Acta Math., 166 (1991), 163–221

The topology of rational functions and divisors of surfaces

by

F. R. COHEN

University of Rochester
Rochester, NY, U.S.A.

R. L. COHEN

Stanford University
Stanford, CA, U.S.A.

B. M. MANN

University of New Mexico
Albuquerque, NM, U.S.A.

R. J. MILGRAM[1]

Stanford University
Stanford, CA, U.S.A.

§ 1. Introduction

Spaces of holomorphic maps between complex manifolds have played a fundamental role in such diverse branches of mathematics as analysis, differential geometry, topology, mathematical physics, and linear control theory. In seminal work, Segal [Seg] studied the homotopy types of the spaces of holomorphic functions of the 2-sphere S^2, of closed surfaces of higher genus, and of the spaces of divisors of these surfaces. In particular he showed that the space of holomorphic functions of degree k fills out the homotopy type of an appropriate function space in a stable range of dimensions (roughly up to dimension $k-2g$, where g is the genus). In this paper we continue Segal's program by describing the entire stable homotopy types of these spaces in terms of the homotopy types of more familiar spaces.

A. The spaces $\mathrm{Rat}_k(\mathbf{CP}^n)$

One of our basic results concerns Rat_k[2], the space of based holomorphic self-maps of the Riemann sphere having degree k. In [Seg] Segal determined the homotopy type of

[1] During the preparation of this work each of the authors were supported by NSF grants, the second author by an NSF-PYI award, and the first and fourth authors by the S.F.B. 170 in Göttingen.

[2] This terminology is used because an element $f\in \mathrm{Rat}_k$ is given by a rational function of one complex variable having degree k. That is, $f(z)=p(z)/q(z)$, where p and q are degree k monic polynomials in one complex variable with no common roots.

Rat_k through dimension k (the topological dimension of Rat_k is $4k$). More recently, F. Cohen described a certain multiplicative structure on the Rat_k's which, via the theory of homology operations, yields additional homological information. This structure is presented in [BoMa] where it is used to study the homology of $SU(2)$ monopoles. It is natural to conjecture that the homology classes yielded by these operations detect the entire homology of Rat_k. The truth of this conjecture is a consequence of the following theorem.

THEOREM 1.1. *There is a stable homotopy equivalence*

$$\mathrm{Rat}_k \underset{s}{\simeq} K(\beta_{2k}, 1).$$

Here β_n is Artin's braid group on n-strings and $K(\beta_n, 1)$ is the corresponding Eilenberg–MacLane space. By stable homotopy equivalence we mean a homotopy equivalence of the corresponding suspension spectra. In this case Rat_k and $K(\beta_{2k}, 1)$ become homotopy equivalent after suspending $2k$ times. Thus, Theorem 1.1 implies that these spaces have the same homology groups as well as the same generalized homology theories (e.g. K-theory, stable homotopy theory, and bordism theory).

The stable homotopy type of $K(\beta_{2k}, 1)$ is well understood and we show how these two spaces are related to each another. We need to establish further notation to state our results in more detail. Given a space X consider the configuration space

$$F(X, k) = \{(x_1, \dots, x_k) \in X^k \text{ such that } x_i \neq x_j \text{ if } i \neq j\}.$$

The symmetric group $\mathcal{S}_k$ acts freely on $F(X, k)$ and the associated orbit space $DP^k(X) = F(X, k)/\mathcal{S}_k$ is a subspace of the symmetric product space $SP^k(X) = X^k/\mathcal{S}_k$. $DP^k(X)$ is often referred to as the *deleted symmetric product*. Notice that there is a canonical k-dimensional vector bundle $\gamma_k(X)$ over $DP^k(X)$ given by the natural projection

$$\gamma_k: F(X, k) \times_{\mathcal{S}_k} \mathbf{R}^k \to DP^k(X)$$

where $\mathcal{S}_k$ acts on $\mathbf{R}^k$ by permuting coordinates. For X a Riemann surface with one boundary component these bundles were studied in detail in [C²M²]. In the case $X = \mathbf{R}^2$, the deleted symmetric product is an Eilenberg–MacLane space,

$$DP^k(\mathbf{R}^2) = K(\beta_k, 1)$$

where β_k is Artin's braid group on k-strings [FaN]. Moreover, in this case the bundle

$\gamma_k(\mathbf{R}^2)$ is induced by the permutation representation of β_k given by sending a braid to the associated permutation of the endpoints of the strings. The Thom space $T\gamma_k(\mathbf{R}^2)$ of this bundle over the braid group is given by

$$T\gamma_k(\mathbf{R}^2) = F(\mathbf{R}^2, k)_+ \wedge_{\mathcal{G}_k} (S^1)^{(k)}$$

and denoted $D_k(S^1)$ for short, where the subscript + denotes a disjoint basepoint and the superscript (k) denotes the k-fold smash product.

In [Seg] Segal defined a natural inclusion map

$$i_k\colon \mathrm{Rat}_{k-1} \hookrightarrow \mathrm{Rat}_k.$$

Let $\mathrm{Rat}_k/\mathrm{Rat}_{k-1}$ denote the mapping cone of i_k. The proof of Theorem 1.1 actually shows the following

THEOREM 1.2. (a) *There is a homotopy equivalence*

$$\mathrm{Rat}_k/\mathrm{Rat}_{k-1} \to D_k(S^1) = D_k.$$

(b) Rat_k *is stably homotopy equivalent to* $\bigvee_{j=1}^{k} D_j$.

Remark 1.3. It is well known [BP1, CMT] that

$$K(\beta_{2k}, 1) \underset{s}{\simeq} \bigvee_{j=1}^{k} D_j$$

and thus 1.1 is really 1.2(b).

The spaces D_k have well-understood homotopy types [Sn, Ma, BP1, Co]. When localized at any fixed prime p, D_k is the Brown–Gitler spectrum [BG, C]. Consequently, their mod p cohomology groups are certain explicit quotients of the mod p Steenrod algebra $\mathcal{A}_p$, [Ma, C]. They have had basic applications in topology including embedding and immersion theory [BP2, BP3, C2], classical stable homotopy theory [Mah], and classification of manifolds [MdMi]. A more complete discussion occurs in [C3]. In addition, the cohomology of β_n was first given by Fuks [Fu] and by [C].

To prove Theorem 1.1 we first identify a generic set E_0 in Rat_k, consisting of elements with no repeated zeros (see 3.2), and show that the natural inclusion of $E_0 \hookrightarrow \mathrm{Rat}_k \hookrightarrow \Omega_k^2 S^2$ homologically surjects onto a well-known piece of $H_*(\Omega_k^2 S^2)$. This establishes a lower bound for the size of the homology of Rat_k (see 3.5). Next, we analyze Leray spectral sequences associated to a bifiltration of related spaces to show that our lower bound is, in fact, an upper bound (see (4.9)). In fact, the generic set E_0

contains far more homology than Rat_k and hence our arguments show that as the lower strata of Rat_k, consisting of points with more and more repeated zeros, are added to the generic set they precisely cone off the extra homology in E_0.

One can also consider "real rats", the subspace RRat_k of Rat_k which consists of rational functions $p(z)/q(z)$ where p and q are now assumed to have real coefficients. Brockett [Br1] showed that RRat_k has $k+1$ connected components $\mathrm{RRat}_{k,r}$ indexed by the set of $r \equiv k \pmod 2$ with $|r| \leq k$. Let s and t be given by $s+t=k$ and $s-t=r$ and set $m=\min(s, t)$. Segal [Seg] showed that $\mathrm{RRat}_{k,r}$ is homeomorphic to Rat_m and hence we have

COROLLARY 1.4. *There is a stable homotopy equivalence*

$$\mathrm{RRat}_{k,r} \underset{s}{\simeq} \bigvee_{j=1}^{m} D_j.$$

Our analysis for $\mathrm{Rat}_k = \mathrm{Rat}_k(\mathbf{CP}^1)$ extends immediately, with only trivial modification, to $\mathrm{Rat}_k(\mathbf{CP}^n)$, the space of based holomorphic maps of degree k from S^2 to $\mathbf{CP}^n$. Thus we obtain

THEOREM 1.5. *There is a stable homotopy equivalence*

$$\mathrm{Rat}_k(\mathbf{CP}^n) \underset{s}{\simeq} \bigvee_{j=1}^{k} \Sigma^{(2n-2)j} D_j$$

where Σ^q *denotes the q-fold (reduced) suspension.*

Certain spaces of holomorphic maps are also of interest in control theory. Indeed, many basic questions in linear control theory are related to the orbit space $\mathcal{M}_{n,m,k}$ of observable, controllable systems having n inputs, m outputs, and MacMillan degree k:

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

here u and y represent the input and output functions of the system respectively. Basically, such a system is observable if two different input functions give two different output functions and controllable if any output function can be obtained by a suitable choice of input function. For the precise definition see, for example, [Kai]. Furthermore, it is well known that any observable and controllable system has a minimal realization (again see [Kai]) where the size of the square matrix A is the MacMillan degree.

Elements of $\mathcal{M}_{n,m,k}$ are known to be in one to one correspondence, via the Laplace transform, with proper rational transfer matrices

$$G(s) = C(sI-A)^{-1}B+D \in \mathrm{Mat}_{n\times m}(\mathbf{C}(s))$$

of degree k. In turn, proper rational transfer matrices are known to be [HM] in one to one correspondence with elements of $\mathrm{Rat}_k(G_{n,n+m})$. Hence $\mathrm{Rat}_k(G_{n,n+m})$ denotes the degree k holomorphic maps from the Riemann sphere, S^2, to the Grassmann manifold of n-planes in $m+n$ space, $G_{n,n+m}(\mathbf{C})$, with the compact-open topology.

In the special case of n inputs and one output (or one input and n outputs) we have

COROLLARY 1.6. *The moduli space of observable, controllable linear dynamical systems $\mathcal{M}_{n,1,k}$ of tridegree $(n, 1, k)$ is stably homotopy equivalent to $\bigvee_{j=1}^{k}\Sigma^{(2n-2)j}D_j$.*

Previous work on the topology of $\mathcal{M}_{n,m,k}=\mathrm{Rat}_k(G_{n,m+n})$ includes [Br1, Br2], [BD], [Del], [G], [He1, He2], [K] and, of course, [Seg]. Note that the most recent computational results cited ([Del] and [He1, He2]) are quite different in focus from 1.6 in that they consider only the rational homology in tridegrees (n, m, k) for general n and m but small values of k. Finally, we note that the homology of $\mathcal{M}_{n,m,k}=\mathrm{Rat}_k(G_{n,m+n})$ is now known for all values of n, m and k [MM]. This computation uses Theorem 1.5 in an essential way.

For practical applications it is more important to understand the cell structure of $\mathcal{M}_{n,m,k}$ rather than its stable homotopy type. In section 6 we give a conceptually simple cell decomposition for these spaces.

We conclude our remarks concerning $\mathrm{Rat}_k(\mathbf{CP}^n)$ by noting that there is an alternate proof of Theorems 1.1, 1.2 and 1.5 that is geometric and combinatorial in nature, rather than homological. It is presented in [$\mathrm{C^2M^22}$]. This second method has the advantage that it gives a much simpler approach to the Rat_k spaces. On the other hand, the approach presented in the current paper has the advantage that we can simultaneously analyze a much wider class of "divisor spaces" of which Rat_k is the simplest example. We now discuss one particular family of such spaces.

B. The Div spaces

For any space X, let $\mathrm{Div}_k(X)$ be the subspace of $SP^k(X)\times SP^k(X)$ defined by

$$\mathrm{Div}_k(X) = \{(\xi, \eta) \in SP^k(X)\times SP^k(X) |\ \text{the coordinates of } \xi \text{ and } \eta \text{ are disjoint}\}.$$

Let M_g be a closed Riemann surface of genus g, and let M'_g denote the punctured surface; i.e. M_g with a point deleted. The points of $\mathrm{Div}_k(M'_g)$ are thus the classical divisors (k roots, k poles) on the surface M'_g, hence the name $\mathrm{Div}_k(X)$ for the general construction. Clearly, when $g=0$, we have that $M'_0=\mathbf{C}$ and so $\mathrm{Div}_k(M'_0)=\mathrm{Rat}_k$.

The methods of § 2–§ 4 give a systematic procedure for computing the homology of the bifiltered Div-spaces, $\mathrm{Div}_{k,n}(X)$, (k roots, n poles) of which Theorems 1.1 and 1.2 are among the easiest cases. The spaces $\mathrm{Div}_k(M'_g)$ represent the next level of examples and are somewhat more complicated. They are also discussed in § 4, but the results there are far from complete. Consequently, in the last four sections we concentrate on them exclusively. Segal shows that in the limit as $n\to\infty$ the space of based holomorphic maps of degree n, $\mathrm{Hol}^*_n(M_g, S^2)$ has the homotopy type of $\mathrm{Map}^*_0(M_g, S^2)$, the space of basepoint preserving maps homotopic to the constant map. We strengthen this result by identifying the homotype type of $\mathrm{Map}^*_0(M_g, S^2)$. In particular, in § 7 we show

THEOREM 1.7. *There is a homotopy equivalence*

$$\mathrm{Map}^*_0(M_g, S^2)\to(\Omega S^2)^{2g}\times X_g$$

where X_g is the total space of a fibration

$$\Omega^2S^3\to X_g\to(S^1)^{4g}.$$

This is 7.8 of § 7 and is a useful first step because $\mathrm{Hol}_n(M_g, S^2)_*$ is a critical subspace of $\mathrm{Div}_n(M'_g)$. When coupled with further results in [Seg] and the results of § 4, it leads to an effective procedure to determine the structure of $\mathrm{Div}_k(M'_g)$ (see in particular 7.15). As an application, in § 8 we determine the rational homology of the $\mathrm{Div}_k(M'_g)$.

In § 9 we give a geometric bifiltration of $\mathrm{Div}_n(M'_g)$ which parallels the homotopy decomposition presented in the previous sections. The various quotients of this bifiltration are smash products of Rat_l spaces with products of spheres whose "leading edges" appear to correspond to the cells in the algebraic decomposition described in § 4. However, at this time we do not have sufficient control over the identifications in the geometric model to make this correspondence precise.

In § 10 we study a certain "nonsingular" subspace $\Sigma_n(M'_g)$ of $\mathrm{Div}_n(M'_g)$. It is naturally the total space of a fibration over $DP^n(M'_g)$ with fiber $SP^k(M'_g-\{n\})$, where $\{n\}$ denotes a set of n distinct points. $\Sigma_n(M'_g)$ consists of those divisors that have distinct roots and is a subspace of generic points in $\mathrm{Div}_n(M'_g)$. As such it will be particularly important in analyzing the structure of $\mathrm{Div}_k(M'_g)$. We compute the holonomy of this fibration and show how, in the genus zero case, one recovers a classical faithful representation of Artin's braid group β_n in the automorphism group of the free group on n-generators.

C. Connection with monopoles and harmonic maps

There are well-known connections between $\text{Rat}_k(\mathbf{CP}^n)$, harmonic maps, and Yang–Mills–Higgs $SU(2)$ monopoles and we end this introduction by stating another corollary of Theorem 1.1.

First, there is a natural energy functional on the space of C^∞ maps from $\mathbf{CP}^1$ to $\mathbf{CP}^n$ given by

$$E(\phi) = \frac{1}{2}\int_{\mathbf{CP}^1} |d\phi(x)|^2 \, d\text{vol}$$

where $|\cdot|$ denotes the norm with respect to the Fubini–Study metric on $\mathbf{CP}^n$ and $d$vol is the standard volume form on $\mathbf{CP}^1$. The critical points of E satisfy $\operatorname{tr} Dd\phi = 0$ where D denotes the induced connection on $\phi^* T\mathbf{CP}^n \otimes T^*\mathbf{CP}^1$. Solutions to this equaton are called *harmonic maps*.

The complex structure on $\mathbf{CP}^n$ implies that the above energy functional can be written as

$$E(\phi) = \frac{1}{2}\int_{\mathbf{CP}^1} (|\partial\phi(x)|^2 + |\bar{\partial}\phi(x)|^2) \, d\text{vol}$$

and that the difference

$$\frac{1}{2}\int_{\mathbf{CP}^1} (|\partial\phi(x)|^2 - |\bar{\partial}\phi(x)|^2) \, d\text{vol}$$

depends only on the degree k. Thus the absolute minima are precisely the holomorphic maps (or antiholomorphic maps) from $\mathbf{CP}^1$ to $\mathbf{CP}^n$. When $n=1$ all critical points are global minima and hence rational maps [EW1, W]. When $n>1$ there exist critical points (equivalently harmonic maps) which are not rational maps; however, it is still true that all minima are given by rational maps and hence are global minima [EW2].

Taubes has shown that the Yang–Mills–Higgs functional associated to the $SU(2)$ Yang–Mills equations on $\mathbf{R}^4$ reduced under "time" translation symmetry has, as its domain, a space that is homotopy equivalent to the space of C^∞ maps from $\mathbf{CP}^1$ to $\mathbf{CP}^1$ [T1]. The global minima for this functional are precisely the solutions to the Bogomol'nyi equations [B] and are called *monopoles*. Furthermore, the moduli space of such monopoles (up to based gauge equivalence), $\hat{\mathcal{M}}_k$ is a $4k$ dimensional manifold. Based on work of Taubes [T1, T2], Hitchin [H1, H2], and Nahm [N], Donaldson proved the following theorem.

THEOREM 1.8 [D]. *For all k there is a natural one-to-one correspondence between equivalence classes of SU(2) Euclidean monopoles and rational functions; that is,*

$$\hat{\mathcal{M}}_k \cong \mathrm{Rat}_k.$$

In [BoMa] it was shown that the Donaldson correspondence preserves the topologies and is actually a homeomorphism. This implies the following corollary of Theorem 1.1.

COROLLARY 1.9. *There is a stable homotopy equivalence*

$$\hat{\mathcal{M}}_k \underset{s}{\simeq} K(\beta_{2k}, 1).$$

It is interesting to observe that Donaldson's theorem shows that there exist two natural functionals on $\Omega_k^2 S^2$ both of which have Rat_k as the solution space of global minima. However, these functionals are globally quite different. For example, while all harmonic self maps of the Riemann sphere are holomorphic, Taubes [T1] has shown that there are non-minimal critical points for the Yang–Mills–Higgs functional.

We would like to thank S. Boyd, C. Boyer, R. Mazzeo, B. Osgood and R. Schoen for helpful discussions during the preparation of this paper.

§ 2. Preliminaries on Rat_k, symmetric products, and loop spaces

Let Rat_k be the space of (degree k) rational maps $f: S^2 \to S^2$ normalized by assuming f has exactly k roots, k poles, and $f(\infty)=1$. Such a function $f(z)$ can be written uniquely in the form

$$f(z) = \frac{z^k + a_1 z^{k-1} + \cdots + a_k}{z^k + b_1 z^{k-1} + \cdots + b_k} = \frac{h(z)}{m(z)} \tag{2.1}$$

where the polynomials $h(z)$ and $m(z)$ have no common roots. A monic polynomial $h(z)$ is, of course, completely specified by its roots, and these comprise a set of k not necessarily distinct unordered points in $\mathbf{C}$.

The space of all unordered k-tuples of points in a CW complex X is called the k-fold symmetric product of X and is written $SP^k(X)$. It is given the quotient topology from the identification map $X^k \to SP^k(X)$, $(x_1, \ldots, x_k) \mapsto x_1 \cdots x_k$, where we write the unordered k-tuple $\langle x_1, \ldots, x_k \rangle$ as the (formal) product $x_1 \cdots x_k$. There is a commutative and associative pairing

$$SP^k(X) \times SP^r(X) \to SP^{k+r}(X), \quad (x_1 \cdots x_k, y_1 \cdots y_r) \mapsto x_1 \cdots x_k \cdot y_1 \cdots y_r \tag{2.2}$$

and we can write an arbitrary point of $SP^k(X)$ as

$$x_1^{i_1} \cdots x_l^{i_l}, \quad i_1 + \cdots + i_l = k.$$

If a basepoint $* \in X$ is chosen then there is an embedding

$$SP^k(X, *) \to SP^{k+1}(X, *), \quad x_1 \cdots x_k \mapsto x_1 \cdots x_k \cdot *,$$

and in this case the space $SP^\infty(X) = SP^\infty(X, *) = \lim_{k \to \infty} SP^k(X)$ is defined.

The pairings in 2.2 fit together to give a commutative associative pairing

$$SP^\infty(X, *) \times SP^\infty(X, *) \to SP^\infty(X, *)$$

with $\lim_{k \to \infty}(*) = *$ acting as a unit. This gives $SP^\infty(X, *)$ the structure of the free commutative, associative monoid with unit $*$, generated by X, and we have

THEOREM 2.3 (Dold–Thom). *Let X be a connected complex, then*

$$SP^\infty(X, *) \simeq \prod_{i=1}^{\infty} K(\tilde{H}_i(X, \mathbf{Z}), i)$$

where $K(\pi, n)$ is the Eilenberg–MacLane space.

THEOREM 2.4 (Steenrod). *Let X be as above, and suppose A is any untwisted coefficient group, then*

(a) $H_*(SP^n(X), A) \cong \coprod_{k=1}^{n} H_*(SP^k(X), SP^{k-1}(X), A)$.

(b) $H_*(SP^\infty(X, *), A) = \lim_{n \to \infty} H_*(SP^n(X, *), A)$.

THEOREM 2.5 (Steenrod). *$H_*(SP^n(X), A)$ is a functor only of $H_*(X, A)$ for all n, with X, A as above.*

This functor is given explicitly in [M].

