficult to show that

$$\mathcal{P}(\Xi^V_{\lambda,a}\text{ covers }K)\to\exp(-2e^{-u/2}-e^{-u})\tag{7.26}$$

and thus the left hand side of (1.1) converges to a random variable with this distribution function.

If $d>2$, the situation is even worse. If λ is as in (7.7), the intensity on an edge is $(a/2)^{d-1}\lambda\sim d2^{1-d}\log a^{-1}$, which is too small to cover. In fact, the correct result is

$$\mathcal{P}(a^dN_{a,1}-2^{d-1}\log a^{-1}-2^{d-1}\log\log a^{-d}\leqslant u)\to\exp\left(-2^{d-1}e^{-u/2^{d-1}}-\frac{d-1}{d}2^{d-1}e^{-u/2^{d-2}}\right).\tag{7.27}$$

The asymptotic behaviour is governed exclusively by the edges and the two-dimensional facets of K, the interior being covered much sooner.

8. Covering a manifold

In this section we suppose that K is a C^2 compact Riemannian manifold. We denote the geodetic distance by d and the Riemannian measure by v. Let, for $\lambda>0$ and R a positive random variable with distribution μ, $\Xi^K_{\lambda,R}$ be the Poisson process on $K\times[0,\infty)$ with intensity $\lambda\,dv\times d\mu$ and identify it with the corresponding Poisson process of geodesic balls $\{B(x,r)\colon (x,r)\in\Xi^K_{\lambda,R}\}$. The argument of Section 3 shows that the case $m=1$ of Theorem 1.2 is equivalent to the following lemma. The case $m>1$ is entirely similar, but we omit the details.

LEMMA 8.1. *Suppose that* $\mathscr{E}R^{d+\varepsilon}<\infty$ *for some* $\varepsilon>0$. *Let*

$$b=\pi^{d/2}\Gamma\left(\frac{d}{2}+1\right)^{-1}\mathscr{E}R^d/v(K)$$

and let a *be given by* (9.24). *If* $a\to 0$ *and*

$$ba^d v(K)\lambda-\log\frac{1}{ba^d}-d\log\log\frac{1}{ba^d}-\log\alpha\to u, \tag{8.1}$$

then

$$\mathscr{P}(\Xi^K_{\lambda,aR}\text{ covers }K)\to\exp(-e^{-u}). \tag{8.2}$$

Proof. For simplicity we assume that $\sup R<\infty$; the general case is treated by truncation as in the proof of Lemma 7.3.

The manifold K may be covered by a finite number of maps (U_i,φ_i). Furthermore, there are compact sets $K_i\subset U_i$ such that $K=\bigcup K_i$, the interiors K_i° are disjoint, and each boundary ∂K_i has measure zero. By shrinking U_i somewhat, we may also assume that the geodetic distance on $\varphi_i(U_i)$ is equivalent to the Euclidean distance. We let $\mathscr{F}_t$ be the mesh of cubes defined in Section 7 and put

$$\{Q_{ij}\}_{j=1}^{n_i}=\{Q\in\mathscr{F}_t\colon Q\subset\varphi_i(K_i)\},\ \{Q_{ij}\}_{j=n_i+1}^{n_i+m_i}=\{Q\in\mathscr{F}_t\colon Q\cap\partial\varphi_i(K_i)\neq\varnothing\}.$$

Thus

$$\sum_{j=1}^{n_i}v(Q_{ij})\to v(K_i)$$

and

$$\sum_{j=1}^{n_i+m_i}v(Q_{ij})\to v(K_i)\quad\text{as } t\to 0,$$

cf. (7.12)–(7.14).

For each cube Q_{ij} we select a point $x_{ij}\in Q_{ij}$ and define a new map $(U_i,\psi_{ij}\circ\varphi_i)$, where ψ_{ij} is the linear map of $\mathbf{R}^d$ onto itself that is defined by $g_i(x_{ij})^{1/2}$, where g_i is the coordinate representation of the metric tensor on the map (U_i,φ_i). Thus, on the map $(U_i,\psi_{ij}\circ\varphi_i)$, the metric tensor is given by the identity matrix at $\psi_{ij}(x_{ij})$ and hence by $I+O(d(x,x_{ij}))$ at $\psi_{ij}(x), x\in U_i$.

