THE PRODUCT OF n REAL HOMOGENEOUS LINEAR FORMS

By

C. A. ROGERS

of LONDON.

1. Let

$$x_r = \sum_{\sigma=1}^{n} a_{r\sigma} u_\sigma, \qquad r = 1, 2, \ldots, n$$

be n real homogeneous linear forms in the integral variables $u_1, \ldots, u_n$. Let $\mathfrak{M}(a_{rs})$ be the lower bound of the product $|x_1 \ldots x_n|$ for all sets of integers $u_1, \ldots, u_n$ other than $0, \ldots, 0$. Let $\mathfrak{M}_n$ be the upper bound of $\mathfrak{M}(a_{rs})$ for all sets of linear forms $x_1, \ldots, x_n$ with determinant 1. A well known result of Minkowski[1] implies that

$$\mathfrak{M}_n \leq \frac{n!}{n^n}. \tag{1}$$

Hence, if

$$m = \overline{\lim_{n \to \infty}} \{\mathfrak{M}_n\}^{1/n}, \tag{2}$$

we have

$$m \leq \frac{1}{e} = \frac{1}{2{\cdot}71828\ldots}. \tag{3}$$

The stronger inequalities

$$m \leq \frac{1}{\sqrt{\frac{3}{2} e \pi}} = \frac{1}{3{\cdot}57905\ldots}, \tag{4}$$

$$m \leq \frac{1}{3{\cdot}65931\ldots}, \tag{5}$$

$$m \leq \frac{1}{e\sqrt{e}} = \frac{1}{4{\cdot}48168\ldots}, \tag{6}$$

[1] Minkowski proved in *Geometrie der Zahlen*, (Leipzig, 1910), § 40 that one can find integers $u_1, \ldots, u_n$, not all zero, satisfying $|x_1| + \cdots + |x_n| \leq (n!)^{1/n}$ and (1) follows from this by the inequality of the arithmetic and geometric means.

have been obtained successively by Blichfeldt[1], Rankin[2] and myself[3].

But I have recently noticed that the inequality

$$\mathfrak{m} \le \frac{1}{\sqrt{2\pi e\sqrt{e}}} = \frac{1}{5{\cdot}30653\ldots} \tag{7}$$

is implicit in a paper by Blichfeldt[4] on the minimum value of the discriminant of a totally real algebraic field.

The main object of this paper is to obtain a new upper bound for $\mathfrak{M}_n$ and to prove that

$$\mathfrak{m} \le \frac{\pi}{4e\sqrt{e}} = \frac{1}{5{\cdot}70626\ldots}. \tag{8}$$

The method of proof is based on my method for proving (6) which is itself based on Blichfeldt's method of proving (4).

As Minkowski[5] pointed out, if D_n is the least discriminant of any totally real algebraic field of degree n, then

$$D_n \ge \frac{1}{\mathfrak{M}_n^2}.$$

Hence, if

$$d = \varliminf_{n\to\infty} \{D_n\}^{1/n},$$

we have

$$d \ge \frac{1}{\mathfrak{m}^2}.$$

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[1] H. F. Blichfeldt, *Monatshefte für Math. und Phys.*, 43 (1936), 410–414.

[2] R. A. Rankin, *Proc. Kon. Ned. Akad. v. Wet., Amsterdam*, 51 (1948), 846–853 (848).

[3] C. A. Rogers, *Journal London Math. Soc.*, 24 (1949), 31—39.

[4] H. F. Blichfeldt, *Monatshefte für Math. und Phys.*, 48 (1939), 531–533. Blichfeldt considers the linear forms

$$w_{k1}x_1 + \cdots + w_{kn}x_n, \quad k = 1, \ldots, n,$$

of determinant Δ, where $w_{11}, \ldots, w_{1n}$ is a basis of a totally real algebraic field of discriminant $D = \Delta^2$ and $w_{k1}, \ldots, w_{kn}$, $k = 2, \ldots, n$ are conugate bases of the conjugate fields. But Blichfeldt proves, without use of his assumption concerning the nature of $w_{11}, \ldots, w_{nn}$, that for any integer $m > 1$ there are integers $u_1, \ldots, u_n$ not all zero such that

$$\Pi_{st} = \prod_{k=1}^{n} (w_{k1}u_1 + \cdots + w_{kn}u_n)^2 \le \Delta^2 \left[\frac{1}{\pi n(m-1)}\right]^n (m-1)^2 [1 \cdot 2^2 \cdot 3^3 \cdot \ldots \cdot m^m]^{\frac{2n}{m(m-1)}} [\Gamma(1+\tfrac{1}{2}n)]^2$$

Taking $m = [n \log n]$ and letting n tend to infinity one obtains (7).

[5] H. Minkowski, *Geometrie der Zahlen*, (1910), § 42.

Thus, by our result (8),

$$d \geq \frac{16e^3}{\pi^2} = 32{\cdot}561\ldots. \tag{9}$$

This inequality is better than the result

$$d \geq 2\pi e\sqrt{e} = 28{\cdot}159\ldots$$

of Blichfeldt[1] corresponding to (7).

I am grateful to Mr. L. A. Wigglesworth for finding for me the function $g(x)$ satisfying the integral equation (18) and playing a central part in this work. I am also grateful to Professor Davenport for a number of useful suggestions.

2. My proof of (6) was based on the following lemma.

Lemma. *If* $m > 1$ *and*

$$z_1 < z_2 < \ldots < z_m, \tag{10}$$

then

$$\left\{ \prod_{1 \leq \varrho < \sigma \leq m}^{\varrho, \sigma} |z_\sigma - z_\varrho| \right\}^{2/\{m(m-1)\}} \leq \frac{k_m}{m} \sum_{\varrho=1}^{m} |z_\varrho|, \tag{11}$$

where

$$k_m = \frac{m}{\frac{1}{2}\sqrt{e}(m - \frac{1}{2} - \log 2)}. \tag{12}$$

In this paper we prove the stronger inequality[2] obtained from (11) by replacing k_m by

$$\varkappa_m = \frac{\pi}{2\sqrt{e}} \left(\frac{e^3 \pi^2 m(m-1)^2}{16} \right)^{1/(2m-2)}. \tag{13}$$

The simple ideas on which the proof is based are obscured by the detailed calculations which will be found in the next section. In the present section we give a brief explanation of the ideas behind the proof.

It is convenient to introduce the following convention. In any sum where the variables of summation and their ranges of variation are not stated in the usual way, the sum will be taken over all the sets of values of all the Greek suffixes and superfixes occurring explicitly in the summand, which satisfy the conditions stated under the sigma and for which the summand is defined. A similar convention will be used for products.

[1] H. F. Blichfeldt, (1939), *loc. cit.*

[2] This result should be compared with the result obtained by C. L. Siegel (*Annals of Math.*, 46(1945), 302–312) for the case when $0 < z_1 < z_2 < \ldots < z_m$. The proofs are quite different.

Let K_m be the maximum of

$$\frac{\left\{\prod_{\varrho<\sigma} |z_\sigma - z_\varrho|\right\}^{2/\{m(m-1)\}}}{\frac{1}{m}\sum |z_\varrho|} \tag{14}$$

for all numbers $z_1, \ldots, z_m$ satisfying (10). By use of the transformation

$$\alpha_r = \tfrac{1}{2}(z_r - z_{m-r+1})$$

we show (see Lemma 1) that K_m is the maximum of (14) for all numbers $z_1, \ldots, z_m$ satisfying both (10) and

$$z_r = -z_{m-r+1}\,, \qquad r = 1, \ldots, m\,.$$

Now we are primarily interested in K_m for large values of m. But, if m is large, it is reasonable to suppose that K_m is close to the upper bound K of

$$\frac{\exp\left\{\int_0^1 dx \int_0^1 \log|\alpha(x) - \alpha(\xi)|\,d\xi\right\}}{\int_0^1 |\alpha(x)|\,dx}, \tag{15}$$

for all strictly increasing functions $\alpha(x)$ satisfying

$$\alpha(x) = -\alpha(1-x)\,, \quad \text{for} \quad 0 \leq x \leq 1\,.$$

In fact we only show (see Lemmas 1 and 2) that

$$K_m \leq \left(\tfrac{1}{4}e^4 m(m-1)^2\right)^{1/(2m-2)} K^{m/(m-1)}\,. \tag{16}$$