In order to describe these groups we first recall the definition of a multi-graded algebra over the field $\mathbf{F}$.

Definition 2.6. Let $\mathcal{I}$ be commutative monoid with unit, then the **F**-algebra A is an $\mathcal{I}$-graded algebra if $A = \amalg_{I \in \mathcal{I}} A_I$ as an additive group, and $A_I \cdot A_J \subset A_{I+J}$.

Also, if $\phi: \mathcal{I} \to \mathbf{Z}$ is a monoid homomorphism, then A is said to be ϕ-commutative if $a_J \cdot a_I = (-1)^{\phi(I)\phi(J)} a_I \cdot a_J$. An $\mathcal{I}$-graded **F**-algebra A is said to be free ϕ-commutative if it is the tensor product of a polynomial algebra on generators b_I with $\phi(b_I)$ even, and an exterior algebra on generators e_J with $\phi(e_J)$ odd, provided the characteristic of **F** is either 0 or odd. If the characteristic is 2, then A is simply the polynomial algebra on the stated generators. The number $\phi(e)$ is usually called the dimension of e.

Example 2.7. $A = H^*(X \times Y; \mathbf{F}) = \amalg_n \amalg_{i+j=n} H^i(X; \mathbf{F}) \otimes H^j(Y; \mathbf{F})$ is naturally a $\mathbf{Z}^+ \times \mathbf{Z}^+$-graded algebra, which is ϕ-commutative where $\phi(m, s) = m + s$, and, similarly, $H^*(X_1 \times \cdots \times X_n; \mathbf{F})$ is graded by $(\mathbf{Z}^+)^n$.

Example 2.8. A different type of example occurs when we consider $SP^\infty(X, *)$ using (2.2), (2.4), (2.5). Indeed, writing

$$H^*(SP^\infty(X, *); \mathbf{F}) = \coprod_n H^*(SP^n(X), SP^{n-1}(X); \mathbf{F})$$

we have that $H^*(SP^\infty(X, *); \mathbf{F})$ is bigraded by $(*, n)$, dimension and filtration degree. This algebra was studied and completely determined in [M] for all locally finite CW-complexes. First, since $SP^\infty(X \vee Y, *) = SP^\infty(X, *) \times SP^\infty(Y, *)$, it should not be unexpected that each time we add a direct summand homology class α in dimension $n > 0$, the resulting graded algebra becomes the tensor product of the algebra associated to $SP^\infty(M(\alpha), *)$ and the algebra associated to the previous classes. Here $M(\alpha)$ is a Moore space of dimension n. That is, $H_i(M(\alpha); \mathbf{Z}) = 0$ if $i \neq n$, and $H_n(M(\alpha); \mathbf{Z})$ is the direct summand in $H_*(X; \mathbf{Z})$ associated to α. Moreover, the basic bigradings of $H^*(SP^\infty(M(\alpha)), \mathbf{F})$ are completely determined in [M].

In what follows we only need to consider three examples along with their tensor products. These are

(2.9) $$H^*(SP^\infty(S^1); \mathbf{Z}) = E[e_{(1,1)}],$$

the exterior algebra on a 1-dimensional generator of bidegree $(1, 1)$,

(2.10) $$H^*(SP^\infty(S^2); \mathbf{Z}) = \mathbf{Z}[b_{(2,1)}],$$

the polynomial algebra on the 2-dimensional generator of bidegree $(2, 1)$, and

(2.11) $$H^*(SP^\infty(S^3);\mathbf{Z}/2)=\mathbf{Z}/2[\iota_{(3,1)},f_{(5,2)},f_{(9,4)},\dots,f_{(2^{2i}+1,2^i)},\dots]$$

with $Sq^{2^i}\cdots Sq^4Sq^2(\iota)=f_{(2^{2i}+1,2^i)}$ where Sq^i is the ith Steenrod square. On the other hand, for odd p we have

(2.12)
$$H^*(SP^\infty(S^3);\mathbf{Z}/p)=E[\iota_{(3,1)},h_{(2p+1,p)},\dots,h_{(2p^i+1,p^i)},\dots]\otimes\mathbf{Z}/p[b_{(2p+2,p)},\dots,b_{(2p^i+2,p^i)},\dots].$$

In this case, as before, the generators are given as iterations of mod(p) Steenrod operations on ι_3. Specifically, $h_{(2p^i+1,p^i)}=\mathcal{P}^{p^{i-1}}\mathcal{P}^{p^{i-2}}\cdots\mathcal{P}^1\iota_3$, while $b_{(2p^i+2,p^i)}=\beta h_{(2p^i+1,p^i)}$ and β is the mod(p) cohomology Bockstein.

Returning to geometry, we will be dealing exclusively with the case where X is a 2 dimensional manifold. For such spaces the following result is well known [And].

THEOREM 2.13. *Let X be a 2 dimensional manifold without boundary, then $SP^n(X)$ is a $2n$ dimensional manifold for all n.*

We return to our consideration of the Rat_k spaces. By identifying a rational function with its collection of zeros and poles we may identify Rat_k with an open subset and hence an open $4k$ dimensional submanifold

$$\mathrm{Rat}_k\subset SP^k(S^2)\times SP^k(S^2).$$

Specifically,

$$\mathrm{Rat}_k=\{(x_1\cdots x_k,y_1\cdots y_k)|\infty\neq x_i\neq y_j\neq\infty \text{ for all } 1\leq i,j\leq k\}.$$

Thus, using Alexander–Poincaré duality we have

(2.14) $$\tilde{H}^{4k-*}\left(\frac{SP^k(S^2)\times SP^k(S^2)}{[SP^{k-1}(S^2)\times SP^k(S^2)]\cup[SP^k(S^2)\times SP^{k-1}(S^2)]\cup V_{k,k}};\mathbf{F}\right)\cong H_*(\mathrm{Rat}_k;\mathbf{F})$$

where the *singular set*

$$V_{k,k}\subset SP^k(S^2)\times SP^k(S^2)$$

is the closed subset of points $(x_1\cdots x_k,y_1\cdots y_k)$ with $x_i=y_j$ for some i,j. In particular, Rat_k is the complement of the resultant locus.

At this point, we return to the natural inclusion

$$\mathrm{eval}:\mathrm{Rat}_k\to(\Omega^2S^2)_k=\mathrm{Map}_{\infty\mapsto1}(S^2,S^2)_k.$$

It is well known that each component of $\Omega^2 S^2$ has the homotopy type of $\Omega^2 S^3$ due to the Hopf fibering

$$S^1 \to S^3 \to S^2.$$

The May–Milgram model for $\Omega^2\Sigma^2 X$ (for X any connected CW-complex) is given as follows

$$J_2(X) = \coprod_{k=1}^{\infty} F(\mathbf{C}, k) \times_{\mathcal{S}_k} X^k/\text{equivalence} \tag{2.15}$$

where $F(\mathbf{C}, k)$ is the set of k-tuples of distinct *ordered* points in $\mathbf{C}$, and the equivalence relation is given by

$$(z_1, \ldots, z_k, \theta_1, \ldots, \theta_k) \sim (z_1, \ldots, \hat{z}_j, \ldots, z, \theta_1, \ldots, \hat{\theta}_j, \ldots, \theta_k)$$

if and only if $\theta_j = * \in X$ is a fixed basepoint. $J_2(X)$ is naturally a filtered space with a filtration preserving H-space pairing map

$$\mu\colon (J_2(X) \times J_2(X) \to J_2(X).$$

See [May, Seg2] for the definition of μ. Moreover, under the homotopy equivalence $e\colon J_2(X) \to \Omega^2\Sigma^2 X$ of [May, M2], the following diagram homotopy commutes

$$\begin{array}{ccc} J_2(X)\times J_2(X) & \xrightarrow{\mu} & J_2(X) \\ \downarrow{\scriptstyle e\times e} & & \downarrow{\scriptstyle e} \\ \Omega^2\Sigma^2 X \times \Omega^2\Sigma^2 X & \xrightarrow{*} & \Omega^2\Sigma^2 X \end{array}$$

where $*$ is the usual loop sum. Now recall

THEOREM 2.16 [Sn]. *For any space X the natural projection map*

$$F(\mathbf{R}^m, k) \times X^k \to F(\mathbf{R}^m, k)_+ \wedge X^k$$

from the equivariant cartesian product to the equivariant smash product has a stable section. In particular, this induces a stable splitting

$$J_2(X) \underset{s}{\simeq} \bigvee_{0}^{\infty} F(\mathbf{C}, k)_+ \wedge_{\mathcal{S}_k} \underbrace{X \wedge \cdots \wedge X}_{k}.$$

Furthermore, the groups $H_*(J_2^k(S^1), J_2^{k-1}(S^1); \mathbf{F})$ are isomorphic to $\bar{H}_*(D_k; \mathbf{F})$ where it is *customary to denote the* k-th *summand in the above decomposition for* $X=S^1$,

$$F(\mathbf{C},k)_k \wedge_{\mathcal{S}_k} S^k$$

as D_k. *Finally, writing*

$$H_*(J_2(X);\mathbf{F}) \cong \coprod_1^\infty H_*(J_2^k(X), J_2^{k-1}(X);\mathbf{F})$$

makes $H_*(J_2(X);\mathbf{F})$ *into an algebra over the set* $\mathbf{Z}^+\times\mathbf{Z}^+$.

The following calculation for the homology of $J_2(S^1)$ (as an algebra over $\mathbf{Z}^+\times\mathbf{Z}^+$) is given in [M2]. The result is

(2.17)

$$H_*(J_2(S^1);\mathbf{Z}/2) = \mathbf{Z}/2[e_{(1,1)}, q_{(3,2)}, \ldots, q_{(2^{i+1}-1,2^i)}, \ldots]$$

$$H_*(J_2(S^1);\mathbf{Z}/p) = E[e_{(1,1)}, e_{(2p^{i+1}-1,p^i)}, \ldots]\otimes\mathbf{Z}/p[q_{(2p-2,p)}, \ldots, q_{(2p^i-2,p^i)}, \ldots].$$

Here, $q_{(2p^i-2,p^i)}$ is the homology Bockstein of $e_{(2p^i-1,p^i)}$. Note the close connection between these results and the results of (2.11) and (2.12). Recall that the homology groups of the spaces D_k are given explicitly by the subgroups of (2.17) consisting of elements with second grading degree exactly k.

Example 2.18. When $p=2$ the generators for the first few of these groups $H_*(D_k,\mathbf{Z}/2)$ are as follows. For $k=2$ the generators are $e^2_{(1,1)}$ in dimension 2, and $f_{(3,2)}$ in dimension 3. For $k=4$ we have $e^4_{(1,1)}$ in dimension 4, $e^2_{(1,1)}f_{(3,2)}$ in dimension 5, $f^2_{(3,2)}$ in 6, and $f_{(7,4)}$ in 7. By comparison, in (2.11) we see that the groups with second degree 2 have generators ι_3^2 in dimension 6 and $Sq^2(\iota_3)$ in dimension 5, while the groups with second degree 4 have generators ι_3^4 in dimension 12, $\iota_3^2 Sq^2(\iota_3)$ in dimension 11, $Sq^2(\iota_3)^2$ in dimension 10 and $Sq^4Sq^2(\iota_3)$ in dimension 9. In [BCM] it is shown that there is a duality isomorphism between the groups $H_*(D_k;\mathbf{F})$ and the groups $H^{4k-*}(SP^k(S^3), SP^{k-1}(S^3);\mathbf{F})$. The process by which this isomorphism is constructed is similar to the duality calculation of (2.14).

As is the case with the Steenrod operations in cohomology theory, in keeping with the implicit duality indicated in 2.18, there are homology operations in the category of 2nd loop spaces. They are given in terms of (Dyer–Lashof) maps

$$\psi_k\colon F(\mathbf{C},k)\times_{\mathcal{S}_k}(\Omega^2X)^k \to \Omega^2X$$

given by

$$\psi_k(z_1, \ldots, z_k, f_1, \ldots, f_k)(\zeta) = \begin{cases} f_i((\zeta - z_i)N) & \text{if } |\zeta - z_i| \leq N \\ * & \text{otherwise} \end{cases}$$

where $N = (1/10^6)\cdot \min|z_i - z_j|$. Using $J_2(X)$ as a model for $\Omega^2\Sigma^2 X$ we can approximate ψ_k by the maps

$$F(\mathbf{C}, k)\times_{\mathcal{S}_k}\prod_1^k F(\mathbf{C}, j)\times_{\mathcal{S}_j}\underbrace{X\times\cdots\times X}_{j}\to F(\mathbf{C}, w)\times_{\mathcal{S}_{jk}}\underbrace{X\times\cdots\times X}_{jk}.$$

Here the map on the products of X's is just a shuffle map according to the wreath product embedding $\mathcal{S}_k \int \mathcal{S}_j \subset \mathcal{S}_{jk}$, and the map

$$F(\mathbf{C}, k)\times_{\mathcal{S}_k}\underbrace{F(\mathbf{C}, j)\times\cdots\times F(\mathbf{C}, j)}_{k}\to F(\mathbf{C}, kj)$$

is defined by

$$(2.19)\qquad (z_1, \ldots, z_k, \zeta_{11}, \ldots, \zeta_{1j}, \ldots, \zeta_{kj}) \to \left(z_1 + \frac{1}{N}\zeta_{11}, \ldots, z_i + \frac{1}{N}\zeta_{il}, \ldots, z_k + \frac{1}{N}\zeta_{kj}\right)$$

where N is an appropriately large number depending on $(\min|z_i - z_j|, \max|\zeta_{il}|)$, which will guarantee that the points in (2.19) are disjoint. This construction was first given, in this form, in [BV] but such constructions have a long history.

There is a homotopy commutative diagram for the Dyer–Lashof map

$$(2.20)\qquad \begin{array}{ccc} F(\mathbf{C}, k)\times_{\mathcal{S}_k}(\Omega^2\Sigma^2X)^k & \xrightarrow{\psi} & \Omega^2\Sigma^2X \\ \uparrow{\scriptstyle 1\times h^k} & & \uparrow{\scriptstyle h} \\ F(\mathbf{C}, k)\times_{\mathcal{S}_k}(J_2(X))^k & \xrightarrow{\psi} & J_2(X) \\ \uparrow{\scriptstyle 1\times(\mathrm{incl})^k} & & \uparrow{\scriptstyle \mathrm{incl}} \\ F(\mathbf{C}, k)\times_{\mathcal{S}_k}X^k & \longrightarrow & J_2^k(X) \end{array}$$

where h is the homotopy equivalence described in [May] and the lower map is the surjection onto $J_2^k(X) \subset J_2(X)$.

§3. Generic subspaces of Rat_k and Theorem 1.1

In §2 we saw that there is a natural embedding $\mathrm{Rat}_k \subset SP^k(\mathbf{C})\times SP^k(\mathbf{C}) \subset SP^k(S^2)\times SP^k(S^2)$ as an open $4k$-dimensional submanifold. A first attempt at studying Rat_k proceeds by projecting onto the first factor $SP^k(\mathbf{C})$,

$$p_1: \mathrm{Rat}_k \to SP^k(\mathbf{C}), \quad (x_1 \cdots x_k, y_1 \cdots y_k) \mapsto x_1 \cdots x_k.$$

There is a natural decreasing filtration of $SP^k(Y)$ given by $x_1 \cdots x_k \in \mathbf{F}_r(SP^k(Y))$ if and only if we can write

$$x_1 \cdots x_k = (z_1 \cdots z_r)^2 x_{2r+1} \cdots x_k.$$

In particular

$$\mathbf{F}_0(SP^k(Y)) - \mathbf{F}_1(SP^k(Y)) = DP^k(Y) = \{x_1 \cdots x_k | x_i \neq x_j \text{ for all } i \neq j\}$$

is the k-fold deleted symmetric product of Y. This space has been studied intensively when Y is a manifold in [BCT, BCM, LM], though there have been previous studies in special cases.

When we look at the inverse image of $\mathbf{F}_r(SP^k(\mathbf{C})) - \mathbf{F}_{r+1}(SP^k(\mathbf{C}))$ we see that it is a fiber bundle with fiber $SP^k(\mathbf{C} - \{x_1, \ldots, x_{k-r}\})$ over the point $(x_1 \cdots x_r)^2 x_{r+1} \cdots x_{k-r}$. Thus, we have a sequence of subspaces, E_r, $0 \leq r \leq [k/2]$, filling out Rat_k each given as a fibering

$$SP^k(\mathbf{C} - \{x_1, \ldots, x_{k-r}\}) \to E_r \xrightarrow{p_1} \mathbf{F}_r(SP^k(\mathbf{C})) - \mathbf{F}_{r+1}(SP^k(\mathbf{C})). \tag{3.1}$$

The fiber $SP^k(\mathbf{C} - \{x_1, \ldots, x_t\})$ is homotopy equivalent to the torus $(S^1)^t$ as long as $k \geq t$ (see e.g. [M]) and a specific homotopy equivalence can be given as follows. Let $e = \frac{1}{4}\min\{|x_i - x_j| \mid i \neq j\}$ and embed t copies of $S^1 \hookrightarrow \mathbf{C} - \{x_1, \ldots, x_t\}$ as the circles of radius e about the points $x_1, x_2, \ldots, x_t$, respectively. Then $(S^1)^t \subset SP^k(\mathbf{C} - \{x_1, \ldots, x_t\})$ is the set of points $(y_1, \ldots, y_k)$, $y_i \in S^1_i$ centered about x_i for $i \leq t$, and $y_i = *$ for $i > t$. This equivalence naturally extends in the maximal case to give the following:

LEMMA 3.2. *E_0 is homotopy equivalent to the total space of the fibering*

$$(S^1)^k \to F(\mathbf{C}, k) \times_{\mathcal{S}_k} (S^1)^k \xrightarrow{p_1} DP^k(\mathbf{C}).$$

E_0 is a generic subspace of Rat_k and, as we have seen in (2.15) it occurs both as the basic building block in the May–Milgram model for the double loop space $\Omega^2 S^3$ and in the Dyer–Lashof maps of (2.19) and (2.20). When $k = 1$, E_0 is all of Rat_1 and we recover the classical decomposition $\mathrm{Rat}_1 \cong \mathbf{C}^* \times \mathbf{C} \sim S^1$. Let $\vartheta: S^1 \hookrightarrow E_0$ be the naural inclusion of the fiber.

LEMMA 3.3. *The composite* $\mathrm{eval}\circ\vartheta\colon S^1\hookrightarrow \mathrm{Rat}_1\to\Omega_1^2 S^2$ *maps onto a generator of* $\pi_1(\Omega_1^2 S^2;\mathbf{Z})=\mathbf{Z}$.

Proof. The generator of $\pi_3(S^2)=\mathbf{Z}$ is well known to be the Hopf map $\eta\colon S^3\to S^2$, defined by regarding S^2 as $\mathbf{CP}^1$, which is the orbit space of S^3 under the complex circle action $(z_1, z_2)\to(e^{i\theta}z_1, e^{i\theta}z_2)$. Given a differentiable map $f\colon S^3\to S^2$ we can determine its homotopy class by calculating the linking number of $f^{-1}(a), f^{-1}(b)$, for a, b any two regular points of f in S^2.

Hence, we study the adjoint of $\mathrm{eval}\circ\vartheta$ viewed as a map $e(\mathrm{eval}\circ\vartheta)=e\colon S_+^1\wedge S^2\to S^2$. This is given on coordinates as $(\zeta, z)\mapsto\{z-\zeta, z\}$ where the coordinates in the image S^2 are homogeneous. What we will do is construct a map $\mu\colon S^3\to S_+^1\wedge S^2$ so that the composition is seen to be ± 1 times the Hopf map. However, we do have to be careful about one point. $S_+^1\wedge S^2\simeq S^2\vee S^3$, and $e|S^2\to S^2$ is homotopically the identity. Consequently, by modifying μ by a multiple of the Hopf map into the S^2 wedge summand we can modify the image of the composition $e\circ\mu$ in any way we choose. However, we will construct μ so that the composition

$$p\circ\mu\colon S^3\to S_+^1\wedge S^2\xrightarrow{p} S^2$$

is homotopically trivial where $p\colon S_+^1\wedge S^2\to S^2$ is given by $(\zeta, z)\mapsto z$. p is associated to "component shifting", sending h to $h*n(\mathrm{id})$, and it is direct to verify that if $e\circ\mu\simeq\pm h$ while $p\circ\mu\simeq 0$, then, in fact 3.3 will follow.

We regard S^3 as the join $S^1*S^1=S^1\times I\times S^1$ with $(\zeta, 0, \omega)\sim\zeta$, $(\zeta, 1, \omega)\sim\omega$, and define μ by

$$\mu(\zeta, t, \omega)=\begin{cases}(\zeta, \phi(t)\,\omega) & t\neq 1\\ + & t=1\end{cases}\quad\text{where}\quad \phi(t)=\frac{t}{1-t}.$$

Now, $(p\circ\mu)^{-1}(0)=\{(\zeta, 0, \omega)\}\sim\{(\zeta)\}=S^1$ is the boundary of the disk D^2 given as the set of points $\{(\zeta, t, 1)\}$. Also, $(p\circ\mu)^{-1}(i)=\{(\zeta, 1/2, i)\}=S^1$ as well, and the D^2 above is disjoint from this circle. Hence, the linking number of the two inverse images is 0 and $p\circ\mu\simeq 0$.

