THE MINIMUM OF A BINARY QUARTIC FORM (II).

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Introduction.

This paper is a continuation of Part I of a paper with the same title[1] which deals with binary quartic forms having four complex roots. Paragraphs, equations, etc. are numbered consecutively in the two parts.

In order to make this paper intelligible to a reader who has not seen Part I, we repeat here a few definitions.

The binary quartic form

$$\psi(\xi, \eta) = a\,\xi^4 + 4\,b\,\xi^3\eta + 6\,c\,\xi^2\eta^2 + 4\,d\,\xi\eta^3 + e\,\eta^4$$

has two irreducible invariants

$$\mathcal{I} = a\,e - 4\,b\,d + 3\,c^2,$$
$$\mathcal{J} = a\,c\,e + 2\,b\,c\,d - a\,d^2 - e\,b^2 - c^3,$$

and discriminant $\mathcal{D} = \mathcal{I}^3 - 27\,\mathcal{J}^2$. We define

$$\mathcal{H} = b^2 - a\,c, \qquad \mathcal{K} = 2\,\mathcal{H}\mathcal{I} + 3\,a\,\mathcal{J}.$$

We say that $\psi(\xi, \eta)$ is *transformable* into the *standard form* $f(x, y)$ if it is obtainable from $f(x, y)$ by a real linear substitution.

We write k for a number such that every lattice of determinant $\Delta \neq 0$ has a point other than the origin in the region defined by

$$|f(x, y)| \leq (k + \varepsilon)\,\Delta^2,$$

where ε is any positive number. The lower bound of such numbers k is k^*.

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[1] Acta math., 84, p. 263.

The lower bound of the determinants of lattices admissible with respect to the region $\mathcal{R}$ defined by $|f(x, y)| \leq 1$, that is lattices that have no point other than the origin as an inner point of $\mathcal{R}$, is written Δ^*. It is easily seen that $k^* = 1/\Delta^{*2}$.

III. Quartics with $\mathcal{D} = 0$.

23. We next consider forms with $\mathcal{D} = 0$. If $\mathcal{I} = 0$ the solution of our problem is trivial, but we include it for the sake of completeness.

Theorem 5. *If $\psi(\xi, \eta)$ is a binary quartic form with real coefficients, and $\mathcal{I} = \mathcal{J} = 0$, then there exist integers ξ, η, not both zero, such that*

$$|\psi(\xi, \eta)| < \varepsilon,$$

where ε is any positive number.

We have $\mathcal{D} = 0$, $\mathcal{I} = 0$, and so

$$\psi(\xi, \eta) = a(\xi - \omega\eta)^3(\xi - \omega'\eta),$$

where ω, ω' are real. Suppose first $\omega \neq \omega'$. Then by Minkowski's theorem on linear forms (this is an immediate consequence of Lemma 15) there exist integers $\xi, \eta \neq 0, 0$ with

$$|\xi - \omega\eta| \leq \varepsilon', \quad |\xi - \omega'\eta| \leq \frac{|\omega - \omega'|}{\varepsilon'},$$

for any $\varepsilon' > 0$, and so

$$|\psi(\xi, \eta)| \leq a|\omega - \omega'|\varepsilon'^2 < \varepsilon$$

for any $\varepsilon > 0$, by appropriate choice of ε'. If $\omega = \omega'$, we have similarly

$$|\psi(\xi, \eta)| \leq a\varepsilon'^4 < \varepsilon.$$

24. If $\mathcal{D} = \mathcal{K} = 0$, $\mathcal{I} \neq 0$, the form $\psi(\xi, \eta)$ is the square of a binary quadratic and it is necessary only to express in a suitable notation the classical results for binary quadratics given in § 1.

Theorem 6. *If $\psi(\xi, \eta)$ is a binary quartic form with real coefficients, and $\mathcal{D} = \mathcal{K} = 0$, $\mathcal{I} \neq 0$, then there exist integers $\xi, \eta \neq 0, 0$ such that*

$$|\psi(\xi, \eta)| \leq \frac{2}{\chi}\sqrt{(3\mathcal{I})},$$

where $\chi = 5$ if $\mathcal{H} > 0$ and $\chi = 3$ if $\mathcal{H} < 0$. These are the best possible results, the *sign of equality being required if, and only if, ψ is equivalent to a multiple of the form* $(\xi^2-\xi\eta-\eta^2)^2$ *if* $\mathcal{H}>0$, *or of the form* $(\xi^2-\xi\eta+\eta^2)^2$ *if* $\mathcal{H}<0$.

It is sufficient to note that the discriminant of the quadratic $(\xi-\omega\eta)(\xi-\omega'\eta)$ is $(\omega-\omega')^2$, while for the quartic form $\psi(\xi,\eta)=a(\xi-\omega\eta)^2(\xi-\omega'\eta)^2$ we have $\mathcal{J}=\frac{1}{12}a^2(\omega-\omega')^4$. Hence there exist integers $\xi,\eta\neq 0,0$ with

$$|\psi(\xi,\eta)|\leq a\frac{(\omega-\omega')^2}{\chi}=\frac{1}{\chi}\sqrt{12\mathcal{J}}=\frac{2}{\chi}\sqrt{3\mathcal{J}}.$$

The respective critical forms are of course well known.

25. We next investigate the case $\mathcal{D}=0$, $\mathcal{H}<0$. Here we are concerned with the region $\mathcal{R}$ defined by $x^2(x^2+y^2)\leq 1$, which is depicted in Fig. 2. In this case we find the best possible result, and indeed a good deal more, as we are able to give an infinity of successive minima (corresponding to the Markoff chain for an indefinite binary quadratic form). It is of interest to note that none of these successive minima are attained;[1] in particular, none of the critical lattices of $\mathcal{R}$ has a point on $\mathcal{C}$, the boundary of $\mathcal{R}$.[2]

We first recall some classical results due essentially to Markoff; see, for example, Dickson (19) or Cassels (20). We write d_n for the discriminant of the nth Markoff form

$$Q_n(\xi,\eta)=(\xi-\theta_n\eta)(\xi-\phi_n\eta) \qquad (n=1,2,3,\ldots),$$

so that $d_n=(\theta_n-\phi_n)^2$. The numbers d_n increase monotonically to the limit 9, the first three values being

$$d_1=5,\quad d_2=8,\quad d_3=\tfrac{221}{25}.$$

The θ_n are quadratic irrationals of the field $k(\sqrt{d_n})$, no two of which are equi valent.[3] The first three values are

$$\theta_1=\tfrac{1}{2}(\sqrt{5}+1),\quad \theta_2=\sqrt{2}-1,\quad \theta_3=\tfrac{1}{10}(\sqrt{221}-11).$$

Then it is known that $|Q_n(\xi,\eta)|\geq 1$ for integers $\xi,\eta\neq 0,0$; and that there is

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[1] Strictly speaking they are thus only lower bounds; we use the word minimum in the wide sense.

