CRITICAL POINTS AND GRADIENT FIELDS OF SCALARS IN HILBERT SPACE.

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1. Introduction.

The paper is concerned with certain aspects of a theory of critical points of a scalar (i.e., a real valued function) $i(\mathfrak{x})$ defined in a Hilbert space H, especially with the relation of the critical points to the gradient field of $i(\mathfrak{x})$. Moreover, applications to the theory of non-linear integral equations are made.

Let V be a bounded open convex set of H, and S its boundary. We suppose that $i(\mathfrak{x})$ is defined in an open region V' containing $V + S$ in its interior. If $i(\mathfrak{x})$ is written in the form

$$i(\mathfrak{x}) = \|\mathfrak{x}\|^2/2 + I(\mathfrak{x}) \tag{1.1}$$

where $\|\mathfrak{x}\|$ denotes the norm in the space H we will always assume that $G(\mathfrak{x}) = \text{grad } I(\mathfrak{x})$ (Definition 2.2) exists and is completely continuous. Moreover, if a critical point is defined as a point $\mathfrak{x}$ for which

$$\text{grad } i(\mathfrak{x}) = \mathfrak{g}(\mathfrak{x}) = \mathfrak{x} + G(\mathfrak{x}) = \mathfrak{o} \tag{1.2}$$

it will be assumed that such a point is not degenerate (definition 3.2) and does not lie on the boundary S of V.

Under these assumptions it can be proved (theorem 3.1) that there are at most a finite number of critical points in V, say $\mathfrak{a}_1, \mathfrak{a}_2, \ldots \mathfrak{a}_s$. For each critical point $\mathfrak{a}_\sigma$ there will be defined a non-negative integer r_σ, the type of the critical point $\mathfrak{a}_\sigma$ (definition 4.1). It will be proved that the "quadratic form" giving the second differential at $\mathfrak{a}_\sigma$ can, by the use of a proper base of H, be written as a sum of squares multiplied by ± 1, the number of those multiplied by -1 being r_σ (theorem 4.1). If $j_\sigma = j(\mathfrak{a}_\sigma)$ is the index[1] of the point $\mathfrak{x} = \mathfrak{a}_\sigma$ as solution of equation (1.2), then theorem 5.1 asserts that $j(\mathfrak{a}_\sigma) = (-1)^{r_\sigma}$. In agreement with the definitions used in the finite dimensional case[2] we define for $\sigma = 1, 2, \dots s$ the i-th type number m_σ^i of the critical point $\mathfrak{a}_\sigma$ by

$$(1.3) \qquad m_\sigma^i = \delta_{r_\sigma}^i \qquad (\delta_j^i \text{ Kronecker symbol}).$$

Moreover, we call

$$(1.4) \qquad M^i = \sum_{\sigma=1}^{s} m_\sigma^i$$

the i-th Morse number of the scalar $i(\mathfrak{x})$ in V. If now $\chi = \chi(\mathfrak{g}, S)$ denotes the characteristic of the gradient field $\mathfrak{g}$ on the boundary S of V, if $u = u(\mathfrak{g}, S, \mathfrak{o})$ denotes the order of the zero point $\mathfrak{o}$ of H with respect to the image of S under the mapping $\mathfrak{g}(\mathfrak{x})$, and if finally $\gamma = \gamma(\mathfrak{g}, S, \mathfrak{o})$ is the mapping degree of $\mathfrak{g}(\mathfrak{x})$ (considered as mapping of V) in the point $\mathfrak{o}$, then as is well known,[3]

$$(1.5) \qquad \chi = u = \gamma.$$

Theorem 6.1 of the present paper asserts then

$$(1.6) \qquad \chi = u = \gamma = \sum_{\sigma=1}^{s} (-1)^{r_\sigma} = \sum_i (-1)^i M^i.$$

This connection between the Morse numbers of $i(\mathfrak{x})$ in V and the characteristic χ of the gradient field on S has a number of consequences. As an immediate consequence, we note the estimate

$$(1.7) \qquad s \geqq |\chi|$$

for the number s of critical points in V.

More definite statements can be made in special cases. If

$$(1.8) \qquad r_1 \equiv r_2 \equiv \cdots \equiv r_s \pmod 2,$$

(1.6) yields

$$(1.9) \qquad s = |\chi|.$$

[1] For the definition of an index see [1], p. 470 in the finite dimensional case, and [9], p. 54 or [10], p. 188 in the case of a Banach space. (Numbers in brackets refer to the bibliography at the end of the paper.)

[2] [16], p. 33.

[3] [1], chapter 12 in the finite dimensional case, and [10], Satz 4 and the beginning of § 3, in the Banach space case.

Let $d_2(\mathfrak{x}, \mathfrak{h}, \mathfrak{k})$ be the second differential[1] of the scalar $i(\mathfrak{x})$. Then it follows immediately from the definition of the r_σ that (1.8) (and therefore (1.9)) is certainly true if the quadratic form $d_2(\mathfrak{x}, \mathfrak{h}, \mathfrak{h})$ is non-negative in all critical points $\mathfrak{x} = \mathfrak{a}_\sigma$ since in this case all $r_\sigma = 0$. (Corollary 6. 1).

If $d_2(\mathfrak{x}, \mathfrak{h}, \mathfrak{k})$ satisfies the stronger condition of being uniformly positive definite in V, i.e., if there exists a positive c such that for all $\mathfrak{x}$ in V

$$d_2(\mathfrak{x}, \mathfrak{h}, \mathfrak{h}) \geqq c\,\|\mathfrak{h}\|^2 \tag{1.10}$$

it will be proved that $\chi = 1$ if V is a sphere which has the origin $\mathfrak{o}$ as center and whose radius R is greater than $\|\mathfrak{g}(\mathfrak{o})\|/c$ (lemma 6. 1). This together with (1.9) shows that in such a sphere there is exactly one critical point.

In section 7, the preceding theory is applied to uniqueness and existence questions concerning a system of non-linear integral equations of the form

$$y_j^*(s) + \int\limits_{D_0} \sum_{i=1}^{n} K_{ij}(t, s) f_i(t, y_1(t), \ldots y_n(t))\, dt = 0 \qquad (j = 1, 2, \ldots n) \tag{1.11}$$

for the "conjugate n-tuples" (definition 7. 1) $y_j(t)$, $y_j^*(t)$ $(j = 1, \ldots n)$. If (besides certain regularity conditions concerning the f_i and K_{ij}) $\sum\limits_{i=1}^{n} f_i(t, u_1, \ldots u_n)\, du_i$ is a total differential then a certain scalar $i(\mathfrak{x})$ the "Hammerstein scalar"[2] (definition 7.2) can be defined in a suitable Hilbert space of elements $\mathfrak{x}$ together with two mappings $\Phi : \mathfrak{x} \to (y_1(t), \ldots y_n(t))$ and $\Phi^* : \mathfrak{x} \to (y_1^*(t), \ldots y_n^*(t))$ such that the conjugate n-tuple $\Phi(\mathfrak{x})$, $\Phi^*(\mathfrak{x})$ is a solution of (1.11) if and only if $\mathfrak{x}$ is a critical point of the scalar $i(\mathfrak{x})$. Thus the question of existence and uniqueness of a solution of (1.11) is reduced to the investigation of the critical points of $i(\mathfrak{x})$. Now under the assumptions of theorem 7.1[3] the second differential of $i(\mathfrak{x})$ turns out to be uniformly positive definite. Therefore the characteristic of the gradient field of $i(\mathfrak{x})$ on the surface of a large enough sphere V is 1 and such a sphere contains one and only one critical point $\mathfrak{a}$ of $i(\mathfrak{x})$, and $\mathfrak{a}$ turns out to be an absolute minimum (theorem 7.1). Since an estimate for the radius of such a V

[1] In the sense of Fréchet. See [2], [6] or [8].

[2] For the motivation of this name see the introduction to [15]. In the case that the system (1.11) consists of one single equation ($n = 1$), the condition about the total differential is obviously always satisfied.

[3] In the case of one equation ($n = 1$) the main condition (7.21) of this theorem states that there exists a positive constant $\bar{c}$ such that min $\partial f / \partial u \geqq -\bar{c}\, 1/\lambda_1$ where λ_1 is the greatest eigenvalue of the (not necessarily symmetric) kernel of the integral equation.

can be given, theorem 7.2 concerning the system 1.11 (including an estimate for the solution) follows now easily.

