EXTREMAL DEVIATION IN A GEOMETRY BASED ON THE NOTION OF AREA.

By

BUCHIN SU

of HANGCHOW, CHEKIANG.

1. Introduction.

The extension of Levi-Civita's work on geodesic deviation[1] has been carried out by Berwald[2], Duschek and Mayer[3], Knebelman[4], Davies[5] and others in the geometry of Finsler-Cartan. Geodesics in such a space are naturally the extremals of the variation problem

$$\delta \int F(x^1, x^2, \ldots, x^n;\ \dot{x}^1, \dot{x}^2, \ldots, \dot{x}^n)\, dt = 0, \tag{1}$$

where $F(x^1, x^2, \ldots, x^n;\ \dot{x}^1, \dot{x}^2, \ldots, \dot{x}^n)$ denotes a function of x^i, $\dot{x}^i = \frac{dx^i}{dt}$ and is positively homogeneous of degree one in $\dot{x}^i$. On the other hand E. Cartan has obtained a geometry based on the notion of area.[6] In an n-dimensional manifold of coordinates x^i let

$$x^i = x^i(v^1, v^2, \ldots, v^{n-1}) \tag{2}$$

be the parametric representation of a hypersurface and let

---

[1] T. LEVI-CIVITA, Sur l'écart géodésique, Math. Annalen, 97 (1926), 291—320.

[2] L. BERWALD, Una forma normale invariante della seconda variazione, Atti dei Lincei, Rend. (VI) 7 (1928), 301—306.

[3] A. DUSCHEK and W. MAYER, Zur geometrischen Variationsrechnung, Monatsh. f. Math. u. Phys., 40 (1933), 294—308.

[4] M. S. KNEBELMAN, Collineations and motions in generalized space, American Journ. Math., 51 (1929), 527—564.

[5] E. T. DAVIES, Lie derivation in generalized metric spaces, Annali di Mat., (IV) 18 (1939), 261—274.

[6] E. CARTAN, Les espaces métriques fondés sur la notion d'aire. Actualités scientifiques et industrielles 72. Paris, Hermann et Cie., 1933. 47 pages.

$$\int_{(n-1)} \psi\left(x^i, \frac{\partial x^i}{\partial v^\alpha}\right) dv^1\, dv^2 \ldots dv^{n-1}\ {}^{1} \tag{3}$$

be an $(n-1)$-ple integral over a domain of the hypersurface, which is supposed to be invariant with regard to the parameter transformation, where $\psi > 0$. As the curve-length of a curve in a space of Finsler is defined by the integral in the left-hand side of (1), E. Cartan has taken (3) as $(n-1)$-dimensional surface-area of the hypersurface piece. The geometry of Cartan is uniquely determined only in the case where a certain tensor H^{ij} has the rank n. In this case we follow Berwald[2] in calling the manifold behaving the Cartan geometry a *regular Cartan space.*

In the present paper we propose to solve the following question:

How depends the deviation of an extremal hypersurface in a regular Cartan space upon the curvature and torsion of the space, when the extremal hypersurface is deformed to a nearby extremal hypersurface?

In order to express the equation of extremal deviation in an invariantive form we have first to give preliminaries about the infinitesimal deformation of a general hypersurface (§ 2).[3] The variation of the *mean curvature* H of a hypersurface is calculated in § 3, which corresponds to the formula of Duschek and Mayer concerning the variation of Eulerian vectors in a Finsler space. We establish in § 4 the above formula in tensor form and in § 5 reach the extremal deviation of a minimal hypersurface by setting $H = 0$. Finally, a generalization is briefly stated.

Throughout the present paper the notations and formulae in Berwald, Acta are utilized without explanation.

2. Preliminaries.

Let (2) be the parametric representation of a hypersurface in the Cartan space, so that the matrix

---

[1] Latin indices are in the range $1, 2, \ldots, n$ and Greek $1, 2, \ldots, n-1$.

[2] L. BERWALD, Über die n-dimensionalen Cartanschen Räume und eine Normalform der zweiten Variation eines $(n-1)$-fachen Oberflächenintegrals, Acta mathematica, 71 (1939), 191—248. This paper will be referred to as Berwald, Acta.

[3] The infinitesimal deformation of X_m immersed in V_n has been considered by many authors. Cf. for example, E. T. DAVIES, On the deformation of a subspace, Journ. London Math. Soc., 11 (1936), 295—301.

$$(4) \qquad \left(\frac{\partial x^i}{\partial v^\alpha}\right) = \begin{pmatrix} \frac{\partial x^1}{\partial v^1} & \cdots & \frac{\partial x^n}{\partial v^1} \\ \cdots & \cdots & \cdots \\ \frac{\partial x^1}{\partial v^{n-1}} & \cdots & \frac{\partial x^n}{\partial v^{n-1}} \end{pmatrix}$$

is of rank $n - 1$. By $(-1)^{k+1} p_k$ we mean, as in Berwald, Acta, the determinant formed by striking out the kth column of the matrix (4). It is known that the $(n-1)$-dimensional surface-area of a domain of the oriented hypersurface (2) is given by the $(n-1)$-ple integral of the form

$$(5) \qquad O = \int_{(n-1)} L(x, p)\, d v^1\, dv^2 \cdots d v^{n-1},$$

where the integration is calculated over the domain.

Consider the infinitesimal deformation

$$(6) \qquad \bar{x}^i = x^i + \xi^i(x)\, \delta t,$$

which carries on the point (x) into the point $(\bar{x})$ infinitely near (x), δt being an infinitesimal. In (6), $\xi^i(x)$, $i = 1, 2, \ldots, n$, denotes an analytic function of position. The hypersurface S given by (2) is now infinitesimally deformed into another hypersurface $\bar{S}$ of the equations

$$(7) \qquad \bar{x}^i = \bar{x}^i\ (v^1, v^2, \ldots, v^{n-1}),$$

and consequently, the matrix (4) is transformed to

$$(8) \qquad \left(\frac{\partial \bar{x}^i}{\partial v^\alpha}\right) = \left(\frac{\partial x^i}{\partial v^\alpha} + \frac{\partial \xi^i}{\partial x^j} \frac{\partial x^j}{\partial v^\alpha} \delta t\right).$$

Denoting the corresponding variables of p_k by $\bar{p}_k$ $(k = 1, 2, \ldots, n)$ and taking account of (8), we can easily show that

$$(9) \qquad \bar{p}_k = p_k + \left(\frac{\partial \xi^h}{\partial x^h} p_k - \frac{\partial \xi^h}{\partial x^k} p_h\right) \delta t \qquad (k = 1, 2, \ldots, n),$$

where the summation convention for repeating indices is used and higher powers of δt than the first are neglected.

In virtue of (6) and (9) there is no difficulty in expressing the corresponding quantity $\bar{A}{:}{:}{:}$ of any geometrical being $A{:}{:}{:}$ in terms of $A{:}{:}{:}$, ξ^i, p_k and their derivatives. Thus we obtain

$$(10)\quad \begin{cases} \bar{L} = L(\bar{x}, \bar{p}) = L(x, p) + \left\{\dfrac{\partial L}{\partial x^i}\xi^i + \dfrac{\partial L}{\partial p_k}\left(\dfrac{\partial \xi^h}{\partial x^h} p_k - \dfrac{\partial \xi^h}{\partial x^k} p_h\right)\right\}\delta t, \\ \bar{g}_{ik} = g_{ik}(\bar{x}, \bar{p}) = g_{ik}(x, p) + \left\{\dfrac{\partial g_{ik}}{\partial x^l}\xi^l + \dfrac{\partial g_{ik}}{\partial p_l}\left(\dfrac{\partial \xi^h}{\partial x^h} p_l - \dfrac{\partial \xi^h}{\partial x^l} p_h\right)\right\}\delta t, \\ \bar{g} = g(\bar{x}, \bar{p}) = g(x, p) + \left\{\dfrac{\partial g}{\partial x^l}\xi^l + \dfrac{\partial g}{\partial p_l}\left(\dfrac{\partial \xi^h}{\partial x^h} p_l - \dfrac{\partial \xi^h}{\partial x^l} p_h\right)\right\}\delta t, \end{cases}$$

to within terms of higher order in δt.