[2] The existence of such regions is stated in Theorem 2 of Mahler (18), and in fact our region $\mathcal{R}$ satisfies the conditions he postulates in his proof. However, this proof contains some errors which invalidate its application to our problem, though his conclusions are true in our case.

[3] We say ω is equivalent to θ, and write $\omega\sim\theta$, if $\omega=(a\theta+b)/(c\theta+d)$, where a,b,c,d are integers with $ad-bc=\pm 1$.

an infinity of integer solutions of $Q_n(\xi, \eta) = \pm 1$ (with either sign), $|\xi - \theta_n \eta| < \varepsilon$, for any $\varepsilon > 0$.

Now let

$$x = \alpha(\xi - \omega\eta), \qquad y = \gamma(\xi - \omega'\eta)$$

be an admissible lattice of determinant $\Delta > 0$. Then the corresponding form $\psi(\xi, \eta) = x^2(x^2 + y^2)$ has the double real root ω; we may suppose that ω is irrational, since otherwise ψ has the minimum zero. By a theorem of Davenport and Rogers (21), if ω is not equivalent to any one of $\theta_1, \ldots, \theta_{n-1}$ (the case $n = 1$ being interpreted to mean that ω is an arbitrary irrational), then there exist integers $\xi, \eta \neq 0, 0$ such that

$$|xy| \leq \Delta / \sqrt{d_n}, \quad |x| < \varepsilon,$$

for arbitrary $\varepsilon > 0$. For these integers we have

$$x^2(x^2 + y^2) < \Delta^2 / d_n + \varepsilon^4,$$

and it follows that $\Delta \geq \sqrt{d_n}$, since $x^2(x^2 + y^2) \geq 1$ for every point $(x, y) \neq (0, 0)$ of an admissible lattice.

We note that any lattice with

$$x = \alpha(\xi - \theta_n \eta), \quad y = \gamma(\xi - \phi_n \eta), \quad |\alpha\gamma| = 1 \tag{25.1}$$

has $\Delta = \sqrt{d_n}$ and is admissible, since $xy = Q_n(\xi, \eta)$ and so for integers $\xi, \eta \neq 0, 0$ we have

$$x^2(x^2 + y^2) \geq 1 + x^4 > 1. \tag{25.2}$$

Hence $\sqrt{d_n}$ is the true minimum value of Δ under the conditions stated; in particular, $\Delta^* = \sqrt{d_1} = \sqrt{5}$.[1] From (25.2) we see that none of the lattices (25.1) has a point on $\mathcal{C}$.

We show now that, with ω as above, the only admissible lattices with $\Delta = \sqrt{d_n}$ are those given by (25.1). We remark first that $\omega \sim \theta_n$, for otherwise $\Delta \geq \sqrt{d_{n+1}} > \sqrt{d_n}$ for admissible lattices. By a unimodular substitution on ξ, η, which amounts merely to selecting another basis for the lattice, we may assume $\omega = \theta_n$. Hence any admissible lattice L with $\Delta = \sqrt{d_n}$ may be written

$$x = \alpha(\xi - \theta_n \eta), \quad y = \beta(\xi - \theta_n \eta) + \gamma(\xi - \phi_n \eta),$$

where $|\alpha\gamma| = 1$ and β is some real number. We write this, for brevity,

[1] This last result was proved by Mahler (9).

$$x = \alpha u, \quad y = \beta u + \gamma v;$$

so that

$$u v = Q_n(\xi, \eta).$$

Then

$$x^2(x^2+y^2) = \alpha^2 u^2 \{(\alpha^2+\beta^2) u^2 + 2\beta\gamma u v + \gamma^2 v^2\}.$$

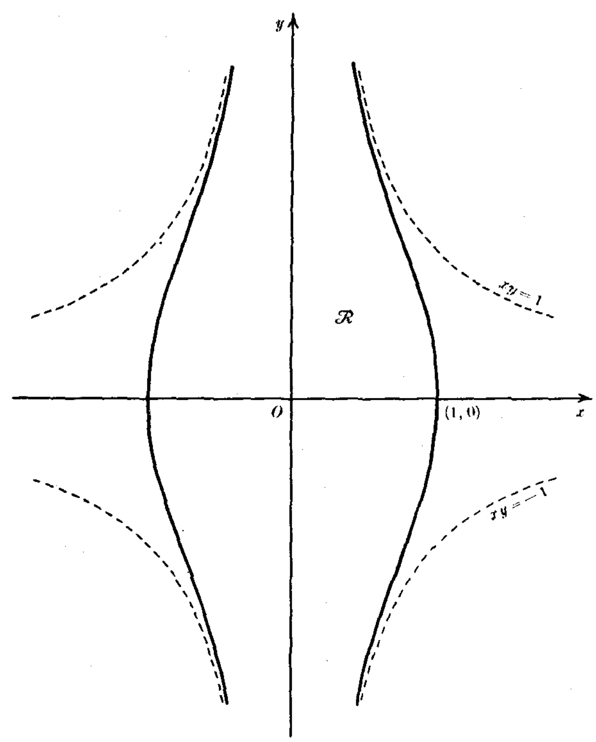

Fig. 2. The region $|x^2(x^2+y^2)| \leq 1$.

Now, if $\beta \neq 0$, there exist integers $\xi, \eta \neq 0, 0$ with $uv = -\operatorname{sgn}(\beta\gamma)$ and $u^2 < \varepsilon$, for any $\varepsilon > 0$. Choose $\varepsilon = 2|\beta\gamma|/(\alpha^2+\beta^2)$. Then, for these ξ, η,

$$x^2(x^2+y^2) = \alpha^2 u^2 \{(\alpha^2+\beta^2) u^2 - 2|\beta\gamma|\} + 1 < 1,$$

which is false, since L is admissible. Hence $\beta = 0$, and so L is one of the lattices (25.1).

We remark further that the above proof shows that, if $\omega \sim \theta_1, \ldots, \theta_{n-1}$, every lattice of determinant $\sqrt{d_n}$ has not merely one but an infinity of points satisfying $x^2(x^2+y^2) < 1+\varepsilon$ for arbitrary $\varepsilon > 0$; and that we may take $\varepsilon = 0$ here unless the lattice is one of those given by (25.1).