If the system (1.11) is symmetric and positive definite then $y_j^*(t) = y_j(t)$ which proves that the system

$$(1.12)\qquad y_j(s) + \int_{D_0} \sum_{i=1}^{n} K_{ij}(t, s) f_i(t, y_1(t), \ldots y_n(t))\, dt = 0 \qquad (j = 1, 2, \ldots n)$$

has one and only one solution (theorem 7.3), a result which was first obtained by M. Golomb.[1]

2. Differentials and Gradients.

The following notations will be used throughout: points of the real Hilbert space H will be denoted by German letters; $\mathfrak{o}$ especially denotes the zero point of H. Correspondingly $\mathfrak{f}(\mathfrak{x})$ is a mapping of the point $\mathfrak{x} < H$ into the point $\mathfrak{f}(\mathfrak{x}) < H$ while $i(\mathfrak{x})$ or $I(\mathfrak{x})$ denote scalars, i.e., real valued functions. The vector field associated with the mapping $\mathfrak{f}(\mathfrak{x})$[2] will also be denoted by $\mathfrak{f}(\mathfrak{x})$. V is a convex open bounded set of H, and S its boundary. All mappings, vector fields, and scalars will be supposed to be defined in some open set V' which contains $V + S$ in its interior. $(\mathfrak{x}, \mathfrak{x})$ is the scalar product of the points $\mathfrak{x}$ and $\mathfrak{y}$ of H, and $\|\mathfrak{x}\| = +\sqrt{(\mathfrak{x}, \mathfrak{x})}$ the norm of $\mathfrak{x}$.

Definition 2.1. The scalar $i(\mathfrak{x})$ is said to be differentiable in the point $\mathfrak{x}_0$ if there exists a linear[3] functional $d(i, \mathfrak{x}_0, \mathfrak{h}) = d(\mathfrak{x}_0, \mathfrak{h})$ of $\mathfrak{h}$ such that if $r(\mathfrak{x}_0, \mathfrak{h})$ is defined by the equation

$$(2.1)\qquad i(\mathfrak{x}_0 + \mathfrak{h}) - i(\mathfrak{x}) = d(\mathfrak{x}_0, \mathfrak{h}) + r(\mathfrak{x}_0, \mathfrak{h})$$

then

$$(2.2)\qquad \lim_{\mathfrak{h} \to 0} \frac{r(\mathfrak{x}_0, \mathfrak{h})}{\|\mathfrak{h}\|} = 0.$$

$d(\mathfrak{x}_0, \mathfrak{h})$ is called the (Fréchet-)differential of $i(\mathfrak{x})$ at $\mathfrak{x}_0$. If $d(\mathfrak{x}, \mathfrak{h})$ exists for all points $\mathfrak{x}$ of a neighborhood of $\mathfrak{x}_0$ and, as function of $\mathfrak{x}$, admits a differential $d_2(\mathfrak{x}, \mathfrak{h}, \mathfrak{k})$ in $\mathfrak{x}_0$, then $d_2(\mathfrak{x}_0, \mathfrak{h}, \mathfrak{k})$ is called the second differential of the $i(\mathfrak{x})$ at $\mathfrak{x}_0$.[4]

[1] [4], Satz 1.
[2] [10], §§ 1 and 3.
[3] Linear means additive and continuous.
[4] [2], [6], [8].

Since $d(\mathfrak{x}_0, \mathfrak{h})$ is linear in h it can be written (in a unique way) as a scalar product, i.e., there exists one and only one $\mathfrak{g} = \mathfrak{g}(\mathfrak{x}_0)$ in H such that

$$d(\mathfrak{x}_0, \mathfrak{h}) = (\mathfrak{g}(\mathfrak{x}_0), \mathfrak{h}). \tag{2.3}$$

Definition 2.2. The $\mathfrak{g}(\mathfrak{x}_0)$ defined by (2.3) is called the gradient of $i(\mathfrak{x})$ at $\mathfrak{x}_0$. If $i(\mathfrak{x})$ is differentiable in V (i.e., at all points of V) then the mapping $\mathfrak{y} = \mathfrak{g}(\mathfrak{x})$ defined by (2.3) for all $\mathfrak{x}$ in V is called the gradient mapping of $i(\mathfrak{x})$. The vectorfield associated with the mapping[1] is called the gradient field of $i(\mathfrak{x})$.[2]

Lemma 2.1. $\mathfrak{g}(\mathfrak{x})$ *is completely continuous if and only if* $d(\mathfrak{x}, \mathfrak{h})$ *is completely continuous considered as mapping of* $\mathfrak{x} < H$ *into the element* $l(\mathfrak{h}) = d(\mathfrak{x}, \mathfrak{h})$ *of the space of linear functionals.*

Proof. The proof follows easily from the fact that on the one hand by (2.3): $|d(\mathfrak{x} - \mathfrak{x}', \mathfrak{h})| \leqq \|\mathfrak{g}(\mathfrak{x}) - \mathfrak{g}(\mathfrak{x}')\| \|\mathfrak{h}\|$, and that on the other hand the inequality $|d(\mathfrak{x} - \mathfrak{x}', \mathfrak{h})| \leqq \varepsilon \|\mathfrak{h}\|$ for all $\mathfrak{h}$ implies, again by (2.3):

$$(\mathfrak{g}(\mathfrak{x}) - \mathfrak{g}(\mathfrak{x}'), \mathfrak{g}(\mathfrak{x}) - \mathfrak{g}(\mathfrak{x}')) \leqq \varepsilon \|\mathfrak{g}(\mathfrak{x}) - \mathfrak{g}(\mathfrak{x}')\|, \text{ i.e., } \|\mathfrak{g}(\mathfrak{x}) - \mathfrak{g}(\mathfrak{x}')\| \leqq \varepsilon.$$

Definition 2.3. The mapping $\mathfrak{y} = \mathfrak{f}(\mathfrak{x})$ (or the associated vectorfield) is said to be differentiable at the point $\mathfrak{x}_0$ if there exists a mapping $\mathfrak{l}(\mathfrak{x}_0, \mathfrak{k}) = \mathfrak{l}(\mathfrak{f}, \mathfrak{x}_0, \mathfrak{k})$, linear in $\mathfrak{k}$, such that if $\mathfrak{r}(\mathfrak{x}, \mathfrak{k})$ is defined by the equation

$$\mathfrak{f}(\mathfrak{x}_0 + \mathfrak{k}) - \mathfrak{f}(\mathfrak{x}_0) = \mathfrak{l}(\mathfrak{x}_0, \mathfrak{k}) + \mathfrak{r}(\mathfrak{x}_0, \mathfrak{k}), \tag{2.4}$$

then

$$\lim_{\mathfrak{k} \to 0} \frac{\mathfrak{r}(\mathfrak{x}_0, \mathfrak{k})}{\|\mathfrak{k}\|} = 0. \tag{2.5}$$

$\mathfrak{l}(\mathfrak{x}_0, \mathfrak{k})$ is called the differential of $\mathfrak{f}(\mathfrak{x})$ at $\mathfrak{x}_0$.

Lemma 2.2. *If the mapping* $\mathfrak{f}(\mathfrak{x})$ *has the differential* $\mathfrak{l}(\mathfrak{x}_0, \mathfrak{k})$ *at* $\mathfrak{x}_0$, *then the scalar* $f(\mathfrak{x}) = (\mathfrak{f}(\mathfrak{x}), \mathfrak{h})$ *has (for each* $\mathfrak{h}$*) a differential at* $\mathfrak{x}_0$, *and this differential equals the scalar product* $(\mathfrak{l}(\mathfrak{x}_0, \mathfrak{k}), \mathfrak{h})$.

Proof. By 2.4:

$$f(\mathfrak{x}_0 + \mathfrak{k}) - f(\mathfrak{x}_0) = (\mathfrak{f}(\mathfrak{x}_0 + \mathfrak{k}) - \mathfrak{f}(\mathfrak{x}_0), \mathfrak{h}) = (\mathfrak{l}(\mathfrak{x}_0, \mathfrak{k}), \mathfrak{h}) + (\mathfrak{r}(\mathfrak{x}_0, \mathfrak{k}), \mathfrak{h}). \tag{2.6}$$

Since $(\mathfrak{l}(\mathfrak{x}_0, \mathfrak{k}), \mathfrak{h})$ is linear in $\mathfrak{k}$ and since because of (2.5), $r(\mathfrak{x}_0, \mathfrak{k}) = (r(\mathfrak{x}_0, \mathfrak{k}), \mathfrak{h})$ divided by $\|\mathfrak{k}\|$ approaches zero as $\mathfrak{k} \to 0$, (2.6) show that the lemma is an im-

[1] [10], § 3.
[2] [3], § 6 and [13].

mediate consequence of Definition 2.1 together with the fact that the differential is uniquely determined.[1]

Lemma 2.3. *If* $\mathfrak{g}(\mathfrak{x}) = \operatorname{grad} i(\mathfrak{x})$ *has a differential* $\mathfrak{l}(\mathfrak{x}_0, \mathfrak{k})$ *at* $\mathfrak{x}_0$, *then* $i(\mathfrak{x})$ *has a second differential* $d_2(\mathfrak{x}_0, \mathfrak{h}, \mathfrak{k})$ *at* $\mathfrak{x}_0$, *and*

$$d_2(\mathfrak{x}_0, \mathfrak{h}, \mathfrak{k}) = (\mathfrak{l}(\mathfrak{x}_0, \mathfrak{k}), \mathfrak{h}). \tag{2.7}$$

Proof. Because of definition 2.2 we have only to apply lemma 2.2 with $\mathfrak{f}(\mathfrak{x}) = \mathfrak{g}(\mathfrak{x})$.