On the other hand when we consider the composite $e\circ\mu$ we have

$$(e\circ\mu)^{-1}(\{0, *\})=\{(\zeta, 1/2, \zeta)\}$$

while

$$(e\circ\mu)^{-1}(\{*, 0\})=\{(\zeta, 0, \omega)\}\sim\{(\zeta)\}=S^1,$$

the same S^1 as above. Consequently, it bounds the same disk D^2. But the circle $\{(\zeta, 1/2, \zeta)\}$ in S^3 intersects this disk in general position, and in a single point $(1, 1/2, 1)$. □

It is routine to verify that the following diagram homotopy commutes.

$$\begin{array}{ccccc} F(\mathbf{C},k)\times_{\mathcal{S}_k}(S^1)^k & \longrightarrow & \mathrm{Rat}_k & \xrightarrow{\text{eval}} & \mathrm{Map}_{\infty\mapsto 1}(S^2,S^2)_k \\ \downarrow{=} & & & & \uparrow{\simeq} \\ F(\mathbf{C},k)\times_{\mathcal{S}_k}(S^1)^k & & \xrightarrow{\quad\psi\quad} & & J_2(S^1). \end{array} \tag{3.4}$$

Here, the map ψ is the Dyer–Lashof map of (2.19) and (2.20), the first map in the upper row is the inclusion of 3.2 and the second map in the upper row is the evaluation map. The point is that the analytic functions in Rat_k obtained as the image of $F(\mathbf{C},k)\times_{\mathcal{S}_k}(S^1)^k$ can be thought of as having k poles determined by the points of $F(\mathbf{C},k)$ and, for each pole, a root lying on a *very small circle* centered at the corresponding pole. Relatively far from each of these k circles the value of the function is uniformly close to 1, and, up to homotopy we can pinch off the maps to be equal to 1 in the complement of appropriate neighborhoods of the circles. But this is just the model for the loop sum, which is given by the lower map ψ.

Thus, using the stable splitting maps of 2.16 we obtain a collection of maps $\Phi_j\colon \Sigma^\infty \mathrm{Rat}_k \to \Sigma^\infty D_j$ as *eval* followed by the inverse of the composition of homotopy equivalences $J_2(S^1)\to\Omega^2S^3$ and $\Omega^2S^3\to\mathrm{Map}_{\infty\mapsto 1}(S^2,S^2)_k$.

COROLLARY 3.5. *For any finite $k>0$ the composite stable map*

$$\bigvee_{j=1}^{k}\Phi_j\colon \mathrm{Rat}_k\to\bigvee_{1}^{k}\mathbf{F}_j(\mathbf{C})_+\wedge_{\mathcal{S}_j}(S^j)$$

has a stable section.

Proof. The fact that $\psi_k\colon \mathrm{Rat}_k\to D_k(S^1)$ has a stable section follows from (2.16) and (3.4) since (3.4) implies that the projection $F(\mathbf{C},k)\times(S^1)^k\to D_k(S^1)$ factors through Rat_k. The corollary is then proved by induction using the fact that in [Seg] Segal showed that there are imbeddings

$$i_q\colon \mathrm{Rat}_q\to\mathrm{Rat}_{q+1}$$

that are compatible, up to homotopy, with the embeddings

$$*[1]\colon \mathrm{Map}_q(S^2, S^2) \to \mathrm{Map}_{q+1}(S^2, S^2)$$

given by taking the loop sum with the identity map.

Of course, (3.5) does not assert more than the fact that the stable homotopy type of Rat_k is at least as big as that of $J_2^k(S^1)$. At this point, in fact, it could be much bigger. In § 4 we will show, however, that $H_*(J_2^k(S^1); \mathbf{F})$ is also an upper bound for the homology of Rat_k and, from this, Theorem 1.1 is immediate.

§ 4. The homology of Rat_k and $\mathrm{Div}_k(M_g - *)$

We now describe a general method to study the homology of the divisor spaces. As Rat_k is the simplest example of such spaces, we will obtain $H_*(\mathrm{Rat}_k)$ as the easiest special case. In [C²M²2] we give a more direct geometric determination of $H_*(\mathrm{Rat}_k)$; however, the methods there, unlike the methods here, do not extend to cover general divisor spaces. In fact, the techniques of this section extend without modification to cover more general types of spaces; specifically, two colour configuration spaces [Böd].

A. Dold and R. Thom introduced the space $AG(X, *)$, the free Abelian group generated by the points of X with $*$ as identity and (compactly generated) quotient topology, in [DT]. Moreover, they showed

THEOREM 4.1 [DT]. *Let X be a connected cell complex with $*$ a vertex, then the natural inclusion*

$$SP^\infty(X, *) \hookrightarrow AG(X, *)$$

is a homotopy equivalence.

We can write each point of $AG(X, *)$ uniquely in the form $*$ or

$$\{x_1 \cdots x_r y_1^{-1} \cdots y_s^{-1} \mid * \neq x_i, y_j,\ x_i \neq y_j,\ 1 \le i \le r,\ 1 \le j \le s\}$$

and we define $AG_{m,n}(X, *) \subset AG(X, *)$ as the set of points which can be written in the form above with $r \le m, s \le n$. Clearly, when $X = S^2$ we have

$$\mathrm{Rat}_k = AG_{k,k}(S^2, \infty) - \{AG_{k,k-1}(S^2, \infty) \cup AG_{k-1,k}(S^2, \infty)\}.$$

Similarly, when $X=M_g$, a closed Riemann surface of genus g, we have

$$\mathrm{Div}_k(M_g-*)=AG_{k,k}(M_g,*)-\{AG_{k,k-1}(M_g,*)\cup AG_{k-1,k}(M_g,*)\}.$$

More generally, we introduce the notation

$$\mathrm{Div}_{m,n}(X-*)=AG_{m,n}(X,*)-\{AG_{m,n-1}(X,*)\cup AG_{m-1,n}(X,*)\}. \tag{4.2}$$

Our main object in this section is to obtain spectral sequences converging to the homology of the spaces $\mathrm{Div}_{m,n}(M_g-*)$.

Although the doubly filtered space $AG(X,*)$ contains the information we want, it is well hidden in the structure. We will now modify the construction (without changing the homotopy type) to make the homological information we desire easier to extract. We regard $AG(X,*)$ as a quotient of $SP^\infty(X,*)\times SP^\infty(X,*)$ by factoring out the diagonal copy of $SP^\infty(X,*)$. In a manner similar to ([BCM], §4), we replace $AG(X,*)$ by the quasifibering

$$SP^\infty(X,*)\times SP^\infty(X,*)\to DY(X)\to SP^\infty(\Sigma X,*). \tag{4.3}$$

We can write $DY(X)$ explicitly as

$$DY(X)=SP^\infty(X,*)\times SP^\infty(X,*)\times_T SP^\infty(c(X,*))$$

where $c(X,*)$ is the reduced cone on X. The twisting is given by the diagonal inclusion $SP^\infty(X,*)\subset SP^\infty(X,*)^2$. More precisely, the points of $DY(X)$ are triples

$$\{(x_1\cdots x_r, y_1\cdots y_s, (t_1,z_1)\cdots(t_l,z_l))\}$$

with the identification that when $t_i=0$ the point above becomes equal to

$$\{(x_1\cdots x_r\cdot z_i, y_1\cdots y_s\cdot z_i, (t_1,z_1)\cdots(t_l,z_l))\}$$

with (t_i,z_i) deleted from the last set of coordinates. The projection onto $SP^\infty(\Sigma X)$ is a quasi-fibration with fiber $SP^\infty(X)\times SP^\infty(X)$. In particular, this means that there is an exact sequence in homotopy and it is not hard to see that the map

$$p\colon DY(X)\to AG(X,*),\quad (x_1\cdots x_r, y_1\cdots y_s,\cdots)\mapsto(x_1\cdots x_r y_1^{-1}\cdots y_s^{-1})$$

is a homotopy equivalence. In fact, more is true, and by properly bifiltering $DY(X)$ we can assume that p induces equivalences of bifiltered spaces. Exactly as in the proof of (4.7) of [BCM], we have

THEOREM 4.4. *If we bifilter $DY(X)$ by setting*

$$DY_{n,m}(X) = \{(x_1 \cdots x_r, y_1 \cdots y_s, (t_1, z_1) \cdots (t_l, z_l)\}$$

with $r+l \leq n$, $s+l \leq m$, then for all $0<m<\infty$, $0<n<\infty$, we have natural homotopy equivalences

$$AG_{n,m}(X, *) \simeq DY_{n,m}(X).$$

For example, the space $DY_{1,1}(X) = X \times X \cup cX/(0,x) \sim (x,x)$ clearly has the homotopy type of $X \times X/(x,x) \sim * = AG_{1,1}(X,*)$. It is instructive to work out the next few cases as an exercise.

Since $DY_{n-1,m}(X)$ and $DY_{n,m-1}(X)$ are both closed subsets of the closed (relative) manifold $DY_{n,m}(X)$ Alexander–Poincaré duality can be applied and gives

COROLLARY 4.5. *Let X be a compact oriented surface without boundary. Then*

$$H_i(DY_{n,m}(X)/\{DY_{n-1,m}(X) \cup DY_{n,m-1}(X)\}; \mathbf{F}) \cong \tilde{H}^{2(n+m)-i}(\mathrm{Div}_{n,m}(X-*); \mathbf{F})$$

for any field $\mathbf{F}$.

Our goal is to compute the homology of these quotients by studying the Leray–Serre spectral sequences for the quasi-fibration (4.3) (compare [BCM], § 5). This is where the advantage of replacing $AG(X,*)$ by $DY(X)$ becomes apparent. For example, when we filter by cells in $SP^\infty(\Sigma X)$ the inverse images become products, and so the E_2-terms become, by (2.11) and (2.12), quite easy to calculate.

Now we assume the coefficients $\mathbf{F}$ are the field $\mathbf{F}_p$, or the rationals $\mathbf{Q}$. The formulae in (2.11) and (2.12) show that the E^2-terms for the entire spaces $DY(X)$ are trigraded differential Hopf algebras of the form

$$(4.6) \qquad \prod_1^{2g} (E[e_{i,0}, e_{0,i}] \otimes \Gamma[h_i]) \otimes \Gamma[f_1, f_2] \otimes E[g, \ldots, g_i, \ldots] \otimes \Gamma[\ldots, \beta g_i, \ldots].$$

Here, h_i is the suspension of the ith generator of $H_1(M_g; \mathbf{Z})$ in $\Sigma M_g \subset SP^\infty(\Sigma M_g)$, and $g, g_i, \beta g_i$ map to the corresponding classes in $H_*(SP^\infty(S^3); \mathbf{F})$ under the suspension of the pinching map $M_g \to S^2$. In other words, they come from the suspension of the orientation class $[M_g]$ in $H_3(SP^\infty(\Sigma M_g);\mathbf{F})$. Of course, $e_{i,0}, e_{0,i}, f_1$ and f_2 correspond to the classes on the two copies of $SP^\infty(M_g)$ in the fiber. The trigrading degrees are given by

(4.7)

Generator	Trigrading
$e_{i,0}$	$(1,1,0)$
$e_{0,i}$	$(1,0,1)$
$h_{i,0}$	$(2,1,1)$
f_1	$(2,0,1)$
f_2	$(2,1,0)$
g	$(3,1,1)$
g_i	$(2p^i+1,p^i,p^i)$
βg_i	$(2p^i+2,p^i,p^i)$

with the first trigrading degree being the dimension of the element, and the last two degrees, of course, index the grading in $DY_{n,m}$. Also, g_i, βg_i are zero if $\mathbf{F}=\mathbf{Q}$. The algebra $\Gamma[\varkappa_{(2r,s,t)}]$ is the divided power algebra. It has a copy of $\mathbf{F}$ with generator $\gamma_j(\varkappa)$ in each tridegree $(2rj, sj, tj)$, is zero elsewhere, and

$$\gamma_j(\varkappa)\circ\gamma_v(\varkappa) = \binom{j+v}{j}\gamma_{v+j}(\varkappa).$$

In the special case where $X=S^2$ the terms $e_{i,j}$ and h_i are not present so we get the much simpler form

(4.8)

Generator	Trigrading
f_1	$(2,0,1)$
f_2	$(2,1,0)$
g	$(3,1,1)$
g_i	$(2p^i+1,p^i,p^i)$
βg_i	$(2p^i+2,p^i,p^i)$

Here is the process for obtaining the information about $H_*(\text{Rat})$ and $H_*(\text{Div})$ from these spectral sequences. We use the following diagram to obtain a filtration of the spaces given in 4.5 which are homotopy equivalent to Rat_k and Div_k, respectively, with explicit E^2-terms.

$$\begin{array}{ccccc} & & & & SP^{\infty}\times SP^{\infty} \\ & & & & \downarrow \\ E_{i,i}/(E_{i-1,i}\cup E_{i,i-1}) & \longleftarrow & E_{i,i} & \longrightarrow & E \\ & & \downarrow{\scriptstyle \pi_i} & & \downarrow{\scriptstyle \pi} \\ & & B_i & \longrightarrow & SP^{\infty}(\Sigma X) \end{array}$$

Here $E_{i,i}$ is the subspace of $E=DY(X)$ consisting of points having filtration degree $\leqslant(i, i)$ and $B_i\subset SP^{\infty}(\Sigma X)$ is the subspace of $SP^i(\Sigma X)$. Note that while the projection map $\pi: E_{i,i}\to B_i$ is not, in this context, to be regarded as a fibering, we can still examine the associated Leray spectral sequence for $H_*(E_{i,i})$ induced from the natural filtration of cells in B_i. Indeed, we have that

$$\pi_i^{-1}(SP^j(\Sigma X)-SP^{j-1}(\Sigma X)) = SP^{i-j}(X)\times SP^{i-j}(X)\times(SP^j(\Sigma X)-SP^{j-1}(\Sigma X))$$

in $E_{i,j}$. This is why the E_2 term for the spectral sequence for $H_*(E_{i,i}/(E_{i-1,i}\cup E_{i,i-1}))$ is the direct summand of $E_2(E_{i,i})$ consisting of those groups of filtration degree $(*, i, i)$.

Consider the terms in tridegrees $(*, k, k)$ appearing in (4.8). They give rise to a quotient spectral sequence converging to $\tilde{H}^*(\overline{\mathrm{Rat}_k}, \mathbf{F})$ as in 4.5, where $\overline{\mathrm{Rat}_k}$ denotes the one point compactification of the open manifold Rat_k (recall (2.14)). This can be written, on comparing with (2.11) and (2.12), as

$$\coprod_j (f_1f_2)^j H^{*-4j}(SP^{k-j}(S^3), SP^{k-j-1}(S^3); \mathbf{F}).$$

Moreover, from 2.18, we can identify the jth piece above with the dual of $H_*(D_j; \mathbf{F})$. But this implies that the lower bound of 3.5 is actually an upper bound. Consequently, we have

THEOREM 4.9. *The composite maps* $\bigvee_{j=1}^k \Phi_j$ *of* 3.5 *induces a homology isomorphism,*

$$H_*(\mathrm{Rat}_k; \mathbf{F})\to \coprod_{j=1}^{k} H_*(D_j; \mathbf{F}).$$

This completes the proof of 1.2 and thus 1.1.

Next we return to a general discussion of the spectral sequences in the cases of the Div spaces. The Hopf algebra structure implies that the dual of $\Gamma|\varkappa|$ is the polynomial algebra $A[\varkappa^*]$. There are differentials given by

(4.9)
$$d_2(h_i) = e_{i,0}+e_{0,i}$$
$$d_2(g) = \sum_1^g (e_{i,0}e_{0,i+g}-e_{i+g,0}e_{0,i})+f_1+f_2$$
$$d_2(g_i) = d_2(\beta g_i) = 0.$$

Moreover $e_{i,j}$, f_1, f_2 are infinite cycles.

For h_i and g, this follows from the diagram

$$\begin{array}{ccc} H_*(cM_g \cup_{\Delta M_g} M_g\times M_g, M_g\times M_g; \mathbf{F}) & \xrightarrow{\partial} & H_{*-1}(M_g\times M_g; \mathbf{F}) \\ \uparrow \cong & & \uparrow \Delta \\ H_*(cM_g, M_g; \mathbf{F}) & \xrightarrow{\partial} & H_{*-1}(M_g; \mathbf{F}) \end{array}$$

which describes, as we have seen in 4.4, the structure of the (1, 1) filtration piece of the spectral sequence. The triviality of the differentials on $g_i, \beta g_i$ follows from naturality and comparison with the spectral sequence of the entire quasifibration (4.3).

There are also higher differentials on the g_i, βg_i which depend in part on the genus g. For a further discussion see § 5 of [BCM].

When we take the associated spectral sequences for the subquotients $\mathrm{Div}_{m,n}(M_g-*)/\partial$, the associated E^2-term is the direct summand of the E^2-term given in 4.6 above consisting of those elements of exactly tridegree $(*, m, n)$. By naturality, the differentials for the subquotients, when they first appear are generated by the differentials above.

Since $d_2(h_i)$ has lower bidegree than h_i, this differential becomes 0 in the subquotient spectral sequence. Moreover, since it has lower filtration degree $d_2(h_i)=d_\infty(h_i)$ and it must be an infinite cycle, as are the $\gamma_j(h_i)$ for all j, i. However,

$$d_2(g) = \sum_1^g e_{i,0}e_{0,i+g}-e_{i+g,0}e_{0,i}$$

is non-zero unless $g=0$, i.e. $M_g=S^2$. The injection

$$H_*(\mathrm{Div}_{m-1,n}(M_g-*))\rightarrow H_*(\mathrm{Div}_{m,n}(M_g-*))$$

corresponds to multiplication (in cohomology) by f_1^*. Similarly, multiplication by f_2^* corresponds to increasing the second index by 1. Again, what is happening is very similar to the discussion in § 5 of [BCM].

In more detail, the terms of degree (k, k) have the form

$$\sum h_1^{v_1}\cdots h_g^{v_g}(f_1f_2)^w e_{I,0}\,e_{0,J}\,K(I,J)\,H_*(SP^r(S^3), SP^{r-1}(S^3); \mathbf{F}) \tag{4.11}$$

where

$$r=k-\sum v_i-w-|I|-|K| \quad \text{and} \quad K(I,J)=\begin{cases} f_1^{|I|-|J|} & \text{if } |J|\leq|I| \\ f_2^{|J|-|I|} & \text{otherwise.}\end{cases}$$

Thus the E_2 term looks like the effect of a sum of dual D_r's, specifically a sum of terms of the form

$$\Sigma^{2(v_1+\cdots+v_g)+2w+|I|+|K|}(D_r^*) \quad \text{or} \quad \Sigma^{2(v_1+\cdots+v_g)+2w+|J|+|K|}(D_r^*).$$

However, the space cannot split since, as we have seen, the differentials are non-trivial, although those we have been able to find only affect the $e_{I,0}e_{0,J}$ terms in the decomposition above, not the v_j or w indices. This suggests the possibility of a wedge decomposition over the v_i and w indices. (See the remarks after 7.15.)

In the special case of S^2, the cycle embedding techniques of [Car] can be used to show that $E^2=E^\infty$ for all the relative sequences above. However, the arguments in the proof of (4.9) give a direct proof of this fact.

More study is needed to completely determine the differential structure for these spectral sequences when $g>0$. The geometric information in [M] should be useful for this. In sections seven through nine we take a somewhat different approach to the study of $\mathrm{Div}_*(M_g-*)$ by imputting the homological results obtained in this section into an analysis of a function space point of view begun by Segal [Seg].

§5. The homology of $\mathrm{Rat}_k(\mathbf{CP}^n)$

In this section we extend the analysis of the previous sections, with only minor modification, to cover the spaces $\mathrm{Rat}_k(\mathbf{CP}^n)$ and establish Theorem 1.4. Using homogeneous coordinates, a base point preserving holomorphic map $S^2\to\mathbf{CP}^n$ of degree k is given by an $n+1$ tuple

$$(f_0(z), \ldots, f_n(z))$$

where each $f_i(z)$ is a degree k monic polynomial in one complex variable and the f_i's have no common zero. Consequently, the natural embedding

$$\mathrm{Rat}_k(\mathbf{CP}^n) \hookrightarrow SP^k(\mathbf{C})^{n+1} \tag{5.1}$$

cataloging the roots of the $f_j(z)$'s exhibits $\mathrm{Rat}_k(\mathbf{CP}^n)$ as the complement of the subspace

$$(\Delta^n\mathbf{C})\circ(SP^{k-1}(\mathbf{C}))^n \subset (SP^k(\mathbf{C}))^{n+1}.$$

That is, $\mathrm{Rat}_k(\mathbf{CP}^n)$ is the complement of those points in $(SP^k(\mathbf{C}))^{n+1}$ where at least one point in each k-tuple is a particular point $z \in \mathbf{C}$, so there is a "diagonal" image point, along with $k-1$ others in each k-tuple, which explains the $\Delta^n\mathbf{C}$ term in the expression above. In strict analogy with (2.14), we have

$$H_*(\mathrm{Rat}_k(\mathbf{CP}^n)/\partial) = H_*(SP^k(S^2)^{n+1}/\{SP^{k-1}(S^2)^n \cup \Delta^n S^2 \circ SP^{k-1}(S^2)^n\}). \tag{5.2}$$

As before we decompose $\mathrm{Rat}_k(\mathbf{CP}^n)$ by projecting onto the first factor of the natural inclusion (5.1). Again, the inverse image of $\mathbf{F}_r(SP^k(\mathbf{C}))-\mathbf{F}_{r+1}(SP^k(\mathbf{C}))$ is a fiber bundle $E_r(n)$ over the point $(x_1\cdots x_r)^2x_{r+1}\cdots x_{k-r}$ with fiber $(SP^k(\mathbf{C}))^n-\Delta^n(x_1,\ldots,x_{k-r})$. Since $\mathbf{C}^n-\Delta^n(z)\simeq S^{2n-1}$ the fiber has the homotopy type of the space $(S^{2n-1})^{k-r}$. A specific homotopy equivalence can be constructed exactly as in (3.1).