On account of the homogeneity of L, g_{ik}, g and the definition of A^l_{ik} [Berwald, Acta, (5.4)] we can rewrite (10) in the form

$$(11)\quad \begin{cases} \bar{L} = L\left\{1 + \left(\dfrac{1}{L}\dfrac{\partial L}{\partial x^i}\xi^i + \dfrac{\partial \xi^h}{\partial x^h} - \dfrac{\partial \xi^h}{\partial x^k} l^k l_h\right)\delta t\right\}, \\ \bar{g}_{ik} = g_{ik} + \left\{\dfrac{\partial g_{ik}}{\partial x^l}\xi^l - 2 A^l_{ik}\dfrac{\partial \xi^h}{\partial x^l} l_h\right\}\delta t, \\ \sqrt{\bar{g}} = \sqrt{g}\left\{1 + \left(\dfrac{\partial \log \sqrt{g}}{\partial x^l}\xi^l - A^k \dfrac{\partial \xi^h}{\partial x^k} l_h\right)\delta t\right\}. \end{cases}$$

In a similar way we can determine $\bar{l}_i$ at the point $(\bar{x})$ of $\bar{S}$, either by the definition $\bar{l}_i = (\sqrt{\bar{g}}\,\bar{p}_i)/\bar{L}$ and (11), or directly by the formula of expansion as well as the relation [Berwald, Acta, (8.11)]

$$(12)\quad \frac{\partial \log \sqrt{g}}{\partial x^h} = \Gamma^{*m}_{mh} - A^m \Gamma^*_{moh}.$$

The result of computation is as follows:

$$(13)\quad \bar{l}_i = l_i - \left\{\frac{\partial \xi^h}{\partial x^i} l_h + (A^m - l^m)\left(\Gamma^*_{moh}\xi^h + \frac{\partial \xi^h}{\partial x^m} l_h\right) l_i\right\}\delta t.$$

It is convenient to give here the corresponding formula for the contravariant components $\bar{l}^i$:

$$(14)\quad \bar{l}^i = l^i + \left\{l^i(\Gamma^*_{ooh} + A^m \Gamma^*_{moh})\xi^h - (\Gamma^{*io}_{h} + \Gamma^{*oi}_{h})\xi^h + \frac{\partial \xi^h}{\partial x^k} l_h (A^k l^i + l^i l^k - g^{ik})\right\}\delta t.$$

This may also be obtained if use is made of (13) and the relation

$$(15)\quad \bar{g}^{ij} = g^{ij} + \left\{\xi^h(2 A^{ijm}\Gamma^*_{moh} - \Gamma^{*ji}_{h} - \Gamma^{*ij}_{h}) + 2 A^{ijk}\frac{\partial \xi^h}{\partial x^k} l_h\right\}\delta t,$$

a relation deduced from known identities [Berwald, Acta, (5.5), (8.8)].

For the subsequent use we also give the following formulae:

$$\bar{\Gamma}^{*h}_{ik} = \Gamma^{*h}_{ik} + \left\{\frac{\partial \Gamma^{*h}_{ik}}{\partial x^j}\xi^j - \Gamma^{*h}_{ik}\|^m \frac{\partial \xi^j}{\partial x^m} l_j\right\}\delta t, \tag{16}$$

$$\bar{\Gamma}^{*}_{iok} = \Gamma^{*}_{iok} + \left\{l_h \frac{\partial \Gamma^{*h}_{ik}}{\partial x^j}\xi^j - l_h \Gamma^{*h}_{ik}\|^m \frac{\partial \xi^j}{\partial x^m} l_j \right. \tag{17}$$
$$\left. - \Gamma^{*h}_{ik}\frac{\partial \xi^j}{\partial x^h} l_j - (A^m - l^m)\left(\Gamma^{*}_{mor}\xi^r + \frac{\partial \xi^r}{\partial x^m} l_r\right)\Gamma^{*}_{iok}\right\}\delta t,$$

the latter being a consequence of (13) and (16).

3. Variation of the Mean Curvature.

We are now in a position to consider the variation of the mean curvature H of a hypersurface S in the Cartan space.

Since the equations of S are given by (2), we have

$$\bar{x}^i_\varrho = x^i_\varrho + \frac{\partial \xi^i}{\partial x^j} x^j_\varrho \,\delta t, \tag{18}$$

so that

$$\bar{\Gamma}^{*}_{\varrho o \sigma} = \Gamma^{*}_{\varrho o \sigma} + \left\{l_h x^i_\varrho x^k_\sigma \frac{\partial \Gamma^{*h}_{ik}}{\partial x^j}\xi^j - \Gamma^{*h}_{\varrho\sigma}\frac{\partial \xi^r}{\partial x^h} l_r \right. \tag{19}$$
$$- l_h x^i_\varrho x^k_\sigma \Gamma^{*h}_{ik}\|^m \frac{\partial \xi^r}{\partial x^m} l_r$$
$$- (A^m - l^m)\left(\Gamma^{*}_{mor}\xi^r + \frac{\partial \xi^r}{\partial x^m} l_r\right)\Gamma^{*}_{\varrho o \sigma}$$
$$\left. + \Gamma^{*}_{\varrho o k} x^i_\sigma \frac{\partial x^k}{\partial x^i} + \Gamma^{*}_{i o \sigma} x^j_\varrho \frac{\partial \xi^i}{\partial x^j}\right\}\delta t.$$

Now, the second differentiation of (6) gives

$$\frac{\partial^2 \bar{x}^i}{\partial v^\varrho \partial v^\sigma} = \frac{\partial^2 x^i}{\partial v^\varrho \partial v^\sigma} + \left\{\frac{\partial^2 \xi^i}{\partial x^j \partial x^k} x^j_\sigma x^k_\varrho + \frac{\partial \xi^i}{\partial x^j}\frac{\partial^2 x^j}{\partial v^\varrho \partial v^\sigma}\right\}\delta t, \tag{20}$$

and the *second grundtensor* $a_{\varrho\sigma}$ of the hypersurface S is given by [Berwald, Acta, (24. 5)]

$$a_{\varrho\tau} = a_{\tau\varrho} = l_i \frac{\partial^2 x^i}{\partial v^\varrho \partial v^\tau} + \Gamma^{*}_{\varrho o \tau}. \tag{21}$$

Hence the second grundtensor $\bar{a}_{\varrho\sigma}$ of the deformed hypersurface $\bar{S}$ turns out as a necessary consequence of (13), (19) and (20), namely,

$$(22)\qquad \bar{a}_{\varrho\sigma} = a_{\varrho\sigma} + \Big\{ l_h\, x^i_\varrho\, x^k_\sigma \frac{\partial \Gamma^{*h}_{ik}}{\partial x^j}\xi^j - l_h\, x^i_\varrho\, x^k_\sigma\, \Gamma^{*h}_{ik}\|^m \frac{\partial \xi^r}{\partial x^m} l_r - \Gamma^{*h}_{\varrho\sigma}\frac{\partial \xi^r}{\partial x^h} l_r - (A^m - l^m)\Big(\Gamma^{*}_{moh}\,\xi^h + \frac{\partial \xi^h}{\partial x^m} l_h\Big) a_{\varrho\sigma} + l_h\, x^j_\varrho\, x^k_\sigma \frac{\partial^2 \xi^h}{\partial x^j \partial x^k} + \Gamma^{*}_{\varrho o k}\, x^j_\sigma \frac{\partial \xi^k}{\partial x^j} + \Gamma^{*}_{\sigma o k}\, x^j_\varrho \frac{\partial \xi^k}{\partial x^j}\Big\}\,\delta t.$$