Definition 2.4. The real valued function $q(\mathfrak{h}, \mathfrak{k})$ of the pair of points $\mathfrak{h}, \mathfrak{k}$ of H is called bilinear if it is linear in $\mathfrak{h}$ and in $\mathfrak{k}$. q is called degenerate if there exists a $\mathfrak{k}_0 \neq \mathfrak{o}$ such that $q(\mathfrak{h}, \mathfrak{k}_0) = 0$ for all $\mathfrak{h} < H$. If no such $\mathfrak{k}_0$ exists q is called non-degenerate.[2]

Lemma 2.4. *If* $q(\mathfrak{h}, \mathfrak{k})$ *is positive definite (i.e., if* $q(\mathfrak{h}, \mathfrak{h}) \geqq 0$ *and the equal sign holds only for* $\mathfrak{h} = \mathfrak{o}$*) then* $q(\mathfrak{h}, \mathfrak{k})$ *is not degenerate.*

Proof. If there would exist a $\mathfrak{k}_0 \neq \mathfrak{o}$ such that $q(\mathfrak{h}, \mathfrak{k}_0) = 0$ for all $\mathfrak{h}$, then we would have $q(\mathfrak{k}_0, \mathfrak{k}_0) = 0$ which is impossible if $q(\mathfrak{h}, \mathfrak{k})$ is positive definite.

Definition 2.5. The differential $\mathfrak{l}(\mathfrak{x}, \mathfrak{k})$ of definition 2.3 is called non-singular at $\mathfrak{x}_0$ if the equation for $\mathfrak{k}$

$$\mathfrak{l}(\mathfrak{x}_0, \mathfrak{k}) = 0 \tag{2.8}$$

has only the solution $\mathfrak{k} = \mathfrak{o}$.

Lemma 2.5. *The differential* $\mathfrak{l}(\mathfrak{x}, \mathfrak{k})$ *of* $\mathfrak{g}(\mathfrak{x}) = \operatorname{grad} i(\mathfrak{x})$ *is non-singular at* $\mathfrak{x}_0$ *if and only if the second differential* $d_2(\mathfrak{x}_0, \mathfrak{h}, \mathfrak{k})$ *of* $i(\mathfrak{x})$ *considered as bilinear form in* $\mathfrak{h}, \mathfrak{k}$ *is non-degenerate.*

Proof. The lemma is an immediate consequence of (2.6).

3. Critical Points.

Definition 3.1. Let the scalar $i(\mathfrak{x})$ be differentiable at $\mathfrak{x} = \mathfrak{a}$. $\mathfrak{a}$ is said to be a critical point of $i(\mathfrak{x})$ if the differential $d(\mathfrak{a}, \mathfrak{h})$ of $i(\mathfrak{x})$ at $\mathfrak{a}$ is the zero functional, i.e., if $d(\mathfrak{a}, \mathfrak{h}) = 0$ for all $\mathfrak{h} < H$. Clearly this is equivalent to saying that

[1] [6], lemma 11.1.

[2] It is immediately seen that in the finite dimensional case of a form $\sum_{i=1}^{n} \sum_{j=1}^{n} q_{ij} h_i k_j$ the above definition of degeneracy coincides with the usual one that the determinant of the matrix (q_{ij}) is zero.

$$\mathfrak{g}(\mathfrak{a}) = \mathfrak{o} \tag{3.1}$$

where $\mathfrak{g}(\mathfrak{x})$ is the gradient of $i(\mathfrak{x})$.

Definition 3.2. The critical point $\mathfrak{a}$ of $i(\mathfrak{x})$ is called non-degenerate if the gradient $\mathfrak{g}(\mathfrak{x})$ of $i(\mathfrak{x})$ has a differential $l(\mathfrak{a}, \mathfrak{k})$ at $\mathfrak{x} = \mathfrak{a}$ and if the second differential $d_2(\mathfrak{a}, \mathfrak{h}, \mathfrak{k})$ of $i(\mathfrak{x})$ at $\mathfrak{a}$ (which by lemma 2.2 exists) considered as bilinear form in $\mathfrak{h}$ and $\mathfrak{k}$ is non-degenerate in the sense of definition 2.4.

Using the notations explained in the first paragraph of section 2 we make now the following assumptions: if $i(\mathfrak{x})$ is written in the form

$$i(\mathfrak{x}) = \|\mathfrak{x}\|^2/2 + I(\mathfrak{x}) \tag{3.2}$$

then:

Hypothesis 3.1. $I(\mathfrak{x})$ has a differential $D(\mathfrak{x}, \mathfrak{h})$ for all $\mathfrak{x} < V'$.

By definition 2.2, $I(\mathfrak{x})$ has then also a gradient $\mathfrak{G}(\mathfrak{x})$ for all $\mathfrak{x} < V'$ and

$$D(\mathfrak{x}, \mathfrak{h}) = (\mathfrak{G}(\mathfrak{x}), \mathfrak{h}) \qquad (\mathfrak{x} < V'). \tag{3.3}$$

Hypothesis 3.2. The gradient $\mathfrak{G}(\mathfrak{x})$ is completely continuous in V'.

Hypothesis 3.3. If $\mathfrak{a} < V$ is a critical point of $i(\mathfrak{x})$ then $\mathfrak{a}$ is a non-degenerate critical point (definition 3.2).

Clearly the existence of differentials for $i(\mathfrak{x})$, $\mathfrak{g}(\mathfrak{x})$ is equivalent to the existence of differentials for $I(\mathfrak{x})$, $\mathfrak{G}(\mathfrak{x})$ respectively. Therefore Hypothesis 3.3 implies the existence of the differential $\mathfrak{L}(\mathfrak{a}, \mathfrak{k})$ of the gradient $\mathfrak{G}(\mathfrak{x})$ at $\mathfrak{x} = \mathfrak{a}$.

Hypothesis 3.4. The linear operator on $\mathfrak{k}$, $\mathfrak{L}(\mathfrak{a}, \mathfrak{k})$ giving the differential of $\mathfrak{G}(\mathfrak{x})$ at a critical point $\mathfrak{a}$ is completely continuous.

Hypothesis 3.5. There are no critical points of $i(\mathfrak{x})$ on the boundary S of V.

Lemma 3.1. *Under the hypotheses 3.1—3.5 we have in a critical point $\mathfrak{a}$ of V*

$$\mathfrak{l}(\mathfrak{a}, \mathfrak{k}) = \mathfrak{k} + \mathfrak{L}(\mathfrak{a}, \mathfrak{k}). \tag{3.4}$$

Moreover, $\mathfrak{l}(\mathfrak{a}, \mathfrak{k})$ is non-singular (definition 2.5), and $\mathfrak{L}(\mathfrak{a}, \mathfrak{k})$ is completely continuous in $\mathfrak{k}$.

Proof. Since grad $\|\mathfrak{x}\|^2/2 = \mathfrak{x}$ and since the differential of $\mathfrak{x}$ is $\mathfrak{k}$, equation (3.4) is an immediate consequence of the definitions. That $\mathfrak{L}(\mathfrak{a}, \mathfrak{k})$ is completely continuous is a restatement of hypothesis 3.4. Finally the non-singularity of $\mathfrak{l}(\mathfrak{a}, \mathfrak{k})$ follows from hypothesis 3.3 and definition 3.2 together with lemma 2.5.

Lemma 3.2. *Under the hypotheses* 3.1—3.5 *a critical point* $\mathfrak{a}$ *of* V *is isolated.*

Proof. $\mathfrak{g}(\mathfrak{x})$ has at $\mathfrak{a}$ a differential $\mathfrak{l}(\mathfrak{a}, \mathfrak{k})$ of the properties indicated in lemma 3.1. By [12], lemma 3 these properties imply that $\mathfrak{a}$ is an isolated root of the equation (3.1).

Theorem 3.2. *Under the hypotheses* 3.1—3.5 *the scalar* $i(\mathfrak{x})$ *has at most a finite number of critical points in* V.