Writing

$$\delta(x) = -\alpha(\tfrac{1}{2}x),$$

it is easy to show that K is the upper bound of

$$\frac{\exp\left\{\tfrac{1}{2}\int_0^1 dx \int_0^1 \log\,|(\delta(x))^2 - (\delta(\xi))^2|\,d\xi\right\}}{\int_0^1 \delta(x)\,dx} \tag{17}$$

for all decreasing functions $\delta(x)$ with $\delta(1) \geq 0$. By a straightforward application of the calculus of variations it is not difficult to show that the expression (17) assumes its maximum K when $\delta(x) = g(x)$, where $g(x)$ is the solution of the integral equation

$$\int_0^1 \frac{2g(x)}{(g(x))^2 - (g(\xi))^2}\,d\xi = \pi, \quad \text{for} \quad 0 < x < 1\,. \tag{18}$$

Once the function $g(x)$ has been determined there is little difficulty in proving that

$$K = \frac{\pi}{2\sqrt{e}}. \tag{19}$$

The inequality obtained from (11) by replacing k_m by $\varkappa_m$ follows from the definition of K_m and (16) and (19).

3. **Lemma 1.** *For any real numbers* $z_1, \ldots, z_m$, *which do not all vanish,*

$$\frac{\left\{\prod\limits_{\varrho<\sigma} |z_\sigma - z_\varrho|\right\}^{2/\{m(m-1)\}}}{\frac{1}{m}\sum |z_\varrho|} \leq \frac{\left\{\prod\limits_{\varrho<\sigma} (\alpha_\sigma - \alpha_\varrho)\right\}^{2/\{m(m-1)\}}}{\frac{1}{m}\sum |\alpha_\varrho|}, \tag{20}$$

for certain real numbers $\alpha_1, \ldots, \alpha_m$ *satisfying*

$$\alpha_1 < \alpha_2 < \ldots < \alpha_m, \tag{21}$$

$$\alpha_r + \alpha_{m-r+1} = 0, \quad \textit{for } r = 1, \ldots, m, \tag{22}$$

and

$$\left\{\frac{1}{m-1}\sum_{\varrho \neq r} \frac{1}{\alpha_r - \alpha_\varrho}\right\}\left\{\frac{1}{m}\sum |\alpha_\varrho|\right\} = \tfrac{1}{2}\operatorname{sgn}\alpha_r, \quad \textit{for} \quad r = 1, \ldots, m. \tag{23}$$

Proof. It is clear that we may suppose without loss of generality that

$$z_1 \leq z_2 \leq \ldots \leq z_m. \tag{24}$$

Write

$$f(z_1, \ldots, z_m) = \frac{\left\{\prod\limits_{\varrho<\sigma} |z_\sigma - z_\varrho|\right\}^{2/\{m(m-1)\}}}{\frac{1}{m}\sum |z_\varrho|}, \tag{25}$$

Then $f(z_1, \ldots, z_m)$ is a homogeneous function of degree 0 in $z_1, \ldots, z_m$, which is a continuous function of $z_1, \ldots, z_m$ except when $z_1 = z_2 = \ldots = z_m = 0$. Thus, as $z_1, \ldots, z_m$ vary subject to the condition (24) and the condition

$$\sum |z_\varrho| = m,$$

the function $f(z_1, \ldots, z_m)$ will assume its maximum value for some numbers $\zeta_1, \ldots, \zeta_m$. Clearly

$$\zeta_1 < \zeta_2 < \ldots < \zeta_m \tag{26}$$

and by the homogeneity of $f(z_1, \ldots, z_m)$

$$f(z_1, \ldots, z_m) \leq f(\zeta_1, \ldots, \zeta_m) \tag{27}$$

for all values of $z_1, \ldots, z_m$, which are not all zero.

Write

$$\alpha_r = \tfrac{1}{2}(\zeta_r - \zeta_{m-r+1}), \quad \text{for} \quad r = 1, \ldots, m\,.$$

Then, using (26),

$$\alpha_1 < \alpha_2 < \ldots < \alpha_m\,, \tag{28}$$

$$\alpha_r + \alpha_{m-r+1} = 0\,, \quad \text{for} \quad r = 1, \ldots, m\,, \tag{29}$$

and

$$\sum |\alpha_\varrho| \leq \sum |\zeta_\varrho|\,. \tag{30}$$

Further, since

$$0 < \zeta_s - \zeta_r = \alpha_s - \alpha_r + \tfrac{1}{2}(\zeta_s - \zeta_r + \zeta_{m-s+1} - \zeta_{m-r+1})\,,$$

$$0 < \zeta_{m-r+1} - \zeta_{m-s+1} = \alpha_s - \alpha_r - \tfrac{1}{2}(\zeta_s - \zeta_r + \zeta_{m-s+1} - \zeta_{m-r+1})\,,$$

we have

$$0 < (\zeta_s - \zeta_r)(\zeta_{m-r+1} - \zeta_{m-s+1}) \leq (\alpha_s - \alpha_r)^2\,,$$

for all integers r, s with $1 \leq r < s \leq m$. Consequently

$$\prod_{\varrho<\sigma} (\zeta_\sigma - \zeta_\varrho)^2 = \prod_{\varrho<\sigma} (\zeta_\sigma - \zeta_\varrho)(\zeta_{m-\varrho+1} - \zeta_{m-\sigma+1}) \leq \prod_{\varrho<\sigma} (\alpha_\sigma - \alpha_\varrho)^2\,. \tag{31}$$

Now by (30) and (31)

$$f(\zeta_1, \ldots, \zeta_m) \leq f(\alpha_1, \ldots, \alpha_m)\,.$$

Hence by (27)

$$f(z_1, \ldots, z_m) \leq f(\alpha_1, \ldots, \alpha_m) \tag{32}$$

for all values of $z_1, \ldots, z_m$, which are not all zero. Consequently

$$\frac{\partial}{\partial \alpha_r} \log f(\alpha_1, \ldots, \alpha_m) = 0$$

for $r = 1, \ldots, m$, provided $\alpha_r \neq 0$. Thus

$$\frac{2}{m(m-1)} \sum_{\varrho \neq r} \frac{1}{\alpha_r - \alpha_\varrho} - \frac{\operatorname{sgn} \alpha_r}{\sum |\alpha_\varrho|} = 0$$

for $r = 1, \ldots, m$, provided $\alpha_r \neq 0$. Hence (23) is satisfied provided $\alpha_r \neq 0$. But, by (28) and (29), it is clear that $\alpha_r = 0$ if and only if $2r = m+1$, and in this case it is clear from (29) that (23) is still satisfied provided we take sgn $0 = 0$. We have now proved that the numbers $\alpha_1, \ldots, \alpha_m$ satisfy all the conditions in the enunciation of the lemma.

Lemma 2. *If $\alpha_1, \ldots, \alpha_m$ are numbers satisfying*

$$\alpha_1 < \alpha_2 < \ldots < \alpha_m \,, \tag{33}$$

$$\alpha_r + \alpha_{m-r+1} = 0 \,, \quad \textit{for} \quad r = 1, \ldots, m \,, \tag{34}$$

and

$$\left\{\frac{1}{m-1} \sum_{\varrho \neq r}' \frac{1}{\alpha_r - \alpha_\varrho}\right\} \left\{\frac{1}{m} \sum' |\alpha_\varrho|\right\} = \tfrac{1}{2} \operatorname{sgn} \alpha_r, \quad \textit{for} \quad r = 1, \ldots, m \,, \tag{35}$$

then

$$\frac{\left\{\prod_{\varrho<\sigma} (\alpha_\sigma - \alpha_\varrho)\right\}^{2/\{m(m-1)\}}}{\frac{1}{m} \sum' |\alpha_\varrho|} \leq \left(\tfrac{1}{4} e^4 m (m-1)^2\right)^{1/(2m-2)} \left\{\frac{\exp\left\{\int_0^1 dx \int_0^1 \log|\alpha(x) - \alpha(\xi)|\, d\xi\right\}}{\int_0^1 |\alpha(x)|\, dx}\right\}^{m/(m-1)}, \tag{36}$$

where $\alpha(x)$ is a certain bounded strictly increasing function defined for $0 \leq x \leq 1$ and such that

$$\alpha(x) + \alpha(1-x) = 0 \,, \tag{37}$$

if $0 \leq x \leq 1$.