On the other hand the second equation (11) and the equation (18) gives

$$(23)\qquad \bar{g}_{\varrho\sigma} = g_{\varrho\sigma} + \Big\{ x^i_\varrho\, x^k_\sigma\, \xi^h \frac{\partial g_{ik}}{\partial x^h} - 2 A^h_{\varrho\sigma}\frac{\partial \xi^k}{\partial x^h} l_k + g_{ik}\, x^i_\varrho\, x^j_\sigma \frac{\partial \xi^k}{\partial x^j} + g_{ik}\, x^k_\sigma\, x^j_\varrho \frac{\partial \xi^i}{\partial x^j}\Big\}\,\delta t,$$

so that

$$(24)\qquad \det.\,(\bar{g}_{\varrho\sigma}) = L^2(x, p)\,(1 + g^{\lambda\mu} h_{\lambda\mu}\,\delta t),$$

$$(25)\qquad \operatorname{adj.}(\bar{g}_{\varrho\sigma}) = \operatorname{adj.}(g_{\varrho\sigma}) + h_{\lambda\mu}\,\operatorname{adj.}\begin{vmatrix} g_{\varrho\sigma} & g_{\varrho\mu} \\ g_{\lambda\sigma} & g_{\lambda\mu}\end{vmatrix}\,\delta t.$$

Consequently, the reciprocal of $g_{\varrho\sigma}$ subjects to the transformation

$$(26)\qquad \bar{g}^{\varrho\sigma} = g^{\varrho\sigma} + \Big\{ L^{-2} h_{\lambda\mu}\,\operatorname{adj.}\begin{vmatrix} g_{\varrho\sigma} & g_{\varrho\mu} \\ g_{\lambda\sigma} & g_{\lambda\mu}\end{vmatrix} - g^{\varrho\sigma}\Big(\frac{\partial \log g}{\partial x^h}\xi^h - l^i\, l^k\, \xi^h \frac{\partial g_{ik}}{\partial x^h} + 2\frac{\partial \xi^h}{\partial x^h} - 2\, l^h\, l_k \frac{\partial \xi^k}{\partial x^h}\Big)\Big\}\,\delta t.$$

In deriving these equations we have put

$$(27)\qquad h_{\lambda\mu} = x^i_\lambda\, x^k_\mu\, \xi^h \frac{\partial g_{ik}}{\partial x^h} - 2 A^h_{\lambda\mu}\frac{\partial \xi^k}{\partial x^h} l_k + g_{ik}\, x^i_\lambda\, x^j_\mu \frac{\partial \xi^k}{\partial x^j} + g_{ik}\, x^k_\mu\, x^j_\lambda \frac{\partial \xi^i}{\partial x^j}$$

and utilized the relation [Berwald, Acta, (23.6)]

$$(28)\qquad \det.\,(g_{\varrho\sigma}) = L^2(x, p).$$

Thus, upon substituting (22) and (26) into the expression

$$(29)\qquad \bar{a}^\varrho_\varrho = \bar{g}^{\varrho\sigma}\,\bar{a}_{\varrho\sigma} = (n-1)\,\bar{H},$$

where $\bar{H}$ denotes the mean curvature of the transform $\bar{S}$, and setting

$$(30)\qquad \delta\, a^\varrho_\varrho = \bar{a}^\varrho_\varrho - a^\varrho_\varrho,$$

we are led to the following equation:

$$(31)\qquad \frac{\delta a_\varrho^\varrho}{\delta t} = l_h g^{\varrho\sigma} x_\varrho^i x_\sigma^k \frac{\partial \Gamma_{ik}^{*h}}{\partial x^j} \xi^j - g^{\varrho\sigma} \Gamma_{\varrho\sigma}^{*h} \frac{\partial \xi^r}{\partial x^h} l_r$$
$$- l_h g^{\varrho\sigma} x_\varrho^i x_\sigma^k \Gamma_{ik}^{*h} \|^m \frac{\partial \xi^r}{\partial x^m} l_r + l_h g^{\varrho\sigma} x_\varrho^j x_\sigma^k \frac{\partial^2 \xi^h}{\partial x^j \partial x^k}$$
$$- \left\{ (A^m - l^m) \left(\Gamma_{m0h}^* \xi^h + \frac{\partial \xi^h}{\partial x^m} l_h \right) + \frac{\partial \log g}{\partial x^h} \xi^h \right.$$
$$\left. - l^i l^k \xi^h \frac{\partial g_{ik}}{\partial x^h} + 2 \frac{\partial \xi^h}{\partial x^h} - 2 l^h l_k \frac{\partial \xi^k}{\partial x^h} \right\} a_\varrho^\varrho$$
$$+ g^{\varrho\sigma} \Gamma_{\varrho 0 k}^* x_\sigma^j \frac{\partial \xi^k}{\partial x^j} + g^{\varrho\sigma} \Gamma_{\sigma 0 k}^* x_\varrho^j \frac{\partial \xi^k}{\partial x^j}$$
$$+ L^{-2} a_{\varrho\sigma} h_{\lambda\mu} \,\text{adj.} \begin{vmatrix} g_{\varrho\sigma} & g_{\varrho\mu} \\ g_{\lambda\sigma} & g_{\lambda\mu} \end{vmatrix}.$$

In order to carry out the computation of the last term in the right-hand side of (31), it shall be mentioned that by the known result of determinant theory [1]

$$(32)\qquad \text{adj.} \begin{vmatrix} g_{\varrho\sigma} & g_{\varrho\mu} \\ g_{\lambda\sigma} & g_{\lambda\mu} \end{vmatrix} = \det. (g_{\varrho\sigma}) \cdot (g^{\varrho\sigma} g^{\lambda\mu} - g^{\lambda\sigma} g^{\varrho\mu}).$$

Therefore we have

$$(33)\qquad L^{-2} a_{\varrho\sigma} h_{\lambda\mu} \,\text{adj.} \begin{vmatrix} g_{\varrho\sigma} & g_{\varrho\mu} \\ g_{\lambda\sigma} & g_{\lambda\mu} \end{vmatrix}$$
$$= \tfrac{1}{2} (a_{\varrho\sigma} h_{\lambda\mu} - a_{\lambda\sigma} h_{\varrho\mu}) (g^{\varrho\sigma} g^{\lambda\mu} - g^{\lambda\sigma} g^{\varrho\mu})$$
$$= a_\varrho^\varrho h_\lambda^\lambda - a_\lambda^\varrho h_\varrho^\lambda,$$

whence follows the equation:

$$(34)\qquad \frac{\delta a_\varrho^\varrho}{\delta t} = E - a_\varrho^\varrho G.$$

In (34) we have put

$$(35)\qquad E \equiv l_h g^{\varrho\sigma} x_\varrho^i x_\sigma^k \frac{\partial \Gamma_{ik}^{*h}}{\partial x^j} \xi^j - g^{\varrho\sigma} \Gamma_{\varrho\sigma}^{*h} \frac{\partial \xi^r}{\partial x^h} l_r$$
$$- g^{\varrho\sigma} x_\varrho^i x_\sigma^k l_h \Gamma_{ik}^{*h} \|^m \frac{\partial \xi^r}{\partial x^m} l_r + g^{\varrho\sigma} x_\varrho^j x_\sigma^k \frac{\partial^2 \xi^h}{\partial x^j \partial x^k} l_h$$
$$+ 2 g^{\varrho\sigma} \Gamma_{\varrho 0 k}^* x_\sigma^j \frac{\partial \xi^k}{\partial x^j} - a_\varrho^\lambda h_\lambda^\varrho$$

[1] Cf. e. g. SCOTT and MATHEWS, Theory of determinants, Cambridge (1904), p. 63.

and

$$(36)\qquad G = \frac{\partial \log g}{\partial x^h}\xi^h - l^i l^k \xi^h \frac{\partial g_{ik}}{\partial x^h} + 2\frac{\partial \xi^h}{\partial x^h} - 2 l^h l_k \frac{\partial \xi^k}{\partial x^h}$$

$$+ (A^m - l^m)\left(\Gamma^*_{m o h}\xi^h + \frac{\partial \xi^h}{\partial x^m} l_h\right) - h^\lambda_\lambda.$$