Proof. For each critical point $\mathfrak{a}$ equation (3.1), i.e., $\mathfrak{a} + \mathfrak{G}(\mathfrak{a}) = \mathfrak{o}$, holds. The complete continuity of $\mathfrak{G}(\mathfrak{x})$ shows immediately that a bounded set of solutions of this equation is compact. Therefore if there were infinitely many solutions in V they would have a limit point $\mathfrak{a}_0$ is $V + S$. Because of the continuity of $\mathfrak{G}(\mathfrak{x})$, $\mathfrak{a}_0$ would also be a solution of $\mathfrak{g}(\mathfrak{x}) = \mathfrak{o}$, i.e., be a critical point. By hypothesis 3.5 $\mathfrak{a}_0$ could not lie on S. Therefore $\mathfrak{a}_0$ would be a non-isolated critical point of V in contradiction to lemma 3.2.

4. Type Numbers of a Critical Point.

We first add to the hypotheses 3.1—3.5 the following:

Hypothesis 4.1. If $\mathfrak{a} < V$ is a critical point of $i(\mathfrak{x})$ then there exists a neighborhood $N_{\mathfrak{a}}$ of $\mathfrak{a}$ such that at all points of $N_{\mathfrak{a}}$ the gradient $\mathfrak{G}(\mathfrak{x})$ of $I(\mathfrak{x})$ has a differential $\mathfrak{L}(\mathfrak{x}, \mathfrak{k})$ which, moreover, is continuous in $\mathfrak{x}$.

Lemma 4.1. *Under the hypotheses* 3.1—3.5 *and* 4.1 *the differential* $\mathfrak{L}(\mathfrak{a}, \mathfrak{k})$ *of the gradient* $\mathfrak{G}(\mathfrak{x})$ *of* $I(\mathfrak{x})$ *at the critical point* $\mathfrak{a} < V$ *is a linear, completely continuous and symmetric operator in* $\mathfrak{k}$.

Proof. The linearity and complete continuity are obvious from Hypothesis 3.4. To prove the symmetry we note that $\mathfrak{G}(\mathfrak{x}) = \operatorname{grad} I(\mathfrak{x})$ and that therefore by lemma 2.3

$$(4.1) \qquad D_2(\mathfrak{x}, \mathfrak{h}, \mathfrak{k}) = (\mathfrak{L}(\mathfrak{x}, \mathfrak{k}), \mathfrak{h})$$

is the second differential of $I(\mathfrak{x})$ in each point $\mathfrak{x}$ in which $\mathfrak{L}(\mathfrak{x}, \mathfrak{k})$ exists. By Hypothesis this is the case for all $\mathfrak{x} < N_{\mathfrak{a}}$, and, by the same hypothesis, $\mathfrak{L}(\mathfrak{x}, \mathfrak{k})$, and therefore by (4.1) also $D_2(\mathfrak{x}, \mathfrak{h}, \mathfrak{k})$, is continuous in $\mathfrak{x}$ for $\mathfrak{x} < N_{\mathfrak{a}}$. But it is well known that the continuity of a second differential in the neighborhood $N_{\mathfrak{a}}$ of a point $\mathfrak{a}$ implies its symmetry in $\mathfrak{h}, \mathfrak{k}$ for the point $\mathfrak{a}$.[1] Therefore, we have from (4.1)

[1] [5], theorem 8; [8], Satz 1.

$$(\mathfrak{L}(\mathfrak{a}, \mathfrak{k}), \mathfrak{h}) = (\mathfrak{L}(\mathfrak{a}, \mathfrak{h}), \mathfrak{k}) = (\mathfrak{k}, \mathfrak{L}(\mathfrak{a}, \mathfrak{h}))$$

which, by definition, is the symmetry of $\mathfrak{L}(\mathfrak{a}, \mathfrak{k})$ as operator on $\mathfrak{k}$.

It follows from lemma 4.1 that $\mathfrak{L}(\mathfrak{a}, \mathfrak{k})$ has a finite or countably infinite number of eigenvalues and that in the latter case 0 is their only limit point.

In any case there are at most a finite number of eigenvalues less then -1. Moreover, we prove

Lemma 4.2. -1 *is not an eigenvalue of* $\mathfrak{L}(\mathfrak{a}, \mathfrak{k})$.

Proof. Otherwise there would exist a $\mathfrak{k}_0 \neq \mathfrak{o}$ such that $\mathfrak{L}(\mathfrak{a}, \mathfrak{k}_0) = -\mathfrak{k}_0$, or $\mathfrak{l}(\mathfrak{a}, \mathfrak{k}_0) = \mathfrak{k}_0 + \mathfrak{L}(\mathfrak{a}, \mathfrak{k}_0) = \mathfrak{o}$. By lemma 2.5 this is a contradiction to hypothesis 3.3.

Definition 4.1. Let r be the number of eigenvalues of $\mathfrak{L}(\mathfrak{a}, \mathfrak{k})$ which are less than -1, each counted according to its multiplicity.[1] Then r is called the type of the critical point $\mathfrak{a}$, and $m^i = m^i(\mathfrak{a}) = \delta^i_r$ (δ^i_r the Kronecker symbol; $i = 1, 2, \ldots$) is called the i-th typenumber of the critical point $\mathfrak{a}$. (By lemma 4.2, r may be also defined as the number of eigenvalues not greater than -1.)

Theorem 4.1. *With the assumptions and notations of lemma* 4.1 *and definition* 3.2 *there exists a normed orthogonal basis* $\mathfrak{e}_1, \mathfrak{e}_2, \ldots$ *of* H *and positive numbers* $p_1, p_2, \ldots$ *such that*

$$\mathfrak{l}(\mathfrak{a}, \mathfrak{k}) = -\sum_{\nu=1}^{r} p_\nu k_\nu \mathfrak{e}_\nu + \sum_{\nu=r+1}^{\infty} p_\nu k_\nu \mathfrak{e}_\nu \tag{4.2}$$

$$d_2(\mathfrak{a}, \mathfrak{k}, \mathfrak{k}) = -\sum_{\nu=1}^{r} p_\nu k_\nu^2 + \sum_{\nu=r+1}^{\infty} p_\nu k_\nu^2 \tag{4.3}$$

where $k_\nu = (\mathfrak{e}_\nu, \mathfrak{k})$ *and where* r *is the type number of the critical point* $\mathfrak{a}$. (*If* $r = 0$, *the symbol* $\sum_{\nu=1}^{r}$ *is understood to mean* $\mathfrak{o}$ *in* (4.2) *and* 0 *in* (4.3)).

Proof. Let $\mathfrak{e}'_1, \mathfrak{e}'_2, \ldots$ be a complete normed system of eigenelements of the linear symmetric and completely continuous operator $\mathfrak{L}(\mathfrak{a}, \mathfrak{k})$, and $\mu'_1, \mu'_2, \ldots$ the corresponding eigenvalues. Then

$$\mathfrak{L}(\mathfrak{a}, \mathfrak{k}) = \sum_{\nu} \mu'_\nu (\mathfrak{e}'_\nu, \mathfrak{k}) \mathfrak{e}'_\nu. \tag{4.4}$$

[1] The multiplicity of an eigenvalue is the number of linearly independent eigenfunctions belonging to it.

Let now $\mathfrak{e}_1'', \mathfrak{e}_2'', \ldots$ be a normed ortogonal system such that the $\mathfrak{e}_\nu', \mathfrak{e}_\nu''$ together span H and such that $(\mathfrak{e}_\nu', \mathfrak{e}_\varrho'') = 0$ for all $\mathfrak{e}_\nu', \mathfrak{e}_\varrho''$. We set $\mu_\nu'' = 0$. We now bring the $\mathfrak{e}_\nu', \mathfrak{e}_\nu''$ and, correspondingly, the μ_ν', μ_ν'' in a simple order and call them $\mathfrak{e}_1, \mathfrak{e}_2, \ldots$ and $\mu_1, \mu_2, \ldots$ in this order. (4.4) remains then true if we replace $\mathfrak{e}_\nu', \mu_\nu'$ by $\mathfrak{e}_\nu, \mu_\nu$ respectively. If we add to the equation thus obtained the equation $\mathfrak{k} = \sum_{\nu=1}^{\infty} (\mathfrak{e}_\nu, \mathfrak{k})\, \mathfrak{e}_\nu$ we obtain

$$\mathfrak{l}(\mathfrak{a}, \mathfrak{k}) = \mathfrak{k} + \mathfrak{L}(\mathfrak{a}, \mathfrak{k}) = \sum_\nu (1 + \mu_\nu)(\mathfrak{e}_\nu, \mathfrak{k})\, \mathfrak{e}_\nu. \tag{4.5}$$

This proves that $\mathfrak{l}(\mathfrak{a}, \mathfrak{k})$ can be written in the form (4.2) since by definition 4.1 exactly r of the numbers $1 + \mu_\nu$ are negative and since by lemma 4.2 none of them is zero. Finally, (4.3) is a consequence of (4.2) and (2.7).