Proof. It is clear that we may suppose without loss of generality that

$$\sum' |\alpha_\varrho| = m \,. \tag{38}$$

We first use (34), (35) and (38) to evaluate the sum

$$\sum_{\varrho<\sigma} \frac{1}{(\alpha_\sigma - \alpha_\varrho)^2} = \frac{1}{2} \sum_{\varrho \neq \sigma} \frac{1}{(\alpha_\sigma - \alpha_\varrho)^2}$$

$$= \frac{1}{2} \sum_{\substack{\varrho \neq \sigma \\ \varrho \neq \tau}} \frac{1}{(\alpha_\varrho - \alpha_\sigma)(\alpha_\varrho - \alpha_\tau)} - \frac{1}{2} \sum_{\substack{\sigma \neq \tau \\ \tau \neq \varrho \\ \varrho \neq \sigma}} \frac{1}{(\alpha_\varrho - \alpha_\sigma)(\alpha_\varrho - \alpha_\tau)}$$

$$= \frac{1}{2} \sum_{r=1}^{m} \left(\sum_{\sigma \neq r} \frac{1}{\alpha_r - \alpha_\sigma}\right) \left(\sum_{\tau \neq r} \frac{1}{\alpha_r - \alpha_\tau}\right)$$

$$- \frac{1}{6} \sum_{\substack{\sigma \neq \tau \\ \tau \neq \varrho \\ \varrho \neq \sigma}} \left\{\frac{1}{(\alpha_\varrho - \alpha_\sigma)(\alpha_\varrho - \alpha_\tau)} + \frac{1}{(\alpha_\sigma - \alpha_\tau)(\alpha_\sigma - \alpha_\varrho)} + \frac{1}{(\alpha_\tau - \alpha_\varrho)(\alpha_\tau - \alpha_\sigma)}\right\}$$

$$= \frac{1}{2} \sum_{r=1}^{m} \left\{\frac{1}{2}(m-1) \operatorname{sgn} \alpha_r\right\}^2 + \frac{1}{6} \sum_{\substack{\sigma \neq \tau \\ \tau \neq \varrho \\ \varrho \neq \sigma}} \frac{(\alpha_\sigma - \alpha_\tau) + (\alpha_\tau - \alpha_\varrho) + (\alpha_\varrho - \alpha_\sigma)}{(\alpha_\sigma - \alpha_\tau)(\alpha_\tau - \alpha_\varrho)(\alpha_\varrho - \alpha_\sigma)}$$

$$= \frac{1}{8}(m-1)^2 \sum \{\operatorname{sgn} \alpha_\varrho\}^2$$

$$= \begin{cases} \frac{1}{8}(m-1)^3\,, & \text{if } m \text{ is odd,} \\ \frac{1}{8}m(m-1)^2\,, & \text{if } m \text{ is even} \end{cases} \tag{39}$$

Write

$$\eta = \frac{2}{(m-1)\sqrt{m}}. \tag{40}$$

Then by (39)

$$\sum_{\varrho<\sigma} \frac{1}{(\alpha_\sigma-\alpha_\varrho)^2} \leq \frac{1}{2\eta^2}, \tag{41}$$

so that

$$\alpha_{r+1}-\alpha_r \geq \sqrt{2}\eta\,, \quad \text{for} \quad r = 1,\ldots, m-1\,. \tag{42}$$

Now define the function $\alpha(x)$ by the conditions

$$\alpha(0) = \alpha_1 - \tfrac{1}{2}\eta\,, \qquad \alpha(1) = \alpha_m + \tfrac{1}{2}\eta\,,$$

$$\alpha\left(\frac{r}{m}\right) = \tfrac{1}{2}(\alpha_r+\alpha_{r+1})\,, \quad \text{for} \quad r = 1,\ldots, m-1\,, \tag{43}$$

and

$$\alpha(x) = \alpha_r + (mx-r+\tfrac{1}{2})\eta\,, \quad \text{if } \frac{r-1}{m} < x < \frac{r}{m}, \tag{44}$$

for $r = 1,\ldots, m$. Then, by (42) and (34), $\alpha(x)$ is uniquely defined for $0 \leq x \leq 1$ and is in this interval a bounded strictly increasing function of x satisfying

$$\alpha(x)+\alpha(1-x) = 0\,, \quad \text{for} \quad 0 \leq x \leq 1\,. \tag{45}$$

If m is even it is clear from (42) and (44) that

$$\int_0^1 |\alpha(x)|dx = \sum \int_{-1/2m}^{1/2m} |\alpha_\varrho + m\eta\xi|d\xi = \frac{1}{m}\sum |\alpha_\varrho| = 1\,.$$

If m is odd, then $m \geq 3$ and

$$\int_0^1 |\alpha(x)|dx = \sum \int_{-1/2m}^{1/2m} |\alpha_\varrho + m\eta\xi|d\xi = \frac{1}{m}\sum |\alpha_\varrho| + \int_{-1/2m}^{1/2m} |m\eta\xi|d\xi = 1+\frac{\eta}{4m},$$

so that

$$\log \int_0^1 |\alpha(x)|dx = \log\left(1+\frac{\eta}{4m}\right) = \log\left(1+\frac{1}{2m(m-1)\sqrt{m}}\right) < \frac{1}{2m(m-1)\sqrt{m}} \leq \frac{1}{4m\sqrt{3}} < \frac{1}{6m}.$$

Thus, in any case,

$$\log \int_0^1 |\alpha(x)| dx < \frac{1}{6m}. \tag{46}$$

Now, using (44),

$$\begin{aligned}
&\sum_{\varrho<\sigma} \log (\alpha_\sigma - \alpha_\varrho) - m^2 \int_0^1 dx \int_x^1 \log \{\alpha(\xi) - \alpha(x)\} d\xi \\
&= \sum_{\varrho<\sigma} \log (\alpha_\sigma - \alpha_\varrho) - m^2 \sum_{\varrho=1}^{m} \int_{(\varrho-1)/m}^{\varrho/m} dx \int_x^1 \log \{\alpha(\xi) - \alpha(x)\} d\xi \\
&= - \sum_{\varrho<\sigma} m^2 \int_{(\varrho-1)/m}^{\varrho/m} dx \int_{(\sigma-1)/m}^{\sigma/m} \log \frac{\alpha(\xi) - \alpha(x)}{\alpha_\sigma - \alpha_\varrho} d\xi \\
&\quad - \sum_{\varrho=1}^{m} m^2 \int_{(\varrho-1)/m}^{\varrho/m} dx \int_x^{\varrho/m} \log \{\alpha(\xi) - \alpha(x)\} d\xi \\
&= - \sum_{\varrho<\sigma} \int_0^1 dx \int_0^1 \log \left\{1 + \frac{\eta(\xi - x)}{\alpha_\sigma - \alpha_\varrho}\right\} d\xi \\
&\quad - m \int_0^1 dx \int_x^1 \log \eta(\xi - x) d\xi \\
&= - \sum_{\varrho<\sigma} \int_0^1 dx \int_0^x \log \left\{1 - \frac{\eta^2 (x-\xi)^2}{(\alpha_\sigma - \alpha_\varrho)^2}\right\} d\xi \\
&\quad - \tfrac{1}{2} m \log \eta - m \int_0^1 dx \int_x^1 \log (\xi - x) d\xi \, .
\end{aligned} \tag{47}$$