Consequently, for some C not depending on i,j or t (less than some t_0), a geodesic ball $\{y: d(x,y)<r\}$ with $r<t$ on this map that intersects $\psi_{ij}(Q_{ij})$ lies between the two Euclidean balls $\{y: |x-y|<r(1-Ct)\}$ and $\{y: |x-y|<r(1+Ct)\}$. Further, the intensity of the centers is $\lambda dv=\lambda(1+O(d(x,x_{ij}))\,dx$.

Consequently, if $\Xi_{\lambda,a}$ denotes the Poisson process on $\mathbf{R}^d$ defined in Section 3 for the random set $A=\{x: |x|<R\}$ (i.e. a sphere with random radius R),

$$\begin{aligned}\mathscr{P}(\psi_{ij}(Q_{ij}) \text{ is covered by } \Xi_{(1-Ct)\lambda,(1-Ct)a}) &\leq \mathscr{P}(\varphi_i^{-1}(Q_{ij}) \text{ is covered by } \Xi^K_{\lambda,aR}) \\ &\leq \mathscr{P}(\psi_{ij}(Q_{ij}) \text{ is covered by } \Xi_{(1+ct)\lambda,(1+Ct)a}).\end{aligned} \tag{8.3}$$

We note that $\alpha(A)=\alpha$ by (9.24) and that $\mathscr{E}|A|=bv(K)$.

Let $s=t^2$ and apply Lemma 7.2 to each $\psi_{ij}(Q_{ij})$, noting that

$$s^d n_s \leq |\psi_{ij}(Q_{ij})| \leq s^d(n_s+m_s), \quad s^d m_s \leq C\frac{s}{t}|\psi_{ij}(Q_{ij})|$$

and

$$(1-Ct)\,v(Q_{ij}) \leq |\psi_{ij}(Q_{ij})| \leq (1+Ct)\,v(Q_{ij}),$$

whence

$$(1-Ct)\,v(Q_{ij}) \leq s^d n_s \leq s^d(n_s+m_s) \leq (1+Ct)\,v(Q_{ij}). \tag{8.4}$$

Since

$$\begin{aligned}\gamma((1-Ct)\,aA,(1-Ct)\,\lambda) &\leq (1-Ct)\,\lambda^d(v_d\mathscr{E}R^d a^d)^{d-1}e^{-(1-Ct)\lambda a^d v_d \mathscr{E}R^d} \\ &\leq \gamma(aA,\lambda)\,e^{Ct\lambda a^d},\end{aligned} \tag{8.5}$$

(7.2) yields

$$\mathscr{P}(\varphi_i^{-1}(Q_{ij}) \text{ is covered}) \geq \exp(-\gamma(aA,\lambda)\,e^{Ct\lambda a^d}\alpha_+(1+Ca/s)(1+Ct)\,v(Q_{ij})) \tag{8.6}$$

and hence, by the correlation inequality,

$$\begin{aligned}\mathscr{P}(K \text{ is covered}) &\geq \prod_{i,j}\mathscr{P}(\varphi_i^{-1}(Q_{ij}) \text{ is covered}) \\ &\geq \exp(-\gamma(aA,\lambda)\,e^{Ct\log 1/a}\alpha_+(1+C\,a/s)(1+Ct)\sum_{i,j}v(Q_{ij})).\end{aligned} \tag{8.7}$$

With $t=a^\varkappa$, $1/2>\varkappa>0$, $t\log 1/a\to 0$ as $a\to 0$ and the right-hand side of (8.7) converges, cf. (7.15) and (7.20), to $\exp(-v(K)^{-1}a^{-1}e^{-u}\alpha(A)\,v(K))=\exp(-e^{-u})$.

Similar estimates in the other direction are obtained by studying slightly smaller cubes $\tilde{Q}_{ij}$ with sides $t-Ca$, cf. the proof of Lemma 7.2. Q.E.D.

9. The constant term

To avoid trivial complications we assume henceforth that $A^\circ \neq \emptyset$ (a.s.). We recall that $\alpha(A, v)=\alpha(A)$ a.e., where $\alpha(A, v)$ is given by (5.3). We begin with a criterion for a given v to satisfy this equality. (It follows that the condition in Lemma 7.3 that v be admissible is superfluous.)