As expected the generic set $E_0(n)$ for $\mathrm{Rat}_k(\mathbf{CP}^n)$ is easily identified with the basic building block both in the May–Milgram model and in the Dyer–Lashof maps for the double loop space Ω^2S^{2n+1}.

LEMMA 5.3. *$E_0(n)$ is homotopy equivalent to the total space of the fibering*

$$F(\mathbf{C},k)\times_{\mathcal{S}_k}(S^{2n-1})^k.$$

The analogs of (3.3), (3.4) and (3.5) are now immediate.

LEMMA 5.4. *When $k=1$,*

$$E_0(n)=\mathrm{Rat}_1(\mathbf{CP}^n)\cong S^{2n-1}\times\mathbf{R}^{2n+3}$$

and the composite

$$S^{2n-1}\hookrightarrow \mathrm{Rat}_1(\mathbf{CP}^n)\xrightarrow{\text{eval}}\Omega_1^2S^{2n+1}$$

maps onto a generator of $H_{2n-1}(\Omega_1^2S^{2n+1};\mathbf{Z})=\mathbf{Z}$.

COROLLARY 5.5. *For any finite $k>0$ the composite stable map*

$$\bigvee_{j=1}^{k} \Phi_j\colon \mathrm{Rat}_k \mathbf{CP}^n \to \bigvee_{1}^{k} \mathbf{F}_j(\mathbf{C})_+ \wedge_{\mathcal{S}_j} (S^{(2n-1)j})$$

has a stable section.

As $\Sigma^{(2n-2)j} D_j(S^1) \simeq D_j(S^{2n-1})$ [CMM], 5.5 gives the lower bound needed to establish Theorem 1.4.

Hence, as before, we consider the space

$$DY^{n+1}(X) = SP^\infty(X, *)^{n+1} \times_T SP^\infty(c(X, *))$$

where the twisting is given by the $(n+1)$-fold diagonal inclusion

$$\Delta^{n+1}\colon SP^\infty(X, *) \to SP^\infty(X, *)^{n+1}.$$

The resulting space $DY^{n+1}(X)$ is $(n+1)$-graded, the grading degree $(i_0, \ldots, i_n)$ corresponding to maps

$$\mathbf{C} \to \mathbf{C}^n, \quad z \mapsto (f_0(z), \ldots, f_n(z))$$

with f_j monic of degree $\leq i_j$, and, as before, there is an equivalence

$$(SP^\infty(X, *)^{n+1} / \Delta^{n+1}(SP^\infty(X, *)))_{(i_0, \ldots, i_n)} \simeq DY^{n+1}(X)_{(i_0, \ldots, i_n)}$$

where we $(n+1)$-grade $DY^{n+1}(X)$ by setting the $(n+1)$-grading of

$$(x_1^0 \cdots x_{r_0}^0, \ldots, x_1^n \cdots x_{r_n}^n, (t_1, z_1), \ldots, (t_l, z_l))$$

equal to $(r_0+l, r_1+l, \ldots, r_n+l)$. The resulting Leray spectral sequence has E^2-term

$$\Gamma[f_0, \ldots, f_n] \otimes E[g, \ldots, g_i, \ldots] \otimes \Gamma[\ldots, \beta g_i, \ldots]$$

where the $(n+2)$-grading of f_1 is $(2, 0, \ldots, 1, 0 \ldots 0)$ with the 1 in the $(i+1)$st position, of g is $(3, 1, 1, \ldots, 1)$, of g_i is $(2p^i+1, p^i, \ldots, p^i)$ and of βg_i is $(2p^i+2, p^i, \ldots, p^i)$. Therefore the terms of bidegree $(*, n+1, n+1, \ldots, n+1)$ must have the form

$$\gamma_m(f_0, f_1 \cdots f_n) \circ \nu(g, P^{[i]}(g), \beta P^{[i]}(g), \ldots)$$

where $m \leq n+1$ and $\nu(\ \ )$ is a "homogeneous" polynomial of $(n+1)$-degree

$$(n-m+1, \ldots, n-m+1).$$

Now Theorem 1.4 follows by a direct comparison argument.

§6. A cell decomposition for $\mathrm{Rat}_k(\mathbf{CP}^n)$

We now give a cell decomposition for $\mathrm{Rat}_k(\mathbf{CP}^n)\simeq\mathcal{M}_{n,1,k}\simeq\mathcal{M}_{1,n,k}$, the moduli space of observable, controllable linear dynamical systems of tridegree $(n,1,k)$ and $(1,n,k)$, respectively.

$\mathrm{Rat}_k(\mathbf{CP}^n)$ is an open manifold and any finite decomposition into regularly embedded open balls of varying dimensions must, of necessity, contain balls whose closures are not compact. Specifically, we now construct a decomposition into a finite number of cells of the one-point compactification $\mathrm{Rat}_k(\mathbf{CP}^n)^+$, having the ideal point $+$ as its only vertex. The adjoined point $+$ will denote the limit of any sequence of rational maps $[r_0^m(z),\dots,r_n^m(z)]$ having the property that the moduli of the zeroes of some $r_j^m(z)$ tend to ∞ as m tends to infinity. These cells, with $+$ deleted, give a decomposition of $\mathrm{Rat}_k(\mathbf{CP}^n)$ into a finite number of convex regions and should be useful in applications to control theory.

The following cell decomposition, while conceptually quite simple, is not minimal. Furthermore, the combinatorics used to describe the attaching maps are sufficiently complicated so that it is non-trivial to recover the homological calculations given in the previous sections. However, our decomposition does have the advantage that the cells we give are convex and very easy to describe. Also, it may be quite hard to find a smaller decomposition which consists of *convex* cells.

We begin with the simplest case, $\mathrm{Rat}_k=\mathrm{Rat}_k(\mathbf{CP}^1)$, as the extension to the general case will be evident. Recall from §2 (particularly the discussion preceding Proposition 2.14) that we may view Rat_k as the open $4k$ dimensional submanifold of $SP^k(\mathbf{C})\times SP^k(\mathbf{C})\subset SP^k(S^2)\times SP^k(S^2)$ which misses the points at infinity and the singular set $V_{k,k}$. Thus, Alexander–Poincaré duality, on the simplicial level, implies that it is sufficient to give a cell decomposition of

$$\mathrm{Rat}_k^+=\frac{SP^k(S^2)\times SP^k(S^2)}{[SP^{k-1}(S^2)\times SP^k(S^2)]\cup[SP^k(S^2)\times SP^{k-1}(S^2)]\cup V_{k,k}}.$$

Let $I^2=\{(t_1,t_2)\mid 0\leq t_1,t_2\leq 1\}$ denote the closed unit square in $\mathbf{C}$ and $\dot{I}^2=\{(t_1,t_2)\mid 0<t_1,t_2<1\}$ denote the interior of I^2. By fixing a homeomorphism between $\mathbf{C}$ and the open unit square $\dot{I}^2$ in $\mathbf{C}$ one obtains a homeomorphic model of Rat_k where all of the roots and poles lie inside the open unit square. Thus we may work with the homeomorphic model

$$\mathrm{Rat}_k^+\cong\frac{SP^k(I^2)\times SP^k(I^2)}{[SP^{k-1}(I^2)\times SP^k(I^2)]\cup[SP^k(I^2)\times SP^{k-1}(I^2)]\cup V_{k,k}}.$$

where, by abuse of notation, the entire boundary $\partial I^2 = I^2 - \dot{I}^2$ in I^2 is identified to the distinguished base point.

Hence, for each point $(x_1, \ldots, x_k, y_1, \ldots, y_k) \in \mathrm{Rat}_k$ we can write each $x_i = (t_{1,i}, t_{2,i})$ and each $y_j = (\tau_{1,j}, \tau_{2,j})$ where $0 < t_{l,i}, \tau_{l,j} < 1$ for every i, j and l. Since the x_i's and y_j's are unordered in their respective symmetric products, it follows that we can use the following obvious lexicographical order to uniquely represent each point in Rat_k by the pair

(6.1) $$\left\{\left\langle\begin{pmatrix} t_{1,1} \\ t_{2,1}\end{pmatrix}, \ldots, \begin{pmatrix} t_{1,k} \\ t_{2,k}\end{pmatrix}\right\rangle, \left\langle\begin{pmatrix} \tau_{1,1} \\ \tau_{2,1}\end{pmatrix}, \ldots, \begin{pmatrix} \tau_{1,k} \\ \tau_{2,k}\end{pmatrix}\right\rangle\right\}$$

where

(1) $t_{1,1} \leq t_{1,2} \leq \ldots \leq t_{1,k}$ and if $t_{1,i} = t_{1,i+1}$ then $t_{2,i} \leq t_{2,i+1}$.

(2) $\tau_{1,1} \leq \tau_{1,2} \leq \ldots \leq \tau_{1,k}$ and if $\tau_{1,i} = \tau_{1,i+1}$ then $\tau_{2,i} \leq \tau_{2,i+1}$.

Next consider a pair of k-tuples given by (6.1). If one forgets the difference between the t's and τ's and simply regards the point as a $2k$-tuple of points in $\dot{I}^2$ there is a permutation of the entries which brings the $2k$-tuple into proper lexicographical order. For example, the permutation (243) corresponds to bringing the point

$$\left\{\left\langle\begin{pmatrix} 1/3 \\ 2/3\end{pmatrix}, \begin{pmatrix} 1/2 \\ 1/3\end{pmatrix}\right\rangle, \left\langle\begin{pmatrix} 5/12 \\ 1/2\end{pmatrix}, \begin{pmatrix} 11/24 \\ 7/9\end{pmatrix}\right\rangle\right\}$$

to the 4-tuple

$$\left\{\left\langle\begin{pmatrix} 1/3 \\ 2/3\end{pmatrix}, \begin{pmatrix} 5/12 \\ 1/2\end{pmatrix}\right\rangle, \left\langle\begin{pmatrix} 11/24 \\ 7/9\end{pmatrix}, \begin{pmatrix} 1/2 \\ 1/3\end{pmatrix}\right\rangle\right\}$$

which is in proper lexicographic order.

Of course, the process of bringing any point in $SP^k(I^2) \times SP^k(I^2)$ represented as in (6.1) to a lexicographically ordered $2k$-tuple may correspond to several permutations. However, for any $p \in \mathrm{Rat}_k$, the condition that $(t_{1,i}, t_{2,i}) \neq (\tau_{1,j}, \tau_{2,j})$ for all i and j insures that the associated permutation is unique. Formally we have

Definition 6.2. Let

$$p = \left\{\left\langle\begin{pmatrix} t_{1,1} \\ t_{2,1}\end{pmatrix}, \ldots, \begin{pmatrix} t_{1,k} \\ t_{2,k}\end{pmatrix}\right\rangle, \left\langle\begin{pmatrix} \tau_{1,1} \\ \tau_{2,1}\end{pmatrix}, \ldots, \begin{pmatrix} \tau_{1,k} \\ \tau_{2,k}\end{pmatrix}\right\rangle\right\}$$

be a point in

$$\mathrm{Rat}_k \subset SP^k(\dot{I}^2) \times SP^k(\dot{I}^2) \to \mathrm{Rat}_k^+$$

where the t's and τ's are ordered as above. We say that $\sigma \in \mathscr{S}_{2k}$ is the kth order shuffle of p if σ is that unique permutation of the columns of p which preserves two properties. First, it preserves the internal ordering in the t's and the τ's respectively. Second, it maps p to the ordered $2k$-tuple

$$\left\langle \begin{pmatrix} r_{1,1} \\ r_{2,1} \end{pmatrix}, \ldots, \begin{pmatrix} r_{1,2k} \\ r_{2,2k} \end{pmatrix} \right\rangle$$

where $r_{1,1} \leq r_{1,2} \leq \ldots \leq r_{1,2k}$ and if $r_{1,i} = r_{1,i+1}$ then $r_{2,i} \leq r_{2,i+1}$.

That is, σ is the kth order shuffle of p if it intertwines the t's and τ's into a lexicographically ordered $2k$-tuple while preserving the respective internal orderings. As mentioned above it is manifest that to each $p \in \mathrm{Rat}_k$ there corresponds a unique kth order shuffle $\sigma(p) \in \mathscr{S}_{2k}$. In this way one may partition the points of Rat_k into disjoint sets labelled by elements of the symmetric group $\mathscr{S}_{2k}$.

It is quite natural to try to use this partition of Rat_k indexed by the kth order shuffles to build a cell decomposition for Rat_k. However, the kth order shuffle σ itself does not keep track of whether a strict inequality or equal sign occurs at each position in the $2k$-tuple $r_{1,1} \leq \cdots \leq r_{1,2k}$ as the t's and τ's are lexicographically intertwined. To index the cell structure and associated boundary maps it is necessary to incorporate this additional bookkeeping, as even the most trivial example demonstrates.

Example 6.3. Rat_1^+ has a cell decomposition with:

(1) Two open 4-cells

$$e_4^1 = \left\{ \left\langle \begin{pmatrix} t_{11} \\ t_{21} \end{pmatrix} \begin{pmatrix} \tau_{11} \\ \tau_{21} \end{pmatrix} \right\rangle \middle| \begin{array}{l} t_{11} < \tau_{11} \\ t_{21} \text{ and } \tau_{21} \text{ arbitrary} \end{array} \right\}$$

$$e_4^2 = \left\{ \left\langle \begin{pmatrix} \tau_{11} \\ \tau_{21} \end{pmatrix} \begin{pmatrix} t_{11} \\ t_{21} \end{pmatrix} \right\rangle \middle| \begin{array}{l} \tau_{11} < t_{11} \\ t_{21} \text{ and } \tau_{21} \text{ arbitrary} \end{array} \right\}.$$

(2) Two open 3-cells:

$$e_3^1 = \left\{ \left\langle \begin{pmatrix} t_{11} \\ t_{21} \end{pmatrix} \begin{pmatrix} \tau_{11} \\ \tau_{21} \end{pmatrix} \right\rangle \middle| \begin{array}{l} t_{11} = \tau_{11} \\ t_{21} < \tau_{21} \end{array} \right\}$$

$$e_3^2 = \left\{ \left\langle \begin{pmatrix} \tau_{11} \\ \tau_{21} \end{pmatrix} \begin{pmatrix} t_{11} \\ t_{21} \end{pmatrix} \right\rangle \middle| \begin{array}{l} \tau_{11} = t_{11} \\ \tau_{21} < t_{21} \end{array} \right\}.$$

(3) The 0-cell $+$.

(4) The attaching maps in the cell structure are obtained by computing the closures of each cell, which, in turn, are obtained by considering limit points of sequences in each open cell. Notice that in the induced chain complex the boundary homomorphisms are given by

$$\partial(e_4^1) = e_3^1 - e_3^2 \quad \partial(e_3^1) = 0$$

$$\partial(e_4^2) = e_3^2 - e_3^1 \quad \partial(e_3^2) = 0.$$

This is to be expected from (2.14) as $\mathrm{Rat}_1 \simeq S^1$. Thus, the cell structure and boundary maps may be completely described by the indexed $\mathcal{S}_2$ permutations (keeping track of whether or not the t and τ columns are interchanged and whether or not $t_{1,1} = \tau_{1,1}$)

$$(\mathrm{id})\begin{pmatrix} < \\ * \end{pmatrix}, \quad (12)\begin{pmatrix} < \\ * \end{pmatrix}, \quad (\mathrm{id})\begin{pmatrix} = \\ < \end{pmatrix} \quad \text{and} \quad (12)\begin{pmatrix} = \\ < \end{pmatrix}$$

corresponding to e_4^1, e_4^2, e_3^1, and e_3^2 respectively.

More generally, to each kth order shuffle $\sigma \in \mathcal{S}_{2k}$ there corresponds a family of indexed kth order shuffles

$$\sigma\begin{pmatrix} \circ_{1,1} \cdots \circ_{1,2k-1} \\ \circ_{2,1} \cdots \circ_{1,2k-1} \end{pmatrix} \tag{6.4}$$

where:

(1) Each $\circ_{1,j}$ is either a $<$ or $=$ sign.

(2) If $\circ_{1,j}$ is an $<$ sign then there is no condition on $\circ_{2,j}$. In what follows we will write $\circ_{2,j}$ as $*$ (meaning no equality/inequality condition) in this case.

(3) If $\circ_{1,j}$ is an $=$ sign then $\circ_{2,j}$ is either a $<$ or $=$ sign.

Of course, many index kth order shuffles represent points in $SP^k(I^2) \times SP^k(I^2)$ that are collapsed to the base point $+$. However, once again, to each interior point $p \in \mathrm{Rat}_k \subset \mathrm{Rat}_k^+$ there corresponds a unique indexed kth order shuffle

$$\sigma\begin{pmatrix} \circ \cdots \circ \\ \circ \cdots \circ \end{pmatrix}.$$

That is, while every point in $[SP^{k-1}(I^2) \times SP^k(I^2)] \cup [SP^k(I^2) \times SP^{k-1}(I^2)] \cup V_{k,k}$ (which may formally correspond to many degenerate indexed shuffles) is collapsed to the base point, the indexed kth order shuffles corresponding to non-base points in Rat_k^+ further partition Rat_k into disjoint sets. We shall see that these indexed kth order shuffles can be used to label our cell decomposition. Before proceeding to the general case it is instructive to work through the following example.

Example 6.5. Rat_2^+ has a cell decomposition with

(1) Six 8-cells indexed by

$$\sigma\begin{pmatrix} < & < & < \\ * & * & * \end{pmatrix}$$

where $\sigma \in \mathcal{S}_4$ runs through id, (23), (243), (123), (1243), and (13)(24).

(2) Eighteen 7-cells indexed by

$$\sigma\begin{pmatrix} \circ_1 & \circ_2 & \circ_3 \\ \circ_4 & \circ_5 & \circ_6 \end{pmatrix}$$

where $\sigma \in \mathcal{S}_4$ runs through id, (23), (243), (123), (1243), and (13)(24) and the indices associated to each permutation run through

$$\begin{pmatrix} < & < & = \\ * & * & < \end{pmatrix}, \quad \begin{pmatrix} < & = & < \\ * & < & * \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} = & < & < \\ < & * & * \end{pmatrix}.$$

(3) 24 6-cells.

(4) 18 5-cells.

(5) Eight 4-cells indexed by

$$\mathrm{id}\begin{pmatrix} = & < & = \\ = & * & = \end{pmatrix} \quad (13)(24)\begin{pmatrix} = & < & = \\ = & * & = \end{pmatrix}$$

$$\mathrm{id}\begin{pmatrix} = & = & = \\ = & < & < \end{pmatrix} \quad (13)(24)\begin{pmatrix} = & = & = \\ = & < & < \end{pmatrix}$$

$$\mathrm{id}\begin{pmatrix} = & = & = \\ < & < & = \end{pmatrix} \quad (13)(24)\begin{pmatrix} = & = & = \\ < & < & = \end{pmatrix}$$

$$(123)\begin{pmatrix} = & = & = \\ < & = & < \end{pmatrix} \quad (243)\begin{pmatrix} = & = & = \\ < & = & < \end{pmatrix}.$$

(6) Two 3-cells given by

$$\mathrm{id}\begin{pmatrix} = & = & = \\ = & < & = \end{pmatrix} \quad \text{and} \quad (13)(24)\begin{pmatrix} = & = & = \\ = & < & = \end{pmatrix}.$$

(7) The base point + (which contains all 86 degenerate σ's).

(8) The following table summarizes the number of cells corresponding to each σ and 27 associated indexes.

dim \ σ	id	(23)	(234)	(123)	(1243)	(13)(24)	Total
8	1	1	1	1	1	1	6
7	3	3	3	3	3	3	18
6	5	3	4	4	3	5	24
5	5	1	3	3	1	5	18
4	3		1	1		3	8
3	1					1	2
degenerate	9	19	15	15	19	9	86
Total	27	27	27	27	27	27	162

(9) The rules for computing the boundary homomorphisms in the associated chain complex can also be given. For example,

$$\partial\left(\mathrm{id}\begin{pmatrix} < & < & < \\ * & * & * \end{pmatrix}\right) = \mathrm{id}\begin{pmatrix} < & = & < \\ * & < & * \end{pmatrix} - (23)\begin{pmatrix} < & = & < \\ * & < & * \end{pmatrix}$$

$$\begin{aligned}\partial\left((23)\begin{pmatrix} < & < & < \\ * & * & * \end{pmatrix}\right) &= (23)\begin{pmatrix} = & < & < \\ < & * & * \end{pmatrix} + (23)\begin{pmatrix} < & = & < \\ * & < & * \end{pmatrix} \\ &\quad + (23)\begin{pmatrix} < & < & = \\ * & * & < \end{pmatrix} - (123)\begin{pmatrix} = & < & < \\ < & * & * \end{pmatrix} \\ &\quad - \mathrm{id}\begin{pmatrix} < & = & < \\ * & < & * \end{pmatrix} - (243)\begin{pmatrix} < & < & = \\ * & * & < \end{pmatrix}\end{aligned}$$

$$\partial(243)\begin{pmatrix} = & < & < \\ < & * & * \end{pmatrix} = (243)\begin{pmatrix} = & < & = \\ < & * & < \end{pmatrix} - (23)\begin{pmatrix} = & < & = \\ < & * & < \end{pmatrix}$$

$$\partial\left((13)(24)\begin{pmatrix} = & = & = \\ < & < & = \end{pmatrix}\right) = (13)(24)\begin{pmatrix} = & = & = \\ = & < & = \end{pmatrix}$$

$$\partial\left((243)\begin{pmatrix} = & < & = \\ < & * & < \end{pmatrix}\right) = 0.$$

Definition 6.6. Let σ_s be an indexed kth-order shuffle, and let $e(\sigma_s)$ be the set of $p \in \mathrm{Rat}_k^+$ whose unique indexed kth order shuffle is σ_s; that is,

$$e(\sigma_s) = \{p \in \mathrm{Rat}_k^+ \mid \alpha_s(p) = \sigma_s\}.$$

Furthermore, we say σ_s and $e(\sigma_s)$ are proper if $e(\sigma_s)$ contains a point of Rat_k^+ different from the base point $+$.