But, if $0 \leq x \leq \frac{1}{2}$,

$$-\log (1-x) \leq \frac{x}{1-x} \leq 2x \, .$$

Hence, using (42) and (41) and writing $x - \xi = X$, $x + \xi = Y$,

$$\begin{aligned}
&- \sum_{\varrho<\sigma} \int_0^1 dx \int_0^x \log \left\{1 - \frac{\eta^2 (x-\xi)^2}{(\alpha_\sigma - \alpha_\varrho)^2}\right\} d\xi \\
&\leq \sum_{\varrho<\sigma} \int_0^1 dx \int_0^x \frac{2\eta^2 (x-\xi)^2}{(\alpha_\sigma - \alpha_\varrho)^2} d\xi \\
&\leq \int_0^1 dx \int_0^x (x-\xi)^2 d\xi = \tfrac{1}{2} \int_0^1 dX \int_X^{2-X} X^2 dY = \tfrac{1}{12} \leq \tfrac{1}{24} m \, .
\end{aligned} \tag{48}$$

Also, writing $\xi-x=X$, $\xi+x=Y$,

$$\int_0^1 dx \int_x^1 \log (\xi-x) d\xi = \tfrac{1}{2}\int_0^1 dX \int_X^{2-X} \log X\, dY = -\tfrac{3}{4}. \tag{49}$$

Thus, combining (48), (40) and (49) with (47),

$$\begin{aligned}
&\sum_{\varrho<\sigma} \log (\alpha_\sigma-\alpha_\varrho) - m^2 \int_0^1 dx \int_x^1 \log \{\alpha(\xi)-\alpha(x)\} d\xi \\
&\leq \tfrac{1}{24} m - \tfrac{1}{2} m \log \frac{2}{(m-1)\sqrt{m}} + \tfrac{3}{4} m \\
&= \tfrac{19}{24} m + \tfrac{1}{4} m \log \tfrac{1}{4} m(m-1)^2 .
\end{aligned} \tag{50}$$

Finally by (50), (38) and (46)

$$\begin{aligned}
&\frac{2}{m(m-1)} \sum_{\varrho<\sigma} \log (\alpha_\sigma-\alpha_\varrho) - \log \frac{1}{m} \sum |\alpha_\varrho| \\
&- \frac{m}{m-1} \int_0^1 dx \int_0^1 \log |\alpha(x)-\alpha(\xi)| d\xi + \frac{m}{m-1} \log \int_0^1 |\alpha(x)| dx \\
&\leq \frac{1}{m-1} \left(\tfrac{19}{12} + \tfrac{1}{2} \log \tfrac{1}{4} m(m-1)^2 + \tfrac{1}{6}\right) \\
&= \frac{1}{2m-2} \left(\tfrac{7}{2} + \log \tfrac{1}{4} m(m-1)^2\right) \\
&< \frac{1}{2m-2} \left(4 + \log \tfrac{1}{4} m(m-1)^2\right),
\end{aligned}$$

so that (36) is satisfied. This proves the lemma.

Before we determine the upper bound, for all integrable functions $\alpha(x)$, of the expression

$$\frac{\exp \left\{ \int_0^1 dx \int_0^1 \log |\alpha(x)-\alpha(\xi)| d\xi \right\}}{\int_0^1 |\alpha(x)| dx}, \tag{51}$$

occurring in the right hand side of (36), we investigate the properties of Wigglesworth's function $g(x)$. We shall see eventually that the expression (51) attains its upper bound when $\alpha(x)$ is in a certain special relationship to $g(x)$.

Lemma 3. *There is a strictly decreasing function $g(x)$, defined for $0 \leq x \leq 1$, and such that*[1]

(a)
$$\int_0^1 \frac{2g(x)}{g^2(x)-g^2(\xi)}\,d\xi = \pi\,, \quad \textit{for } 0 < x < 1\,, \tag{52}$$

(b)
$$\int_0^1 g(x)\,dx = \frac{1}{\pi}, \tag{53}$$

and

(c)
$$\int_0^1 dx \int_0^1 \log|g^2(x)-g^2(\xi)|\,d\xi = -\log 4e\,. \tag{54}$$

Further, if $0 < x < 1$,

(d)
$$\left.\begin{aligned} \tfrac{1}{8}(1-x)^2 &< g(x) < 1\,, \\ g'(x) &< -\tfrac{1}{10}\sqrt{1-x}\,, \\ g''(x) &> 0\,. \end{aligned}\right\} \tag{55}$$

Proof. For all g with $0 \leq g \leq 1$ write
$$f(g) = \cos^{-1} g - g\cosh^{-1}\frac{1}{g},$$
where
$$0 \leq \cos^{-1} g \leq \tfrac{1}{2}\pi\,, \quad 0 \leq \cosh^{-1}\frac{1}{g} \leq +\infty\,.$$
Then
$$f(0) = \tfrac{1}{2}\pi, \quad f(1) = 0$$
and
$$f'(g) = -\cosh^{-1}\frac{1}{g}.$$
Thus, if $0 \leq x \leq 1$, the equation
$$f(g) = \tfrac{1}{2}\pi x$$
has a unique solution for g satisfying $0 \leq g \leq 1$; we use $g(x)$ to denote this solution. We prove that this function $g(x)$ defined for $0 \leq x \leq 1$ satisfies the requirements of the lemma. Clearly $g(x)$ is a bounded continuous strictly decreasing function of x for $0 \leq x \leq 1$ and
$$\frac{1}{g'(x)} = -\frac{2}{\pi}\cosh^{-1}\frac{1}{g(x)}\,, \quad \text{for} \quad 0 < x < 1\,. \tag{56}$$
Note that $g(0) = 1$ and $g(1) = 0$.

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[1] We work throughout with the principal values of our integrals; we use $g^2(x)$ to denote $(g(x))^2$ and $g'(x)$ to denote the derivative of $g(x)$.

Now, if $0 < x < 1$ and $g = g(x) \leq \frac{1}{8}(1-x)^2$, then

$$\begin{aligned} f(g) &= \cos^{-1} g - g \cosh^{-1}\frac{1}{g} \\ &\geq \cos^{-1} \tfrac{1}{8}(1-x)^2 - \tfrac{1}{8}(1-x)^2 \cosh^{-1}\frac{8}{(1-x)^2} \\ &> \frac{\pi}{2}\left\{1-\tfrac{1}{8}(1-x)^2\right\} - \tfrac{1}{8}(1-x)^2 \log\frac{16}{(1-x)^2} \\ &> \tfrac{1}{2}\pi - \tfrac{1}{4}(1-x) - \tfrac{1}{8}(1-x)^2 \cdot 2\frac{4}{1-x} \\ &= \tfrac{1}{2}\pi x + (1-x)(\tfrac{1}{2}\pi - \tfrac{1}{4} - 1) > \tfrac{1}{2}\pi x\,. \end{aligned}$$

This is contrary to the definition of $g(x)$, and so we have

$$\tfrac{1}{8}(1-x)^2 < g(x) < 1\,, \quad \text{for} \quad 0 < x < 1\,.$$

Using this result in (56), if $0 < x < 1$,

$$\begin{aligned} -g'(x) &= \frac{\tfrac{1}{2}\pi}{\cosh^{-1}\dfrac{1}{g(x)}} > \frac{\tfrac{1}{2}\pi}{\log\dfrac{2}{g(x)}} > \frac{\tfrac{1}{2}\pi}{\log\dfrac{16}{(1-x)^2}} \\ &= \frac{\pi}{8\log\sqrt{\dfrac{4}{1-x}}} > \frac{\pi}{8}\sqrt{\frac{1-x}{4}} > \tfrac{1}{10}\sqrt{1-x}\,. \end{aligned}$$

We have now proved the first two of the inequalities (55). The last of these inequalities can be obtained immediately by differentiating (56).