LEMMA 9.1. *If $\alpha(A, v)=\alpha_0(A, v)$, then $\alpha(A)=\alpha(A, v)$. The corresponding result for β holds too.*

Proof. In fact, we show that for every $v \neq 0$,

$$\alpha_0(A, v) \leq \alpha(A) \leq \alpha(A, v), \tag{9.1}$$

$$\beta_0(A_1, \ldots, A_d, v) \leq \beta(A_1, \ldots, A_d) \leq \beta(A_1, \ldots, A_d, v). \tag{9.2}$$

Let B be a ball. By Fubini's theorem and (5.1),

$$\begin{aligned} \beta(A_1, \ldots, A_d) &= |B|^{-1} \int_B \beta(A_1, \ldots, A_d, v)\, dv \\ &= \int_{\partial A_1} \cdots \int_{\partial A_d} |B \cap \operatorname{Cone}(n_1(y_1), \ldots, n_d(y_d))|/|B|\, d\bar{\omega}. \end{aligned} \tag{9.3}$$

Let B shrink to v; then

$$\begin{aligned} I(v \in \operatorname{Cone}^\circ((n_i)_1^d)) &\leq \liminf |B \cap \operatorname{Cone}((n_i)_1^d)|/|B| \\ &\leq \limsup |B \cap \operatorname{Cone}((n_i)_1^d)|/|B| \\ &\leq I(v \in \operatorname{Cone}((n_i)_1^d)), \end{aligned}$$

and (9.2) follows. (9.1) follows by (5.3), (5.4). Q.E.D.

Let $\omega_d = 2\pi^{d/2}/\Gamma(d/2)$ be the area $\omega(S^{d-1})$ of the unit sphere. We note the following analogue of (9.3)

$$\beta(A_1, \ldots, A_d) = \omega_d^{-1} \int_{\partial A_1} \cdots \int_{\partial A_d} \omega(S^{d-1} \cap \operatorname{Cone}(n_i(y_i))_1^d)\, d\bar{\omega}. \tag{9.4}$$

When the random set A has a centrosymmetric distribution, we may simplify the expression for α. (In particular, this is true if each small set is centrosymmetric, but that is not necessary.) We define

$$\beta^*(A_1,\ldots,A_d)=\int_{\partial A_1}\cdots\int_{\partial A_d} d\bar{\omega}=\bar{\omega}(\partial A_1\times\ldots\times\partial A_d). \tag{9.5}$$

LEMMA 9.2. *If $-A$ and A are equidistributed, then*

$$\alpha(A)=(d!)^{-1}2^{-d}(\mathscr{E}|A|)^{-(d-1)}\mathscr{E}\beta^*(A_1,\ldots,A_d). \tag{9.6}$$

Proof. Let $\varepsilon_i=\pm1, i=1,\ldots,d$. If $n_1,\ldots,n_d$ are linearly independent, then a.e. $v\in S^{d-1}$ belongs to $\operatorname{Cone}(\varepsilon_1 n_1,\ldots,\varepsilon_d n_d)$ for exactly one choice of $\{\varepsilon_i\}$. Hence, summing over the 2^d possible choices and using (9.4), $(\operatorname{Det}(n_i(y_i))\neq 0$ $\bar{\omega}$-a.e. by the definition of $\bar{\omega})$

$$\begin{aligned}\sum_{\{\varepsilon_i\}}\beta(\varepsilon_1A_1,\ldots,\varepsilon_dA_d)&=\int_{\partial A_1}\cdots\int_{\partial A_d}\omega_d^{-1}\sum_{\{\varepsilon_i\}}\omega(S^{d-1}\cap\operatorname{Cone}((\varepsilon_i n_i(y_i))_1^d))\,d\bar{\omega}\\&=\int_{\partial A_1}\cdots\int_{\partial A_d}I(\operatorname{Det}(n_i(y_i))\neq0)\,d\bar{\omega}=\beta^*(A_1,\ldots,A_d).\end{aligned} \tag{9.7}$$

Thus $\mathscr{E}\beta^*(A_1,\ldots,A_d)=2^d\mathscr{E}\beta(A_1,\ldots,A_d)$. Q.E.D.

When the distribution of the small sets is isotropic, i.e. when their orientations are random, we may simplify further. We begin with a preparatory lemma.