Since $e(\sigma_s)$ is described by a series of order relations on the t's and the τ's we have

LEMMA 6.7. *For each proper indexed k-th order shuffle σ_s, $e(\sigma_s)$ is an open convex cell.*

Thus we have

PROPOSITION 6.8. *There is a cell decomposition of* Rat_k^+ *with one zero cell, namely* $+$, *and one open convex cell for every proper indexed k-th order shuffle*

$$\sigma\begin{pmatrix} \circ \cdots \circ \\ \circ \cdots \circ \end{pmatrix}.$$

To describe the closure of each cell one takes limit points of sequences in the open cells and from this analysis it is easy to recover the attaching maps and the boundary maps on the chain level. It is routine to see precisely how such limit points arise. First, if one replaces an arbitrary $<$ sign in the indexed shuffle σ_s by an $=$ sign we will obtain a new indexed shuffle $\sigma_{s'}$ and it is easy to see that $e(\sigma_{s'})$ must form part of the boundary of $\overline{e(\sigma_s)}$. However, if the $<$ sign occurs between two t's (or two τ's), it is easy to see that $e(\sigma_{s'})$ occurs twice in the boundary with opposite orientations and thus the two copies combinatorially cancel when computing the boundary map ∂ on the chain level. Second, for certain indexed shuffles σ_s, there are sequences of points $p_n \in e(\sigma_s)$ which converge to a point $q \notin e(\sigma_{s'})$. In these cases it is straightforward to check that $q \in e(\sigma'_{s'})$ for some other permutation σ' (with the same index set s'). This is most easily seen by considering the possible reordering of the second row of points in $e(\sigma_s)$. In this case, there is an element $\theta \in \mathcal{S}_{2k}$ such that $\theta\sigma' = \sigma$. Let $c(\sigma', s')$ be the number of times, counting orientations, that $e(\sigma'_{s'})$ occurs in the boundary of $e(\sigma_s)$. It is instructive to verify part 9 of example 6.5 in this way.

LEMMA 6.9. *The boundary map on the chain level for the cell decomposition given in* 6.7 *and* 6.8 *is given by the formula*

$$\partial(e(\sigma_s)) = \sum_{\sigma', s'} c(\sigma', s')\, e(\sigma'_{s'})$$

where

(1) *s' runs through index sets obtained from s by replacing one $<$ sign by an $=$ sign.*

(2) *(σ', θ) runs through all pairs such that $\theta\sigma' = \sigma$ as described in the paragraph above. Notice that σ' remains equal to σ in many cases.*

In addition, the closue of each cell is given by

$$\overline{e(\sigma_s)} = \bigcup_{\sigma', s'} e(\sigma'_{s'})$$

where σ' and s' are as given above.

Finally, if we extend the indexed kth order shuffles to ordered $n+1$ tuples of points in $SP^k(I^2)$ in the manner described in definition 6.2 and (6.4) above we obtain

PROPOSITION 6.10. *There is a cell decomposition of* $\mathrm{Rat}_k(\mathbf{CP}^n)^+$, *with one vertex* $+$, *and with one open convex cell $e(\sigma)$ for every proper indexed k-th order shuffle of the $n+1$ ordered tuples in $[SP^k(I^2)]^{n+1}$.*

Remark 6.11. This cell decomposition specializes to that given by Fox and Neuwirth [FoN] for $(F(\mathbf{R}^2, k)/\mathcal{S}_k)^+$. In addition, Fuks [Fu] used that cell decomposition to give the mod 2 homology of the braid groups.

§7. The homotopy type of $\mathbf{Map}^*_{n,n}(M_g, \mathbf{CP}^\infty \vee \mathbf{CP}^\infty)$

In the remainder of this paper we concentrate on the spaces $\mathrm{Div}_k(M'_g)$ where, to ease notation, we are writing $M_g - * = M'_g$. To begin we review some facts from [Seg] relating to the connection between holomorphic maps of degree n from $M_g \to S^2$, points on the divisor space, continuous maps $M_g \to S^2$, and continuous maps $M_g \to \mathbf{CP}^\infty \vee \mathbf{CP}^\infty$. Then we prove a splitting theorem, 7.8, for the first and give partial results (see, in particular, 7.15) for the second. We remark that we are extending Segal's work here only in that we obtain better control of the relevant mapping spaces. We hope that by comparing the results for the Div spaces given in sections 4 and 5 with the structure of the mapping spaces described in this section it will eventually be possible to determine the exact structure of the Div spaces.

Definition 7.1 [Seg]. $\mathrm{Map}^*_n(M_g, S^2)$ is the space of based maps $(* \to 1)$ of degree n from $M_g \to S^2$, $F^*_n(M_g) \subset \mathrm{Map}^*_n(M_g, S^2)$ is the subspace consisting of holomorphic maps.

Associated to $\alpha \in F^*_n(M_g)$ are its roots and poles, hence an element in $\mathrm{Div}_n(M'_g)$, and the basepoint condition assures that this defines an embedding $F^*_n(M_g) \hookrightarrow \mathrm{Div}_n(M'_g)$. Segal also constructs a map $\mathrm{Div}_n(M'_g) \to \mathrm{Map}^*_{n,n}(M_g, \mathbf{CP}^\infty \vee \mathbf{CP}^\infty)$, together with various commutative diagrams (see § 4 of [Seg]). (The components of $\mathrm{Map}^*(M_g, \mathbf{CP}^\infty \vee \mathbf{CP}^\infty)$ are indexed by the degrees of the map on the top cell to the to $\vee$-summands of $\mathbf{CP}^\infty \vee \mathbf{CP}^\infty$, and consequently are isomorphic to $\mathbf{Z}\times\mathbf{Z}$.) The salient points for our discussion here may be summarized in 7.2 and 7.3 which are key steps in the proof of 7.8:

PROPOSITION 7.2. (a) *The inclusion* $F_n^*(M_g)\hookrightarrow \mathrm{Map}_n^*(M_g, S^2)$ *is a homotopy equivalence through dimension* $n-2g$.

(b) *The map* $\mathrm{Div}_n(M_g')\rightarrow \mathrm{Map}_{n,n}^*(M_g, \mathbf{CP}^\infty\vee\mathbf{CP}^\infty)$ *is a homotopy equivalence through a range which increases with* n.

(This is Proposition 4.1 and 4.2 of [Seg].)

PROPOSITION 7.3. *There are homotopy fibration sequences*

$$S^3\rightarrow \mathbf{CP}^\infty\vee\mathbf{CP}^\infty\hookrightarrow \mathbf{CP}^\infty\times\mathbf{CP}^\infty$$

$$S^2\rightarrow \mathbf{CP}^\infty\vee\mathbf{CP}^\infty\xrightarrow{\theta}\mathbf{CP}^\infty$$

where $\theta(a,*)=a$, $\theta(*,b)=b^{-1}$. *Moreover, the following is a commutative diagram of fibrations*

$$\begin{array}{ccccc} S^3 & \xrightarrow{=} & S^3 & \longrightarrow & * \\ \downarrow h & & \downarrow & & \downarrow \\ S^2 & \longrightarrow & \mathbf{CP}^\infty\vee\mathbf{CP}^\infty & \xrightarrow{\theta} & \mathbf{CP}^\infty \\ \downarrow & & \downarrow & & \downarrow = \\ \mathbf{CP}^\infty & \xrightarrow{\Delta} & \mathbf{CP}^\infty\times\mathbf{CP}^\infty & \xrightarrow{\bar\theta} & \mathbf{CP}^\infty \end{array} \tag{7.4}$$

where h *is the Hopf fibering,* Δ *is the diagonal map and* $\bar\theta(a,b)=ab^{-1}$.

Proof. This proposition is a special case of T. Ganea's theorem [Ga] that the homotopy fibre of $A\vee B\hookrightarrow A\times B$ is $\Sigma(\Omega A\wedge\Omega B)$. We check that the fiber of $\theta\colon\mathbf{CP}^\infty\vee\mathbf{CP}^\infty\rightarrow\mathbf{CP}^\infty$ is S^2. Indeed, the fiber is the total space of the fibration over $\mathbf{CP}^\infty\vee\mathbf{CP}^\infty$ induced from the path, loop fibration

$$S^1\rightarrow S^\infty\rightarrow\mathbf{CP}^\infty$$

over $\mathbf{CP}^\infty$. But this is just

$$S^\infty\cup_{S^1}S^\infty\simeq\Sigma S^1=S^2.$$

The proof that the middle column is also a fibering is identical, and (7.4) follows easily. □

COROLLARY 7.5. *There is a commutative diagram of homotopy fibration sequences*

$$\begin{array}{ccccc}
\mathrm{Map}^*(M_g, S^3) & \xrightarrow{=} & \mathrm{Map}^*(M_g, S^3) & \longrightarrow & * \\
\downarrow{\scriptstyle h_*} & & \downarrow & & \downarrow \\
\mathrm{Map}^*_n(M_g, S^2) & \longrightarrow & \mathrm{Map}^*_{n,n}(M_g, \mathbf{CP}^\infty \vee \mathbf{CP}^\infty) & \xrightarrow{\theta_*} & \mathrm{Map}^*_n(M_g, \mathbf{CP}^\infty) \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle =} \\
\mathrm{Map}^*_n(M_g, \mathbf{CP}^\infty) & \xrightarrow{\Delta_*} & \mathrm{Map}^*_{n,n}(M_g, \mathbf{CP}^\infty \times \mathbf{CP}^\infty) & \xrightarrow{\bar{\theta}_*} & \mathrm{Map}^*_n(M_g, \mathbf{CP}^\infty).
\end{array}$$

In 7.5 we can identify most of these space without difficulty. Thus we have

LEMMA 7.6. (a) $\mathrm{Map}^*_n(X, Y\times Z)\simeq \mathrm{Map}^*_n(X, Y)\times \mathrm{Map}^*_n(X, Z)$.
(b) $\mathrm{Map}^*_n(M_g, \mathbf{CP}^\infty)\simeq (S^1)^{2g}$.
(c) $\mathrm{Map}^*_n(M_g, S^3)\simeq \Omega^2S^3\times(\Omega S^3)^{2g}$.

Proof. (a) is clear. For (b) we have that $\mathrm{Map}^*_k(X, \mathbf{CP}^\infty)$ is an Abelian monoid using the multiplication in $\mathbf{CP}^\infty = SP^\infty(S^2)$ to define a product structure on the mapping space. It also has the homotopy type of a CW complex by a theorem of Milnor [Mil]. On the other hand $\pi_i(\mathrm{Map}^*_k(X, \mathbf{CP}^\infty)) = H^2(\Sigma^i X, \mathbf{Z})$ since, by adjointing, a based map

$$f\colon S^i \to \mathrm{Map}^*_k(X, \mathbf{CP}^\infty)$$

corresponds to a map $S^i\wedge X \to \mathbf{CP}^\infty$. Thus $\pi_i(\mathrm{Map}^*_k(X, \mathbf{CP}^\infty)) = H^{2-i}(X; \mathbf{Z})$ and (b) follows.

To prove (c) we can analyze $\mathrm{Map}^*(M_g, S^3)$ by using the cofibering sequence

$$S^1 \xrightarrow{f} \bigvee_1^{2g} S^1 \to M_g \to S^2 \xrightarrow{0} \bigvee_1^{2g} S^2 \to \dots \tag{7.7}$$

where $f_*([S^1]) = [x_1, x_2][x_3, x_4]\dots[x_{2g-1}, x_{2g}] \in \pi_1(\vee_1^{2g} S^1) = \mathbf{F}\langle x_1, \dots, x_{2g}\rangle$. This gives the fibration sequence

$$\mathrm{Map}^*(S^2, S^3) \to \mathrm{Map}^*(M_g, S^3) \to \mathrm{Map}^*\left(\bigvee_1^{2g} S^1, S^3\right) \xrightarrow{f^*} \mathrm{Map}^*(S^1, S^3)$$

whence a principal fibering $\Omega^2 S^3 \to \mathrm{Map}^*(M_g, S^3) \to (\Omega S^3)^{2g}$ with classifying map

$$f^*: (\Omega S^3)^{2g} = \mathrm{Map}\left(\bigvee_{2g} S^1, S^3\right) \to \mathrm{Map}(S^1, S^3) = \Omega S^3.$$

To prove (c) we will show that f^* is null homotopic. Since S^3 is a group, f^* is given by

$$f^*: \mathrm{Map}\left(\bigvee_{2g} S^1, \Omega B S^3\right) \to \mathrm{Map}(S^1, \Omega B S^3)$$

which is equal to

$$\Sigma f^*: \mathrm{Map}\left(\bigvee_{2g} S^2, S^3\right) \to \mathrm{Map}(S^2, S^3)$$

where $\Sigma f: S^2 \to \bigvee_{2g} S^2$ is the suspension of f. But Σf is a commutator in an Abelian group, and is hence zero. □

Corollary 7.8. $\mathrm{Map}_n^*(M_g, S^2) \simeq X_g \times (\Omega S^3)^{2g}$ *where* X_g *is the total space of a fibration*

$$\Omega^2 S^3 \to X_g \to (S^1)^{2g}.$$

Proof. From (7.7) there is a fibering

(7.9) $$\Omega^2(S^2)_n \to \mathrm{Map}_n^*(M_g, S^2) \to (\Omega S^2)^{2g},$$

and from 7.6 we have the fibering

$$\Omega^2 S^3 \times (\Omega S^3)^{2g} \to \mathrm{Map}_n^*(M_g, S^2) \to (S^1)^{2g},$$

Of course $(\Omega^2 S^2)_n \simeq \Omega^2 S^3$, and $\Omega S^2 \simeq S^1 \times \Omega S^3$, so we can rewrite 7.9 as

$$\Omega^2 S^3 \to \mathrm{Map}^*(M_g, S^2) \to (S^1)^{2g} \times (\Omega S^3)^{2g}.$$

Now (7.9) is classified by $f^*: (\Omega S^2)^{2g} \to \Omega S^2$, which can, by the use of the second fibration, be seen to be homotopic to the composite

$$(S^1)^{2g} \times (\Omega S^3)^{2g} \to (S^1)^{2g} \to \Omega S^2.$$

7.8 follows. □

This completes our discussion of the space $\mathrm{Map}_n^*(M_g, S^2)$ with path components that are the limits of the based holomorphic maps from M_g to S^2. We now turn to the space $\mathrm{Map}^*_{\mathbf{Z}\times\mathbf{Z}}(M_g, \mathbf{CP}^\infty \vee \mathbf{CP}^\infty)$ with components approximated by the spaces of divisors $\mathrm{Div}_n(M_g)$.

We apply (7.7) again to obtain the fibration sequence

$$(7.10)\quad \Omega^2(\mathbf{CP}^\infty \vee \mathbf{CP}^\infty) \to \mathrm{Map}^*_{\mathbf{Z}\times\mathbf{Z}}(M_g, \mathbf{CP}^\infty \vee \mathbf{CP}^\infty) \to \Omega(\mathbf{CP}^\infty \vee \mathbf{CP}^\infty)^{2g} \xrightarrow{f^*} \Omega(\mathbf{CP}^\infty \vee \mathbf{CP}^\infty).$$

The key space here is $\Omega(\mathbf{CP}^\infty \vee \mathbf{CP}^\infty)$ and we have

LEMMA 7.11. $\Omega(\mathbf{CP}^\infty \vee \mathbf{CP}^\infty) \simeq S^1 \times S^1 \times \Omega S^3$. *There are generators in homology, $e_1, e_2 \in H_1(\Omega(\mathbf{CP}^\infty \vee \mathbf{CP}^\infty); \mathbf{Z})$, and, on these classes, the multiplication*

$$H_*(\Omega X; \mathbf{Z}) \otimes H_{*'}(\Omega X; \mathbf{Z}) \to H_{*+*'}(\Omega X; \mathbf{Z})$$

induced from the loop sum is given by $e_1^2 = e_2^2 = 0$, while $e_1 e_2 \neq e_2 e_1$. In fact $e_1 e_2 + e_2 e_1$ represents the class in the Hurewicz image of the generating sphere in $H_2(\Omega(\mathbf{CP}^\infty \vee \mathbf{CP}^\infty); \mathbf{Z})$ coming from $\pi_2(\Omega S^3)$. Note that this Hurewicz image gives a $\mathbf{Z}$ direct summand in homology.

Proof. From 7.3 there is a fibration sequence

$$\Omega S^3 \xrightarrow{j} \Omega(\mathbf{CP}^\infty \vee \mathbf{CP}^\infty) \xrightarrow{P} \Omega(\mathbf{CP}^\infty \times \mathbf{CP}^\infty) = S^1 \times S^1.$$

Let $e_1: S^1 \to \Omega(\mathbf{CP}^\infty \vee \mathbf{CP}^\infty)$ be chosen so that $\sigma e_1: S^2 \to \mathbf{CP}^\infty \vee \mathrm{pt} \subset \mathbf{CP}^\infty \vee \mathbf{CP}^\infty$ represents a generator for the first summand $\pi_2(\mathbf{CP}^\infty)$. Similarly, choose e_2 so that σe_2 represents a generator for the second summand. Using the loop sum map, $(*)$, we can lift P to a map $l = e_1 * e_2: S^1 \times S^1 \to \Omega(\mathbf{CP}^\infty \vee \mathbf{CP}^\infty)$, and, again using loop sum, we exend it to a map

$$l': \Omega S^3 \times S^1 \times S^1 \to \Omega(\mathbf{CP}^\infty \vee \mathbf{CP}^\infty) \times \Omega(\mathbf{CP}^\infty \vee \mathbf{CP}^\infty) \xrightarrow{*} \Omega(\mathbf{CP}^\infty \vee \mathbf{CP}^\infty).$$

This gives the desired homotopy equivalence.

Next, since the generator $\tau_3 \in \pi_3(\mathbf{CP}^\infty \vee \mathbf{CP}^\infty)$ is the Whitehead product $[\sigma e_1, \sigma e_2]$ it follows that the class of $\Omega(\tau_3)$ is given by the Samelson product of e_1 and e_2. Passing to homology, the second part of 7.11 follows. □

We now look more closely at the map f^* in (7.10). From the definition of f as the product of commutators $[x_1, x_2][x_3, x_4] \ldots [x_{2g-1}, x_{2g}]$ it follows that f^* is given as the following composition

$$\Omega(\mathbf{CP}^\infty \vee \mathbf{CP}^\infty)^{2g} \xrightarrow{\Delta} (\Omega(\mathbf{CP}^\infty \vee \mathbf{CP}^\infty)^{2g})^2 \xrightarrow{1\times\chi^{2g}} (\Omega(\mathbf{CP}^\infty \vee \mathbf{CP}^\infty)^{2g})^2$$

$$\xrightarrow{\text{shuff}} \Omega(\mathbf{CP}^\infty \vee \mathbf{CP}^\infty)^{4g} \xrightarrow{*^{4g}} \Omega(\mathbf{CP}^\infty \vee \mathbf{CP}^\infty)$$

where χ is the inverse map with respect to the loop sum $\chi(f)(t)=f(1-t)$, (cannonical anti-automorphism in homology) and

$$\begin{aligned}&\text{shuff}(v_1, \dots, v_{2g}, \chi(v_1), \dots, \chi(v_{2g}))\\ &= (v_1, v_2, \chi(v_1), \chi(v_2), v_3, v_4, \chi(v_3), \chi(v_4), \dots, v_{2g-1}, v_{2g}, \chi(v_{2g-1}), \chi(v_{2g})).\end{aligned}$$

COROLLARY 7.12. *The composite map*

$$(\Omega S^3)^{2g} \xrightarrow{j^{2g}} \Omega(\mathbf{CP}^\infty \vee \mathbf{CP}^\infty)^{2g} \xrightarrow{f^*} \Omega(\mathbf{CP}^\infty \vee \mathbf{CP}^\infty)$$

is homotopically trivial.