The equation (34) for the variation of the mean curvature of a general hypersurface in the Cartan space evidently furnishes an analogue of the formula in a Finsler space due to Duschek and Mayer.[1]

4. Tensor Form of *E* and *G*.

Before we proceed further, expressions h^ϱ_λ and h^λ_λ should be calculated by using (27), namely,

$$(37)\qquad h^\varrho_\lambda = g^{\varrho\mu} x^i_\lambda x^k_\mu \xi^h \frac{\partial g_{ik}}{\partial x^h} - 2 A^{\varrho h}_\lambda \frac{\partial \xi^k}{\partial x^h} l_k + g_{ik} x^i_\lambda x^j_\mu g^{\varrho\mu}\frac{\partial \xi^k}{\partial x^j} + g_{ik} x^k_\mu x^j_\lambda g^{\varrho\mu}\frac{\partial \xi^i}{\partial x^j}.$$

In particular, taking account of the relations [Berwald, Acta, (23. 21), (26. 16)]

$$(38)\qquad A^{\varrho h}_\varrho = 0,\quad g^{\varrho\sigma} x^h_\varrho x^j_\sigma = g^{hj} - l^h l^j,$$

we have

$$(39)\qquad h^\lambda_\lambda = 2\frac{\partial \xi^h}{\partial x^h} - 2 l^j l_k \frac{\partial \xi^k}{\partial x^j} + \frac{\partial \log g}{\partial x^h}\xi^h - l^i l^k \frac{\partial g_{ik}}{\partial x^h}\xi^h.$$

Substitution of (37) and (39) into (35) and (36) respectively gives

$$(35)'\qquad E = g^{\varrho\sigma} x^i_\varrho x^k_\sigma l_h \frac{\partial \Gamma^{*h}_{ik}}{\partial x^j}\xi^j - g^{\varrho\sigma} x^i_\varrho x^k_\sigma l_h \Gamma^{*h}_{ik}\|^m \frac{\partial \xi^r}{\partial x^m} l_r$$

$$- g^{\varrho\sigma}\Gamma^{*h}_{\varrho\sigma}\frac{\partial \xi^r}{\partial x^h} l_r + g^{\varrho\sigma} x^j_\varrho x^k_\sigma l_h \frac{\partial^2 \xi^h}{\partial x^j \partial x^k}$$

$$+ 2 g^{\varrho\sigma}\Gamma^*_{\varrho o k} x^j_\sigma \frac{\partial \xi^k}{\partial x^j} - a^{\lambda\mu} x^i_\lambda x^k_\mu \xi^h \frac{\partial g_{ik}}{\partial x^h}$$

$$+ 2 a^\lambda_\varrho A^{\varrho h}_\lambda \frac{\partial \xi^k}{\partial x^h} l_k - 2 g_{ik} x^i_\lambda x^j_\mu a^{\lambda\mu}\frac{\partial \xi^k}{\partial x^j},$$

$$(36)'\qquad G = (A^m \Gamma^*_{m o h} - \Gamma^*_{o o h})\xi^h + \frac{\partial \xi^h}{\partial x^m} l_h (A^m - l^m),$$

---

[1] Duschek and Mayer, loc. cit., formula (30), p. 301.

provided that the formulae (8.11) and (8.13) in Berwald, Acta are used for simplification.

We come now to rewrite E and G in an invariantive form. For this purpose the covariant derivatives of ξ^h shall be utilized. Noticing that ξ^h is a function of position we have

$$(40) \qquad \xi^h|_j = \frac{\partial \xi^h}{\partial x^j} + \xi^r \Gamma^{*h}_{rj},$$

$$(41) \qquad \xi^h|_{jk} = \frac{\partial^2 \xi^h}{\partial x^j \partial x^k} + (\xi^r|_k - \xi^m \Gamma^{*r}_{mk}) \Gamma^{*h}_{rj} + \xi^r \left(\frac{\partial \Gamma^{*h}_{rj}}{\partial x^k} + \Gamma^{*h}_{rj}\|^m \Gamma^{*}_{m o k} \right) + \Gamma^{*h}_{mk} \xi^m|_j - \Gamma^{*m}_{jk} \xi^h|_m,$$

so that partial derivatives $\partial \xi^h / \partial x^j$ and $\partial^2 \xi^h / \partial x^j \partial x^k$ in E and G can be substituted by the corresponding covariant derivatives.

In the first place we shall give a reduction of E in tensor form. After (40) and (41) the expression may be written as

$$(42) \qquad E = l_h g^{\varrho\sigma} x^j_\varrho x^k_\sigma \xi^h|_{jk} + C^m_p \xi^p|_m + l_h g^{\varrho\sigma} x^j_\varrho x^k_\sigma \xi^r \bar{R}^h_{jkr} + \xi^r \left(2 g_{ik} x^i_\lambda x^j_\mu a^{\lambda\mu} \Gamma^{*k}_{rj} - x^i_\lambda x^j_\mu a^{\lambda\mu} \frac{\partial g_{ik}}{\partial x^r} - 2 a^\lambda_\varrho A^{\varrho h}_\lambda \Gamma^{*}_{roh} \right),$$

where [cf. Berwald, Acta (12.7)]

$$(43) \qquad \bar{R}^h_{jkr} = \frac{\partial \Gamma^{*h}_{jk}}{\partial x^r} - \frac{\partial \Gamma^{*h}_{jr}}{\partial x^k} + \Gamma^{*h}_{rm} \Gamma^{*m}_{jk} - \Gamma^{*h}_{km} \Gamma^{*m}_{rj} + \Gamma^{*h}_{jk}\|^m \Gamma^{*}_{mor} - \Gamma^{*h}_{jr}\|^m \Gamma^{*}_{mok}$$

and

$$(44) \qquad C^m_p = 2 a^\lambda_\varrho A^{\varrho m}_\lambda l_p - 2 g_{ip} x^i_\varrho x^m_\sigma a^{\varrho\sigma} - l_h g^{\varrho\sigma} x^i_\varrho x^k_\sigma \Gamma^{*h}_{ik}\|^m l_p.$$

The expression in the parenthesis of (42) vanishes, because

$$2 a^\lambda_\varrho A^{\varrho h}_\lambda = 2 a^{\lambda\mu} A^h_{\lambda\mu} = a^{\lambda\mu} x^i_\lambda x^k_\mu g_{ik}\|^h,$$

$$2 g_{ik} x^i_\lambda x^j_\mu a^{\lambda\mu} \Gamma^{*k}_{rj} = (g_{im} \Gamma^{*m}_{kr} + g_{mk} \Gamma^{*m}_{ir}) x^i_\lambda x^k_\mu a^{\lambda\mu}$$

and consequently the expression reduces to

$$- a^{\lambda\mu} x^i_\lambda x^k_\mu \left(\frac{\partial g_{ik}}{\partial x^r} + g_{ik}\|^m \Gamma^{*}_{mor} - g_{im} \Gamma^{*m}_{kr} - g_{mk} \Gamma^{*m}_{ir} \right) = - a^{\lambda\mu} x^i_\lambda x^k_\mu g_{ik}|_r = 0.$$

It remains for us to simplify the expression C_p^m. In virtue of a formula [Berwald, Acta (16.8)] we have

(45) $$x_\varrho^i x_\sigma^k l_h \Gamma_{ik}^{*h} \|^m = - x_\varrho^i x_\sigma^k A_{ik}^m |_o - K_s^g (A^m |_g - A^r A_{rg}^m |_o) A_{\varrho\sigma}^s,$$

(46) $$g^{\varrho\sigma} x_\varrho^i x_\sigma^k l_h \Gamma_{ik}^{*h} \|^m = - g^{\varrho\sigma} x_\varrho^i x_\sigma^k A_{ik}^m |_o,$$

since $l_{h|r} = 0$, $A^{iko} = 0$ and $A_{\varrho\sigma}^s g^{\varrho\sigma} = A_\varrho^{\varrho s} = 0$ [Berwald, Acta (8.6), (5.5) and (23.21)]. From (44) it follows that