Noting that $\mathcal{J} = \frac{1}{12}$ for the form $x^2(x^2+y^2)$, the results we have proved may be collected together in the following:

Theorem 7. *Let $\psi(\xi,\eta)$ be a binary quartic form with real coefficients and $\mathcal{D}=0$, $\mathcal{K}<0$, and so having a double real root ω, say. Then if ω is not equivalent to one of $\theta_1, \ldots, \theta_{n-1}$ there exists, for arbitrary $\varepsilon>0$, an infinity of integer pairs ξ, η such that*

$$(25.3) \qquad |\psi(\xi,\eta)| < \frac{2}{d_n}\sqrt{3\mathcal{J}} + \varepsilon.$$

If $\psi(\xi,\eta)=x^2(x^2+y^2)$, where xy is equivalent to a multiple of the Markoff form $Q_n(\xi,\eta)$, the relation (25.3) with $\varepsilon=0$ has no integer solutions except $\xi=\eta=0$, even if equality be admitted. Otherwise there is an infinity of integer solutions with $\varepsilon=0$.

26. We turn now to the case $\mathcal{D}=0$, $\mathcal{K}>0$. Here the standard form we consider is $f(x,y)=x^2(x^2-y^2)$.[1] We define a region $\mathcal{R}$ by $|f(x,y)|\le 1$; this region is bounded by the curves $f(x,y)=\pm 1$. The region is symmetrical about both axes, so it suffices to consider the first quadrant. Consider first the curve $f(x,y)=1$, which passes through the point $(1,0)$. Here we have $y^2=x^2-1/x^2$, so $x\ge 1$ and $y=x$ is an asymptote. Next consider the curve $f(x,y)=-1$. Here $y^2=x^2+1/x^2$, and so $y\ge\sqrt{2}$ and the lines $x=0$, $y=x$ are asymptotes. The point $(1,\sqrt{2})$ is a minimum. Neither curve has a real finite point of inflection, since their Hessian is $x=0$. The region is illustrated in Fig. 3.

We are unable to give the best possible result for this region, but an estimate is obtained by inscribing a region of known critical determinant Δ^*. Let $\mathcal{R}'$ be the bounded region

$$|xy|\le 1, \qquad \left|tx+\frac{y}{t}\right|\le\sqrt{5},$$

where t is any positive number. We show that $\mathcal{R}'$ lies strictly within $\mathcal{R}$ for sufficiently large t. Clearly every point of $\mathcal{R}'$ satisfies $|x|\le\frac{1}{2}$ for sufficiently large t. Then every point of the boundary of the region $|xy|\le 1$, $|x|\le\frac{1}{2}$, which contains $\mathcal{R}'$, lies strictly within $\mathcal{R}$. For if (x,y) is such a point, $x\ne 0$ and so

$$f(x,y)=x^2(x^2-y^2)\le x^4<1, \quad f(x,y)=-1+x^4>-1.$$

Now, by a theorem of MAHLER (19), $\Delta^*(\mathcal{R}')=\sqrt{5}$ and it follows that $\Delta^*(\mathcal{R})>\sqrt{5}$. We might strengthen this estimate a little by expanding the region $\mathcal{R}'$ until $\mathcal{R}$

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[1] Since writing the above, Professor DAVENPORT has informed me that he has found the critical determinant and critical lattices of the region $|x^2(x^2-y^2)|\le 1$. The critical determinant is $\sqrt{1+2\sqrt{5}}$. His work is to be published in the Quarterly Journal of Mathematics.

and $\mathcal{R}'$ first have a boundary point in common, but the small improvement thus obtainable hardly seems worth the labour involved. Noting that $\mathcal{J}=\frac{1}{12}$ for the form $x^2(x^2-y^2)$, the estimate we have found leads to

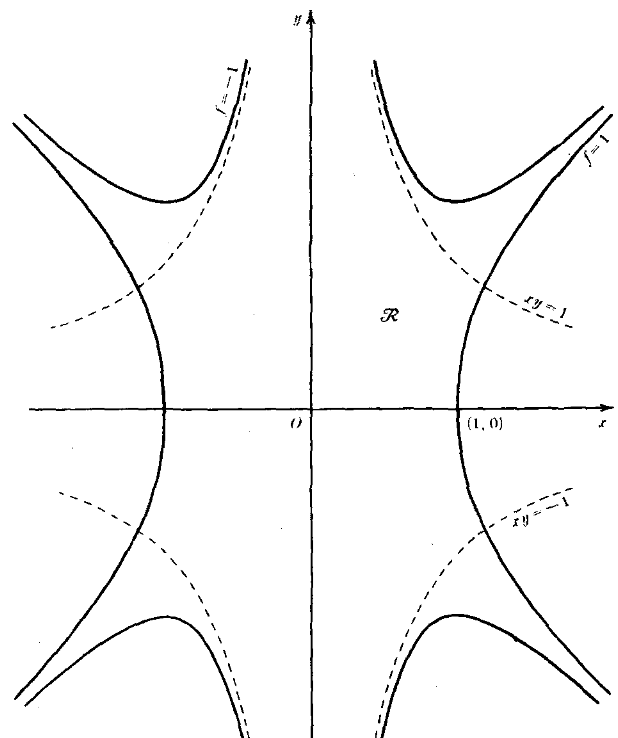

Fig. 3. The region $|x^2(x^2-y^2)| \leq 1$.

Theorem 8. *If $\psi(\xi,\eta)$ is a binary quartic form with real coefficients and $\mathcal{D}=0$, $\mathcal{K}>0$, then integers ξ,η, not both zero, exist such that*

$$|\psi(\xi,\eta)| < \tfrac{2}{5}\sqrt{3\mathcal{J}}.$$

IV. Quartics with $\mathcal{D}>0$ (Four Real Roots).

27. We now deal with the case of quartics with four distinct real roots. We obtain an estimate for the lower bound in all cases, and prove that this estimate is best possible for an infinity of cases.

Again we may take

$$f(x,y) = x^4 + 6mx^2y^2 + y^4$$

as standard form, and we define the region $\mathcal{R}$ by $|f(x,y)| \leq 1$. The roots of

$f(t, 1) = 0$ must be real and distinct; referring back to (4.3) we see that $m < -\frac{1}{3}$. If we substitute

$$x = \nu(X - Y), \qquad y = \nu(X + Y),$$

where $\nu^4 = -\frac{1}{2}(1 + 3m)^{-1} > 0$, we find

$$-f(x, y) = X^4 + 6MX^2Y^2 + Y^4,$$

where $M = \frac{1-m}{1+3m}$. If $m < -1$ this gives $-1 < M < -\frac{1}{3}$, so it is enough to suppose $-1 \leq m < -\frac{1}{3}$.