Proof. We can factor the composition in 7.12 as

$$\Omega(S^3)^{2g} \xrightarrow{\Delta^{2g}} (\Omega(S^3)^{2g})^2 \xrightarrow{1\times\chi^{2g}} (\Omega(S^3)^{2g})^2 \xrightarrow{\text{shuff}} (\Omega(S^3)^{2g})^2 \xrightarrow{*^{4g}} \Omega(S^3) \xrightarrow{\text{incl}} \Omega(\mathbf{CP}^\infty \vee \mathbf{CP}^\infty).$$

As we observed above, S^3 is a topological group so the loop sum on $\Omega(S^3)$ is homotopy commutative. Consequently we can replace shuff by the map

$$\text{shuff}': (v_1, \dots, v_{2g}, \chi(v_1), \dots, \chi(v_{2g})) \to (v_1, \chi(v_1), v_2, \chi(v_2), \dots, v_{2g}, \chi(v_{2g})),$$

and the composition of this map with $*^{4g}$ is homotopy trivial. □

We next analyze the map f^* on the piece $(S^1)^{4g} \subset \Omega(\mathbf{CP}^\infty \vee \mathbf{CP}^\infty)^{2g}$. To do this, note that it is just the loop sum of g copies of the composition for two factors,

$$(S^1 \times S^1)^2 \subset \Omega(\mathbf{CP}^\infty \vee \mathbf{CP}^\infty)^2 \xrightarrow{[x_1, x_2]^*} \Omega(\mathbf{CP}^\infty \vee \mathbf{CP}^\infty).$$

Let e_{11}, e_{12} represent the two generators in 7.11 for the first term, and e_{21}, e_{22} represent the corresponding generators for the second term. Now $\chi_*(e_{ij}) = -e_{ij}$ and, as $\chi_*(\alpha\circ\beta) = (-1)^{|\alpha||\beta|}\chi_*(\beta)\circ\chi_*(\alpha)$, we have that $\chi_*(e_{ij}e_{sk}) = -e_{sk}e_{ij}$. Thus, we must calculate the effect in homology of the composition

$$(S^1 \times S^1)^2 \xrightarrow{\Delta} ((S^1 \times S^1)^2)^2 \xrightarrow{\text{id}^2\times\chi^2} ((S^1 \times S^1)^2)^2 \xrightarrow{*} S^1 \times S^1 \times \Omega(S^3).$$

First recall that

(a) $\Delta(e_{ij})=e_{ij}\otimes 1+1\otimes e_{ij}$.

(b) $\Delta(e_{ij}\otimes e_{kl})=\Delta(e_{ij})\otimes\Delta(e_{kl})$.

Next, using these facts, 7.11, and the known effect of χ in homology, it is a direct calculation to verify that first $[x_1,x_2]^*(e_{ij})=0$ for all i,j. Second, in dimension two we have $[x_1,x_2]^*(e_{ij}\otimes e_{ik})=[x_1,x_2]^*(e_{ij}\otimes e_{kj})=0$, while

$$e_{11}\otimes e_{22}\mapsto e_1e_2+e_2e_1 \quad\text{and}\quad e_{12}\otimes e_{21}\mapsto e_1e_2+e_2e_1. \tag{7.13}$$

Furthermore $[x_1,x_2]^*$ is trivial on $H_3((S^1)^4;\mathbf{Z})$, however, in dimension 4 the map is non-trivial and we have

$$e_{11}e_{22}e_{12}e_{21}\to(e_1e_2+e_2e_1)^2 \tag{7.14}$$

where the squaring operation above is with respect to loop sum. This term generates a $\mathbf{Z}$-direct summand $\mathbf{Z}$ in $H_4(\Omega(\mathbf{CP}^\infty\vee\mathbf{CP}^\infty);\mathbf{Z})$ coming from $H_4(\Omega(S^3);\mathbf{Z})$. From this we have

COROLLARY 7.15. *The map $(S^1)^{4g}\to\Omega(\mathbf{CP}^\infty\vee\mathbf{CP}^\infty)$ obtained by restricting f^* to $(S^1)^{4g}$ satisfies the property that there is no (non-trivial) product decomposition $(S^1)^{4g}=A\times B$ so that f^* restricted to $(S^1)^{4g}$ is a composition*

$$A\times B\xrightarrow{p_2}B\xrightarrow{h}\Omega(\mathbf{CP}^\infty\vee\mathbf{CP}^\infty).$$

Additionally, f^* factors through a map $\phi\colon((\Omega S^3)\times(S^1)^2)^2\to\Omega S^3\hookrightarrow\Omega S^3\times(S^1)^2$.

Proof. From (7.14), the image of $H_{4g}((S^1)^{4g};\mathbf{Z})=\mathbf{Z}$ is generated by $(e_1e_2+e_2e_1)^{2g}$ where the product is induced in homology from the loop sum. But since $H_*(\Omega(S^3);\mathbf{Z})=\mathbf{Z}[p_2]$ is a polynomial algebra under loop sum, this class is non-zero and the first statement follows.

The only thing that remains to be checked is that the map

$$(S^1)^{4g}\to\Omega(\mathbf{CP}^\infty\vee\mathbf{CP}^\infty)=S^1\times S^1\times\Omega(S^3)$$

factors through $\Omega(S^3)$. But maps into $S^1\times S^1$, (which is the Eilenberg–MacLane space $K(\mathbf{Z}^2,1)$) are completely determined by the induced map in cohomology in dimension 1. Since we have already seen that the homology map in dimension 1 is zero, and there is no torsion in $H_*(\Omega(\mathbf{CP}^\infty\vee\mathbf{CP}^\infty);\mathbf{Z})$, the map in cohomology is just the dual map, on Hom groups. Consequently, it is also zero in dimension 1, so 7.15 follows.

It should be true that the map $f^*\colon (\Omega(\mathbf{CP}^\infty \vee \mathbf{CP}^\infty))^{2g} \to \Omega(\mathbf{CP}^\infty \vee \mathbf{CP}^\infty) = (S^1)^2 \times \Omega(S^3)$ factors as a composition

$$\Omega(S^3)^{2g} \times (S^1)^{4g} \xrightarrow{p_2} (S^1)^{4g} \xrightarrow{h} \Omega(S^3) \xrightarrow{\text{incl}} (S^1)^2 \times \Omega(S^3) = \Omega(\mathbf{CP}^\infty \vee \mathbf{CP}^\infty).$$

Consequently, there should be a decomposition

$$\mathrm{Map}^*_{\mathbf{Z}\times\mathbf{Z}}(M_g, \mathbf{CP}^\infty \vee \mathbf{CP}^\infty) \simeq (\mathbf{Z})^2 \times \Omega(S^3)^{2g} \times Y_g$$

where Y_g is the total space of a (principal) fibering

$$\Omega^2(S^3) \to Y_g \to (S^1)^{4g}.$$

However, the above argument does not quite prove this since the map on the terms of the product does not generally determine the map on the entire product space.

But in homology the map is, in fact, determined by the discussion above, since, rationally we have $H^*(\Omega(\mathbf{CP}^\infty \vee \mathbf{CP}^\infty); \mathbf{Q}) = E(e_1, e_2) \otimes \mathbf{Q}[h^*]$, and, consequently the cohomology map is determined. Dualizing and using the fact that $H_*(\Omega(\mathbf{CP}^\infty \vee \mathbf{CP}^\infty); \mathbf{Z})$ is torsion free, the cohomology map determines the map in homology as well.

§8. The rational homology of $\mathrm{Div}_k(M'_g)$ and Y_g

Segal proved that $H_*(\mathrm{Div}_k(M'_g); \mathbf{Z}) \hookrightarrow H_*(\mathrm{Div}_{k+1}(M'_g); \mathbf{Z})$ is a monomorphism for all k under the "collar" inclusion, and adjoining this fact to Segals' proposition 4.2, we have

$$\varinjlim \mathrm{Div}_k(M'_g) \simeq \mathrm{Map}^*_{0,0}(M^2_g, \mathbf{CP}^\infty \vee \mathbf{CP}^\infty).$$

Let $p \in H^2(\Omega(S^3); \mathbf{Z})$ be a generator. Then $p^*\colon \Omega(S^3) \to \mathbf{CP}^\infty$ defined by $p^*(\iota) = p$ is a rational homotopy equivalence so 7.10 and 7.15 give

COROLLARY 8.1. $\mathrm{Map}_{0,0}(M^2_g, \mathbf{CP}^\infty \vee \mathbf{CP}^\infty)$ *has the rational homotopy type of the product* $(\Omega S^3)^{2g} \times W_g$ *where* W_g *is the total space of the principal fibration*

$$S^1 \to W_g \to (S^1)^{4g} \tag{8.2}$$

with k-invariant $e^1 \cup e^2 + e^3 \cup e^4 + \cdots + e^{4g-1} \cup e^{4g}$.

Notice that W_g is a $K(\pi, 1)$ where π is a free nilpotent group of class 2. The Serre spectral sequence for (8.2) has only 2 rows, the 0-row and the 1-row, so only the d_2-

differential is non-trivial, and $H^*(W_g;\mathbf{Q})=H_*(E[e^1,\dots,e^{4g},f])$ with $\delta e^i=0$, $1\leqslant i\leqslant 4g$, and $\delta f=e^1e^2+e^3e^4+\dots+e^{4g-1}e^{4g}$. We can write this cochain complex schematically as

$$E[e^1,\dots,e^{4g}]f\to E[e^1,\dots,e^{4g}]$$

so

$$H^*(W_g;\mathbf{Q})=\Sigma^1\,\mathrm{Ker}(\cup(e^1e^2+\dots+e^{4g-1}e^{4g})) \oplus E[e^1,\dots,e^{4g}]/(E[e^1,\dots,e^{4g}](e^1e^2+\dots+e^{4g-1}e^{4g})). \tag{8.3}$$

This complex was studied in [BCM]. In particular it was proved there that

LEMMA 8.4. *The map*

$$\cup(e^1e^2+\dots+e^{4g-1}e^{4g})\colon E[e^1,\dots,e^{4g}]\to E[e^1,\dots,e^{4g}]$$

is injective in degrees $\leqslant 2g$, *and surjective in degrees* $\geqslant 2g$.

In particular this shows that $H_i(W_g;\mathbf{Q})=\mathbf{Q}^{v(i,g)}$ where $v(i,g)$ is given explicitly as

$$v(i,g)=\begin{cases}\binom{4g}{i}-\binom{4g}{i-2} & \text{for } i\leqslant 2g\\ \binom{4g}{i-1}-\binom{4g}{i+1} & \text{for } 2g<i\leqslant 4g+1\\ 0 & \text{for } i>4g+1.\end{cases} \tag{8.5}$$

For example, in the first few cases this gives the table

g \ i	0	1	2	3	4	5	6	7	8	9
1	1	4	5	5	4	1				
2	1	8	27	38	22	22	38	27	8	1
3	1	12	65	208	429	572	429	429	572	429
4	1	16	119	544	1 600	3 808	6 188	7 072	4 862	4 862

Of course the determination of the rational cohomology of the space $\mathrm{Div}_k(M_g')$ now is given by putting a bigrading on

$$H_*(W_g;\mathbf{Q})\times H_*((\Omega S^2)^{2g};\mathbf{Q})\simeq H_*(W_g;\mathbf{Q})\otimes\mathbf{Q}[h_1,\dots,h_{2g}].$$

But, using (4.7), this is easily accomplished. h_i has bidegree $(2,1)$, and the elements in $H_j(W_g;\mathbf{Q})$ have bidegree (j,j). Then, the rational homology of $\mathrm{Div}_k(M_g')$ is given by all the classes above with second degree $\leqslant k$.

We conclude this section by noting that the rational homology of W_g (see (8.5)) is the rational homology of the classical configuration spaces (equivalently, braid groups) of M'_g.

§9. A geometric bifiltration of $\mathrm{Div}_n(M'_g)$

In this section we construct a geometric bifiltration of $\mathrm{Div}_n(M'_g)$ where the quotients are "recognizable" smash products of Rat_l with products of spheres. This bifiltration closely corresponds to the homotopy decomposition of $\mathrm{Map}_*(M'_g, \mathbf{CP}^\infty \vee \mathbf{CP}^\infty)$ discussed in sections 7 and 8. It is not sharp however, in that the bifiltration has too many copies of these smash products to give $\mathrm{Div}_n(M'_g)$ without identifications, so in homology there must be considerable cancellation. This cancellation corresponds, in some sense, to the effect of amalgamating roots in our generic subcomplexes.

THEOREM 9.1. *There is a bifiltration by cofibrations*

$$\mathrm{Div}_n^{r,s}(M'_g) \hookrightarrow \mathrm{Div}_n(M'_g)$$

for $0 \leq r, s \leq n$ *satisfying the following properties:*

(1) $\mathrm{Div}_n^{0,0}(M'_g) = \mathrm{Rat}_n$.

(2) $\mathrm{Div}_n^{n,n}(M'_g) = \mathrm{Div}_n(M'_g)$.

(3) *There is a natural homotopy equivalence of the subquotients*

$$\mathrm{Div}_n^{r,s}(M'_g)/\mathrm{Div}_n^{r-1,s} \cup \mathrm{Div}_n^{r,s-1} \simeq (\mathrm{Rat}_{n-m})_+ \wedge \mathrm{Div}_{r,s}(M'_g, D^2)/\mathrm{Div}_{r-1,s} \cup \mathrm{Div}_{r,s-1}$$

where $m = \max(r, s)$.

(4) $\mathrm{Div}_{r,s}(M'_g - D^2)/\mathrm{Div}_{r-1,s} \cup \mathrm{Div}_{r,s-1}$ *is homotopy equivalent to a wedge of spheres of dimensions* $\leq r+s$.

Proof. Let $Y \subset X$ be a subspace. Recall that Segal defined

$$SP^\infty(X, Y) = SP^\infty(X/Y \sim *)$$

and

$$Q(X, Y) \subset SP^\infty(X, Y) \times SP^\infty(X, Y)$$

to be the subspace consisting of pairs (ξ, η) having disjoint coordinates. We define

$$\mathrm{Div}_{n,k}(X, Y)$$

to be the intersection of $Q(X, Y)$ with the image of

$$SP^n(X, Y)\times SP^k(X, Y)\subset SP^\infty(X, Y)\times SP^\infty(X, Y).$$

Of course, when $Y=*$, we recover the Div spaces defined in (4.2). As before, to simplify notation, set $\mathrm{Div}_n(X, Y)=\mathrm{Div}_{n,n}(X, Y)$ and $\mathrm{Div}_{n,k}(X)=\mathrm{Div}_{n,k}(X,\varnothing)$. Thus $\mathrm{Div}_n(D^2)=\mathrm{Div}_n(S^2-p_0)=\mathrm{Rat}_n$. For $0\leq r, s\leq n$ we define $\mathrm{Div}_n^{r,s}(M_g')\subset\mathrm{Div}_n(M_g')$ as follows.

Definition 9.2.

$$\mathrm{Div}_n^{r,s}(M_g')=\{(\xi,\eta)\in\mathrm{Div}_n(M_g') \text{ such that at least } n-r \text{ of the coordinates in } \xi \text{ and } n-s \text{ of the coordinates in } \eta \text{ lie in } \bar{D}^2\subset M_g'\}.$$

Notice that $\mathrm{Div}_n^{0,0}=\mathrm{Div}_n(\bar{D}^2)=\mathrm{Rat}_n$ and $\mathrm{Div}_n^{n,n}(M_g')=\mathrm{Div}_n(M_g')$. The following point set argument finishes the proof of the first two parts of Theorem 9.1.

LEMMA 9.3. *The natural inclusions*

$$\mathrm{Div}_n^{r-1,s}(M_g')\hookrightarrow\mathrm{Div}_n^{r,s}(M_g') \quad \textit{and} \quad \mathrm{Div}_n^{r,s-1}(M_g')\hookrightarrow\mathrm{Div}_n^{r,s}(M_g')$$

are cofibrations.

Proof. We will prove the first of these inclusions is a cofibration as the second case is obviously handled in exactly the same way. It suffices to construct a collar around $\mathrm{Div}_n^{r-1,s}(M_g')\subset\mathrm{Div}_n^{r,s}(M_g')$. To do so notice that $\mathrm{Div}_n^{r-1,s}(M_g')$ is a closed subspace of $\mathrm{Div}_n^{r,s}(M_g')$ and its boundary (i.e., the complement of its interior) is given by

$$\partial\mathrm{Div}_n^{r-1,s}(M_g')=\{(\xi,\eta)\in\mathrm{Div}_n^{r,s}(M_g') \text{ such that } \xi \text{ has at least} n, r+1 \text{ coordinates in } \bar{D}^2 \text{ but no more than } n-r \text{ coordinates in the interior of the disk } \mathrm{int}(D^2)\}.$$

Now let

$$J=\bar{D}^2\cap\mathscr{H}\subset M_g'$$

where $\mathscr{H}$ is the union of the $2g$ closed handles in M_g'. Thus J is the disjoint union of $2g$ closed intervals. Let $\nu_J\subset\mathscr{H}$ be a closed tubular neighborhood. Thus ν_J is a disjoint union of $2g$ closed disks. Consider the closed neighborhood of $\mathrm{Div}_n^{r-1,s}(M_g')$ in $\mathrm{Div}_n^{r,s}(M_g')$ given by

$$U_n^{r-1,s}=\{(\xi,\eta)\in\mathrm{Div}_n^{r,s}(M_g') \text{ such that at least } n-r+1 \text{ of the coordinates of } \xi \text{ lie in } \bar{D}^2\cup\nu_J\}.$$

Notice that there is a natural embedding

$$\mathrm{Div}_n^{r-1,s}(M_g') \cup \partial \mathrm{Div}_n^{r-1,s}(M_g') \times I \hookrightarrow U_n^{r-1,s}$$

thus making a collar around $\mathrm{Div}_n^{r-1,s}(M_g')$ in $\mathrm{Div}_n^{r,s}(M_g')$.

We now complete the proof of Theorem 9.1, part 3. A non-basepoint in

$$\mathrm{Div}_n^{r,s}(M_g')/\mathrm{Div}_n^{r-1,s} \cup \mathrm{Div}_n^{r,s-1}$$

has a unique representation as a pair $(\xi, \eta) \in \mathrm{Div}_n(M_g')$ where ξ and η have exactly $n-r$ and $n-s$ of their respective coordinates lying in $\bar{D}^2$. If ξ_{n-r} and η_{n-s} denote the unordered collection of these coordinates, then (ξ_{n-r}, η_{n-s}) is an element of $\mathrm{Div}_{n-r,n-s}(\bar{D}^2)$. Let $p(\xi, \eta) \in Q(M_g', \bar{D}^2)$ be the natural projection. Clearly $p(\xi, \eta)$ lies in $\mathrm{Div}_{r,s}(M_g', \bar{D}^2) \subset Q(M_g', \bar{D}^2)$. We define

$$h_{r,s}\colon \mathrm{Div}_n^{r,s}(M_g')/\mathrm{Div}_n^{r-1,s} \cup \mathrm{Div}_n^{r,s-1} \to (\mathrm{Div}_{n-r,n-s}(\bar{D}^2))_+ \wedge \mathrm{Div}_{r,s}(M_g' - \bar{D}^2)/\mathrm{Div}_{r-1,s} \cup \mathrm{Div}_{r,s-1}$$

by the formula

$$h_{r,s}(\xi, \eta) = (\xi_{n-r}, \eta_{n-s}) \times p(\xi, \eta).$$

We leave it to the reader to check that $h_{r,s}$ is well defined and continuous. To see that h_{r-s} is actually a homeomorphism consider the map

$$g_{r,s}\colon (\mathrm{Div}_{n-r,n-s}(\bar{D}^2))_+ \wedge \mathrm{Div}_{r,s}(M_g' - \bar{D}^2)/\mathrm{Div}_{r-1,s} \cup \mathrm{Div}_{r,s-1} \to \mathrm{Div}_n^{r,s}(M_g')/\mathrm{Div}_n^{r-1,s} \cup \mathrm{Div}_n^{r,s-1}$$

defined as follows. Represent a point in

$$(\mathrm{Div}_{n-r,n-s}(\bar{D}^2))_+ \wedge \mathrm{Div}_{r,s}(M_g' - \bar{D}^2)/\mathrm{Div}_{r-1,s} \cup \mathrm{Div}_{r,s-1}$$

by a pair $(\xi_{n-r}, \eta_{n-s}) \times (\xi_r, \eta_s) \in \mathrm{Div}_{n-r,n-s}(\bar{D}^2) \times \mathrm{Div}_{r,s}(M_g', \bar{D}^2)$. We define

$$g_{r,s}((\xi_{n-r}, \eta_{n-s}) \times (\xi_r, \eta_s)) = (\xi_{n-r}, \xi_r; \eta_{n-s}, \eta_s).$$

Again one can check that this formula yields a well defined, continuous map which is inverse to h_k. Part 3 of Theorem 9.1 then follows from an observation of Segal [Seg] that $\mathrm{Div}_{p,q}(\bar{D}^2) \simeq \mathrm{Rat}_m$ where $m = \min(p, q)$.

Next, recall that the complex $(\Pi_{2g} J_1(S^2)) \times (S^1)^{4g}$, where $J_1(S^2)$ is the James model for ΩS^3, which appeared in § 7 has a natural bifiltration. Let $F_k(J_1(S^2))$ denote the CW skeletal filtration, and give the torus $M_1 = S^1 \times S^1$ the bifiltration defined by

$$F_{0,0}(M_1)=*,$$

$$F_{1,0}(M_1)=S^1\times *,$$

$$F_{0,1}(M_1)=*\times S^1,$$

$$F_{1,1}(M_1)=S^1\times S^1=M_1$$

where $*$ denotes the appropriate basepoint. The bifiltration on $J(S^2)\times M_1$ is given by

$$F_{p,q}(J(S^2)\times M_1)=\bigcup_{r=0}^{m} F_k(J(S^2))\times F_{p-r,q-r}(M_1)$$

where $m=\min(p,q)$. This bifiltration induces the product bifiltration on $\Pi_{2g}(J_1(S^2)\times M_1)$.