5. Type Number and Topological Index.

Theorem 5.1. *With the same assumptions as in theorem 4.1 let r be the type of the critical point $\mathfrak{a}$. Denote by $j(\mathfrak{a})$ the index of $\mathfrak{a}$ as solution of the equation* (3.1).[1] *(Because of lemma 3.2 the index exists). Then*

$$j(\mathfrak{a}) = (-1)^r. \tag{5.1}$$

Proof. $g(\mathfrak{x})$ has the differential $\mathfrak{l}(\mathfrak{a}, \mathfrak{k})$ at $\mathfrak{a}$. It follows from the properties of this differential described in lemma 3.1 that the index $j(\mathfrak{a})$ of $\mathfrak{a}$ as solution of $\mathfrak{g}(\mathfrak{x}) = \mathfrak{o}$ is the same as the index of $\mathfrak{k} = \mathfrak{o}$ as solution of the equation $\mathfrak{l}(\mathfrak{a}, \mathfrak{k}) = 0$.[2] This latter index is by definition the order $u(\mathfrak{l}, S_\varrho, \mathfrak{o})$ of the point $\mathfrak{o}$ with respect to the image under the mapping $\mathfrak{l} = \mathfrak{l}(\mathfrak{a}, \mathfrak{k})$ of the sphere S_ϱ: $\|\mathfrak{k}\| = \varrho$.[3] Therefore,

$$j(\mathfrak{a}) = u(\mathfrak{l}, S_\varrho, \mathfrak{o}). \tag{5.2}$$

If now $\mathfrak{e}_1, \mathfrak{e}_2, \ldots$ is the normed orthogonal base of H used in theorem 4.1, if $l_\nu = (\mathfrak{e}_\nu, \mathfrak{l}(\mathfrak{a}, \mathfrak{k}))$, and, as is theorem 4.1, $k_\nu = (\mathfrak{e}_\nu, \mathfrak{k})$ then (4.2) shows that the mapping $k \to \mathfrak{l}(\mathfrak{a}, \mathfrak{k})$ is given by

$$\begin{aligned} l_\nu &= -p_\nu k_\nu = k_\nu + (-1 - p_\nu)\, k_\nu, & \nu &= 1, 2, \ldots r \\ l_\nu &= p_\nu k_\nu = k_\nu + (p_\nu - 1)\, k_\nu, & \nu &= r+1, r+2, \ldots \end{aligned} \tag{5.3}$$

[1] For the properties of an index in a Banach space see [9], [10], [12].

[2] [9], p. 55/56. For a detailed proof see [12], lemma 3.

[3] Since $\mathfrak{l}$ is linear, the size of the radius ϱ of S_ϱ does not matter.

Comparison with (4.5) shows that $p_\nu - 1 = \mu_\nu$ for $\nu > r$. Since $\lim\limits_{\nu\to\infty} \mu_\nu = 0$ it follows that there exists a $\nu_0 > r$ such that

$$0 < \mu_\nu = |(p_\nu - 1)| < \pi/\varrho \text{ for } \nu \geqq \nu_0 \tag{5.4}$$

where π is the distance of the image of S_ϱ under the map (4.5) from the point $\mathfrak{o}$.

With such a ν_0 we define a mapping $\mathfrak{k} \to \mathfrak{l}' = \sum\limits_{\nu=1}^{\infty} \mathfrak{l}'_\nu \mathfrak{e}_\nu$ where

$$\begin{aligned} l'_\nu &= -p_\nu k_\nu \text{ for } \nu = 1, \dots r \\ l'_\nu &= p_\nu k_\nu \quad \text{»} \quad \nu = r+1, \dots \nu_0 \\ l'_\nu &= k_\nu \quad \text{»} \quad \nu = \nu_0 + 1, \nu_0 + 2 \dots . \end{aligned} \tag{5.5}$$

(5.3), (5.5) and (5.4) show that for $\mathfrak{k} < S_\varrho$, i.e. $\|\mathfrak{k}\| = \varrho$

$$\|\mathfrak{l}' - \mathfrak{l}\|^2 = \sum_{\nu=1}^{\infty} (l'_\nu - l_\nu)^2 = \sum_{\nu=\nu_0+1}^{\infty} (p_\nu - 1)^2 k_\nu^2 < (\|\mathfrak{k}\|^2 \pi/\varrho)^2 = \pi^2 < \|\mathfrak{l}\|^2.$$

This estimate together with the theorem of Rouché[1] proves that

$$u(\mathfrak{l}, S_\varrho, \mathfrak{o}) = u(\mathfrak{l}', S_\varrho, \mathfrak{o}). \tag{5.6}$$

If E^{ν_0} is the space spanned by $\mathfrak{e}_1, \mathfrak{e}_2, \dots \mathfrak{e}_{\nu_0}$, (5.5) shows that $\mathfrak{l}' - \mathfrak{k} < E^{\nu_0}$, i.e., that $\mathfrak{l}'$ is a "layer mapping" with respect to E^{ν_0}.[2] By the definition of the order of a layer mapping,[3] the order $u(\mathfrak{l}', S_\varrho, \mathfrak{o})$ is therefore equal to the order of $\mathfrak{o}$ with respect to the image of the intersection $S_\varrho^{\nu_0-1} = S_\varrho \wedge E^{\nu_0}$ under the mapping of E^{ν_0} into itself given by the first ν_0 of the equations (5.5). But this order is equal to the sign of the determinant of these equations. This determinant is $(-1)^r \prod\limits_{\nu=1}^{\nu_0} p_\nu$ and its sign is $(-1)^r$ since all p_ν are positive. Thus $u(\mathfrak{l}', S_\varrho, \mathfrak{o}) = (-1)^r$ which because of (5.6) and (5.2) proves (5.1).

6. Morse Numbers and Properties of the Gradient Field.

By theorem 3.1, there are at most a finite number s of critical points of $i(\mathfrak{x})$ in V. We denote them by $\mathfrak{a}_1, \mathfrak{a}_2, \dots \mathfrak{a}_s$.

Definition 6.1. For $i = 1, 2, \dots$ the i-th Morse number M^i of V is defined by

[1] For the finite dimensional case see [1], p. 459, for the Banach space case see [11], theorem 4.

[2] For the definition of a layer mapping see e.g. [11], p. 374.

[3] [11], definition 1.

$$M^i = \sum_{\sigma=1}^{s} m^i_\sigma \tag{6.1}$$

where $m^i_\sigma = m^i(\mathfrak{a}_\sigma)$ is the i-th type number of the critical point $\mathfrak{a}_\sigma$ (definition 4.1). Obviously M^i is the number of critical points in V of type i, and

$$\sum_i M^i = s. \tag{6.2}$$

Theorem 6.1. *With the usual notations assumptions let $\chi = \chi(\mathfrak{g}, S)$ be the characteristic of the gradient field $\mathfrak{g}$ on the boundary S of V, let $u = u(\mathfrak{g}, S, \mathfrak{o})$ be the order*[1] *of $\mathfrak{o}$ with respect to the image of S under the mapping $\mathfrak{g}$, and let $\gamma = \gamma(\mathfrak{g}, S, \mathfrak{o})$ be the mapping degree*[2] *in $\mathfrak{o}$ of the mapping $\mathfrak{g}$ (considered as mapping of V). Then*

$$\chi = u = \gamma = \sum_{\sigma=1}^{s} (-1)^{r_\sigma} = \sum_i (-1)^i M^i \tag{6.3}$$

where r_σ denotes the type of the critical point $\mathfrak{a}_\sigma$ (definition 4.1).

Proof. That $\chi = u = \gamma$ is known.[3] That the two sums in (6.3) are equal is an immediate consequence of definition 6.1. It remains to prove that

$$\chi = \sum_{\sigma=1}^{s} (-1)^{r_\sigma}.$$

But this equation is by (5.1) equivalent to the equation $\chi = \sum_{\sigma=1}^{s} j(\mathfrak{a}_\sigma)$ which is known to be true.[4]

We formulate the following obvious consequences of theorem 6.1 as

Corollary 6.1. Let s_0 be the number of critical points of even type and s_1 the number of critical points of odd type in V. Then

$$\chi = s_0 - s_1, \qquad s = s_0 + s_1. \tag{6.4}$$

Therefore, $s \geqq |\chi|$, and the equality sign hold if and only if the types of all critical points are of the same parity. This is certainly the case if none of the eigenvalues of the linear operators $\mathfrak{L}(\mathfrak{a}_\sigma, \mathfrak{k})$ $(\sigma = 1, 2, \ldots s)$ is less than -1 (or,

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[1] See [10], definition 11 for the definition of "order" in Banach spaces, and [10], beginning of § 3, for the definition of "characteristic" in such spaces.