We may determine m in the same way as we did in § 4 above. The result here is that we define φ by

$$\cos \varphi = -\mathcal{J}\left(\frac{3}{\mathcal{J}}\right)^{3/2}, \qquad 2\pi < \varphi \leq \tfrac{5}{2}\pi,$$

and then

$$m = \frac{1}{\sqrt{3}} \cot \frac{\varphi}{3}. \tag{27.1}$$

The region $\mathcal{R}$ is bounded by the curves $f(x, y) = \pm 1$. Each of these curves is symmetrical about the lines $x = 0$, $y = 0$, $y = \pm x$, and about the origin. The curve $f = 1$ passes through the points $(0, \pm 1)$, $(\pm 1, 0)$. Each of the curves has as asymptotes the four lines $y = \pm \mu x$, $y = \pm x/\mu$, where

$$\mu = \sqrt{\frac{-3m+1}{2}} - \sqrt{\frac{-3m-1}{2}}.$$

The points of inflection of the bounding curves lie on the lines $y = \pm \lambda x$, $y = \pm x/\lambda$, where λ is given by (5.2). The value of λ is imaginary since $m < -\frac{1}{3}$, and so there are no real points of inflection. Thus we see that $\mathcal{R}$ is an infinite star domain bounded by eight arcs, each convex viewed from the origin. In Fig. 4 the region $\mathcal{R}$ is depicted for $m = -\frac{2}{3}$.

28. We first treat the case $-\frac{1}{2} < m < -\frac{1}{3}$. Here we are not able to give the best possible result for any value of m, but we find an estimate which is better than the known one. In this case the region $\mathcal{R}$ contains the square $|x| \leq 1$, $|y| \leq 1$, so we might trivially take $k(m) = 1$. However, we can materially improve this value without much extra labour, by considering an inscribed non-convex region which we have already investigated.

Let $m' = -\frac{1}{3}(3m + 2)$, so that $-\frac{1}{3} < m' < -\frac{1}{6}$, and let $\mathcal{R}_{m'}$ be the region $|f_{m'}(x, y)| \leq 1$, where

$$f_{m'}(x, y) = x^4 + 6m'x^2y^2 + y^4.$$

Then $\mathcal{R}_{m'}$ is a bounded non-convex star domain of the type covered by Theorem 3, so $k^*(m') = \frac{6}{5}(1+m') = \frac{2}{5}(1-3m)$. We show that $\mathcal{R}$ contains $\mathcal{R}_{m'}$, and it follows that $k^*(m) \leq k^*(m')$.

It is clearly sufficient for this to prove that the boundary of $\mathcal{R}_{m'}$, that is

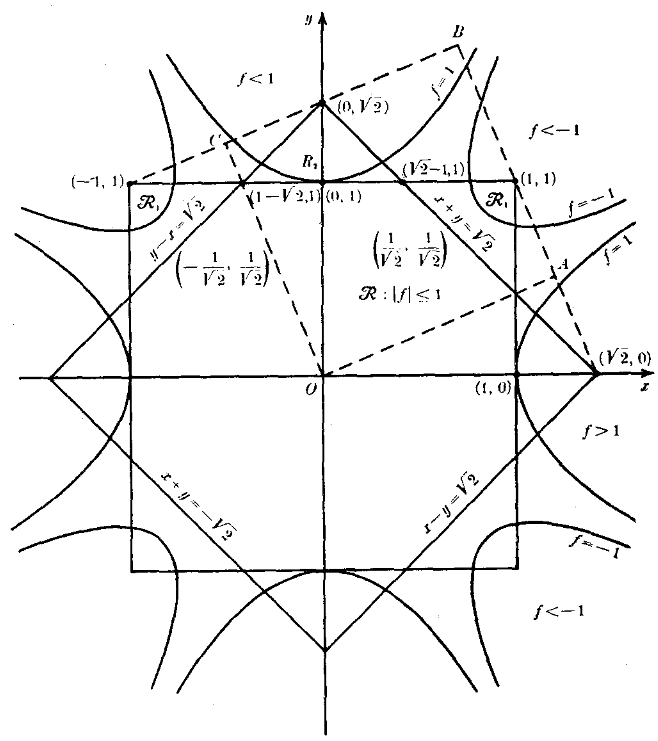

Fig. 4. The region $|x^4 - 4x^2y^2 + y^4| \leq 1$.

$f_{m'}(x, y) = 1$, lies in $\mathcal{R}$. Let (X, Y) be any point on the curve $f_{m'}(x, y) = 1$. We have then

$$f(X, Y) = f_{m'}(X, Y) - 6(m'-m)X^2Y^2 = 1 - 2(6m'+2)X^2Y^2.$$

But $1 = X^4 + 6m'X^2Y^2 + Y^4 = (X^2 - Y^2)^2 + (6m'+2)X^2Y^2$, and so $0 \leq (6m'+2)X^2Y^2 \leq 1$, since $m' > -\frac{1}{3}$. Hence $|f(X, Y)| \leq 1$, that is (X, Y) is a point of $\mathcal{R}$.

We remark further that $|f(X, Y)| < 1$ unless $X = 0$ or $Y = 0$ or $X = Y$, and that none of the points so defined is a point of a critical lattice of $\mathcal{R}_{m'}$. It follows that $k^*(m) < k^*(m')$.

We may improve this estimate for part of the range concerned, by considering the region $\mathcal{R}'$ defined by $|x^2 - y^2| \leq 1$, $|y| \leq \frac{1}{2}\sqrt{5}$, for which $\Delta^* =$

$= \frac{1}{2}\sqrt{5}$.[1] Every point of $|x^2-y^2| \leq 1$ satisfies $f(x,y) \leq 1$, since then $f(x,y) = x^4 + 6mx^2y^2 + y^4 = (x^2-y^2)^2 + (6m+2)x^2y^2 \leq (x^2-y^2)^2 \leq 1$. Further, $f(x,y) = (x^2+3my^2)^2-(9m^2-1)y^4 \geq -(9m^2-1)y^4 \geq -1$, if $(9m^2-1)y^4 \leq 1$. Hence every point with $|y| \leq \frac{1}{2}\sqrt{5}$ satisfies $f(x,y) \geq -1$ if $(9m^2-1)(\frac{1}{2}\sqrt{5})^4 \leq 1$, i.e. if $m \geq -\frac{\sqrt{41}}{15}$. It is easily seen that every critical lattice[2] of $\mathcal{R}'$ has a point in the interior of $\mathcal{R}$, so we have $k^*(m) < \frac{4}{5}$ for $-\frac{\sqrt{41}}{15} \leq m < -\frac{1}{3}$.

We have thus proved

Theorem 9. *Let $\psi(\xi,\eta)$ be a binary quartic form with real coefficients and $\mathcal{D} > 0$, $\mathcal{H} > 0$, $\mathcal{K} > 0$. Further let $3^2\mathcal{J}^3 < 7^3\mathcal{J}^2$, so that m given by (27.1) satisfies $-\frac{1}{2} < m < -\frac{1}{3}$. Then there exist integers ξ,η, not both zero, such that*

$$|\psi(\xi,\eta)| < \frac{2(1-3m)}{5(9m^2-1)^{1/3}}\mathcal{D}^{1/6}.$$

If $-\frac{\sqrt{41}}{15} \leq m < -\frac{1}{3}$, we may replace the right hand side by the smaller number $\frac{4}{5}(9m^2-1)^{-1/3}\mathcal{D}^{1/6}$.