Part 4 of Theorem 9.1 is a consequence of

PROPOSITION 9.5. *There is a homotopy equivalence*

$$\mathrm{Div}_{p,q}(M_g',\bar{D}^2)\simeq F_{p,q}\left(\prod_{2g}(J_1(S^2)\times M_1)\right).$$

Thus $\mathrm{Div}_{p,q}(M_g',\bar{D}^2)/\mathrm{Div}_{p-1,q}\cup\mathrm{Div}_{p,q-1}$ *is a wedge of spheres of dimensions* $\leqslant p+q$.

Proof. There is a homeomorphism of bifiltered spaces

$$\mathrm{Div}_\infty(M_g'-\bar{D}^2)\cong\prod_{2g}\mathrm{Div}_\infty(I^2,\partial_0 I^2)$$

where $I=[0,1]$ and $\partial_0 I^2=[0,1]\times\{0,1\}\in\partial I^2\subset I^2$. The $2g$ copies of I^2 in the above product decomposition corresponds to the $2g$ handles in M_g'. Thus, to prove Proposition 9.5 we are reduced to showing that there is a homotopy equivalence

$$\mathrm{Div}_{p,q}(I^2,\partial_0 I^2)\simeq F_{p,q}(J_1(S^2)\times M_1)$$

where the right hand side is the bifiltration of $J_1(S^2)\times M_1$ described above.

Segal [Seg, Proposition 3.2] showed

LEMMA 9.6. *There is a homotopy equivalence*

$$f\colon Q(I^2,\partial_0 I^2)\xrightarrow{\simeq}\Omega(\mathbf{CP}^\infty\vee\mathbf{CP}^\infty).$$

Recall from 7.3 that there is a homotopy fibration sequence $S^3\to\mathbf{CP}^\infty\vee\mathbf{C}^\infty\to$

$\mathbf{CP}^\infty\times\mathbf{CP}^\infty$. The inclusion $\mathbf{CP}^\infty\vee\mathbf{CP}^\infty\hookrightarrow\mathbf{CP}^\infty\times\mathbf{CP}^\infty$ induces an isomorphism on the second homotopy group $\pi_2(-)$ and hence

$$\Omega(\mathbf{CP}^\infty\vee\mathbf{CP}^\infty)\to\Omega(\mathbf{CP}^\infty\times\mathbf{CP}^\infty)\simeq S^1\times S^1=M_1$$

is an isomorphism on the level of π_1. This fact, and the loop space multiplication defines a section

$$\sigma\colon M_1\to\Omega(\mathbf{CP}^\infty\vee\mathbf{CP}^\infty)$$

which in turn defines a splitting

$$\Omega(\mathbf{CP}^\infty\vee\mathbf{CP}^\infty)\xrightarrow{\simeq}\Omega S^3\times M_1\xrightarrow{\simeq}J_1(S^2)\times M_1.$$

Thus, using 9.3 the $\mathrm{Div}_{p,q}(I^2,\partial_0 I^2)$ defines a bifiltration (up to homotopy) of $J_1(S^2)\times M_1$. We will now prove that it is the filtration claimed in Proposition 9.5. To do this we first describe a certain multiplicative structure on $Q(I^2,\partial_0 I^2)$ that is compatible with the loop space multiplication on $\Omega(\mathbf{CP}^\infty\vee\mathbf{CP}^\infty)$.

Let $I_0^2=[0,1/3]\times[0,1]$, and $I_1^2=[2/3,1]\times[0,1]$. Let $\partial_0 I_j^2$ be $[0,1/3]\times\{0,1\}$ and $[2/3,1]\times\{0,1\}$, respectively. The natural homeomorphisms

$$h_0\colon(I^2,\partial_0 I^2)\to(I_0^2,\partial_0 I_0^2)\quad\text{and}\quad h_1\colon(I^2,\partial_0 I^2)\to(I_1^2,\partial_0 I_1^2)$$

define homeomorphisms

$$h_0\colon\mathrm{Div}_{p,q}(I^2,\partial_0 I^2)\to\mathrm{Div}_{p,q}(I_0^2,\partial_0 I_0^2)$$

and

$$h_1\colon\mathrm{Div}_{p,q}(I^2,\partial_0 I^2)\to\mathrm{Div}_{p,q}(I_1^2,\partial_0 I_1^2)$$

There is a pairing

$$\mu\colon\mathrm{Div}_{p,q}(I^2,\partial_0 I^2)\times\mathrm{Div}_{r,s}(I^2,\partial_0 I^2)\to\mathrm{Div}_{p+q,r+s}(I^2,\partial_0 I^2)$$

defined by the composition

$$\mu\colon\mathrm{Div}_{p,q}(I^2,\partial_0 I^2)\times\mathrm{Div}_{r,s}(I^2,\partial_0 I^2)$$

$$\xrightarrow{h_0\times h_1}\mathrm{Div}_{p,q}(I_0^2,\partial_0 I_0^2)\times\mathrm{Div}_{r,s}(I_1^2,\partial_0 I_1^2)\hookrightarrow\mathrm{Div}_{p+q,r+s}(I^2,\partial_0 I^2)$$

where the last map is the natural inclusion. These pairings extend to an A_∞ operad action on $Q(I^2, \partial_0 I^2)$ in the sense of May [May] and Segal's equivalence $Q(I^2, \partial_0 I^2) \to \Omega(\mathbf{CP}^\infty \vee \mathbf{CP}^\infty)$ preserves this multiplicative structure. We will use this structure to define a filtration preserving equivalence

$$J_1(S^2) \times M_1 \to Q(I^2, \partial_0 I^2).$$

First of all, consider the map $\sigma: S^1 \times S^1 \to \mathrm{Div}_1(I^2, \partial_0 I^2)$ defined by the formula

$$\sigma(s, t) = ((1/3, s), (2/3, t)) \in \mathrm{Div}_{1,1}(I^2, \partial_0 I^2) \subset I^2/\partial_0 I^2 \times I^2/\partial_0 I^2.$$

Clearly the composition

$$S^1 \times S^1 \xrightarrow{\sigma} \mathrm{Div}_{1,1}(I^2, \partial_0 I^2) \hookrightarrow Q(I^2, \partial_0 I^2) \hookrightarrow SP^\infty(I^2, \partial_0 I^2) \times SP^\infty(I^2, \partial_0 I^2) \simeq S^1 \times S^1$$

is a homotopy equivalence, and so σ defines the section of $\Omega(\mathbf{CP}^\infty \vee \mathbf{CP}^\infty)$ discussed above. We now describe a map $f_n: F_n J_1(S^2) \to \mathrm{Div}_{n,n}(I^2, \partial_0 I^2)$ which combinatorially models the inclusion map $f: \Omega S^3 \to \Omega(\mathbf{CP}^\infty \vee \mathbf{CP}^\infty)$ described above. Notice that f factors as a composition

$$f: \Omega S^3 \xrightarrow{\Omega\eta} \Omega S^2 \xrightarrow{\Omega g} \Omega(\mathbf{CP}^\infty \vee \mathbf{CP}^\infty)$$

where $\eta: S^3 \to S^2$ is the Hopf map, and $g: S^2 \to \mathbf{CP}^\infty \vee \mathbf{CP}^\infty$ is the element of $\pi_2(\mathbf{CP}^\infty \vee \mathbf{CP}^\infty) \cong \mathbf{Z} \oplus \mathbf{Z}$ defined to be the sum of the generators of each of the factors $\pi_2(\mathbf{CP}^\infty)$. We give a combinatorial description of Ωg using the James model $J_1(S^1)$ for ΩS^2. Let $\alpha: S^1 \to \mathrm{Div}_1(I^2, \partial_0 I^2)$ be the composition

$$\alpha: S^1 \xrightarrow{\Delta} S^1 \times S^1 \xrightarrow{\sigma} \mathrm{Div}_{1,1}(I^2, \partial_0 I^2).$$

By the definition of σ it is immediate that when composed with the inclusion

$$\mathrm{Div}_{1,1}(I^2, \partial_0 I^2) \hookrightarrow Q(I^2, \partial_0 I^2) \simeq \Omega(\mathbf{CP}^\infty \vee \mathbf{CP}^\infty)$$

that α represents the class $g \in \pi_2(\mathbf{CP}^\infty \vee \mathbf{CP}^\infty) \cong \pi_1(\Omega(\mathbf{CP}^\infty \vee \mathbf{CP}^\infty))$. The H space structure of $Q(I^2, \partial_0 I^2)$ defined above then allows us to extend α to a map of H spaces

$$\alpha: J_1(S^1) \to Q(I^2, \partial_0 I^2)$$

which models the map $\Omega g: \Omega S^2 \to \Omega(\mathbf{CP}^\infty \vee \mathbf{CP}^\infty)$. Notice that α is a filtration preserving map taking the nth James filtration to $\mathrm{Div}_{n,n}(I^2, \partial_0 I^2)$:

$$\alpha_n: F_n(J_1(S^1)) \to \mathrm{Div}_{n,n}(I^2, \partial_0 I^2).$$

There exists a well known combinatorial model for the Hopf map $\Omega\eta: \Omega S^3 \to \Omega S^2$ given as a map $j: J_1(S^2) \to J_1(S^1)$ which doubles the James filtration; that is, given by maps $j_n: J_1^n(S^2) \to J_1^{2n}(S^1)$. Equivalently, it takes the CW skeletal filtration of $J_1(S^2)$ to the James filtration of $J_1(S^1)$; i.e., $j_n: F_n(J_1(S^2)) \to J_1^n(S^1)$. We define

$$f_n: F_n(J_1(S^2)) \to \mathrm{Div}_{n,n}(I^2, \partial_0 I^2)$$

to be the composition

$$f_n: F_n(J_1(S^2)) \xrightarrow{j_n} J_1^n(S^1) \xrightarrow{\alpha_n} \mathrm{Div}_{n,n}(I^2, \partial_0 I^2).$$

By construction this models the map $f: \Omega S^3 \to \Omega(\mathbf{CP}^\infty \vee \mathbf{CP}^\infty)$ described above. Now consider the map

$$h_{p,q}: F_{p,q}(J_1(S^2) \times M_1) \to \mathrm{Div}_{p,q}(I^2, \partial_0 D^2) \tag{9.7}$$

defined to be the union of the map

$$F_k(J_1(S^2)) \times F_{p-k,q-k}(M_1) \xrightarrow{f_k \times \sigma} \mathrm{Div}_{k,k}(I^2, \partial_0 I^2) \times \mathrm{Div}_{p-k,q-k}(I^2, \partial_0 I^2) \xrightarrow{\mu} \mathrm{Div}_{p,q}(I^2, \partial_0 I^2).$$

These maps give a combinatorial model for the splitting

$$\Omega S^3 \times M_1 \simeq \Omega(\mathbf{CP}^\infty \vee \mathbf{CP}^\infty).$$

Finally, Proposition 9.5 will be proved once we verify the following.

LEMMA 9.8. *The map*

$$h_{p,q}: F_{p,q}(J_1(S^2) \times M_1) \to \mathrm{Div}_{p,q}(I^2, \partial_0 I^2)$$

is a homotopy equivalence.

Proof. The classical James–Milnor stable splitting of $J_1(S^2)$ says that $J_1(S^2)$ splits as a wedge of the subquotients of the James filtration and, hence, of the skeletal filtration. This induces a canonical splitting of $J_1(S^2) \times M_1$. Therefore the composition

$$F_{p,q}(J_1(S^2)\times M_1)\xrightarrow{h_{p,q}}\mathrm{Div}_{p,q}(I^2,\partial_0 I^2)\hookrightarrow Q(I^2,\partial_0 I^2)\simeq\Omega(\mathbf{CP}^\infty\vee\mathbf{CP}^\infty)\simeq J_1(S^2)\times M_1$$

has a stable retraction and thus induces a monomorphism in homology. Hence $h_{p,q}$ induces a monomorphism in homology. Let $D_{p,q}$ be the subquotient

$$\mathrm{Div}_{p,q}=F_{p,q}(J_1(S^2)\times M_1)/F_{p-1,q}\cup F_{p,q-1}.$$

Thus $D_{p,q}$ is a wedge of spheres. Let

$$e_{p,q}\colon SP^\infty(D_{p,q})\to SP^\infty(F_{p,q}(J_1(S^2)\times M_1))$$

be the map induced by the James–Milnor splitting. By induction it is enough to show that the composition

$$SP^\infty(D_{p,q})\xrightarrow{e_{p,q}}SP^\infty(F_{p,q}(J_1(S^2)\times M_1))\xrightarrow{h_k}SP^\infty(\mathrm{Div}_{p,q}(I^2,\partial_0 I^2))$$

$$\xrightarrow{\mathrm{proj}}SP^\infty(\mathrm{Div}_{p,q}(I^2,\partial_0 I^2))/\mathrm{Div}_{p-1,q}\cup\mathrm{Div}_{p,q-1}$$

induces a monomorphism in homology. This is proved using the fact that there is a splitting map

$$\tau\colon \mathrm{Div}_{p,q}(I^2,\partial_0 I^2)\to SP^\infty(\mathrm{Div}_{p-1,q}(I^2,\partial_0 I^2)\cup\mathrm{Div}_{p,q-1}(I^2,\partial_0 I^2))$$

as in Segal [Seg]. We leave the details of this argument to the reader.

Remark 9.9. Of course, as seen in § 7, the cofibration sequences in Theorem 9.1 do not stably split. However, just as in the discussion preceeding 7.8, the trifiltration $\mathrm{Div}_n^{r,s}(M_g')$ of $\mathrm{Map}^*(M_g,\mathbf{CP}^\infty\vee\mathbf{CP}^\infty)$ stably lifts to a trifiltration $\widetilde{\mathrm{Div}}_n^{r,s}(M_g')$ of $\mathrm{Map}^*(M_g,S^3)$ where all the natural inclusions do stably split.

§ 10. The generic subspace of $\mathrm{Div}(M_g')$

In our discussion of Rat_k we used a generic subspace to gain control of $H_*(\mathrm{Rat}_k)$. In the case of $\mathrm{Div}_k(M_g')$ there is a similar subspace of generic points. In it we can see many of the structures inferred so far in our discussion—but much work remains to bring our understanding of these new spaces to the level of our understanding of the Rat_k's. We define these "non-singular" subspaces of $\mathrm{Div}_{n,k}(M_g')$ as follows.

Definition 10.1. $\Sigma_{n,k}(M'_g)$ is the subspace of $SP^n((M'_g)\times SP^k(M'_g)$ given by disjoint pairs (A, B) when A lies in $DP^n(M'_g)=F(M'_g, n)/\mathscr{S}_n$.

Since $SP^k(M'_g)$ is a quotient of $(M'_g)^k$, we write $\hat{\Sigma}_{n,k}(M'_g)$ for the pull-back in the diagram

$$\begin{array}{ccc} \hat{\Sigma}_{n,k}(M'_g) & \longrightarrow & DP^n(M'_g)\times(M'_g)^k \\ \downarrow & & \downarrow \\ \Sigma_{n,k}(M'_g) & \longrightarrow & DP^n(M'_g)\times SP^k(M'_g) \end{array}$$

Notice that there are fibrations

(1) $\pi:\Sigma_{n,k}(M'_g)\to DP^n(M'_g)$ with fibre $SP^k(M'_g-\{n\})$ where $\{n\}$ is a subset of M'_g having cardinality n and

(2) $\hat{\pi}:\hat{\Sigma}_{n,k}(M'_g)\to DP^n(M'_g)$ with fibre $(M'_g-\{n\})^k$.

Clearly there are cross-sections for π and $\hat{\pi}$ together with a stabilization map

$$\sigma:\Sigma_{n,k}(M'_g)\to\Sigma_{n,k+1}(M'_g)$$

which induces an isomorphism on π_1 if $k\geq 2$. We study the $K(G, 1)$'s given by $\Sigma_n(M'_g)=\bigcup_k\Sigma_{n,k}(M'_g)$ and $\hat{\Sigma}_{n,k}(M'_g)$.

PROPOSITION 10.2. (i) *There are maps* $I:\Sigma_n(M'_g)\to\Sigma_{n+1}(M'_g)$ *so that* $\Sigma_n(M'_g)$ *is stably equivalent to*

$$\bigvee_{1\leq i\leq n}\Sigma_i(M'_g)/\Sigma_{i-1}(M'_g)$$

(*where* $\Sigma_i(M'_g)/\Sigma_{i-1}(M'_g)$ *is the cofibre of* I).

(ii) *There are fibrations with cross-sections*

$$SP^\infty(M'_g-\{n\})\to\Sigma_n(M'_g)\to DP^n(M'_g)$$

and

$$(M'_g-\{n\})^k\to\Sigma_{n,k}(M'_g)\to DP^n(M'_g)$$

with monodromy given below. Restricting to $M'_g=\mathbf{R}^2$ *in the second fibration with* $k=1$, *the monodromy is Artin's faithful representation of the braid group* B_n *into* $\mathrm{Aut}(F_n)$, *the automorphism group of the free group on n letters* [Ar].

As the Hurewicz map $\pi_1(SP^\infty(M'_g-\{n\}))\to H_1(SP^\infty(M'_g-\{n\}))$ is an isomorphism, we shall compute the action of $\pi_1(DP^n(M'_g))$ on $H_1(SP^\infty(M'_g-\{n\}))$. The situation is slightly different for $\hat{\Sigma}_{n,k}(M'_g)$ as the fundamental group of $M'_g-\{n\}$ is non-Abelian. To fix notation, we first give generators for H_1 and π_1 of $M'_g-\{n\}$. (See figure 10.3.)

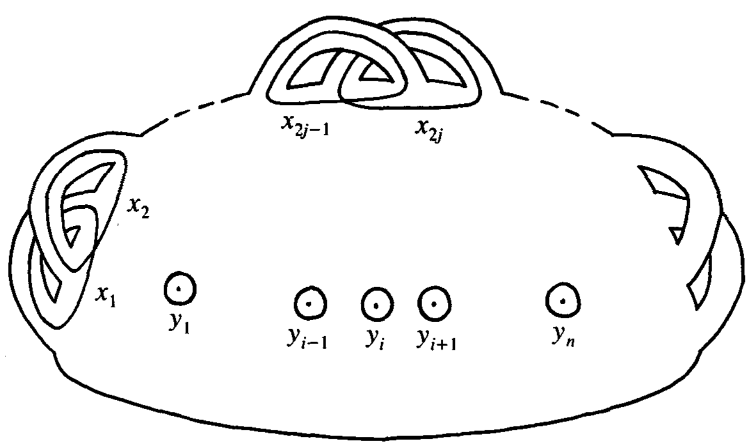

Fig. 10.3. $H_1(M'_g-\{n\};\mathbf{Z})$

A basis for $H_1(M'_g-\{n\};\mathbf{Z})$ is given by $x_1,\ldots,x_{2g},y_1,\ldots,y_n$ where x_i or y_i is the fundamental cycle of the embedded circle by the same name in the picture. (See figure 10.4.)

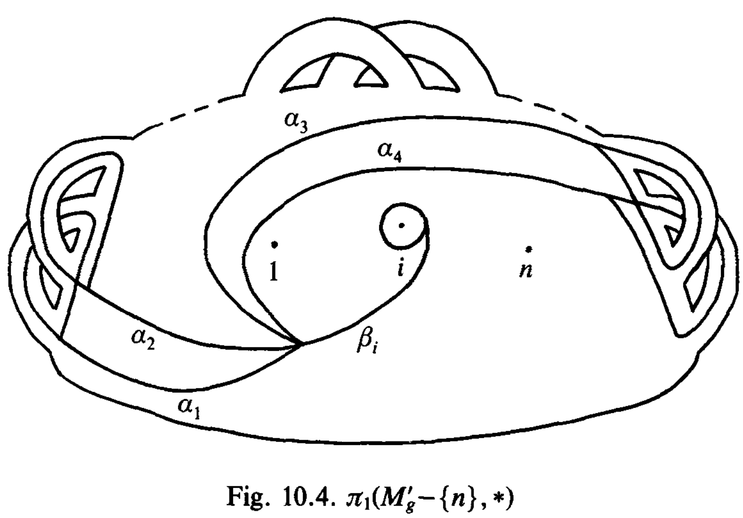

Fig. 10.4. $\pi_1(M'_g-\{n\},*)$

A basis for the free group $\pi_1(M'_g-\{n\},*)$ is represented by loops

$$\alpha_1,\dots,\alpha_{2g},\beta_1,\dots,\beta_n$$

based at $*$.

Next, we give generators for $\pi_1(DP^n(M'_g),*)$. They are named by

(1) $\sigma_1,\dots,\sigma_{n-1}$ and

(2) τ_{ij}, $1\leqslant i\leqslant n$, $1\leqslant j\leqslant 2g$.

"Pictures" are obtained as in figure 10.5.

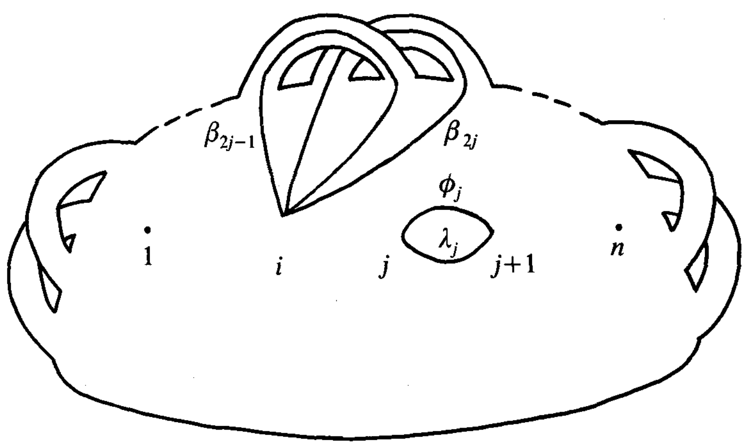

Fig. 10.5

To be precise, define

(i) $\tau_{ij}: S^1\to DP^n(M'_g)$ by $\tau_{ij}(\xi)=(1,2,\dots,i-1,\theta_j(\xi),i+1,\dots,n)$ where θ_j is an embedding of S^1 indicated in our picture and

(ii) $\sigma_j: S^1\to DP^n(M'_g)$ by $\sigma_j(\xi)=(1,2,\dots,j-1,\phi_j(\xi),\lambda_j(\xi),j+2,\dots,n)$ where ϕ_j and λ_j are the pictured embeddings of $[0,1]$.