(44)' $$C_p^m = 2\, a^{\varrho\sigma} (A_{\varrho o}^m l_p - g_{\varrho\mu} x_\sigma^m x_p^\mu) - g^{\varrho\sigma} x_p^i x_\sigma^k A_{ik}^m |_o l_p.$$

A reference to the relation [Berwald, Acta (23.16)]

(47) $$g^{\varrho\sigma} x_\varrho^i x_\sigma^k = g^{ik} - l^i l^k$$

shows immediately

$$g^{\varrho\sigma} x_\varrho^i x_\sigma^k A_{ik}^m |_o = (g^{ik} - l^i l^k) A_{ik}^m |_o = A_i^{im} |_o - A^{oom} |_o = 0.$$

In consequence,

(48) $$C_p^m = 2\, (a^{\varrho\sigma} A_{\varrho\sigma}^\tau l_p - a_\mu^\tau x_\varrho^\mu) x_\tau^m.$$

Thus, noticing that the covariant derivatives of l_h vanish identically we obtain

(49) $$E = g^{\varrho\sigma} x_\varrho^j x_\sigma^k (l_h \xi^h) |_{jk} + 2\, (a^{\varrho\sigma} A_{\varrho o}^\tau l_p - a_\mu^\tau x_p^\mu) \xi^p |_\tau + g^{\varrho\sigma} x_\varrho^j x_\sigma^k \bar{R}_{jokr} \xi^r.$$

The last term in E as given by (49) may further reduce to a simpler form if we notice that, using Berwald, Acta (12.12),

$$\bar{R}_{jokr} = (\delta_j^q + l_j A^q) R_{qokr}$$

and accordingly,

(50) $$g^{\varrho\sigma} x_\varrho^j x_\sigma^k \bar{R}_{jokr} = g^{\varrho\sigma} x_\varrho^j x_\sigma^k R_{jokr} = R_{\varrho or}^\varrho.$$

Hence follows the result:

(51) $$E = g^{\varrho\sigma} (l_h \xi^h) |_{\varrho\sigma} + 2\, (a^{\varrho\sigma} A_{\varrho\sigma}^\tau l_p - a_\mu^\tau x_p^\mu) \xi^p |_\tau + R_{\varrho or}^\varrho \xi^r,$$

where we have used the abbreviation

(52) $$A:::|_{\varrho\sigma} = A:::|_{ij} x_\varrho^i x_\sigma^j$$

for a scalar or tensor $A:::$.

Having thus established the tensor form of E, we have in the next place to consider G, for which (36)' and (40) are sufficient. The result runs as follows:

(53) $$G = A^\tau (l_h \xi^h) |_\tau - (l_h \xi^h) |_o.$$

In case the deviation is in the *normal* direction

(54) $$\xi^i = V l^i$$

the equation (34) takes the form

(55) $$\frac{\delta a^\varrho_\varrho}{\delta t} = E_N - G_N a^\varrho_\varrho,$$

where

(56) $$E_N = g^{\varrho\sigma} V_{|\varrho\sigma} + 2 a^{\varrho\sigma} A^\tau_{\varrho\sigma} V_{|\tau} + R^\varrho_{\varrho o o} V,$$

(57) $$G_N = A^\tau V_{|\tau} - V_{|o}.$$

5. Extremal Deviation of a Minimal Hypersurface.

Suppose that the original hypersurface S which we have to deform into a near one is minimal, namely,

(58) $$a^\varrho_\varrho = 0.$$

In order that the deformed hypersurface $\bar{S}$ be minimal also, it is necessary and sufficient that $E = 0$, namely,

(59) $$g^{\varrho\sigma}(l_h \xi^h)_{|\varrho\sigma} + 2(a^{\varrho\sigma} A^\tau_{\varrho\sigma} l_p - a^\tau_\mu x^\mu_p) \xi^p|_\tau + R^\varrho_{\varrho o r} \xi^r = 0.$$

This we shall call the *equation of extremal deviation.*

Putting

(60) $$\xi^i = \lambda^\alpha x^i_\alpha + V l^i$$

we are led to

(61) $$g^{\varrho\sigma} V_{|\varrho\sigma} + 2 a^{\varrho\sigma} A^\tau_{\varrho\sigma} V_{|\tau} + R^\varrho_{\varrho o o} V + (R^\varrho_{\varrho o \alpha} - 2 a^\tau_\mu x^\mu_p x^p_\alpha |_\tau) \lambda^\alpha - 2 a^\tau_\mu \lambda^\mu |_\tau = 0.$$

For the purpose of rewriting this equation in explicit form it is convenient to prove here the following formulae:

(62) $$V\|^m = V A^m + \lambda^\alpha x^m_\alpha,$$

(63) $$\frac{\partial V}{\partial x^j} x^j_\sigma + V\|^m \Gamma^*_{m o \sigma} = \frac{\partial V}{\partial v^\sigma} + a_{\varrho\sigma} V\|^m x^\varrho_m.$$

In fact, the function ξ^i being of position alone must obey the condition

(64) $$\lambda^\alpha \|^m x^i_\alpha + \lambda^\alpha x^i_\alpha \|^m + V\|^m l^i + V l^i \|^m = 0.$$

By composing this equation with l_i and using the relations

$$l_i x^i_\alpha \|^m = - x^i_\alpha l_i \|^m,$$

$$l_i \|^m = \delta^m_i - l_i (l^m - A^m),$$

$$l^i \|^m = g^{im} - l^i (l^m + A^m),$$

we easily find (62). On account of the equation $A^o = 0$ it follows that $V\|^o = 0$, that is, V is homogeneous of degree zero in p_i. For such a function we have

$$\begin{aligned} \frac{\partial V}{\partial v^\sigma} &= \frac{\partial V}{\partial x^j} x^j_\sigma + \frac{\partial V}{\partial p_m} \frac{\partial p_m}{\partial x^\sigma} \\ &= \frac{\partial V}{\partial x^j} x^j_\sigma + V\|^m \left(\frac{\partial l_m}{\partial v^\sigma} + l_m \frac{\partial}{\partial v^\sigma} \log \frac{L}{\sqrt{g}} \right) \\ &= \frac{\partial V}{\partial x^j} x^j_\sigma + V\|^m (\Gamma^*_{\sigma o m} - a_{\varrho\sigma} x^\varrho_m), \end{aligned}$$

which proves (63).

The last equation implies

$$V_{|\tau} = \frac{\partial V}{\partial v^\tau} + a_{\varrho\tau} A^\varrho V + a_{\varrho\tau} \lambda^\varrho. \tag{65}$$

As $V_{|i}$ is also homogeneous of degree zero in p_i,

$$V_{|i\sigma} = \frac{\partial V_{|i}}{\partial v^\sigma} + a_{\nu\sigma} V_{|i}\|^m x^\nu_m - V_{|p} \Gamma^{*\,p}_{i\sigma}$$

and consequently

$$\begin{aligned} V_{|\varrho\sigma} &= \frac{\partial V_{|i}}{\partial v^\sigma} x^i_\varrho + a_{\nu\sigma} V_{|i}\|^m x^\nu_m x^i_\varrho - V_{|p} \Gamma^{*\,p}_{\varrho\sigma} \\ &= \frac{\partial V_{|\varrho}}{\partial v^\sigma} - V_{|i} \frac{\partial^2 x^i}{\partial v^\varrho \partial v^\sigma} + a_{\nu\sigma} V_{|i}\|^m x^\nu_m x^i_\varrho - V_{|p} \Gamma^{*\,p}_{\varrho\sigma}. \end{aligned}$$