I conjecture that, for values of m sufficiently near to $-\frac{1}{3}$, the lattice L_2, which was critical in Lemma 9, is here a critical lattice if it is admissible, which will probably be true for infinitely many m (cf. Theorems 10 and 12). This rests on a geometrical lemma, similar in nature to Lemma 23, which I have not yet succeeded in proving, though it appears to be true.

29. For the range $-1 \leq m \leq -\frac{1}{2}$ we can say a good deal more, though the results are still far from complete. We shall prove that $k^*(m) \leq 1$, and that 1 is the best possible value if the lattice $x=\xi$, $y=\eta$ is admissible. We show further that this is so for a set of m with positive measure, although the contrary is also true for a set of positive measure.

We require first some simple lemmas.

Lemma 27. *If OABC is a parallelogram of area S, and P, Q are two points*

[1] This is a trivial transformation of the region $|xy| \leq 1$, $|x+y| \leq \sqrt{5}$ used in § 26.

[2] These critical lattices are given by MAHLER (19). They consist of the critical lattices for the infinite region $|x^2-y^2| \leq 1$, together with the lattices generated by the point (1, 0) and any point on the line $y = \frac{1}{2}\sqrt{5}$.

in it, with OPQ in the same sense as $OABC$, then $\triangle OPQ \leq \frac{1}{2} S$.[1] *Further, the sign of equality occurs only if P is at A and Q on BC, or if Q is at C and P on AB.*

Take O as origin and axes of x and y along OA and OC respectively. Let P be the point (x_1, y_1) and Q be the point (x_2, y_2). Let $OA = a$, $OC = b$ and angle $AOC = \omega$. Then $S = ab \sin \omega$, and

$$\triangle OPQ = \tfrac{1}{2} \sin \omega\, (x_1 y_2 - x_2 y_1) \leq \tfrac{1}{2} a b \sin \omega = \tfrac{1}{2} S,$$

with equality only if $x_1 = a$, $y_1 = b$ and $x_2 y_1 = 0$.

Lemma 28. *Let P and Q be two points with coordinates (x_1, y_1) and (x_2, y_2) respectively in a rectangular coordinate system with origin O. Then if $x_1 y_1 \geq \frac{1}{2}$, $y_1 > 0$, $x_2 y_2 \leq -\frac{1}{2}$, $y_2 < 0$, we have $\triangle\ OPQ \geq \frac{1}{2}$, with equality if, and only if, $x_1 y_1 = \frac{1}{2}$, $x_2 y_2 = -\frac{1}{2}$, $x_1 = x_2$.*

For

$$\triangle\ OPQ = \tfrac{1}{2}(x_2 y_1 - x_1 y_2) \geq \tfrac{1}{4}\left(\frac{x_2}{x_1} + \frac{x_1}{x_2}\right) \geq \tfrac{1}{2},$$

by the theorem of the arithmetic-geometric mean. Equality occurs in both places only under the conditions stated.

The substance of this lemma may be expressed in other forms by considering other axes through O. We enunciate it in one such form, which we will require later.

Lemma 29. *Let P: (x_1, y_1) and Q: (x_2, y_2) be two points with $x_1^2 - y_1^2 \geq 1$, $x_1 > 0$, $x_2^2 - y_2^2 \leq -1$, $y_2 > 0$. Then $\triangle OPQ \geq \frac{1}{2}$, with equality if, and only if, $x_1^2 - y_1^2 = 1$, $x_2^2 - y_2^2 = -1$, $x_1 = y_2$.*

We now apply these lemmas to points associated with our region $\mathcal{R}$.

Lemma 30. *Let P_1 be a point of the region $f(x, y) \leq -1$, $x > 0$, $y > 0$, and let P_1' be a point of the region $f(x, y) \leq -1$, $x > 0$, $y < 0$. Then $\triangle\ OP_1P_1' > \frac{1}{2}$ if $m > -1$. If $m = -1$, $\triangle\ OP_1P_1' \geq \frac{1}{2}$, with equality only if $P_1 = (1/\sqrt{2}, 1/\sqrt{2})$ and $P_1' = (1/\sqrt{2}, -1/\sqrt{2})$.*

After Lemma 28 it is clearly sufficient to show that every point (x, y) with $f(x, y) \leq -1$ satisfies $|xy| \geq \frac{1}{2}$, with equality only if $m = -1$, $f(x, y) = -1$, $y = \pm x$.

[1] This use of the symbol $\triangle$ for "the area of the triangle" should cause no confusion with Δ (capital delta).

We have

$$f(x, y) = x^4 + 6\,m\,x^2 y^2 + y^4 = (x^2 - y^2)^2 + (6\,m + 2)\,x^2 y^2 \leq -1,$$

so

$$-(6\,m + 2)\,x^2 y^2 \geq 1 + (x^2 - y^2)^2 \geq 1.$$

But $1 \leq -(6m+2) \leq 4$, since $-1 \leq m \leq -\frac{1}{2}$, and it follows that $x^2 y^2 \geq \frac{1}{4}$, i.e. $|x\,y| \geq \frac{1}{2}$. Equality in each place requires the conditions stated above.

Lemma 31. *Let P_2 be a point of the region $f(x, y) \geq 1$, $|x| < y$, and let P_2' be a point of the region $f(x, y) \geq 1$, $|y| < x$. Then $\triangle\, OP_2P_2' \geq \frac{1}{2}$, with equality only if P_2 is $(0, 1)$ and P_2' is $(1, 0)$.*

We have

$$f(x, y) = (x^2 - y^2)^2 + (6\,m + 2)\,x^2 y^2 \geq 1,$$

giving

$$(x^2 - y^2)^2 \geq 1 - (6\,m + 2)\,x^2 y^2 \geq 1,$$

with final equality only if $f(x, y) = 1$, $x\,y = 0$. The result then follows from Lemma 29.

30. We now show that every lattice of determinant 1 has a point in $\mathcal{R}$, and establish a condition for this to be the best possible result.

Let L be any lattice of determinant 1 which is admissible with respect to $\mathcal{R}$.

Consider the square $|x| \leq 1$, $|y| \leq 1$, of area 4. By Lemma 15 (Minkowski's theorem), it contains a point of L other than O, say P_1. Now every point of this square except $(0, \pm 1)$ and $(\pm 1, 0)$ satisfies $f(x, y) < 1$. For if $x^2 \geq y^2$, we have

$$f(x, y) = x^4 + y^2\,(y^2 + 6\,m\,x^2) \leq x^4 \leq 1,$$

since $6\,m < -1$, with equality in each place only if $y^2 = 0$, $x^4 = 1$; and similarly if $y^2 > x^2$. Hence, by the symmetry about O, we may suppose P_1 is $(0, 1)$ or $(1, 0)$ or that it lies in one of the regions $\mathcal{R}_1$, $\mathcal{R}_1'$, where $\mathcal{R}_1$ is defined by $0 < x \leq 1$, $0 < y \leq 1$, $f(x, y) \leq -1$, and $\mathcal{R}_1'$ is the image of $\mathcal{R}_1$ in the y-axis.