We now prove that $g(x)$ satisfies the integral equation (52). Write

$$\operatorname{sech}\Theta = g(x)\,, \quad \operatorname{sech}\theta = g(\xi)\,, \tag{57}$$

so that by (56)

$$\frac{d\xi}{d\theta} = \frac{dg(\xi)}{d\theta}\Big/\frac{dg(\xi)}{d\xi} = \frac{2}{\pi}\,\theta \operatorname{sech}\theta \tanh\theta\,.$$

Hence

$$\int_0^1 \frac{2g(x)}{g^2(x)-g^2(\xi)}\,d\xi = \int_0^\infty \frac{4}{\pi}\,\frac{\operatorname{sech}\Theta\cdot\theta\operatorname{sech}\theta\tanh\theta}{\operatorname{sech}^2\Theta-\operatorname{sech}^2\theta}\,d\theta$$

$$=\frac{4}{\pi}\int_0^\infty \frac{\theta\sinh\theta\cosh\Theta}{\cosh^2\theta-\cosh^2\Theta}\,d\theta$$

$$=\frac{2}{\pi}\int_0^\infty \frac{\theta\{\sinh(\theta+\Theta)+\sinh(\theta-\Theta)\}}{\sinh(\theta+\Theta)\sinh(\theta-\Theta)}\,d\theta$$

$$=\frac{2}{\pi}\int_0^\infty \left\{\frac{\theta}{\sinh(\Theta+\theta)}-\frac{\theta}{\sinh(\Theta-\theta)}\right\}d\theta$$

$$=\frac{2}{\pi}\int_{-\infty}^\infty \frac{\theta}{\sinh(\Theta+\theta)}\,d\theta$$

$$=\frac{2}{\pi}\int_{-\infty}^\infty \frac{\theta}{\sinh\theta}\,d\theta$$

$$=\pi\,. \tag{58}$$

Also, by the substitution (57) and integration by parts,

$$\int_0^1 g(\xi)d\xi = \frac{2}{\pi}\int_0^\infty \theta\operatorname{sech}^2\theta\tanh\theta\,d\theta$$

$$=\frac{1}{\pi}\int_0^\infty \operatorname{sech}^2\theta\,d\theta=\frac{1}{\pi}. \tag{59}$$

We have now to evaluate the double integral (54). Writing $g(x)=y$, $g(\xi)=\eta$, integrating by parts and using the substitutions $y=\sin\Phi=\operatorname{sech}\Theta$, $\eta=\sin\varphi=\operatorname{sech}\theta$, we have

$$\int_0^1 \log|g^2(x)-g^2(\xi)|\,d\xi=\frac{2}{\pi}\int_0^1 \cosh^{-1}\frac{1}{\eta}\log|y^2-\eta^2|\,d\eta$$

$$=\frac{2}{\pi}\int_0^1 \frac{1}{\eta\sqrt{1-\eta^2}}\{(\eta+y)\log|\eta+y|+(\eta-y)\log|\eta-y|-2\eta\}\,d\eta$$

$$=\frac{2}{\pi}\int_0^1 \frac{1}{\sqrt{1-\eta^2}}\{\log|\eta^2-y^2|-2\}\,d\eta+\frac{2y}{\pi}\int_0^1\frac{1}{\eta\sqrt{1-\eta^2}}\log\left|\frac{\eta+y}{\eta-y}\right|d\eta$$

$$=\frac{2}{\pi}\int_0^{\pi/2}\{\log|\sin^2\varphi-\sin^2\Phi|-2\}\,d\varphi+\frac{2g(x)}{\pi}\int_0^\infty \log\left|\frac{\operatorname{sech}\theta+\operatorname{sech}\Theta}{\operatorname{sech}\theta-\operatorname{sech}\Theta}\right|d\theta\,. \tag{60}$$

Now

$$\int_0^{\pi/2} \log|\sin^2\varphi - \sin^2\Phi|d\varphi = \int_0^{\pi/2} \log|\sin(\varphi+\Phi)\sin(\varphi-\Phi)|d\varphi$$

$$= \tfrac{1}{2}\int_0^{\pi/2} \{\log \sin^2(\Phi+\varphi) + \log\sin^2(\Phi-\varphi)\}\, d\varphi$$

$$= \tfrac{1}{2}\int_0^{\pi} \log\sin^2\varphi d\varphi = -\pi\log 2\,. \tag{61}$$

But also, integrating by parts and using (58),

$$\int_0^\infty \log\left|\frac{\operatorname{sech}\theta + \operatorname{sech}\Theta}{\operatorname{sech}\theta - \operatorname{sech}\Theta}\right| d\theta = \int_0^\infty \frac{2\theta \operatorname{sech}\theta \tanh\theta \operatorname{sech}\Theta}{\operatorname{sech}^2\Theta - \operatorname{sech}^2\theta}\, d\theta = \tfrac{1}{2}\pi^2\,. \tag{62}$$

Substituting from (61) and (62) into (60),

$$\int_0^1 \log|g^2(x) - g^2(\xi)|\, d\xi = -2\log 2 - 2 + \pi g(x)\,.$$

Integrating this result with respect to x and using (59) we have

$$\int_0^1 dx \int_0^1 \log|g^2(x) - g^2(\xi)|d\xi = -\log 4 - 2 + \pi\int_0^1 g(x)dx = -\log 4e\,. \tag{63}$$

We have now proved that $g(x)$ satisfies all the requirements of the lemma.

Lemma 4. *Suppose $\alpha(x)$ is a bounded strictly increasing function satisfying*

$$\alpha(x) + \alpha(1-x) = 0$$

for $0 \le x \le 1$. *Then*

$$\frac{\exp\left\{\int_0^1 dx \int_0^1 \log|\alpha(x) - \alpha(\xi)|\, d\xi\right\}}{\int_0^1 |\alpha(x)|dx} \le \frac{\pi}{2\sqrt{e}}\,. \tag{64}$$

Proof. Write $\delta(x) = -\alpha(\tfrac{1}{2}x)$. Then $\delta(x) + \delta(2-x) = 0$ for $0 \le x \le 2$ and

$$\int_0^1 |\alpha(x)|\, dx = \tfrac{1}{2}\int_0^2 |\delta(x)|\, dx = \int_0^1 \delta(x) dx\,. \tag{65}$$

Further

$$\int_0^1 dx \int_0^1 \log |\alpha(x)-\alpha(\xi)| d\xi$$

$$= \tfrac{1}{4}\int_0^2 dx \int_0^2 \log |\delta(x)-\delta(\xi)| d\xi$$

$$= \tfrac{1}{4}\left\{\int_0^1 dx \int_0^1 + \int_0^1 dx \int_1^2 + \int_1^2 dx \int_0^1 + \int_1^2 dx \int_1^2\right\} \log |\delta(x)-\delta(\xi)| d\xi$$

$$= \tfrac{1}{4}\int_0^1 dx \int_0^1 \log |\delta^2(x)-\delta^2(\xi)|^2 d\xi$$

$$= \tfrac{1}{2}\int_0^1 dx \int_0^1 \log |\delta^2(x)-\delta^2(\xi)| d\xi\,. \tag{66}$$

Thus, by (65) and (66),

$$\frac{\exp\left\{\int_0^1 dx \int_0^1 \log |\alpha(x)-\alpha(\xi)| d\xi\right\}}{\int_0^1 |\alpha(x)| dx} = \frac{\exp\left\{\tfrac{1}{2}\int_0^1 dx \int_0^1 \log |\delta^2(x)-\delta^2(\xi)| d\xi\right\}}{\int_0^1 \delta(x) dx}. \tag{67}$$

We use $g(x)$ to denote the function of Lemma 3. By using first the inequality of the arithmetic and geometric means and then its integral analogue, we obtain

$$\tfrac{1}{2}\int_0^1 dx \int_0^1 \log |\delta^2(x)-\delta^2(\xi)| d\xi - \tfrac{1}{2}\int_0^1 dx \int_0^1 \log |g^2(x)-g^2(\xi)| d\xi$$

$$= \int_0^1 dx \int_0^1 \tfrac{1}{2}\left[\log\left\{\frac{\delta(x)+\delta(\xi)}{g(x)+g(\xi)}\right\} + \log\left\{\frac{\delta(x)-\delta(\xi)}{g(x)-g(\xi)}\right\}\right] d\xi$$

$$\le \int_0^1 dx \int_0^1 \log \tfrac{1}{2}\left\{\frac{\delta(x)+\delta(\xi)}{g(x)+g(\xi)} + \frac{\delta(x)-\delta(\xi)}{g(x)-g(\xi)}\right\} d\xi$$

$$\le \log\left[\int_0^1 dx \int_0^1 \tfrac{1}{2}\left\{\frac{\delta(x)+\delta(\xi)}{g(x)+g(\xi)} + \frac{\delta(x)-\delta(\xi)}{g(x)-g(\xi)}\right\} d\xi\right]$$

$$= \log\left[\int_0^1 dx \int_0^1 \frac{\delta(x)g(x)-\delta(\xi)g(\xi)}{g^2(x)-g^2(\xi)} d\xi\right]. \tag{68}$$