In virtue of (65) and the fundamental equations of the hypersurface [Berwald, Acta (25.6)] we obtain

$$\begin{aligned} V_{|\varrho\sigma} = &\frac{\partial^2 V}{\partial v^\varrho \partial v^\sigma} + a_{\mu\varrho} A^\mu \frac{\partial V}{\partial v^\sigma} + \frac{\partial (a_{\mu\varrho} A^\mu)}{\partial v^\sigma} V \\ &- (\Gamma^\tau_{\varrho\sigma} + A^{\tau\mu}_\varrho a_{\mu\sigma}) \left(\frac{\partial V}{\partial v^\tau} + a_{\mu\tau} A^\mu V \right) \\ &- a_{\varrho\sigma} V_{|o} + a_{\nu\sigma} V_{|i}\|^m x^i_\varrho x^\nu_m \\ &+ a_{\nu\sigma} \frac{\partial \lambda^\nu}{\partial v^\sigma} + \left\{ \frac{\partial a_{\mu\varrho}}{\partial v^\sigma} - a_{\mu\tau} (\Gamma^\tau_{\varrho\sigma} + A^{\tau\nu}_\varrho a_{\nu\sigma}) \right\} \lambda^\mu. \end{aligned}$$

Hence under the assumption (58) follows

$$(66)\qquad g^{\varrho\sigma} V_{|\varrho\sigma} = g^{\varrho\sigma} \frac{\partial^2 V}{\partial v^\varrho \partial v^\sigma} + (a^\tau_\mu A^\mu - \Gamma^{\tau\varrho}_{\varrho} - A^{\sigma\tau\nu} a_{\nu\sigma}) \frac{\partial V}{\partial v^\tau}$$

$$+ \left\{ g^{\varrho\sigma} \frac{\partial (a_{\mu\varrho} A^\mu)}{\partial v^\sigma} - a_{\mu\tau} A^\mu (\Gamma^{\tau\varrho}_{\varrho} + A^{\sigma\tau\nu} a_{\nu\sigma}) \right\} V$$

$$+ a^\varrho_\nu \frac{\partial \lambda^\nu}{\partial v^\varrho} + g^{\varrho\sigma} \left\{ \frac{\partial a_{\mu\varrho}}{\partial v^\sigma} - a_{\mu\tau} (\Gamma^{\tau}_{\varrho\sigma} + A^{\tau\nu}_{\varrho} a_{\nu\sigma}) \right\} \lambda^\mu$$

$$+ a^\varrho_\nu x^i_\varrho x^\nu_m V_{|i} \|^m.$$

To compute the last term we apply the operator $\|^m$ to the equation

$$V_{|i} = \frac{\partial V}{\partial x^i} + V \|^r \Gamma^*_{r0i}.$$

Noticing the interchange law

$$V \|^{rm} - V \|^{mr} = V \|^r \left(\log \frac{L}{\sqrt{g}} \right) \|^m - V \|^m \left(\log \frac{L}{\sqrt{g}} \right) \|^r$$

and (62) we shall first find

$$(67)\qquad V \|^{rm} = (V A^r + \lambda^\alpha x^r_\alpha) l^m + V A^m \|^r + \lambda^\alpha \|^r x^m_\alpha + \lambda^\alpha x^m_\alpha \|^r$$
$$- (l^r - A^r)(V A^m + \lambda^\alpha x^m_\alpha)$$

and, as a consequence of the formulae [Berwald, Acta (8.11), (8.13), (25.6), (25.10)], obtain

$$(68)\qquad x^i_\varrho x^\nu_m V_{|i} \|^m = A^\nu \frac{\partial V}{\partial v^\varrho} + V \frac{\partial A^\nu}{\partial v^\varrho} + \frac{\partial \lambda^\nu}{\partial v^\varrho}$$

$$+ V A^\sigma (\Gamma^\nu_{\varrho\sigma} + A^{\nu\mu}_{\varrho} a_{\mu\sigma} - \Gamma^{*\nu}_{\varrho\sigma})$$

$$+ \lambda^\alpha (\Gamma^\nu_{\alpha\varrho} + A^{\nu\mu}_{\alpha} a_{\mu\varrho} - \Gamma^{*\nu}_{\alpha\varrho})$$

$$+ \Gamma^*_{r0\varrho} x^\nu_m A^m \|^r V + x^i_\varrho x^\nu_m \Gamma^*_{r0i} \|^m (V A^r + \lambda^\alpha x^r_\alpha)$$

$$+ \Gamma^*_{r0\varrho} (\lambda^\nu \|^r + \lambda^\alpha x^\nu_m x^m_\alpha \|^r).$$

On the other hand composition of (64) with x^ν_i gives immediately

$$(69)\qquad \lambda^\nu \|^r + \lambda^\alpha x^\nu_m x^m_\alpha \|^r = - V g^{\nu\mu} x^r_\mu,$$

so that

$$(70)\qquad x^i_\varrho x^\nu_m V_{|i} \|^m = A^\nu \frac{\partial V}{\partial v^\varrho} + V \left\{ \frac{\partial A^\nu}{\partial v^\varrho} + A^\sigma (\Gamma^\nu_{\varrho\sigma} + A^{\nu\mu}_{\varrho} a_{\mu\sigma} - \Gamma^{*\nu}_{\varrho\sigma}) \right.$$

$$\left. + \Gamma^*_{r0\varrho} x^\nu_m A^m \|^r + A^r x^\nu_m x^i_\varrho \Gamma^*_{r0i} \|^m - \Gamma^*_{\varrho 0\mu} g^{\nu\mu} \right\}$$

$$+ \frac{\partial \lambda^\nu}{\partial v^\varrho} + \lambda^\alpha (\Gamma^\nu_{\alpha\varrho} + A^{\nu\mu}_{\alpha} a_{\mu\varrho} - \Gamma^{*\nu}_{\alpha\varrho} + x^r_\alpha x^\nu_m x^i_\varrho \Gamma^*_{r0i} \|^m).$$

For convenience' sake let us introduce

$$X_{\varrho}^{\mu\nu} = l^q x_p^\mu x_m^\nu x_\varrho^i \Gamma_{qi}^{*p} \|^m, \tag{71}$$

$$Y^{\mu\nu} = A^m \|^r x_m^\mu x_r^\nu, \tag{72}$$

so as to express in (68) each term with Latin indices. The expression $X_{\varrho}^{\mu\nu}$ is closely connected with the tensor $P_{o\varrho}^{\varrho\sigma}$ [Berwald, Acta (7), p. 245] by the relation

$$l^q x_p^\varrho x_m^\sigma \Gamma_{qi}^{*p} \|^m = X_{\mu}^{\varrho\sigma} x_i^\mu + P_{oo}^{\varrho\sigma} l_i, \tag{73}$$

which shows that both the tensors $X_{\mu}^{\varrho\sigma}$ and $P_{oo}^{\varrho\sigma}$ stand for the tangential and normal components of a certain covariant vector in space.

As to the second expression (72) we can write the alternative one:

$$Y^{\mu\nu} = -3 A^\mu A^\nu - 2 A_{\varrho\tau}^\mu A^{\varrho\tau\nu} + \frac{1}{2} \frac{L^2}{g} g^{ik} \frac{\partial^2 g_{ik}}{\partial p_r \partial p_m} x_m^\mu x_r^\nu \tag{74}$$

[cf. Berwald, Acta (5.4), (5.14), (5.9), (23.20)].