Suppose that P_1 is the point $(0, 1)$. Then the line $x = 1$ contains points of L spaced at unit intervals, and so every segment of length > 1 of this line contains at least one point of L. We consider the open segment of $x = 1$ lying between the two branches of the curve $f(x, y) = -1$. Every point of this segment is an inner point of $\mathcal{R}$, except the point $(1, 0)$. Further, its length is greater than that of the corresponding intercept made by $x\,y = \pm\frac{1}{2}$, by the proof of Lemma 30, and so > 1. It follows that $(1, 0)$ is a point of L, and therefore

L is the lattice $x=\xi$, $y=\eta$, say L_0. The same conclusion clearly holds if P_1 is the point $(1, 0)$. The lattice L_0 has the point $(1, 1)$ in $\mathcal{R}_1$ and the point $(-1, 1)$ in $\mathcal{R}_1'$. Thus, by symmetry, we may suppose without loss of generality that the point P_1 lies in $\mathcal{R}_1$.

Now consider similarly the square of area 4 defined by $|x+y|\leq\sqrt{2}$, $|x-y|\leq\sqrt{2}$. By Lemma 15 it contains a point of L other than O, say P_2. Every point of this square satisfies $f(x, y)>-1$, except the points $(1/\sqrt{2}, \pm 1/\sqrt{2})$, $(-1/\sqrt{2}, \pm 1/\sqrt{2})$ when $m=-1$. It suffices to show this for the first quadrant; and then if (X, Y) is a point of $f(x, y)\leq -1$ we have, by the proof of Lemma 30, $X+Y\geq 2\sqrt{XY}\geq\sqrt{2}$, with equality only when $m=-1$, $(X, Y)=$ $=(1/\sqrt{2}, 1/\sqrt{2})$. Hence we may suppose that P_2 lies in the region $\mathcal{R}_2$ defined by $0<x+y\leq\sqrt{2}$, $0<y-x\leq\sqrt{2}$, $f(x, y)\geq 1$, unless $m=-1$, when it might be one of the points $(\pm 1/\sqrt{2}, 1/\sqrt{2})$. But if $m=-1$ and $P_2=(\pm 1/\sqrt{2}, 1/\sqrt{2})$, we find as above, using here the proof of Lemma 31, that $(\mp 1/\sqrt{2}, 1/\sqrt{2})$ is a point of L, and hence so is $(0, \sqrt{2})$, which is a point of $\mathcal{R}_2$. Thus we may suppose in any case that P_2 lies in $\mathcal{R}_2$.

Lemma 32.

$$\triangle OP_1P_2=\tfrac{1}{2}.$$

Let A be the mid-point of the line joining the points $(\sqrt{2}, 0)$, $(1, 1)$, and let C be the mid-point of the line joining the points $(-1, 1)$, $(0, \sqrt{2})$. Let these two joins produced meet in B. Then $OABC$ is clearly a square, and it contains the regions $\mathcal{R}_1$ and $\mathcal{R}_2$. Its side is of length $\sqrt{2}\cos\frac{\pi}{8}<\sqrt{2}$, so area $OABC<$ <2. Then by Lemma 27, $\triangle OP_1P_2<1$ and hence $\triangle OP_1P_2=\frac{1}{2}$.

Now let $P_3=P_1-P_2$, and let the point P_i have coordinates (x_i, y_i) for $i=1, 2, 3$. Then $\triangle OP_1P_3=\triangle OP_2P_3=\frac{1}{2}$, since $\triangle OP_1P_2=\frac{1}{2}$ by Lemma 32. We note that $\mathcal{R}_1$ lies in the triangle with vertices $(1, 1)$, $(\sqrt{2}-1, 1)$ and $(1, \sqrt{2}-1)$, excluding the two last-named vertices, and so the coordinates of P_1 satisfy $x_1>\sqrt{2}-1$, $\sqrt{2}-1<y_1\leq 1$. Similarly, we have $x_2>1-\sqrt{2}$, $1\leq y_2\leq\sqrt{2}$. It follows that $x_3=x_1-x_2>0$ and $-1<y_3\leq 0$. Since P_3 is a point of L, it is not an inner point of $\mathcal{R}$. Hence it must lie in the region $f(x, y)\geq 1$, $|y|<x$, or in the region $f(x, y)\leq -1$, $x>0$, $y<0$. In the first case, $\triangle OP_2P_3>\frac{1}{2}$ by Lemma 31, unless P_2 is $(0, 1)$ and P_3 is $(1, 0)$, that is unless L is L_0. Similarly, in the second case we have $\triangle OP_1P_3 > \frac{1}{2}$ by Lemma 30, unless L is L_0 or, if $m = -1$, the lattice L_0' defined by $x = \frac{1}{\sqrt{2}}(\xi - \eta)$, $y = \frac{1}{\sqrt{2}}(\xi + \eta)$.

We have thus proved that every lattice of determinant 1 other than L_0, and L_0' for $m = -1$, has a point other than O in the interior of $\mathcal{R}$. If $m \neq -1$, the lattice L_0 may or may not be admissible. If it is, it follows that $\Delta^* = 1$ and $k^* = 1$, and that L_0 is the only critical lattice. If $m = -1$, both L_0 and L_0' are admissible, since each gives rise to the condition $|\xi^4 - 6\xi^2\eta^2 + \eta^4| \geq 1$ for integers $\xi, \eta \neq 0, 0$, which is true. Thus $k^* = 1$, and L_0, L_0' are both critical in this case. In any case, $k^* \leq 1$.

Theorem 10. *If* $-1 \leq m \leq -\frac{1}{2}$, *there is a point* x, y, *other than the origin, of every lattice* L *of determinant* Δ, *such that*

$$|x^4 + 6mx^2y^2 + y^4| \leq \Delta^2.$$

This is the best possible result if

$$|\xi^4 + 6m\xi^2\eta^2 + \eta^4| \geq 1 \tag{30.1}$$

for all integers $\xi, \eta \neq 0, 0$. *If this is so, the lattice defined by* $x = \xi$, $y = \eta$ *is a critical lattice, and is the only one unless* $m = -1$, *when the lattice* $x = \frac{1}{\sqrt{2}}(\xi - \eta)$, $y = \frac{1}{\sqrt{2}}(\xi + \eta)$ *is also critical. The condition* (30.1) *is satisfied, for example, if* $6m = -3, -4, -5, -6$.