[2] The mapping degree in Banach spaces was introduced in [9].

[3] [10], Satz 4 asserts $u = \gamma$. The equation $\chi = u$ is the definition of χ; cf. [10], § 3.

[4] [10], Satz 5.

what is the same, if all second differentials $d_2(\mathfrak{a}_\sigma, \mathfrak{h}, \mathfrak{k})$ become positive definite quadratic forms for $\mathfrak{h} = \mathfrak{k}$) since then by definition 4.1 the types of all critical points are zero.

Before stating theorem 6.2 we give the following

Definition 6.2. A bilinear form (definition 2.4) $q(\mathfrak{x}, \mathfrak{h}, \mathfrak{k})$ in $\mathfrak{h}, \mathfrak{k}$ is called uniformly positive definite in a set $W < H$ if there exists a positive constant c such that for all $\mathfrak{x} < W$

$$q(\mathfrak{x}, \mathfrak{k}, \mathfrak{k}) \geqq c \|\mathfrak{k}\|^2. \tag{6.5}$$

A linear operator $\mathfrak{l}(\mathfrak{x}, \mathfrak{k})$ of $\mathfrak{k}$ is called uniformly positive definite in W if the associated bilinear form $q(\mathfrak{x}, \mathfrak{h}, \mathfrak{k}) = (\mathfrak{l}(\mathfrak{x}, \mathfrak{k}), \mathfrak{h})$ is uniformly positive definite in W.

Theorem 6.2. *In addition to the hypotheses 3.1—3.4 and 4.1 we assume that the second differential $d_2(\mathfrak{x}, \mathfrak{h}, \mathfrak{k})$ of the scalar $i(\mathfrak{x})$ is uniformly positive definite in $V + S$ and that V is a solid sphere of center $\mathfrak{o}$ and of radius R with*

$$R > \|\mathfrak{g}(\mathfrak{o})\|/c \tag{6.6}$$

where the positive constant c satisfies (6.5) with $q = d_2$. Then $i(\mathfrak{x})$ has exactly one critical points in V and none on the boundary S of V. Moreover, if $R = R_0$ satisfies (6.6) and if $M(R_0)$ is an upper bound for $|i(\mathfrak{x})|$ in $\|\mathfrak{x}\| \leqq R_0$[1]*, then for any $m > 1$*

$$i(\mathfrak{x}) \geqq (1 - 1/m)c\|\mathfrak{x}\|\{R_0 - \|\mathfrak{g}(\mathfrak{o})\|/c\} - M(R_0) \quad \text{for} \quad \|\mathfrak{x}\| \geqq m R_0. \tag{6.7}$$

Before proving this theorem we first state and prove

Lemma 6.1. *Let $\mathfrak{v}(\mathfrak{x}) = \mathfrak{x} + \mathfrak{V}(\mathfrak{x})$ with completely continuous $\mathfrak{V}$ be a vector field*[2] *which is differentiable in $V + S$ (definition 2.3). We suppose that the differential $\mathfrak{l}(\mathfrak{x}, \mathfrak{k})$ of $\mathfrak{v}(\mathfrak{x})$ is uniformly positive definite in $V + S$* (definition 6.2). *If then c is a constant satisfying (6.5) with $q = (\mathfrak{l}(\mathfrak{x}, \mathfrak{k}), \mathfrak{h})$ and if V is a solid sphere with center $\mathfrak{o}$ whose radius R satisfies the inequality*

$$R > \|\mathfrak{v}(\mathfrak{o})\|/c, \tag{6.8}$$

then $\mathfrak{v}(\mathfrak{x})$ does not vanish on S and the characteristic χ of the field $\mathfrak{v}(\mathfrak{x})$ on S equals $+1$. Moreover, for any $\mathfrak{x}$ whose norm $R = \|\mathfrak{x}\|$ satisfies (6.8) we have

$$(\mathfrak{v}(\mathfrak{x}), \mathfrak{x}) \geqq c\|\mathfrak{x}\|(\|\mathfrak{x}\| - \|\mathfrak{v}(\mathfrak{o})\|/c). \tag{6.9}$$

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[1] Such a finite $M(R_0)$ exists since $i(\mathfrak{x}) - \mathfrak{x} = I(\mathfrak{x})$ is completely continuous.

[2] $\mathfrak{v}(\mathfrak{x})$ is not necessarily a gradient field.

Proof. We recall that if $v(\mathfrak{x})$ is an arbitrary scalar which is differentiable in a convex domain containing the two points $\mathfrak{x}_0$ and $\mathfrak{x}$, then

$$v(\mathfrak{x}) - v(\mathfrak{x}_0) = \int_0^1 \delta(\mathfrak{x}_0 + (\mathfrak{x} - \mathfrak{x}_0)t, \mathfrak{x} - \mathfrak{x}_0)\, dt \tag{6.10}$$

where $\delta(\mathfrak{x}, \mathfrak{h})$ is the differential of $v(\mathfrak{x})$.[1] We apply this equality to $\mathfrak{v}(\mathfrak{x}) = (\mathfrak{v}(\mathfrak{x}), \mathfrak{h})$ with $\mathfrak{x}_0 = \mathfrak{o}$, and take for $\mathfrak{x}$ an arbitrary point of S. Lemma 2.2 shows that (6.10) then becomes

$$(\mathfrak{v}(\mathfrak{x}), \mathfrak{h}) - (\mathfrak{v}(\mathfrak{o}), \mathfrak{h}) = \int_0^1 (\mathfrak{l}(t\mathfrak{x}, \mathfrak{x}), \mathfrak{h})\, dt. \tag{6.11}$$

We now set $\mathfrak{h} = \mathfrak{x}$. On account of the assumptions made about c, (6.11) yields then

$$(\mathfrak{v}(\mathfrak{x}), \mathfrak{x}) - (\mathfrak{v}(\mathfrak{o}), \mathfrak{x}) = \int_0^1 (\mathfrak{l}(t\mathfrak{x}, \mathfrak{x}), \mathfrak{x})\, dt \geqq c\,\|\mathfrak{x}\|^2 = c R^2$$

or

$$(\mathfrak{v}(\mathfrak{x}), \mathfrak{x}) \geqq c R^2 - |(\mathfrak{v}(\mathfrak{o}), \mathfrak{x})| \geqq c R^2 - \|\mathfrak{v}(\mathfrak{o})\| R = Rc(R - \|\mathfrak{v}(\mathfrak{o})\|/c).$$

This proves (6.9). Moreover, we see now from (6.8) that $(\mathfrak{v}(\mathfrak{x}), \mathfrak{x}) > 0$ for $\mathfrak{x} < S$. This shows that $\mathfrak{v}(\mathfrak{x}) \neq \mathfrak{o}$ for $\mathfrak{x} < S$. It also shows that in no point of S the field $\mathfrak{v}(\mathfrak{x})$ can have the direction of the interior normal since otherwise by definition of the term "interior normal"[2] there would exist an $\mathfrak{x} < S$ and a positive ϱ such that $\mathfrak{v}(\mathfrak{x}) = -\varrho\,\mathfrak{x}$, and this point would render $(\mathfrak{v}(\mathfrak{x}), \mathfrak{x}) = -\varrho\,\|\mathfrak{x}\|^2$ negative. This proves that $\chi = 1$ since it is known that a vectorfield $\mathfrak{v}(\mathfrak{x}) = \mathfrak{x} + \mathfrak{V}(\mathfrak{x})$ with completely continuous $\mathfrak{V}(x)$ which is non-vanishing on S and which has no interior normal on S has the characteristic $+1$.[3]

Proof of theorem 6.2. On the one hand, the application of lemma 6.1 to $\mathfrak{v}(\mathfrak{x}) = \mathfrak{g}(\mathfrak{x})$ shows that $\mathfrak{g}(\mathfrak{x}) \neq \mathfrak{o}$ for $\mathfrak{x} < S$ and that the characteristic χ of the gradient field $\mathfrak{g}(\mathfrak{x})$ of the scalar $i(\mathfrak{x})$ on S is equal to $+1$. Since, on the other hand, the second differential $d_2(\mathfrak{x}, \mathfrak{h}, \mathfrak{k})$ is uniformly positive definite the quadratic forms $d_2(\mathfrak{a}_\sigma, \mathfrak{k}, \mathfrak{k})$ are certainly positive for all critical points $\mathfrak{a}_\sigma$ $(\sigma = 1, 2, \ldots s)$ of V and corollary 6.1. shows that $s = |\chi|$. Thus $1 = \chi = s$.