A reference to the relation

$$\Gamma_{roi}^* \|^m = \Gamma_{ri}^{*p} \|^m l_p + \Gamma_{ri}^{*m} - \Gamma_{roi}^* (l^m - A^m)$$

suffices to rewrite two terms in (70), namely,

$$A^r x_m^\nu x_\varrho^i \Gamma_{roi}^* \|^m = A^r x_m^\nu x_\varrho^i l_p \Gamma_{ri}^{*p} \|^m + A^\sigma \Gamma_{\sigma\varrho}^{*\nu} + A^\mu A^\nu \Gamma_{\mu o\varrho}^*,$$

$$x_\alpha^r x_m^\nu x_\varrho^i \Gamma_{roi}^* \|^m = x_\alpha^r x_m^\nu x_\varrho^i l_p \Gamma_{ri}^{*p} \|^m + \Gamma_{\alpha\varrho}^{*\nu} + A^\nu \Gamma_{\alpha o\varrho}^*.$$

Moreover, decomposing, as usual, $\Gamma_{ro\varrho}^*$ into components:

$$\Gamma_{ro\varrho}^* = \Gamma_{\sigma o\varrho}^* x_r^\sigma + \Gamma_{oo\varrho}^* l_r$$

and making use of the formula

$$l_r A^m \|^r \equiv A^m \|^o = 0,$$

we have

$$\begin{aligned} x_\varrho^i x_m^\nu V_{|i} \|^m = A^\nu \frac{\partial V}{\partial v^\varrho} + V \Big\{ &\frac{\partial A^\nu}{\partial v^\varrho} + A^\sigma (\Gamma_{\varrho\sigma}^\nu + A_\varrho^{\nu\mu} a_{\mu\sigma}) \\ &+ A^\mu A^\nu \Gamma_{\varrho o\mu}^* - \Gamma_{\varrho o\mu}^* g^{\nu\mu} + \Gamma_{\sigma o\varrho}^* Y^{\nu\sigma} + A^r x_m^\nu x_p^i l_p \Gamma_{ri}^{*p} \|^m \Big\} \\ &+ \frac{\partial \lambda^\nu}{\partial v^\varrho} + \lambda^\alpha \{ \Gamma_{\alpha\varrho}^\nu + A_\alpha^{\nu\mu} a_{\mu\varrho} + A^\nu \Gamma_{\alpha o\varrho}^* + x_\alpha^r x_m^\nu x_\varrho^i l_p \Gamma_{ri}^{*p} \|^m \}. \end{aligned} \tag{75}$$

The only two terms with Latin indices in the right-hand side of (75) can, however, be expressed in terms of $X^{\mu\nu}_{\varrho}$, A_μ and $g_{\alpha\beta}$, as it may easily be shown on account of the relation [Berwald, Acta (15.6)]

$$l_p \Gamma^{*p}_{sm}\|^h = 2\, l_s \Gamma^{*r}_{rm}\|^h - g_{sp}\, l^r \Gamma^{*p}_{rm}\|^h$$

and $A_p = A_\mu x^\mu_\varrho$. The result of carrying out the computation is as follows:

$$A^r x^\nu_m x^i_\varrho\, l_p \Gamma^{*p}_{ri}\|^m = - A_\mu X^{\mu\nu}_\varrho,$$

$$x^r_\alpha x^\nu_m x^i_\varrho\, l_p \Gamma^{*p}_{ri}\|^m = - g_{\alpha\mu} X^{\mu\nu}_\varrho.$$

Thus we have finally

$$(76)\qquad V_{|i}\|^m x^i_\varrho x^\nu_m = A^\nu \frac{\partial V}{\partial v^\varrho} + V \Big\{ \frac{\partial A^\nu}{\partial v^\varrho} + A^\sigma (\Gamma^\nu_{\varrho\sigma} + A^{\nu\mu}_\varrho a_{\mu\sigma}) + \Gamma^*_{\varrho o\mu} A^\mu A^\nu - \Gamma^*_{\varrho o\mu} g^{\nu\mu} - A_\mu X^{\mu\nu}_\varrho + \Gamma^*_{\sigma o\varrho} Y^{\nu\sigma} \Big\} + \frac{\partial \lambda^\nu}{\partial v^\varrho} + \lambda^\alpha \{ \Gamma^\nu_{\alpha\varrho} + A^{\nu\mu}_\alpha a_{\mu\varrho} + A^\nu \Gamma^*_{\alpha o\varrho} - g_{\alpha\mu} X^{\mu\nu}_\varrho \}.$$

Substitution of (76) into (66) and utilization of the equation

$$x^\mu_p x^p_\alpha |_\tau + \lambda^\mu |_\tau = \frac{\partial \lambda^\mu}{\partial v^\tau} + \lambda^\alpha \Gamma^{*\mu}_{\alpha\tau} - \Gamma^*_{\sigma o\tau} g^{\mu\sigma} V$$

give the required form of (61):

$$(77)\qquad g^{\varrho\sigma} \frac{\partial^2 V}{\partial v^\varrho \partial v^\sigma} + (2\, a^{\varrho\sigma} A^\tau_{\varrho\sigma} + 2\, a^\tau_\mu A^\mu - \Gamma^{\tau\varrho}_\varrho - A^{\sigma\tau\nu} a_{\nu\sigma}) \frac{\partial V}{\partial v^\tau} + \mathcal{J} V + \Lambda_\alpha \lambda^\alpha = 0,$$

where we have placed

$$(78)\qquad \mathcal{J} = (a^\varrho_\mu A^\mu)_{(\varrho)} + R^\varrho_{\varrho o o} + a^\varrho_\nu \{ A^\nu_{(\varrho)} + A^\sigma (B^\nu_{\varrho\sigma} + 3\, A^{\nu\mu}_\varrho a_{\mu\sigma}) - A_\mu X^{\mu\nu}_\varrho - A^{\sigma\mu}_\varrho A^\nu a_{\mu\sigma} + \Gamma^*_{\varrho o\sigma} (A^\nu A^\sigma + g^{\nu\sigma} + Y^{\nu\sigma}) \},$$

$$(79)\qquad \Lambda_\alpha = R^\varrho_{\varrho o\alpha} - 2\, a^\tau_\mu \Gamma^{*\mu}_{\alpha\tau} + a^\varrho_{\alpha(\varrho)} + 2\, \Gamma^\tau_{\alpha\varrho} a^\varrho_\tau + a^\sigma_\varrho (A^{\varrho\mu}_\alpha a_{\mu\sigma} - A^{\mu\varrho}_\sigma a_{\mu\alpha} + A^\varrho \Gamma^*_{\alpha o\sigma} - g_{\alpha\mu} X^{\mu\varrho}_\sigma + 2\, A^{\varrho\mu}_\sigma a_{\mu\alpha}).$$

Evidently, $\mathcal{J}$ is an invariant similar to Koschmieder's U^*_o [cf. Berwald, Acta (33.2)], since they have several terms in common. But I have not been able to find the relation between these invariants.

6. A Generalization.

As in the case of a space of K-spreads[1], the infinitesimal deformation so far we have considered can also be generalized to the case where each of the functions ξ^i depends upon the position as well as the normal direction of the hypersurface. Hereafter we shall denote the corresponding formula in this case by means of an equation with the same *number preceded* by prime. Thus, instead of (6) we have to consider the extended infinitesimal deformation

$$(6)' \qquad \bar{x}^i = x^i + \xi^i(x, p)\,\delta t,$$

where the ξ^i is a function of both x^i and p_m, and is homogeneous of degree zero in p_m, so that

$$(80) \qquad \xi^h\|^o = 0.$$

Upon setting

$$(81) \qquad P^{\alpha}_{(k)\,l} = \text{cofactor of } \frac{\partial x^l}{\partial v^\alpha} \text{ in } p_k$$

and

$$(82) \qquad \frac{g}{L^2}\frac{\partial p_m}{\partial v^\alpha} P^{\alpha}_{(k)\,l} = Q_{kml}$$

we can after an easy calculation show that

$$(9)' \qquad \bar{p}_k = p_k + \left(\frac{\partial \xi^h}{\partial x^h} p_k - \frac{\partial \xi^h}{\partial x^k} p_h + Q_{kml}\frac{\partial \xi^l}{\partial p^m}\right)\delta t;$$

$$(11)' \qquad \begin{cases} \bar{L} = L\left\{1 + \left(\dfrac{1}{L}\dfrac{\partial L}{\partial x^i}\xi^i + \dfrac{\partial \xi^h}{\partial x^h} - \dfrac{\partial \xi^h}{\partial x^k} l^k l_h + Q_{kml}\, l^k \xi^l\|^m\right)\delta t\right\}, \\ \bar{g}_{ik} = g_{ik} + \left\{\dfrac{\partial g_{ik}}{\partial x^j}\xi^j - 2A^j_{ik}\left(\dfrac{\partial \xi^h}{\partial x^j} l_h - Q_{jmh}\,\xi^h\|^m\right)\right\}\delta t, \\ \sqrt{\bar{g}} = \sqrt{g}\left\{1 + \left(\dfrac{\partial \log \sqrt{g}}{\partial x^r}\xi^r - A^k\left(\dfrac{\partial \xi^h}{\partial x^k} l_h - Q_{kmh}\,\xi^h\|^m\right)\right)\delta t\right\}; \end{cases}$$

$$(13)' \qquad \bar{l}_i = l_i - \left[\frac{\partial \xi^h}{\partial x^i} l_h + (A^m - l^m)\left(\Gamma^*_{moh}\xi^h + \frac{\partial \xi^h}{\partial x^m} l_h\right) l_i + \{Q_{imh} - l_i(l^j - A^j)\,Q_{jmh}\}\,\xi^h\|^m\right]\delta t;$$