It remains only to prove the last statement in the theorem. In these cases the form $\xi^4 + 6m\xi^2\eta^2 + \eta^4$ has integer coefficients and is not zero for integers $\xi, \eta \neq 0, 0$, since this would imply $\xi^2 = \eta^2$ and thence $2 + 6m = 0$, which is false.

Theorem 10 leads at once to

Theorem 11. *Let* $\psi(\xi, \eta)$ *be a binary quartic form with real coefficients and* $\mathcal{D} > 0$, $\mathcal{H} > 0$, $\mathcal{K} > 0$. *Further, let* $3^2\mathcal{I}^3 \geq 7^3\mathcal{J}^2$, *so that* m *given by* (27.1) *satisfies* $-1 \leq m \leq -\frac{1}{2}$. *Then there exist integers* ξ, η, *not both zero, such that*

$$|\psi(\xi, \eta)| \leq (9m^2 - 1)^{-1/3}\mathcal{D}^{1/6}. \tag{30.2}$$

This is the best possible result if

$$|\xi^4 + 6m\xi^2\eta^2 + \eta^4| \geq 1 \tag{30.3}$$

for all integers $\xi, \eta \neq 0, 0$, *and then the sign of equality is required in* (30.2) *if, and only if*, $\psi(\xi, \eta)$ *is equivalent to a multiple of the form*

$$f(\xi, \eta) = \xi^4 + 6m\,\xi^2\eta^2 + \eta^4.$$

The condition (30.3) *is satisfied, for example, if* $6m = -3, -4, -5, -6$.

We note here that the result of HERMITE (4) for a quartic form

$$\psi(\xi, \eta) = a(\xi - \alpha_1\eta)(\xi - \alpha_2\eta)(\xi - \alpha_3\eta)(\xi - \alpha_4\eta),$$

where $\alpha_1 > \alpha_2 > \alpha_3 > \alpha_4$, is, in the notation of § 22,

$$|A_0| < \tfrac{1}{3}|a|(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_4).$$

This is really equivalent to $k^*(m) < \frac{2}{3}(1 - 3m)$, and in this shape it may be compared with the much sharper results given in Theorems 9 and 11.

31. It would be an interesting question to investigate general conditions under which (30.3) is satisfied; I have not yet seriously considered this problem. Whether or not it is satisfied in a particular case will clearly depend on the arithmetical nature of the parameter m, and the question is a very deep one. Even if we restrict our attention to forms $\psi(\xi, \eta)$ with integer (or rational) coefficients, the number m will in general be an algebraic number of degree 6. If, however, m is a rational number, it should not be difficult to decide the matter. For example, simple considerations of quadratic and biquadratic residues show that (30.3) is satisfied for all m in the range concerned with $6m$ of the form $-\frac{n}{3}$ or $-\frac{n}{4}$ (n an integer), except for the values $6m = -\frac{13}{3}, -\frac{17}{4}$.[1]

Nevertheless, we can easily obtain some metrical information about the set of values of m for which (30.3) either is or is not satisfied. In particular, we can show that both sets are of positive measure. For brevity, write $M = -6m$; then $3 \leq M \leq 6$. The condition (30.3) becomes

$$|\xi^4 - M\xi^2\eta^2 + \eta^4| \geq 1. \tag{31.1}$$

We shall describe M (or m) as admissible if (31.1) is satisfied for all integers $\xi, \eta \neq 0, 0$, and otherwise as non-admissible.

Theorem 12. *If* $A(M)$ *and* $N(M)$ *are the measures of the sets of admissible and non-admissible* M *respectively, then*

[1] That these values of m must in fact be exceptions follows at once from the remark in the next footnote.

$$2.45 < A(M) < 2.50, \qquad 0.50 < N(M) < 0.55.$$

If M is non-admissible, there exist integers r, s such that

$$(31.2) \qquad -1 < r^4 - M r^2 s^2 + s^4 < 1.$$

For this particular pair r, s, all values of M with

$$(31.3) \qquad -\frac{1}{r^2 s^2} < M - \frac{r^4 + s^4}{r^2 s^2} < \frac{1}{r^2 s^2}$$

satisfy (31.2) and so are non-admissible. Also, all non-admissible M arise in this way for some pair r, s. We may clearly suppose $(r, s) = 1$ and $r > s \geq 1$. Thus the set of non-admissible M consists of an enumerable set of intervals. These intervals have centres $M = \frac{r^4 + s^4}{r^2 s^2}$ and lengths $\frac{2}{r^2 s^2}$, where r, s are any integers with

$$(31.4) \qquad r > s \geq 1, \qquad (r, s) = 1, \qquad 3 \leq \frac{r^4 + s^4}{r^2 s^2} \leq 6.$$

These intervals may, of course, overlap, so

$$(31.5) \qquad N(M) \leq \sum \frac{2}{r^2 s^2},$$

where the summation is taken over all r, s satisfying (31.4).

We now examine the last condition in (31.4). Writing $u = r^2/s^2$, this becomes $u^2 - 3u + 1 \geq 0$, $u^2 - 6u + 1 \leq 0$. Since $u > 1$, this yields $\frac{1}{2}(3 + \sqrt{5}) \leq$ $\leq u \leq 3 + 2\sqrt{2}$, and hence

$$\tfrac{1}{2}(1 + \sqrt{5}) \leq \frac{r}{s} \leq 1 + \sqrt{2}.$$

Thus for a fixed value of $s \geq 4$, the number of values of r is at most

$$[\{(1 + \sqrt{2}) - \tfrac{1}{2}(1 + \sqrt{5})\} s] + 1 \leq 0.797\, s + 1 \leq 1.047\, s.$$

We easily find that 2, 1[1] and 5, 3 are the only pairs with $s < 4$. These give $N(M) > 0.5088$ and

$$N(M) < 0.5089 + 2 \sum_{s \geq 4} \frac{1}{r^2 s^2} < 0.5089 + \frac{2 \cdot 1.047}{\frac{1}{4}(1 + \sqrt{5})^2} \sum_{s \geq 4} \frac{1}{s^3} < 0.55.$$

The result follows, since $N(M) + A(M) = 3$.

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[1] This pair makes the whole interval $4 < M < 4\frac{1}{2}$ non-admissible.

This establishes that the results we give in Theorems 10 and 11 are best possible for a set of m of positive measure, and indeed for "most m". We conclude this section by remarking that there are non-admissible m as close as we please to any given value in the range, and, in particular, to admissible values of m.

Theorem 13. *The set of non-admissible M is dense in the interval $3 \leq M \leq 6$.*

Let M be any number in the interval, and let $\mu > 1$ be a root of $\mu^4 - M\mu^2 + 1 = 0$. Then $M = (\mu^4 + 1)/\mu^2$. Now $(\mu^4 + 1)/\mu^2$ is a continuous function of μ for $\mu > 0$, so, given $\varepsilon > 0$, there exists $\varepsilon' > 0$ such that

$$\left| M - \frac{r^4 + s^4}{r^2 s^2} \right| < \varepsilon$$

if $|\mu - r/s| < \varepsilon'$; and the latter inequality always has solutions satisfying (31.4).