To prove (6.7) let $R_0 > \|\mathfrak{g}(\mathfrak{o})\|/c$, let m be a given number > 1 and $\mathfrak{x}$ be such that $\|\mathfrak{x}\| > m R_0$. We set $\mathfrak{x}_0 = \mathfrak{x}/m$. Since $(\mathfrak{g}(\mathfrak{x}), \mathfrak{h})$ is the differential of $i(\mathfrak{x})$ we obtain from the general formula (6.10)

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[1] [5], theorem 5.

[2] [10], § 1, no. 4.

[3] [10], part b of Satz 7.

$$i(\mathfrak{x}) - i(\mathfrak{o}) = \int_0^1 (\mathfrak{g}(t\mathfrak{x}), \mathfrak{x})\,dt = \int_0^{1/m} (g(t\mathfrak{x}), \mathfrak{x})\,dt + \int_{1/m}^1 (\mathfrak{g}(t\mathfrak{x}), \mathfrak{x})\,dt. \tag{6.12}$$

Now using the substitution $s = tm$ one sees easily that

$$\int_0^{1/m} (\mathfrak{g}(t\mathfrak{x}), \mathfrak{x})\,dt = \int_0^1 (\mathfrak{g}(s\mathfrak{x}_0), \mathfrak{x}_0)\,ds = i(\mathfrak{x}_0) - i(\mathfrak{o}).$$

Therefore, we see from (6.12) that

$$i(\mathfrak{x}) = i(\mathfrak{x}_0) + \int_{1/m}^1 (\mathfrak{g}(t\mathfrak{x}), \mathfrak{x})\,dt \geqq \int_{1/m}^1 \frac{1}{t}\,\mathfrak{g}(t\mathfrak{x}, t\mathfrak{x})\,dt - M(R_0). \tag{6.13}$$

Now for $t \geqq 1/m$ we have $t\|\mathfrak{x}\| \geqq \frac{\|\mathfrak{x}\|}{m} \geqq R_0 > \|\mathfrak{g}(\mathfrak{o})\|/c$. Therefore, we can apply lemma 6.1 with $\mathfrak{v}(\mathfrak{x}) = \mathfrak{g}(\mathfrak{x})$ and obtain from (6.9) for $t \geqq \frac{1}{m}$

$$\mathfrak{g}(\mathfrak{x}t, \mathfrak{x}t) \geqq c\|\mathfrak{x}\|t(\|\mathfrak{x}\|t - \|\mathfrak{g}(\mathfrak{o})\|/c) \geqq c\|\mathfrak{x}\|t(R_0 - \|\mathfrak{g}(\mathfrak{o})\|/c).$$

Substituting this in (6.13) we obtain (6.7).

Corollary to theorem 6.2. If $i(\mathfrak{x})$ is defined and satisfies our usual assumptions for all $\mathfrak{x} < H$, and if $d_2(\mathfrak{x}, \mathfrak{h}, \mathfrak{k})$ is uniformly positive definite in H, then $i(\mathfrak{x})$ has one and only one critical point $\mathfrak{a}$ in H, and $i(\mathfrak{a})$ is an absolute minimum.

Proof. It is an obvious consequence of (6.7) that there exists an $R_1 > R_0$ such that

$$i(\mathfrak{x}) > i(\mathfrak{o}) \quad \text{for} \quad \|\mathfrak{x}\| \geqq R_1. \tag{6.14}$$

Now in the sphere $\bar{V}$ defined by $\|\mathfrak{x}\| \leqq R_1$ the scalar $i(\mathfrak{x})$ takes an absolute minimum in some point $\mathfrak{a}'$ by a previous theorem ([14], theorem 4.2). Because of (6.14), $i(\mathfrak{a}')$ is then an absolute minimum for the whole space H. Likewise by (6.14), $\mathfrak{a}'$ is an interior point of $\bar{V}$ and, therefore, a critical point of $i(\mathfrak{x})$. Consequently $\mathfrak{a}'$ must coincide with the unique critical point of theorem 6.2.

7. Applications.

The general purpose of this section has been explained in the introduction. Let E^r be the r-dimensional Euclidean space, let $K_{ij}(s, t)$ $(i, j = 1, 2, \ldots n)$ be admissible kernels defined for pairs of points s and t of an admissible domain

$D_0 < E^r$.[1] Let E^n be the Euclidean space of n-tuples $U = (u_1, u_2, \ldots u_n)$, and $f_i(t, u_1, \ldots u_n)$ $(i = 1, 2, \ldots n)$ be n functions defined and continuous in the productspace $D_0 \times E^n$ and for which

$$F(t, U) = \int_0^U \sum_{i=1}^n f_i(t, v_1, \ldots v_n)\, d v_i \tag{7.1}$$

is a function of the upper limit $U = (u_1, \ldots u_n)$ alone.

We now recall the definition of the Hammerstein scalars connected with the $K_{ij}(s, t)$ and the f_i.[2]

For $i = 0, 1, \ldots n-1$ let D_i denote the domain obtained from D_0 by the translation $i \cdot d_0$ where the translation vector d_0 is such that no two of the domains D_i have a non-zero intersection. We then obtain the admissible kernel $K(s, t)$ defined for $s, t < D = \sum_{i=0}^{n-1} D_i$ by setting (in obvious notation)

$$K(s, t) = K_{i+1, j+1}(s - i d_0, t - j d_0) \text{ for } s < D_i,\ t < D_j \qquad (i, j = 0, 1, \ldots n-1). \tag{7.2}$$

Likewise we obtain a one to one correspondence between the ordered n-tuples $y_1(t), \ldots y_n(t)$ of functions defined in D_0 and the functions $y(t)$ defined in D by setting

$$y(s) = y_{i+1}(s - i d_0) \text{ for } s < D_i, \qquad (i = 0, 1, \ldots n-1). \tag{7.3}$$

This correspondence will be indicated by writing

$$y(s) = (y_1(s), \ldots y_n(s)). \tag{7.4}$$

Let now $\varphi_\nu(s)$, $\varphi_\nu^*(s)$ $(\nu = 1, 2, \ldots)$ be a complete system of pairs of normed orthogonal eigenfunctions of the (not necessarily symmetric) kernel $K(s, t)$, and λ_ν the corresponding eigenvalues.[3] The λ_ν may, and will, be assumed to be positive and to be arranged in not increasing order. The Hilbert space H we deal with will then be the space of all sequences $\mathfrak{x} = (x_1, x_2, \ldots)$ for which $\sum_\nu \lambda_\nu x_\nu^2$ converges with the scalar product $(\mathfrak{x}, \mathfrak{y})$ of $\mathfrak{x}$ with $\mathfrak{y} = (y_1, y_2, \ldots)$ defined by

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[1] The word "admissible" has the same meaning as in [15], definition 2.1.

[2] For details we refer to [15], sections 2 and 3.

[3] The so-called Schmidt eigenvalues of an unsymmetric kernel. The notation differs from the one used by Schmidt, Hammerstein and others in that λ_ν is replaced $1/\lambda_\nu$.

$$(\mathfrak{x}, \mathfrak{y}) = \sum_{\nu} \lambda_\nu x_\nu y_\nu. \tag{7.5}$$

By H_1 we denote the subspace of those $\mathfrak{y} < H$ for which $\sum_\nu y_\nu^2$ converges[1], by L^2 the space of those functions $y(t)$ for which $y^2(t)$ is summable over D, and by M and M^* the subspaces of L^2 spanned by the functions $\varphi_1(s)$, $\varphi_2(s)$, ... and $\varphi_1^*(s)$, $\varphi_2^*(s)$ ... respectively. The mapping $H_1 \to M$ which assigns to the element $\mathfrak{y} = (y_1, y_2, \ldots)$ of H_1 that element $y(t)$ of M whose component with respect to $\varphi_\nu(t)$ is y_ν, is called $\Phi_1 : y(t) = \Phi_1(y)$. Correspondingly a mapping $y^*(t) = \Phi_1^*(y)$ of H onto the space M^* is defined by using the system $\varphi_\nu^*(t)$ instead of the system $\varphi_\nu(t)$. Φ_1, Φ_1^* and $y(t)$, $y^*(t)$ are called pairs of conjugate mappings and functions respectively. Also the ordered n-tuples $y_1(t), \ldots y_n(t)$ and $y_1^*(t), \ldots y_n^*(t)$ associated by the correspondence (7.3), (7.4) with $y(t)$ and $y^*(t)$ respectively are called conjugate. This terminology is in agreement with the following general

Definition 7.1. The functions $y(t)$, $y^*(t)$ of L^2 are called conjugate if (i) $\int_D \varphi_\nu(t)\, y(t)\, dt = \int_D \varphi_\nu^*(t)\, y^*(t)\, dt$ ($\nu = 1, 2, \ldots$), and (ii) $y(t) < M$, $y^*(t) < M^*$. The n-tuples $(y_1(t), \ldots y_n(t))$ and $(y_1^*(t), \ldots y_n^*(t))$ are conjugate if the functions $y(t)$, $y^*(t)$ corresponding to them by the correspondence (7.3), (7.4) are conjugate.