LEMMA 9.3. *If $e_1,\ldots,e_d$ are independent, uniformly distributed unit vectors in $\mathbf{R}^d$, then*

$$\mathscr{E}|\operatorname{Det}(e_1,\ldots,e_d)|=\pi^{-1/2}\Gamma\left(\frac{d}{2}\right)^d\Gamma\left(\frac{d+1}{2}\right)^{-(d-1)}. \tag{9.8}$$

Proof. Let $X_1,\ldots,X_d$ be independent standard normal random vectors in $\mathbf{R}^d$. Then $X_i=|X_i|e_i$, with e_i as above and $|X_i|$ has the chi distribution χ_d, $|X_i|$ and e_i independent. Thus

$$\begin{aligned}\mathscr{E}|\operatorname{Det}(X_1,\ldots,X_d)|&=\mathscr{E}\left(\prod_1^d|X_i||\operatorname{Det}(e_1,\ldots,e_d)|\right)\\&=(\mathscr{E}\chi_d)^d\mathscr{E}|\operatorname{Det}(e_1,\ldots,e_d)|.\end{aligned} \tag{9.9}$$

A standard computation yields

$$\mathscr{E}\chi_d = \sqrt{2}\,\Gamma\left(\frac{d+1}{2}\right)\Big/\Gamma\left(\frac{d}{2}\right).$$

However,

$$|\operatorname{Det}(X_1, \ldots, X_d)| = |X_1||\pi_{X_1}(X_2)||\pi_{X_1X_2}(X_3)| \ldots |\pi_{X_1 \ldots X_{d-1}}(X_d)|,$$

where $\pi_{X_1 \ldots X_k}(X_{k+1})$ is the projection of X_{k+1} onto the orthogonal complement of $X_1, \ldots, X_k$. Given $X_1, \ldots, X_k$, this projection is a $(d-k)$-dimensional standard normal random vector and thus

$$\mathscr{E}(|\pi_{X_1 \ldots X_k}(X_{k+1})||X_1 \ldots X_k) = \mathscr{E}\chi_{d-k} = \sqrt{2}\,\Gamma\left(\frac{d-k+1}{2}\right)\Big/\Gamma\left(\frac{d-k}{2}\right). \tag{9.10}$$

Consequently,

$$\mathscr{E}|\operatorname{Det}(X_1, \ldots, X_d)| = \prod_{k=0}^{d-1} \sqrt{2}\,\Gamma\left(\frac{d-k+1}{2}\right)\Big/\Gamma\left(\frac{d-k}{2}\right) = 2^{d/2}\Gamma\left(\frac{d+1}{2}\right)\Big/\Gamma\left(\frac{1}{2}\right) \tag{9.11}$$

and (9.8) follows from (9.9) and (9.11). Q.E.D.

LEMMA 9.4. *If $R(A)$ and A are equidistributed for every rotation R of $\mathbf{R}^d$, then*

$$\alpha(A) = 2^{-d}\pi^{-1/2}(d!)^{-1}\Gamma\left(\frac{d+1}{2}\right)^{-(d-1)}\Gamma\left(\frac{d}{2}\right)^{d}(\mathscr{E}\omega(\partial A))^d(\mathscr{E}|A|)^{-(d-1)}. \tag{9.12}$$

Proof. For simplicity we assume that Lemma 9.2 applies. (This is necessarily true if d is even. The proof in the general case is similar, using (9.4).) Let $R_1, \ldots, R_d$ be independent, uniformly distributed random elements of the compact group of rotations. Then, for fixed $A_1, \ldots, A_d$,

$$\begin{aligned}\mathscr{E}\beta^*(R_1(A_1), \ldots, R_d(A_d)) &= \mathscr{E}\int_{\partial A_1} \ldots \int_{\partial A_d} |\operatorname{Det}(R_i(n_i(y_i)))|\,d\omega(y_1) \ldots d\omega(y_d) \\ &= \int_{\partial A_1} \ldots \int_{\partial A_d} \mathscr{E}|\operatorname{Det}((e_i)_1^d)|\,d\omega(y_1) \ldots d\omega(y_d) \\ &= \mathscr{E}|\operatorname{Det}((e_i)_1^d)|\,\omega(\partial A_1) \ldots \omega(\partial A_d).\end{aligned} \tag{9.13}$$

Hence

$$\mathscr{E}\beta^*(A_1, \ldots, A_d) = \mathscr{E}\beta^*(R_1(A_1), \ldots, R_d(A_d)) = \mathscr{E}|\operatorname{Det}((e_i)_1^d)|(\mathscr{E}\omega(\partial A))^d \tag{9.14}$$

and (9.12) follows by Lemmas 9.2 and 9.3. Q.E.D.