V. Quartics with $\mathfrak{D} < 0$.

32. We have in the preceding sections dealt with all types of binary quartic form except those with $\mathfrak{D} < 0$, that is those having two real and two complex roots, all distinct. I have not yet attempted to investigate this case systematically, but to complete the discussion of the quartic I give here the known results, and point out a small improvement that can readily be made.

For this case we may take as standard form

$$f(x, y) = x^4 + 6mx^2y^2 - y^4,$$

with discriminant $-(1 + 9m^2)^2$. We define the region $\mathcal{R}$ by $|f(x, y)| \leq 1$, and first investigate its shape. It is bounded by the curves $f(x, y) = 1$, $f(x, y) = -1$, which pass through the points $(\pm 1, 0)$ and $(0, \pm 1)$ respectively. Noting that

$$-f(y, x) = x^4 - 6mx^2y^2 - y^4,$$

we see that we may suppose $m \geq 0$.

The asymptotes to each boundary curve are given by $y = tx$, where t is any root of

$$t^4 - 6mt^2 - 1 = 0.$$

This gives

$$t^2 = 3m \pm \sqrt{1 + 9m^2},$$

and so there are just two real asymptotes $y = \pm \mu x$, where

$$\mu^2 = 3m + \sqrt{1 + 9m^2}, \qquad \mu > 0.$$

Since we may write $f = \pm 1$ as $(y^2 - \mu^2 x^2)(y^2 + x^2/\mu^2) = \mp 1$, we see that $f = 1$ lies in the region $|y| < \mu|x|$, and $f = -1$ in the region $|y| > \mu|x|$.

The points of inflection lie on the Hessian

$$m x^4 - (1 + 3m^2) x^2 y^2 - m y^4 = 0,$$

and so on $y = \pm \lambda x$, where

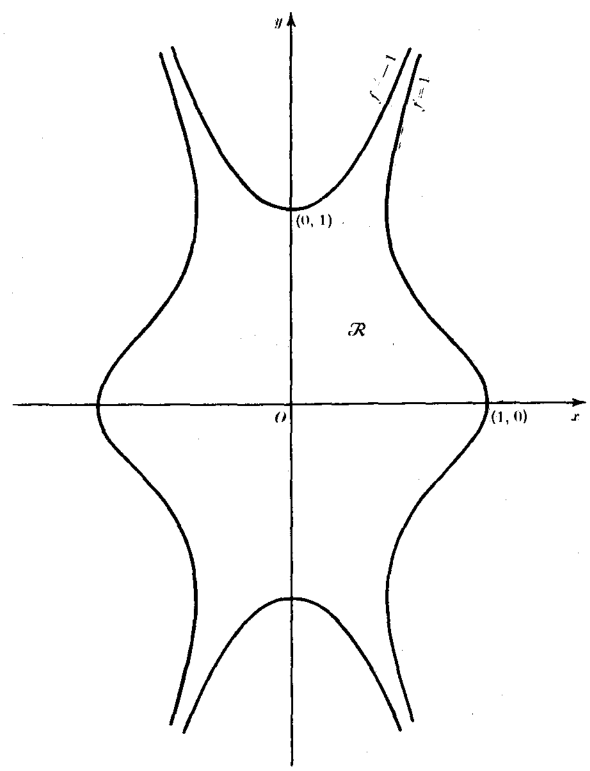

Fig. 5. The region $|x^4 + 8x^2y^2 - y^4| \leq 1$.

$$2m\lambda^2 = -(1 + 3m^2) + \sqrt{(1 + m^2)(1 + 9m^2)} > 0,$$

if $m \neq 0$. If $m = 0$, we find $xy = 0$, but the intersections with the boundary are then points of undulation. We note that the inflections lie on $f(x, y) = 1$, since

$$\begin{aligned} 2m(\mu^2 - \lambda^2) &= 2m\{3m + \sqrt{1 + 9m^2}\} + 1 + 3m^2 - \sqrt{(1 + m^2)(1 + 9m^2)} \\ &= \sqrt{1 + 9m^2}\{\sqrt{1 + 9m^2} - \sqrt{1 + m^2} + 2m\} \\ &> 0, \end{aligned}$$

and so $|\lambda| < \mu$.

The shape of the region $\mathcal{R}$ is illustrated in Fig. 5, which was drawn for $m = \frac{4}{3}$.

33. The best possible result is known only for $m = 0$, in which case MORDELL (6) showed $k^* = 4/\sqrt{17}$. The only other result known, so far as I am aware, is that of JULIA (5). Again using the notation of § 22, his result is essentially that

$$|A_0| < \tfrac{1}{6}|a|(|z_1 - z_2||z_3 - z_4| + |z_1 - z_3||z_2 - z_4| + |z_1 - z_4||z_2 - z_3|),$$

where z_1, z_2, z_3, z_4 are the roots of $\psi(z, 1) = 0$. This is equivalent to

$$k^*(m) < \tfrac{2}{3}\{1 + \sqrt{1 + 9m^2}\}.$$

We can improve this for small values of m by inscribing a convex region in $\mathcal{R}$. Clearly the rectangle defined by $|x| \leq x_0$, $|y| \leq 1$, where x_0 is the minimum positive abscissa of $f(x, y) = 1$, lies in $\mathcal{R}$. The area of this rectangle is $4x_0^2$, so, by Lemma 15, $\Delta^* \geq x_0^2$ and hence $k^*(m) \leq 1/x_0^2$. We easily find $x_0^4 = 1/(1 + 9m^2)$, so we have $k^*(m) \leq \sqrt{1 + 9m^2}$. This estimate is better than Julia's if $|m| < \frac{1}{\sqrt{3}}$.

We collect these results together in

Theorem 14. *Let $\psi(\xi, \eta)$ be a binary quartic form with real coefficients and $\mathfrak{D} < 0$, and so transformable into the form $x^4 + 6mx^2y^2 - y^4$. Then there exist integers ξ, η, not both zero, such that*

$$|\psi(\xi, \eta)| \leq \frac{k(m)}{(1 + 9m^2)^{1/8}}|\mathfrak{D}|^{1/6},$$

where we may take

$$k(0) = \frac{4}{\sqrt{17}} \text{ (best possible)};$$

$$k(m) = \sqrt{1 + 9m^2}, \text{ for } 0 < |m| < \frac{1}{\sqrt{3}};$$

$$k(m) = \tfrac{2}{3}\{1 + \sqrt{1 + 9m^2}\}, \text{ for } |m| \geq \frac{1}{\sqrt{3}}.$$

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