We now establish the formula

$$\int_0^1 dx \int_0^1 \frac{\delta(x)g(x)-\delta(\xi)g(\xi)}{g^2(x)-g^2(\xi)} d\xi = \int_0^1 dx \int_0^1 \frac{2\delta(x)g(x)}{g^2(x)-g^2(\xi)} d\xi\,. \tag{69}$$

Owing to the singularity in the integrand the necessary change in the order of the integrations needs a special justification. As the integrand of the following integral is positive, if $x \neq \xi$, we have

$$\begin{aligned}
&\int_0^1 dx \int_0^1 \frac{\delta(x)g(x)-\delta(\xi)g(\xi)}{g^2(x)-g^2(\xi)}\,d\xi \\
&\qquad = \lim_{\varepsilon\to+0}\int_{0\le x\le 1} dx \int_{\substack{0\le\xi\le 1\\ |\xi-x|\ge\varepsilon}} \frac{\delta(x)g(x)-\delta(\xi)g(\xi)}{g^2(x)-g^2(\xi)}\,d\xi \\
&\qquad = \lim_{\varepsilon\to+0}\int_{0\le x\le 1} dx \int_{\substack{0\le\xi\le 1\\ |\xi-x|\ge\varepsilon}} \frac{2\delta(x)g(x)}{g^2(x)-g^2(\xi)}\,d\xi \\
&\qquad = \lim_{\varepsilon\to+0}\left\{\int_0^1 dx\int_0^1 - \int_0^1 dx\int_{\max(0,\,x-\varepsilon)}^{\min(1,\,x+\varepsilon)}\right\}\frac{2\delta(x)g(x)}{g^2(x)-g^2(\xi)}\,d\xi \\
&\qquad = \int_0^1 dx\int_0^1 \frac{2\delta(x)g(x)}{g^2(x)-g^2(\xi)}\,d\xi
\end{aligned}$$

$$-\lim_{\varepsilon\to+0}\int_0^1\left[\left\{\int_{\max(0,\,x-\varepsilon)}^{\max(0,\,x-\varepsilon,\,2x-1)} + \int_{\max(0,\,x-\varepsilon,\,2x-1)}^{\min(1,\,x+\varepsilon,\,2x)} + \int_{\min(1,\,x+\varepsilon,\,2x)}^{\min(1,\,x+\varepsilon)}\right\}\frac{2\delta(x)g(x)}{g^2(x)-g^2(\xi)}\,d\xi\right]dx. \quad (70)$$

Now, by (55), if $0 < x \le \zeta < 1$ and $0 < \xi \le \zeta$ and $x \neq \xi$, we have

$$\left|\frac{2\delta(x)g(x)}{g^2(x)-g^2(\xi)}\right| \le \left|\frac{2\delta(1)}{g(x)-g(\xi)}\right| \le \frac{2\delta(1)}{|x-\xi|\;|g'(\zeta)|} \le \frac{20\delta(1)}{|x-\xi|\sqrt{1-\zeta}}.$$

Hence, if ε is sufficiently small,

$$\begin{aligned}
0 &\le \int_0^1 dx \int_{\max(0,\,x-\varepsilon)}^{\max(0,\,x-\varepsilon,\,2x-1)} \frac{2\delta(x)g(x)}{g^2(\xi)-g^2(x)}\,d\xi \\
&= \int_{1-\varepsilon}^1 dx \int_{x-\varepsilon}^{2x-1} \frac{2\delta(x)g(x)}{g^2(\xi)-g^2(x)}\,d\xi \\
&\le 20\delta(1)\int_{1-\varepsilon}^1 \frac{dx}{\sqrt{1-x}}\int_{x-\varepsilon}^{2x-1}\frac{d\xi}{x-\xi} \\
&= 20\delta(1)\int_{1-\varepsilon}^1 \frac{1}{\sqrt{1-x}}\log\frac{\varepsilon}{1-x}\,dx \\
&= 80\delta(1)\sqrt{\varepsilon}\,.
\end{aligned}$$

Thus

$$\lim_{\varepsilon\to+0}\int_0^1 dx\int_{\max(0,\,x-\varepsilon)}^{\max(0,\,x-\varepsilon,\,2x-1)}\frac{2\delta(x)g(x)}{g^2(\xi)-g^2(x)}\,d\xi=0\,. \tag{71}$$

A similar (but simpler) calculation shows that

$$\lim_{\varepsilon\to+0}\int_0^1 dx\int_{\min(1,\,x+\varepsilon,\,2x)}^{\min(1,\,x+\varepsilon)}\frac{2\delta(x)g(x)}{g^2(x)-g^2(\xi)}\,d\xi=0\,. \tag{72}$$

It is clear from (52) that the integral

$$\int_0^1 dx\int_0^1\frac{2\delta(x)g(x)}{g^2(x)-g^2(\xi)}\,d\xi$$

is finite. Also it is clear from the above estimate for the integrand that the integrals

$$\left\{\int_\varepsilon^1 dx\int_0^{x-\varepsilon}+\int_0^{1-\varepsilon}dx\int_{x+\varepsilon}^1\right\}\frac{2\delta(x)g(x)}{g^2(x)-g^2(\xi)}\,d\xi\,,$$

$$\int_0^1 dx\int_{\max(0,\,x-\varepsilon)}^{\max(0,\,x-\varepsilon,\,2x-1)}\frac{2\delta(x)g(x)}{g^2(x)-g^2(\xi)}\,d\xi\,.$$

and

$$\int_0^1 dx\int_{\min(1,\,x+\varepsilon,\,2x)}^{\min(1,\,x+\varepsilon)}\frac{2\delta(x)g(x)}{g^2(x)-g^2(\xi)}\,d\xi$$

are all finite. Thus the integral

$$\int_0^1 dx\int_{\max(0,\,x-\varepsilon,\,2x-1)}^{\min(1,\,x+\varepsilon,\,2x)}\frac{2\delta(x)g(x)}{g^2(x)-g^2(\xi)}\,d\xi$$

is finite. But, if $0<x<1$, we have

$$\int_{\max(0,\,x-\varepsilon,\,2x-1)}^{\min(1,\,x+\varepsilon,\,2x)}\frac{2\delta(x)g(x)}{g^2(x)-g^2(\xi)}\,d\xi$$

$$=\int_{x-\min(x,\,\varepsilon,\,1-x)}^{x+\min(x,\,\varepsilon,\,1-x)}\frac{2\delta(x)g(x)}{g^2(x)-g^2(\xi)}\,d\xi$$

$$=2\delta(x)g(x)\int_0^{\min(x,\,\varepsilon,\,1-x)}\frac{g^2(x+\eta)-2g^2(x)+g^2(x-\eta)}{\{g^2(x)-g^2(x+\eta)\}\{g^2(x-\eta)-g^2(x)\}}\,d\eta\,.$$

By (55), the integrand of the integral on the right hand side is positive; and so, for each fixed value of x with $0<x<1$ the integral is a decreasing function of ε tending to the limit zero as ε tends to zero through positive values. Hence

$$\lim_{\varepsilon\to+0}\int_0^1 dx\int_{\max(0,\,x-\varepsilon,\,2x-1)}^{\min(1,\,x+\varepsilon,\,2x)}\frac{2\delta(x)g(x)}{g^2(x)-g^2(\xi)}d\xi = 0\,. \tag{73}$$

Using (71), (72) and (73) in (70) we obtain (69).