Thus, in the isotropic case, the asymptotic distribution of $N_{a,m}$ in Theorem 1 depends only on $\mathscr{E}|A|$ and $\mathscr{E}\omega(\partial A)$. If $d=2$, (9.12) is $\alpha(A)=(4\pi)^{-1}(\mathscr{E}\omega(\partial A))^2/\mathscr{E}|A|$, and if $d=3$, $\alpha(A)=(\pi/384)(\mathscr{E}\omega(\partial A))^3(\mathscr{E}|A|)^{-2}$.

If $d=1$, $\beta(A_1)=1$ for every nonempty convex set A_1. Thus, if $\mathscr{P}(A\neq 0)=1$, $\alpha(A)=1$ and the asymptotic distribution of $N_{a,m}$ depends only on $\mathscr{E}|A|$.

If $d=2$ and A is a non-random centrosymmetric set, the same is true.

LEMMA 9.5. *If A is a fixed centrosymmetric set in $\mathbf{R}^2$, then $\alpha(A)=1$.*

Proof. Let $z(\varphi)$ be the boundary ∂A parametrized e.g. by the direction from the origin. Since A is centrosymmetric, $z(\varphi+\pi)=-z(\varphi)$. Thus

$$\begin{aligned}\beta^*(A,A) &= \int_0^{2\pi}\int_0^{2\pi}|dz(\varphi)\times dz(\psi)| = \int_0^{2\pi} 2\int_\varphi^{\pi+\varphi} dz(\varphi)\times dz(\psi)\\ &= 2\int_0^{2\pi} dz(\varphi)\times(z(\pi+\varphi)-z(\varphi)) = 4\int_0^{2\pi} z(\varphi)\times dz(\varphi) = 8|A|\end{aligned} \tag{9.15}$$

and thus $\alpha(A)=1$ by (9.6). Q.E.D.

It may similarly be shown that if A_1 and $A_2\subset\mathbf{R}^2$ are centrosymmetric, $\frac{1}{8}\beta^*(A_1,A_2)$ equals the mixed volume $V(A_1,A_2)$, cf. [1]. (β^* is not a mixed volume when $d>2$; it has the wrong homogeneity.)

Next, we show that the symmetrization of A to $\pm A$ never increases α for $d=2$. We do not know whether this is true or not for $d\geqslant 3$.

LEMMA 9.6. *Let A be a random convex set in $\mathbf{R}^2$, let R be a random rotation independent of A with an arbitrary distribution (not necessarily uniform) and let $\pm A$ equal A or $-A$ with probability* 1/2 *each (independently of A). Then*

$$\alpha(\pm A)\leqslant\alpha(R(A)). \tag{9.16}$$

In particular,

$$\alpha(\pm A)\leqslant\alpha(A). \tag{9.17}$$

Proof. The mapping $y\to n(y)$ defined a.e. maps ∂A into the unit circle T. Hence, the arc-length measure ω on ∂A induces a measure ν_A on T.
Let k be the periodic function given by

$$k(t)=(2\pi)^{-1}t\sin t\quad\text{for } -\pi\leqslant t\leqslant\pi.$$

It follows immediately from (9.4) that

$$\beta(A_1, A_2) = \int\int k(s-t)\, d\nu_{A_1}(s)\, d\nu_{A_2}(t) = \sum_{-\infty}^{\infty} \hat{k}(n) \overline{\hat{\nu}_{A_1}(n)}\, \hat{\nu}_{A_2}(n) \tag{9.18}$$

and hence

$$\mathscr{E}\beta(A_1, A_2) = \sum_{-\infty}^{\infty} \hat{k}(n) |\mathscr{E}\hat{\nu}_A(n)|^2. \tag{9.19}$$