[1] Cf. B. Su, On the isomorphic transformations in a Douglas space, Science Record, Acad. Sinica, 2 (1947), 11—19; 139—146.

$$(17)' \qquad \overline{\Gamma}^{*}_{iok} = \Gamma^{*}_{iok} + \Big\{ l_h \frac{\partial \Gamma^{*h}_{ik}}{\partial x^j} \xi^j - l_h \Gamma^{*h}_{ik} \|^m \frac{\partial \xi^j}{\partial x^m} l_j$$

$$- \Gamma^{*h}_{ik} \frac{\partial \xi^j}{\partial x^h} l_j - (A^m - l^m) \left(\Gamma^{*}_{mor} \xi^r + \frac{\partial \xi^r}{\partial x^m} l^r \right) \Gamma^{*}_{iok}$$

$$+ (l_r \Gamma^{*r}_{ik} \|^g Q_{gmi} - \Gamma^{*h}_{ik} Q_{hmj} + \Gamma^{*}_{iok} (l^r - A^r) Q_{rmj} \xi^j \|^m) \Big\} \delta t.$$

Proceeding as before and using (8), we have the following relations:

$$(22)' \qquad \bar{a}_{\varrho\sigma} = \cdots + \Big[l_i \frac{\partial \xi^i}{\partial x^j} \|^m x^j_\varrho \frac{\partial l_m}{\partial v^\sigma} + l_i \frac{\partial \xi^i \|^m}{\partial x^j} x^j_\sigma \frac{\partial l_m}{\partial v^\varrho}$$

$$+ l_i \xi^i \|^{mr} \frac{\partial l_m}{\partial v^\varrho} \frac{\partial l_r}{\partial v^\sigma} + l_i \xi^i \|^m \frac{\partial^2 l_m}{\partial v^\varrho \partial v^\sigma}$$

$$+ \Big\{ l_r \Gamma^{*r}_{ik} \|^g x^i_\varrho x^k_\sigma Q_{gmj} - \Gamma^{*h}_{\varrho\sigma} Q_{hmj} + a_{\varrho\sigma} (l^r - A^r) Q_{rmj}$$

$$+ \Gamma^{*}_{jo\sigma} \frac{\partial l_m}{\partial v^\varrho} + \Gamma^{*}_{\varrho oj} \frac{\partial l_m}{\partial v^\varrho} - \frac{\partial^2 x^i}{\partial v^\varrho \partial v^\sigma} Q_{imj} \Big\} \xi^j \|^m \Big] \delta t,$$

$$(23)' \qquad \bar{g}_{\varrho\sigma} = \cdots + \left(g_{hk} x^k_\sigma \frac{\partial l_m}{\partial v^\varrho} + g_{ih} x^i_\varrho \frac{\partial l_m}{\partial v^\sigma} + 2 A^i_{\varrho\sigma} Q_{imh} \right) \xi^h \|^m \delta t.$$

In the right-hand sides of these equations there are omitted all the terms in (22) and (23) respectively.

Similarly, we obtain

$$(26)' \qquad \bar{g}^{\varrho\sigma} = g^{\varrho\sigma} + \left\{ h_{\lambda\mu} \text{ adj.} \begin{vmatrix} g_{\varrho\sigma} & g_{\varrho\mu} \\ g_{\lambda\sigma} & g_{\lambda\mu} \end{vmatrix} : \text{det.} (g_{\varrho\sigma}) - g^{\varrho\sigma} h^\lambda_\lambda \right\} \delta t$$

with the abbreviation

$$(27)' \qquad h_{\lambda\mu} = \cdots + \left(g_{hk} x^k_\mu \frac{\partial l_m}{\partial v^\lambda} + g_{ih} x^i_\lambda \frac{\partial l_m}{\partial v^\mu} + 2 A^j_{\lambda\mu} Q_{jmh} \right) \xi^h \|^m$$

and, in particular,

$$(39)' \qquad h^\lambda_\lambda = \cdots + 2 g^{\lambda\mu} g_{jh} x^j_\mu \frac{\partial l_m}{\partial v^\lambda} \xi^h \|^m,$$

only the terms in addition being exhibited.

It shall be noted that on account of non-vanishing of $\xi^h \|^m$ in the present case appear the derivatives $\frac{\partial l_m}{\partial v^\sigma}, \frac{\partial^2 l_m}{\partial v^\varrho \partial v^\sigma}$ of l_m in each coefficient, for which we have to utilize the formula [Berwald, Acta (25. 11)]

$$(83) \qquad \frac{\partial l_j}{\partial v^\tau} - \Gamma^{*}_{\tau oj} = - a_{\varrho\tau} x^\varrho_j - A^\mu a_{\mu\tau} l_j.$$

Without expressing the way of carrying out the computation precisely we conclude after a somewhat lengthy calculation that the equation for the variation of the mean curvature of a general hypersurface in the Cartan space also takes the form (34) with new coefficients:

$$
\begin{aligned}
(84) \qquad E = {} & g^{\varrho\sigma}(l_h \xi^h)_{|\varrho\sigma} + 2\,(a^{\varrho\sigma} A^{\tau}_{\varrho\sigma} l_p - a^{\tau}_{\mu} x^{\mu}_{\varrho})\, \xi^p |_{\tau} \\
& + R^{\varrho}_{\varrho o r} \xi^r - l_h \xi^h \|^{mr} a^{\sigma}_{\lambda} a_{\sigma\nu} x^{\nu}_m x^{\lambda}_r \\
& - g^{\varrho\sigma} (\Gamma^*_{r o \varrho} \Gamma^*_{m o \sigma} + 2\, a_{\nu\sigma} \Gamma^*_{r o \varrho} x^{\nu}_m)\, \xi^r \|^m \\
& + g^{\varrho\sigma} l_h \xi^h \|^m \Big\{ \Gamma^*_{m o \nu} (\Gamma^{\nu}_{\varrho\sigma} + A^{\nu\mu}_{\varrho} a_{\mu\sigma} - 2\, A^{\mu\nu}_{\sigma} a_{\mu\varrho} + a^{\nu}_{\varrho} A_{\sigma}) \\
& + x^{\tau}_m \left(a_{\varrho\nu} \Gamma^{\nu}_{\sigma\tau} + a_{\varrho\nu} A^{\nu\mu}_{\sigma} a_{\mu\tau} - \frac{\partial a_{\varrho\tau}}{\partial v^{\sigma}} \right) \\
& - a_{\nu\sigma} x^j_{\varrho} x^{\nu}_r \Gamma^{*\prime}_{j o m} \|^r + a_{\varrho\sigma} \Gamma^*_{o o m} - a_{\varrho\nu} \Gamma^{*\,\nu}_{\sigma m} \Big\} \\
& - (\Gamma^{\tau\varrho}_{\varrho} + 2\, a^{\varrho\sigma} A^{\tau}_{\varrho\sigma} + a^{\varrho\sigma} A^{\tau}_{\varrho\sigma})\, Q_{\tau m j}\, \xi^j \|^m,
\end{aligned}
$$

$$
(85) \qquad G = (A^r - l^r)(l_m \xi^m)|_r + (A^r Q_{rmj} - A^r l_j \Gamma^*_{m o r} + l_j \Gamma^*_{o o m})\, \xi^j \|^m .
$$