Now, using the result (52) of Lemma 3,

$$\int_0^1 dx\int_0^1\frac{2\delta(x)g(x)}{g^2(x)-g^2(\xi)}d\xi = \pi\int_0^1\delta(x)dx\,.$$

Combining this with (68) and (69) and using the result (54) of Lemma 3, we obtain

$$\tfrac{1}{2}\int_0^1 dx\int_0^1\log|\delta^2(x)-\delta^2(\xi)|d\xi \le \log\left[\frac{\pi}{2\sqrt{e}}\int_0^1\delta(x)\,dx\right].$$

Hence, by (67),

$$\frac{\exp\left\{\int_0^1 dx\int_0^1\log|\alpha(x)-\alpha(\xi)|d\xi\right\}}{\int_0^1|\alpha(x)|dx}\le\frac{\pi}{2\sqrt{e}}.$$

This proves the lemma.

Lemma 5. *For any numbers $z_1,\dots,z_m$, which do not all vanish,*

$$\frac{\left\{\prod_{\varrho<\sigma}|z_\sigma-z_\varrho|\right\}^{2/\{m(m-1)\}}}{\frac{1}{m}\sum|z_\varrho|}\le\left(\frac{e^3\pi^2m(m-1)^2}{16}\right)^{1/(2m-2)}\frac{\pi}{2\sqrt{e}}. \tag{74}$$

Proof. The inequality (74) is an immediate consequence of Lemmas 1, 2 and 4.

Although it is not necessary for the proof of the main result of this paper, the following lemma seems to be of sufficient intrinsic interest to warrant its inclusion, partly because it is the integral analogue of the inequality (74) and partly because the constant is best possible.

Lemma 6. *For any function $\alpha(x)$, which is integrable in the Lebesgue sense over the interval* (0, 1),

$$\exp\left\{\int_0^1 dx\int_0^1\log|\alpha(x)-\alpha(\xi)|d\xi\right\}\le\frac{\pi}{2\sqrt{e}}\int_0^1|\alpha(x)|dx\,. \tag{75}$$

Further (75) *is satisfied with equality for a certain function $\alpha(x)$ which is not null.*

Proof. We introduce a function $\beta(x)$, which may be regarded as a rearrangement[1] of the values assumed by $\alpha(x)$ in increasing order. Let $m(a)$ be the measure of the set $E_1(a)$ of numbers x with $0 \leq x \leq 1$ for which $\alpha(x) \leq a$. Then the set $E_1(a)$ and the function $m(a)$ do not decrease as a increases. Let $\beta(x)$ be the lower bound of the numbers a for which $m(a) \geq x$. Then $\beta(x)$ is defined for $0 < x < 1$ and is a non-decreasing function of x.

If $\beta(x)$ has the constant value β for $x_1 \leq x \leq x_2$, where $0 \leq x_1 < x_2 \leq 1$, then

$$m(a) < x_1\,, \quad \text{if} \quad a < \beta\,,$$
$$m(b) \geq x_2\,, \quad \text{if} \quad b > \beta\,.$$

Thus the measure of the set $E_1(b)-E_1(a)$ is at least x_2-x_1 if $a < \beta < b$. Consequently $\alpha(x)$ assumes the value β for a set of points x with $0 \leq x \leq 1$ of measure at least x_2-x_1. This would imply that

$$\int_0^1 dx \int_0^1 \log|\alpha(x)-\alpha(\xi)|d\xi = -\infty\,,$$

and in this case (75) is satisfied trivially.

We may suppose now that $\beta(x)$ is a strictly increasing function for $0 < x < 1$ and that $\alpha(x)$ does not assume any constant value in a set of positive measure. Consider the sets $E_1(a)$, $E_2(a)$, $E_3(a)$, $E_4(a)$ defined for any real number a to be the sets of numbers x with $0 < x < 1$ for which

$$(1)\ \alpha(x) \leq a\,,$$
$$(2)\ \alpha(x) < a\,,$$
$$(3)\ \beta(x) \leq a\,,$$
$$(4)\ \beta(x) < a\,,$$

respectively. Clearly $E_1(a)$ and $E_2(a)$ both have measure $m(a)$. Also, as $\beta(x)$ is strictly increasing, $E_3(a)$ and $E_4(a)$ have the same measure; and from the definition of $\beta(x)$ it is clear that $E_3(a)$ is the set of points x with $0 \leq x \leq m(a)$. Hence all the four sets $E_1(a)$, $E_2(a)$ $E_3(a)$, $E_4(a)$ have the same measure. Consequently

$$\int_0^1 |\alpha(x)|\,dx = \int_0^1 |\beta(x)|\,dx \tag{76}$$

and

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[1] See Hardy, Littlewood and Pólya, *Inequalities*, (Cambridge 1934), § 10.12, page 276.

$$\int_0^1 dx \int_0^1 \log|\alpha(x)-\alpha(\xi)|d\xi = \int_0^1 dx \int_0^1 \log|\alpha(x)-\beta(\xi)|d\xi$$
$$= \int_0^1 d\xi \int_0^1 \log|\alpha(x)-\beta(\xi)|dx$$
$$= \int_0^1 d\xi \int_0^1 \log|\beta(x)-\beta(\xi)|dx$$
$$= \int_0^1 dx \int_0^1 \log|\beta(x)-\beta(\xi)|d\xi\,. \tag{77}$$

Now write

$$\gamma(x) = \tfrac{1}{2}\{\beta(x)-\beta(1-x)\}$$

for $0 \leq x \leq 1$, so that $\gamma(x)$ is a strictly increasing function and

$$\gamma(x)+\gamma(1-x) = 0 \tag{78}$$

for $0 < x < 1$. Then

$$\int_0^1 |\gamma(x)|\,dx \leq \int_0^1 |\beta(x)|\,dx \tag{79}$$

and, as in the proof of Lemma 1,

$$0 \leq \{\beta(x)-\beta(\xi)\}\{\beta(1-\xi)-\beta(1-x)\} \leq \{\gamma(x)-\gamma(\xi)\}^2\,.$$

Hence

$$\int_0^1 dx \int_0^1 \log|\beta(x)-\beta(\xi)|d\xi$$
$$= \tfrac{1}{2}\int_0^1 dx \int_0^1 \log\{\beta(x)-\beta(\xi)\}\{\beta(1-\xi)-\beta(1-x)\}\,d\xi$$
$$\leq \int_0^1 dx \int_0^1 \log|\gamma(x)-\gamma(\xi)|d\xi\,.$$

By (76), (77), (79) and this last result,

$$\frac{\exp\left\{\int_0^1 dx \int_0^1 \log|\alpha(x)-\alpha(\xi)|d\xi\right\}}{\int_0^1 |\alpha(x)|dx}$$
$$= \frac{\exp\left\{\int_0^1 dx \int_0^1 \log|\beta(x)-\beta(\xi)|d\xi\right\}}{\int_0^1 |\beta(x)|\,dx} \leq \frac{\exp\left\{\int_0^1 dx \int_0^1 \log|\gamma(x)-\gamma(\xi)|d\xi\right\}}{\int_0^1 |\gamma(x)|dx}. \tag{80}$$

Write

$$\gamma_\varepsilon(x) = \gamma(\varepsilon + x\{1-2\varepsilon\}), \quad \text{for} \quad 0 \le x \le 1,$$

where $0 < \varepsilon < \frac{1}{2}$. Then, by (78),

$$\gamma_\varepsilon(x) + \gamma_\varepsilon(1-x) = 0, \quad \text{for} \quad 0 \le x \le 1,$$

and $\gamma_\varepsilon(x)$ is a bounded strictly increasing function of x for $0 \le x \le 1$. Thus $\gamma_\varepsilon(x)$ satisfies the conditions for the function $\alpha(x)$ in Lemma 4; and so by that Lemma

$$\frac{\exp\left\{\int_0^1 dx \int_0^1 \log|\gamma_\varepsilon(x) - \gamma_\varepsilon(\xi)|\,d\xi\right\}}{\int_0^1 |\gamma_\varepsilon(x)|\,dx} \le \frac{\pi}{2\sqrt{e}}.$$

In the limit as ε tends to zero through positive values, we obtain

$$\frac{\exp\left\{\int_0^1 dx \int_0^1 \log|\gamma(x) - \gamma(\xi)|\,d\xi\right\}}{\int_0^1 |\gamma(x)|\,dx} \le \frac{\pi}{2\sqrt{e}}.$$

Now (75) follows from this result and (80).