We note that $\hat{\nu}_A(-1) = \overline{\hat{\nu}_A(1)} = \int e^{it}\, d\nu_A(t) = \int_{\partial A} n(y)\, d\omega(y) = 0$ by Gauss' theorem. Furthermore, $\mathscr{E}\hat{\nu}_{\pm A}(n) = \mathscr{E}\hat{\nu}_A(n)$ if n is even and 0 if n is odd, and, if λ is the distribution of R, $|\mathscr{E}\hat{\nu}_{R(A)}(n)| = |\hat{\lambda}(n)||\mathscr{E}\hat{\nu}_A(n)| \leq |\mathscr{E}\hat{\nu}_A(n)|$, with equality for $n=0$. An elementary computation shows that $\hat{k}(n) = (-1)^{n+1}(2\pi(n^2-1))^{-1}$ for $n \neq \pm 1$ and thus $\hat{k}(n) < 0$ if $n \neq 0$ is even and $\hat{k}(n) > 0$ if $n \neq \pm 1$ is odd. Hence

$$\begin{aligned} \mathscr{E}\beta(\pm A_1, \pm A_2) &= \sum_{n \text{ even}} \hat{k}(n) |\mathscr{E}\hat{\nu}_A(n)|^2 \leq \sum_{n \text{ even}} \hat{k}(n) |\mathscr{E}\hat{\nu}_{R(A)}(n)|^2 \\ &\leq \mathscr{E}\beta(R_1(A_1), R_2(A_2)). \end{aligned}$$

This yields (9.16), and the special case R=Identity yields (9.17).

With the notation of the proof above,

$$\beta^*(A_1, A_2) = \int\int |\sin(s-t)|\, d\nu_{A_1}(s)\, d\nu_{A_2}(t). \tag{9.20}$$

Furthermore, it may be shown that

$$|A| = \frac{1}{2}\int\int k_1(s-t)\, d\nu_A(s)\, d\nu_A(t), \tag{9.21}$$

where $k_1(t) = (2\pi)^{-1}(\pi - |t|)|\sin t|$, $-\pi \leq t \leq \pi$. Hence, if A is a (non-random) convex set in $\mathbf{R}^2$, $2|A| + \beta(A, A) = \frac{1}{2}\beta^*(A, A)$ and thus

$$1 + \alpha(A) = 2\alpha(\pm A). \tag{9.22}$$

This combined with (9.17) yields

$$\alpha(A) \geq 1, \tag{9.23}$$

where equality holds iff A is centrosymmetric. (This yields an alternative proof of Lemma 9.5.) We will see that (9.22) and (9.23) fail when $d \geq 3$.

We return to arbitrary d in our final lemma. We remarked in Section 5 that α is homogeneous; in fact, a much stronger invariance is true.

LEMMA 9.7. *If T is a linear map of $\mathbf{R}^d$ onto itself,* $\alpha(T(A))=\alpha(A)$.

Proof. An immediate consequence of Theorem 1.1, since $T(K)$ is covered by $\{T(aA+X)\}=\{aT(A)+T(X)\}$ iff K is covered by $\{aA+X\}$. Q.E.D.

Example 1. Let A be a cube with edges parallel to the coordinate axes and random side L. If $A_1,\dots,A_d$ are such cubes with sides $L_1,\dots,L_d$, we choose $v=(1,1,\dots,1)$ and obtain from (5.1), (5.2) and (9.2) $\beta(A_1,\dots,A_d)=d!\,L_1^{d-1}\cdot\ldots\cdot L_d^{d-1}$. Hence

$$\alpha(A)=(\mathscr{E}L^{d-1})^d/(\mathscr{E}L^d)^{d-1}=\left(\frac{\|L\|_{d-1}}{\|L\|_d}\right)^{d^2-d}.$$

In particular, if A is a fixed cube, $\alpha(A)=1$ (for all d). If $d>1$ and the side is random, $\alpha(A)<1$.

Example 2. Let A be a rectangle $[0,L_x]\times[0,L_y]$ in $\mathbf{R}^2$ with edges parallel to the coordinate axes. Then, as in Example 1, if A_i has sides $L_x^i, L_y^i, i=1,2$, then $\beta(A_1,A_2)=L_x^1L_y^2+L_x^2L_y^1$ and $\alpha(A)=\mathscr{E}L_x\,\mathscr{E}L_y/\mathscr{E}L_xL_y$. Hence $\alpha(A)>1$ iff the sides are negatively correlated.