PROPOSITION 10.6 [Sc]. *The maps* $\tau_{ij}, 1\leqslant i\leqslant n$, $1\leqslant j\leqslant 2g$ *and* σ_j, $1\leqslant j\leqslant n$, *represent generators of* $\pi_1(DP^n(M'_g),*)$ *where* $*=(1,2,\dots,n)$.

Proof. Exercise (from the fibrations $DP^n(M'_g)\to B\mathscr{S}_n$ and $F(M'_g-\{n\},j)\to M'_g-\{n\}$ with fibre $F(M'_g-\{n+1\},j-1)$).

To analyze the action of $\pi_1(DP^n(M'_g),*)$ consider the fibration

$$SP^\infty(M'_g)\to\Sigma_n(M'_g)\to DP^n(M'_g).$$

We shall construct an isotopy of M'_g which induces a homotopy commutative diagram

$$\text{(10.7)}\qquad \begin{array}{ccc} I\times SP^{\infty}(M'_g-\{n\}) & \xrightarrow{H} & \Sigma_n(M'_g) \\ \downarrow{\scriptstyle 1\times\text{constant}} & & \downarrow{\scriptstyle \pi} \\ I\times\{*\} & \xrightarrow{\alpha} & DP^n(M'_g) \end{array}$$

where α runs over τ_{ij} and σ_i.

Case I: σ_j. Consider the "half Dehn twist" G interchanging j and $j+1$ which is the identity outside of the outer annulus. (See figure 10.8.)

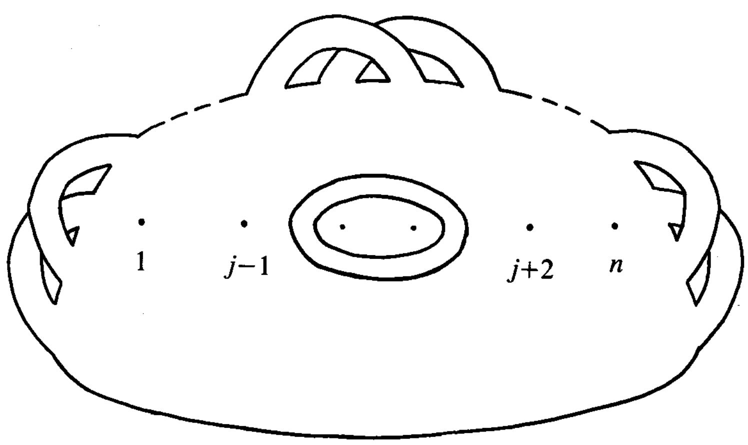

Fig. 10.8

Observe that there is a commutative diagram

$$\text{(10.9)}\qquad \begin{array}{ccc} I\times SP^{\infty}(M'_g-\{n\}) & \xrightarrow{H} & \Sigma_n(M'_g) \\ \downarrow{\scriptstyle 1\times\text{constant}} & & \downarrow{\scriptstyle \pi} \\ I\times\{*\} & \xrightarrow{\sigma_j} & F(M'_g, n) \end{array}$$

where $H(t,A)=((G(t,1), G(t,2), \ldots, G(t,n)), G(t,A))$. Thus there is a homeomorphism of $SP^{\infty}(M'_g-\{n\})$ given by sending $A=H(0,A)$ to $H(1,A)$. Notice that σ_j fixes $x_1, \ldots, x_{2g}$, and y_i for $i\neq j$ or $j+1$. In addition $\sigma_j(y_j)=y_{j+1}$ and $\sigma_j(y_{j+1})=y_j$.

Case II: τ_{ij}. Let G be the isotopy given by the annulus in figure 10.10.

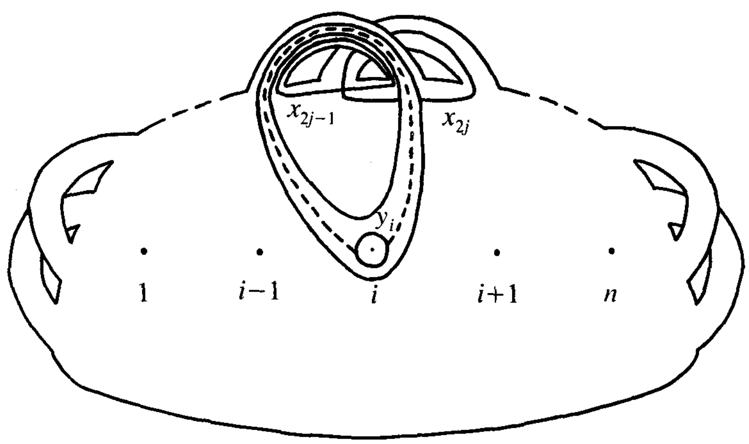

Fig. 10.10

Notice that $\tau_{i,j-1}$ fixes x_k if $k \neq 2j$ and $\tau_{i,j-1}(x_{2j}) = -x_{2j} + y_i$.

A similar picture, see figure 10.11, gives that $\tau_{i,2j}$ fixed y_i, $1 \leq i \leq n$, and x_k if $k \neq 2j-1$, while $\tau_{i,2j}(x_{i,2j-1}) = -x_{2j-1} + y_i$.

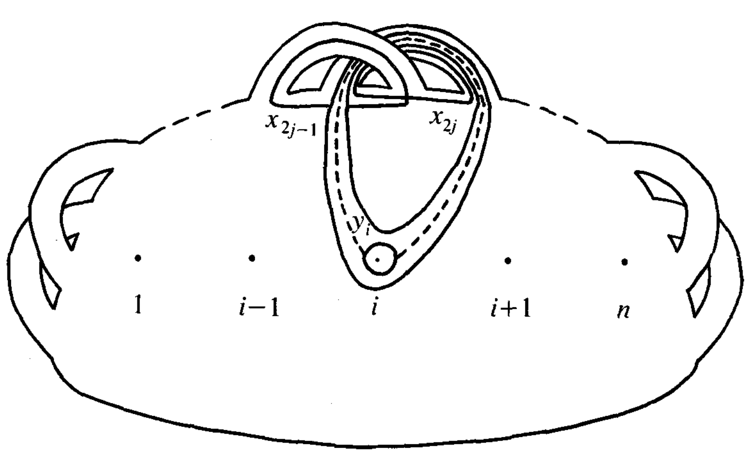

Fig. 10.11

We record the above observations.

PROPOSITION 10.12. (1) *σ_i fixes x_k, $1 \leq k \leq 2g$,*
(2) *$\sigma_i(y_i) = y_{i+1}$ and $\sigma_i(y_{i+1}) = y_i$,*
(3)

$$\tau_{i,2j-1}(x_k) = \begin{cases} x_k & \text{if } k \neq 2j \\ -x_{2j} + y_i & \text{if } k = 2j \end{cases}, \quad \tau_{i,2j}(x_k) = \begin{cases} x_k & \text{if } k \neq 2j-1 \\ -x_k + y_i & \text{if } k = 2j-1 \end{cases} \quad \text{and} \quad \tau_{i,j} \text{ fixes } y_i.$$

Finally, we restrict attention to the monodromy for $\hat{\Sigma}_{n,1}(M'_g) \to DP^n(M'_g)$ as the action obtained for $\hat{\Sigma}_{n,k}$ is the diagonal action on $(M'_g - \{n\})^k$. Thus it suffices to give the action of σ_i and τ_{ij} on α_k and β_k.

Case I: σ_j. Let G be the isotopy $I \times M'_g \to M'_g$ given in case I of the action of σ_i on $H_* SP^\infty(M'_g - \{n\})$ above. Define

$$\hat{H}: I \times M'_g - \{n\} \to \hat{\Sigma}_{n,1}(M'_g)$$

by $\hat{H}(t, m) = ((G(t, 1), \ldots, G(t, n)), G(t, m))$ (as $m \notin \{1, \ldots, n\}$, $\hat{H}$ is well-defined).

Notice that there is a commutative diagram

$$\begin{array}{ccc} I \times (M'_g - \{n\}) & \xrightarrow{\hat{H}} & \Sigma_{n,1}(M'_g) \\ \downarrow{\scriptstyle 1 \times \text{constant}} & & \downarrow{\scriptstyle \pi} \\ I \times \{*\} & \xrightarrow{\sigma_j} & DP^n(M'_g) \end{array}$$

and so there is a homeomorphism of $M'_g - \{n\}$ sending m to $G(1, m)$. Notice that the generators $\alpha_1, \ldots, \alpha_{2g}$ are fixed as is β_i, if $i \neq j$ or $j+1$. Finally σ_i sends β_i to β_{i+1} and β_{i+1} to $\beta_{i+1}^{-1} \beta_i \beta_{i+1}$ by inspecting the picture.

Remark 10.13. By the above restriction of the action of σ_i on $\pi_i(M'_g - \{n\})$ with $M'_g = \mathbf{R}^2$ we get Artin's faithful representation of $B_n \to \text{Aut}(F_n)$ where B_n is Artin's braid group and $\text{Aut}(F_n)$ is the automorphism group of the free group on n letters. This representation is, of course, quite useful and occurs widely in the literature [Ar], [Bi].

Case II: τ_{ij}. Consider the isotopy $G: I \times M'_g \to M'_g$ given in the analysis of the action of τ_{ij} on $H_1(SP^\infty(M'_g - \{n\}))$ given above. Again define $\hat{H}: I \times (M'_g - \{n\}) \to \hat{\Sigma}_{n,1}(M'_g)$ by $\hat{H}(t, m) = ((G(t, 1), \ldots, G(t, n)), G(t, m))$ to get a commutative diagram

$$\begin{array}{ccc} I \times (M'_g - \{n\}) & \xrightarrow{\hat{H}} & \hat{\Sigma}_{n,1}(M'_g) \\ \downarrow{\scriptstyle 1 \times \text{constant}} & & \downarrow{\scriptstyle \pi} \\ I \times \{*\} & \xrightarrow{\tau_{ij}} & DP^n(M'_g). \end{array}$$

The homeomorphism of $M'_g-\{n\}$ obtained by sending m to $\hat{H}(1,m)$ has the following effect on $\pi_1(M'_g-\{n\},*)$:

$$\tau_{ij}(\beta_k)=\begin{cases}\beta_k & \text{if } k\neq i\\ x_{ij}^{-1}\beta_i x_{ij} & \text{if } k=i\end{cases} \tag{1}$$

where $x_{ij}=(\beta_1\cdots\beta_{i-1})^{-1}(\alpha_j)(\beta_1\cdots\beta_{i-1})$,

$$\tau_{i,2j-1}(\alpha_k)=\begin{cases}\alpha_k & \text{if } k\leq 2j-2\\ y^{-1}\alpha_{2j-1}y & \text{if } k=2j-1\\ z^{-1}\alpha_{2j}\alpha_{2j-1}^{-1}\beta^{-1}\alpha_{2j-1}\beta & \text{if } k=2j\\ \lambda^{-1}\alpha_k\lambda & \text{if } k>2j \text{ with } \lambda=\alpha_{2j-1}^{-1}\beta^{-1}\alpha_{2j-1}\beta,\end{cases} \tag{2}$$

where $\beta=(\beta_1\cdots\beta_{i-1})(\beta_i)(\beta_1\cdots\beta_{i-1})^{-1}$ and

$$\tau_{i,2j}(\alpha_k)=\begin{cases}\alpha_k & \text{if } k<2j-1\\ \alpha_{2j-1}\beta_1 & \text{if } k=2j-1\\ \beta^{-1}\alpha_{2j}\beta & \text{if } k=2j\\ \omega^{-1}\alpha_k\omega & \text{if } k>2h \text{ with } \omega=\alpha_{2j}^{-1}\gamma^{-1}\alpha_{2j}\gamma \text{ where } \gamma=\beta_i(\beta_1\cdots\beta_{i-1})^{-1}\end{cases} \tag{3}$$

with β as above.

References

[And] Andreotti, A., On a theorem of Torelli. *Amer. J. Math.*, 80 (1958), 801–828.

[Ar] Artin, E., Theory of braids. *Ann. of Math.*, 48 (1947), 47–72.

[AJ] Atiyah, M. F., & Jones, J. D., Topological aspects of Yang–Mills theory. *Comm. Math. Phys.*, 61 (1978), 97–118.

[Bi] Birman, J. S., *Braids, links and mapping class groups*. Ann. of Math. Stud., 82 (1974), Princeton Univ. Press.

[Böd] Bödigheimer, C.-F., Gefärbte Konfigurationen: Modelle für die stabile Homotopie von Eilenberg–MacLane-Räumen. Dissertation, University of Heidelberg, 1984.

[BCT] Bödigheimer, C.-F., Cohen, F. R. & Taylor, L., On the homology of configuration spaces. To appear in *Topology*.

[BCM] Bödigheimer, C.-F., Cohen, F. R. & Milgram, R. J., On the spaces $TP^n(Y)$. To appear.

[B] Bogomol'nyi, E. B., The stability of classical solutions. *Soviet J. Nuclear Phys.*, 24 (1976), 449.

[BV] Boardman, J. M. & Vogt, R. M., Homotopy-everything H-spaces. *Bull. Amer. Math. Soc.*, 74 (1968), 1117–1122.

[BoMa] Boyer, C. P. & Mann, B. M., Monopoles, non-linear σ models, and two-fold loop spaces. *Comm. Math. Phys.*, 115 (1988), 571–594.

[Br1] Brockett, R. W., Some geometric questions in the theory of linear systems. *IEEE Trans. Automat. Control*, 21 (1976), 449–455.

[Br2] — The geometry of the set of controllable linear systems. *Res. Rep. Autom. Control Lab. Nagoya Univ.*, 24 (1977), 1–7.

[BG] Brown, E. H. & Gitler, S., A spectrum whose cohomology is a certain cyclic module over the Steenrod algebra. *Topology*, 12 (1973), 283–296.

[BP1] Brown, E. H. & Peterson, F. P., On the stable decomposition of $\Omega^2 S^{r+2}$. *Trans. Amer. Math. Soc.*, 243 (1978), 287–298.

[BP2] — Relations among characteristic classes II. *Ann. of Math.*, 81 (1965), 356–363.

[BP3] — A universal space for normal bundles of n-manifolds. *Comment. Math. Helv.*, 54 (1979), 405–430.

[BD] Byrnes, C. I. & Duncan, T., On certain topological invariants arising in system theory. *New Directions in Applied Math.*, Springer-Verlag, New York, 1981, pp. 29–71.

[Car] Cartan, H., Algebres d'Eilenberg–MacLane et homotopie. *Seminaire H. Cartan*, Paris, 1954/55.

[C] Cohen, F. R., The homology of $\mathscr{C}_{n+1}$ spaces. *Lecture Notes in Mathematics*, 533 (1976), 207–352, Springer-Verlag.

[CMT] Cohen, F. R., Taylor, L. & May, J. P., Splitting of certain spaces *CX*. *Math. Proc. Cambridge Philos Soc.*, 84 (1978) 465–496.

[CMM] Cohen, F. R., Mahowald, M. & Milgram, R. J., The stable decomposition of the double loop space of a sphere. *Algebraic and Geometric Topology, Proceedings of Symposia in Pure Mathematics*, 32 (2) (1978), 225–228.

[C²M²] Cohen, F. R., Cohen, R. L., Mann, B. M. & Milgram, R. J., Divisors and configurations on a surface. *Algebraic Topology, Conf. Math., A.M.S.*, 96 (1989), 103–108.

[C²M²2] — The homotopy type of rational functions. To appear in *Math. Z.*

[Co] Cohen, R. L., *Odd primary infinite families in stable homotopy theory*. Mem. Amer. Math. Soc., 242 (1981).

[C2] — The immersion conjecture for differentiable manifolds. *Ann. of Math.*, 122 (1985), 237–328.

[C3] — The homotopy theory of immersions. *Proc. ICM Warsaw*, 1983, 627–639.

[Del] Delchamps, D. F., Global structures of families of multivariable systems with an application to identification. *Math. Systems Theory*, 18 (1985), 329–380.

[DT] Dold, A. & Thom, R., Quasifaserungen und unendliche symmetrische Produkte. *Ann. of Math.*, 67 (1958), 239–281.

[D] Donaldson, S. K., Nahm's equations and the classification of monopoles. *Comm. Math. Phys.*, 96 (1984), 387–407.

[EW1] Eells, J. & Wood, J. C., Restrictions on harmonic maps of surfaces. *Topology*, 17 (1976), 263–266.

[EW2] — Harmonic maps from surfaces to complex projective spaces. *Adv. in Math.*, 49 (1983), 217–263.

[FaN] Fadell, E. & Neuwirth, L., Configuration spaces. *Math. Scand.*, 10 (1962), 111–118.

[FoN] Fox, R. H. & Neuwirth, L., The braid group. *Math. Scand.*, 10 (1962), 119–126.

[Fu] Fuks, D. B., Cohomologies of the braid groups mod 2. *Functional Anal. Appl.*, 4 (1970), 143–151.

[Ga] Ganea, T., A generalization of the homology and homotopy suspension. *Comment. Math. Helv.*, 39 (1965), 295–322.

[G] Guest, M., Topology of the space of absolute minima of the energy functional. *Amer. J. Math.*, 106 (1984), 21–42.

[He1] Helmke, U., The topology of a moduli space for linear dynamical systems. *Comment. Math. Helv.*, 60 (1985), 630–655.

[He2] — Topology of the moduli space for reachable linear dynamical systems: the complex case. *Math. Systems Theory*, 19 (1986), 155–187.

[H1] HITCHIN, N. J., Monopoles and geodesics. *Comm. Math. Phys.*, 83 (1982), 579–602.

[H2] — On the construction of monopoles. *Comm. Math. Phys.*, 89 (1983), 145–190.

[HM] HERMAN, R. & MARTIN, C. F., Applications of algebraic geometry to systems theory: The McMillan degree and Kronecker indices of transfer functions as topological and holomorphic systems invariants. *SIAM J. Control. Optim.*, 16 (1978), 743–755.

[Kai] KAILATH, T., *Linear Systems*. Prentice-Hall, 1980.

[K] KIRWAN, F. C., On spaces of maps from Riemann surfaces to Grassmannians and applications to the cohomology of moduli of vector bundles. *Ark. Mat.*, 24 (1986), 221–275.

[LM] LÖFFLER, P. & MILGRAM, R. J., The structure of deleted symmetric products. *Artin's Braid group and applications, A.M.S. Summer Institutes.*

[MdMi] MADSEN, IB & MILGRAM, R. J., On spherical fiber bundles and their *PL* reductions. *New Developments in Topology*, Cambridge University Press, 1974, 43–59.

[Mah] MAHOWALD, M., A new infinite family in ${}_2\pi^s_*$. *Topology*, 16 (1977), 249–256.

[MM] MANN, B. M. & MILGRAM, R. J., The topology of holomorphic maps from the Riemann sphere to complex Grassmann manifolds. To appear in *J. Differential Geom.*

[May] MAY, J. P., *The Geometry of Iterated Loop Spaces*. Lecture Notes in Mathematics, 271. Springer-Verlag, 1972.

[M] MILGRAM, R. J., The homology of symmetric products. *Trans. Amer. Math. Soc.*, 138 (1969), 251–265.

[M2] — Iterated loop spaces. *Ann. of Math.*, 84 (1966) 386–403.

[Mil] MILNOR, J., On spaces having the homotopy type of a *CW*-complex. *Trans. Amer. Math. Soc.*, 90 (1959), 272–280.

[N] NAHM, W., The algebraic geometry of multimonopoles. *Lecture Notes in Physics*, 180. Springer-Verlag, 1983, pp. 456–466.

[Sc] SCOTT, G. P., Braid groups and the group of homeomorphisms of a surface. *Math. Proc. Camb. Philos. Soc.*, 68 (1970), 605–617.

[Seg] SEGAL, G., The topology of rational functions. *Acta Math.*, 143 (1979), 39–72.

[Seg2] — Configuration spaces and iterated loop spaces. *Invent. Math.*, 21 (1973), 213–221.

[Sn] SNAITH, V. P., A stable decomposition of $\Omega^n\Sigma^n X$. *J. London Math. Soc* (2), 7 (1974), 577–583.

[T1] TAUBES, C. H., The existence of a non-minimal solution to the $SU(2)$ Yang–Mills–Higgs equations on $\mathbf{R}^3$; Part I, *Comm. Math. Phys.*, 86 (1982), 257–298; Part II, *Comm. Math. Phys.*, 86 (1982), 299.

[T2] — Monopoles and maps from S^2 to S^2; the topology of the configuration space. *Comm. Math. Phys.*, 95 (1984), 345–391.

[W] WOO, G., Pseudo-particle configurations in two-dimensional ferromagnets. *J. Math. Phys.*, 18 (1977), 1264.

Received June 6, 1989