It is clear from (53), (54) and (67) that (75) is satisfied with equality when $\alpha(x)$ is defined by

$$\alpha(x) = -g(2x), \quad \text{if} \quad 0 \le x \le \tfrac{1}{2},$$

$$\alpha(x) = g(2-2x), \quad \text{if} \quad \tfrac{1}{2} \le x \le 1,$$

so that

$$\delta(x) = g(x), \quad \text{for} \quad 0 \le x \le 1.$$

4. Before we prove our main result we state the following well known result due to Blichfeldt[1], on which its proof is based.

Lemma 7. *Let S be any closed bounded n-dimensional set with Lebesgue measure (or outer Jordan content) V. Then there is a set of distinct points*

$$X^{(1)} = (x_1^{(1)}, \ldots, x_n^{(1)}), \ldots, X^{(m)} = (x_1^{(m)}, \ldots, x_n^{(m)})$$

of S with $m > V$ such that the differences

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[1] H. F. Blichfeldt, *Trans. American Math. Soc.*, 40 (1914), 227–256. We restate Blichfeldt's Theorem 1 (page 228), in the form he considers in § 7 (page 230).

$$x_k^{(r)} - x_k^{(s)},\ k = 1, \ldots, n$$

are integers for $r, s = 1, \ldots, m$.

Theorem. *Let* $n \geq 3$ *and let*

$$x_r = \sum a_{r\sigma} u_\sigma, \quad r = 1, \ldots, n$$

be n *real homogeneous linear forms in* $u_1, \ldots, u_n$ *with determinant* 1. *Then there exists a set of integers* $u_1, \ldots, u_n$, *other than* $0, \ldots, 0$, *for which*

$$|x_1 \ldots x_n| < \frac{(n!)(1+n\log n)e^{3/2}(2{\cdot}5\log n)^{3/(2\log n)}}{\left(\frac{4\pi\sqrt{e}}{\pi}\right)^n}. \tag{81}$$

Proof. Let (A_{rs}) be the reciprocal matrix to (a_{rs}) and write

$$\left.\begin{aligned} x_r &= \sum a_{r\sigma} y_\sigma, \quad r = 1, \ldots, n, \\ y_r &= \sum A_{r\sigma} x_\sigma, \quad r = 1, \ldots, n. \end{aligned}\right\} \tag{82}$$

Take S to be the set of points $(y_1, \ldots, y_n)$, for which

$$\sum |x_\varrho| \leq \tfrac{1}{2}\{(1+n\log n)(n!)\}^{1/n}.$$

Then, as the determinant of the matrix (a_{rs}) is 1, the volume of S is $1+n\log n$. Hence by Lemma 7 there is a set of distinct points

$$Y^{(1)} = (y_1^{(1)}, \ldots, y_n^{(1)}), \ldots, Y^{(m)} = (y_1^{(m)}, \ldots, y_n^{(m)})$$

of S with $m > 1+n\log n$, such that the differences

$$u_k^{(r,s)} = y_k^{(r)} - y_k^{(s)}, \quad k = 1, \ldots, n \tag{83}$$

are integers for $r, s = 1, \ldots, m$. Thus, if

$$X^{(1)} = (x_1^{(1)}, \ldots, x_n^{(1)}), \ldots, X^{(m)} = (x_1^{(m)}, \ldots, x_n^{(m)})$$

are the points corresponding to $Y^{(1)}, \ldots, Y^{(m)}$ in the transformation (82), we have

$$\sum |x_\varkappa^{(r)}| \leq \tfrac{1}{2}\{(1+n\log n)(n!)\}^{1/n}$$

for $r = 1, \ldots, m$, and so

$$\sum |x_\varkappa^{(\varrho)}| \leq \tfrac{1}{2} m\{(1+n\log n)(n!)\}^{1/n}. \tag{84}$$

Now by Lemma 5

$$\frac{\left\{\prod_{\varrho<\sigma}|x_\varkappa^{(\varrho)}-x_\varkappa^{(\sigma)}|\right\}^{2/\{m(m-1)\}}}{\prod_{k=1}^{n}\left\{\frac{1}{m}\sum|x_k^{(\varrho)}|\right\}}\leq\left(\frac{\pi}{2\sqrt{e}}\right)^n\left(\frac{e^3\pi^2 m(m-1)^2}{16}\right)^{n/(2m-2)} \tag{85}$$

Since

$$m>1+n\log n\geq 1+3\log 3>4\,, \tag{86}$$

we have

$$\frac{m}{m-1}\leq\frac{5}{4},$$

and the right hand side of (85) is less than or equal to

$$\left(\frac{\pi}{2\sqrt{e}}\right)^n\left(\frac{5e^3\pi^2(m-1)^3}{64}\right)^{n/(2m-2)} \tag{87}$$

But $m-1>n\log n>3$, while

$$\frac{d}{d\mu}\log\left(\frac{5e^3\pi^2\mu^3}{64}\right)^{1/\mu}=-\frac{1}{\mu^2}\log\left(\frac{5\pi^2\mu^3}{64}\right)<0,$$

if $\mu>3$. Thus the expression (87) is less than

$$\left(\frac{\pi}{2\sqrt{e}}\right)^n\left(\frac{5e^3\pi^2(n\log n)^3}{64}\right)^{n/(2n\log n)}=\left(\frac{\pi}{2\sqrt{e}}\right)^n e^{3/2}\left(\frac{5e^3\pi^2(\log n)^3}{64}\right)^{1/(2\log n)},$$

and consequently

$$\frac{\left\{\prod_{\varrho<\sigma}|x_\varkappa^{(\varrho)}-x_\varkappa^{(\sigma)}|\right\}^{2/\{m(m-1)\}}}{\prod_{k=1}^{n}\left\{\frac{1}{m}\sum|x_k^{(\varrho)}|\right\}}<\left(\frac{\pi}{2\sqrt{e}}\right)^n e^{3/2}(2\cdot 5\log n)^{3/(2\log n)}\,. \tag{88}$$

Now by the inequality of the arithmetic and geometric means and by (84),

$$\prod_{k=1}^{n}\left\{\frac{1}{m}\sum|x_k^{(\varrho)}|\right\}\leq\left\{\frac{1}{nm}\sum|x_\varkappa^{(\varrho)}|\right\}^n\leq\left(\frac{1}{2n}\right)^n(1+n\log n)(n!)\,. \tag{89}$$

Thus by (88) and (89)

$$\left\{\prod_{\varrho<\sigma}|x_\varkappa^{(\varrho)}-x_\varkappa^{(\sigma)}|\right\}^{2/\{m(m-1)\}}<\frac{(n!)(1+n\log n)e^{3/2}(2\cdot 5\log n)^{3/(2\log n)}}{\left(\frac{4n\sqrt{e}}{\pi}\right)^n}$$

So there are some integers r, s with $1 \leq r < s \leq m$ such that

$$\prod |x_\varkappa^{(r)} - x_\varkappa^{(s)}| < \frac{(n!)(1+n \log n)e^{3/2}(2{\cdot}5 \log n)^{3/(2 \log n)}}{\left(\frac{4n\sqrt{e}}{\pi}\right)^n}.$$

Consequently (81) is satisfied when $u_1, \ldots, u_n$ have the integral values

$$u_k^{(r,s)} = y_k^{(r)} - y_k^{(s)}, \quad k = 1, \ldots, n.$$

These integers are not all zero as the points $Y^{(r)}$ and $Y^{(s)}$ are distinct. This proves the theorem.

It is clear from this theorem that, if $n \geq 3$, then $\mathfrak{M}_n$ is less than or equal to the right hand side of (81), which is asymptotic to

$$\frac{\sqrt{2\pi e^3}\, n^{3/2} \log n}{\left(\frac{4e\sqrt{e}}{\pi}\right)^n},$$

as $n \to \infty$. Thus

$$m = \overline{\lim_{n\to\infty}} \{\mathfrak{M}_n\}^{1/n} \leq \frac{\pi}{4e\sqrt{e}}.$$

This proves (8).