Example 3. If A is a box $[0,L_1]\times[0,L_2]\times\ldots\times[0,L_d]$ in $\mathbf{R}^d$ we similarly obtain

$$\alpha(A)=\prod_{i=1}^{d}\mathscr{E}\Big(\prod_{j\neq i}L_j\Big)\cdot\Big(\mathscr{E}\Big(\prod_{1}^{d}L_i\Big)\Big)^{1-d}.$$

In particular, if $L_1,\dots,L_d$ are independent, $\alpha(A)=1$.

That a fixed (non-random) box always gives $\alpha=1$ also follows from Example 1 and Lemma 9.7.

Example 4. If A is a sphere with random radius $\mathbf{R}$, it follows from Lemma 9.4 that

$$\alpha(A)=\frac{1}{d!}\left(\frac{\sqrt{\pi}\,\Gamma\left(\frac{d}{2}+1\right)}{\Gamma\left(\frac{d+1}{2}\right)}\right)^{d-1}(\mathscr{E}R^{d-1})^d(\mathscr{E}R^d)^{-(d-1)}. \tag{9.24}$$

If the radius R is fixed we obtain for low dimensions: $\alpha=1$ for $d=1$ and $d=2$ (cf. Lemma 9.5), $\alpha=3\pi^2/32$ for $d=3$, $\alpha=64/81$ for $d=4$.

Example 5. If A is a cube of fixed size and random rotation, Lemma 9.4 yields

$$\alpha(A)=\pi^{-1/2}d^d\Gamma\left(\frac{d}{2}\right)^d\Gamma\left(\frac{d+1}{2}\right)^{-(d-1)}(d!)^{-1}.$$

If $d=2$, $\alpha=4/\pi$ and if $d=3$, $\alpha=9\pi/16$.

Example 6. If $d=2$ and A is the triangle with vertices at $(0,0)$, $(1,0)$, $(0,1)$, we obtain from (9.2), with $v=(-1,-1)$, cf. Example 1, $\beta(A,A)=2$ and thus $\alpha(A)=2$. By Lemma 9.7 $\alpha(A)=2$ for every fixed triangle. Similarly

$$\alpha(A)=(d!)^{d-1}((d-1)!)^{-d}=d^{d-1}/(d-1)!$$

for a fixed simplex in $\mathbf{R}^d$.

Example 7. If A is a fixed triangle, it follows from Lemma 9.2 or by (9.22) that $\alpha(\pm A)=3/2$.

Example 8. If A is an equilateral triangle of fixed size and random orientation, Lemma 9.4 yields $\alpha(A)=3\sqrt{3}/\pi$.

We recall that smaller α corresponds to more efficient coverings. Thus, for example, if $d=2$, small squares and discs (of the same area) cover asymptotically equally efficiently, but if $d>2$, $\alpha(\text{sphere})<1=\alpha(\text{cube})$ and small spheres cover better than cubes (i.e. with less overlap), although the difference is minor. Furthermore, if $d\geqslant 2$, small sets of a fixed size cover less efficiently than sets of varying size of the same shape and orientation. On the other hand, long and narrow sets pointing in different directions give a large α and a less efficient covering.

The examples above show that cubes of a fixed size with a fixed orientation cover better than cubes with a random orientation, while equilateral triangles with a random orientation cover better than triangles with a fixed orientation, although triangles with just two opposite orientations cover even better. Lemma 9.6 shows tha this behaviour is typical in two dimensions.

Some of these results are far from obvious and the detailed behaviour of α raises several questions, such as

Problem 1. Does Lemma 9.6 extend to $d\geqslant 3$? In particular, if A is centrosymmetric, and A' is the set A with random orientation, does $\alpha(A)\leqslant\alpha(A')$ always hold?

Problem 2. Lemma 9.4 (or Examples 2 and 3) implies that $\alpha(A)$ may be any positive real number if A is random, but what happens if A is non-random? Thus, what is $\inf\{\alpha(A)$: A is a fixed convex set in $\mathbf{R}^d\}$? What is the supremum? Which convex sets are extremal? Spheres (and thus ellipsoids also) and simplices? (For $d=2$, the infimum is 1 by (9.23), and the supremum is 2 (which is attained by triangles by Example 8). We omit the proof.)

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Received October 3, 1983
Received in revised form February 14, 1985