$\Phi_1(\mathfrak{y})$ and $\Phi_1^*(\mathfrak{y})$ are mappings of the subspace H_1 of H into L^2. We define now mappings $\Phi(\mathfrak{x})$ and $\Phi^*(\mathfrak{x})$ of all of H into L^2 in the following manner: if $\mathfrak{x} = (x_1, x_2, \ldots)$ is an arbitrary element of H we set $\mathfrak{y} = (\lambda_1 x_1, \lambda_2 x_2, \ldots)$. Since $\sum_\nu \lambda_\nu^2$ converges, $\mathfrak{y}$ is in H_1, and $\Phi_1(\mathfrak{y})$, $\Phi_1^*(\mathfrak{y})$ are defined. We set then

$$y(t) = \Phi(\mathfrak{x}) = \Phi_1(\mathfrak{y}),\quad y^*(t) = \Phi^*(\mathfrak{x}) = \Phi_1^*(\mathfrak{y}). \tag{7.6}$$

Definition 7.2. Let $Y(t)$ be the point of E^n whose coordinates are

$$y_1(t),\ \ldots\ y_n(t)$$

where this n-tuple is the one associated by (7.3), (7.4) with $y(t) = \Phi(\mathfrak{x})$, let $Y^*(t)$ be defined in the corresponding way be using $y^*(t) = \Phi^*(\mathfrak{x})$, and let $F(t, U)$ be the function defined by (7.1). Finally set

$$I(\mathfrak{x}) = \int_{D_0} F(t, Y(t))\, dt, \quad I^*(\mathfrak{x}) = \int_{D_0} F(t, Y^*(t)\, dt. \tag{7.7}$$

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[1] We recall that $\lambda_\nu \to 0$ such that H_1 is actually a subspace of H.

Then

$$(7.8)\qquad i(\mathfrak{x}) = \|\mathfrak{x}\|^2/2 + I(\mathfrak{x}), \quad i^*(\mathfrak{x}) = \|\mathfrak{x}\|^2/2 + I^*(\mathfrak{x}),$$ [1]

are called Hammerstein scalars.

$i(\mathfrak{x})$ and $i^*(\mathfrak{x})$ are defined for those $\mathfrak{x} < H$ for which the integrals in (7.7) exist. The conditions imposed in all of the following lemmas and theorems will insure that they exist for all $\mathfrak{x} < H$. It follows moreover from [15], theorem 3.1 that for all $\mathfrak{x} < H$ the scalar $I(\mathfrak{x})$ has a continuous differential $D(\mathfrak{x}, \mathfrak{h})$ given by

$$(7.9)\qquad D(\mathfrak{x}, \mathfrak{h}) = \sum_{i=1}^{n} \int_{D_0} k_i(t) f_i(t, y_1(t), \ldots y_n(t))\, dt$$

$$(k_1(t), \ldots k_n(t)) = k(t) = \Phi(\mathfrak{h})$$

if only the f_i are continuous functions of their arguments $t, u_1 \ldots u_n$ and if the following assumption A) is satisfied:

Assumption A. There exists a constant C such that

$$(7.10)\qquad \sum_{\nu} \lambda_\nu \varphi_\nu^2(t) \leqq C^2 \text{ for all } t < D.$$ [2]

A corresponding statement holds for $I^*(\mathfrak{x})$ with the same constant C.

Our next goal is to show that under certain additional conditions the scalars $i(\mathfrak{x})$ and $i^*(\mathfrak{x})$ satisfy all assumptions of theorem 6.2. We first prove

Lemma 7.1. *Let the f_i be continuous functions of their arguments and let assumption A be satisfied. Then the scalar $I(\mathfrak{x})$ has a completely cintinuous gradient $\mathfrak{G}(\mathfrak{x})$ for all $\mathfrak{x} < H$ given by*

$$(7.11)\qquad \mathfrak{G}(\mathfrak{x}) = (G_1(\mathfrak{x}), G_2(\mathfrak{x}), \ldots)$$

where $G_\nu(\mathfrak{x})$ is the ν-th Fourier coefficient of

$$f(t, y(t)) = (f_1(t, y_1(t), \ldots y_n(t)), \ldots f_n(t, y_1(t), \ldots y_n(t))), \text{ i.e.,}$$

$$(7.12)\qquad G_\nu(\mathfrak{x}) = \int_D \varphi_\nu(t) f(t, y(t))\, dt = \sum_{i=1}^{n} \int_{D_0} \varphi_{\nu_i}(t) f_i(t, y_1(t), \ldots y_n(t))\, dt$$

$$(\varphi_{\nu_1}(t), \ldots \varphi_{\nu_n}(t)) = \varphi_\nu(t), \quad \nu = 1, 2, \ldots).$$

A corresponding statement holds for $I^*(\mathfrak{x})$.

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[1] We recall that $\|\mathfrak{x}\|^2 = \sum_\nu \lambda_\nu x_\nu^2$; cf. (7.5).

[2] As to the significance of the assumption A, see [15], footnote to (3.7).

Proof. We obtain from (7.12), (7.9) using the definition (7.5) of the scalar product and observing that $k(t) = \Phi(\mathfrak{x}) < M$

$$(\mathfrak{G}, \mathfrak{h}) = \sum_{\nu} \lambda_\nu G_\nu h_\nu = \sum_{\nu} G_\nu k_\nu = \int_D f(t, y(t))\, k(t)\, dt =$$

$$= \sum_{i=1}^{n} \int_{D_0} k_i(t) f_i(t, y_1(t), \ldots y_n(t))\, dt = D(\mathfrak{x}, \mathfrak{h}).$$

Thus $(\mathfrak{G}(\mathfrak{x}), \mathfrak{h}) = D(\mathfrak{x}, \mathfrak{h})$ which by definition 2.2 proves that $\mathfrak{G}(\mathfrak{x})$ is the gradient of $I(\mathfrak{x})$. Observing that by [15], theorem 3.2, $D(\mathfrak{x}, \mathfrak{h})$ is completely continuous, the complete continuity of $\mathfrak{G}(\mathfrak{x})$ follows from lemma 2.1. The proof for the statement concerning $I^*(\mathfrak{x})$ is analogous.

Lemma 7.2. *In addition to the assumptions of lemma 7.1 let the derivatives $\partial f_i / \partial u_j$ exist and be continuous in all points $t, u_1, \ldots u_n$ of the product space $D_0 \times E^n$. Then for all $\mathfrak{x} < H$*

(i) *the gradient $\mathfrak{G}(\mathfrak{x})$ of $I(\mathfrak{x})$ has a continuous differential $\mathfrak{L}(\mathfrak{x}, \mathfrak{h}) = (L_1, L_2, \ldots)$* with

$$(7.13) \qquad L_\nu = L_\nu(\mathfrak{x}, \mathfrak{h}) = \sum_{j=1}^{n} \sum_{i=1}^{n} \int_{D_0} \varphi_{\nu_i}(t) \frac{\partial f_i}{\partial u_j}(t, y_1(t), \ldots y_n(t))\, k_j(t)\, dt$$

where the $k_i(t)$ and the $\varphi_{\nu_i}(t)$ have the same meaning as in (7.9) and (7.12).

(ii) *$I(\mathfrak{x})$ has a continuous second differential $D_2(\mathfrak{x}, \mathfrak{h}, \mathfrak{h}')$ given by*

$$(7.14) \qquad D_2(\mathfrak{x}, \mathfrak{h}, \mathfrak{h}') = \sum_{i=1}^{n} \sum_{j=1}^{n} k_i(t)\, k_j'(t) \frac{\partial f_i}{\partial u_j}(t, y_1(t), \ldots y_n(t))$$

where $k'(t) = (k_1'(t), k_2'(t), \ldots k_n'(t)) = \Phi(h')$.

(iii) *For fixed $\mathfrak{x}$, $\mathfrak{L}(\mathfrak{x}, \mathfrak{h})$ is completely continuous in $\mathfrak{h}$.*

Proof. If we define R_ν be setting

$$(7.15) \qquad G_\nu(\mathfrak{x} + \mathfrak{h}) - G_\nu(x) = L_\nu(\mathfrak{x}, \mathfrak{h}) + R_\nu$$

where $L_\nu(\mathfrak{x}, \mathfrak{h})$ is given by (7.13) one sees easily that R_ν may be written in the form

$$(7.16) \qquad R_\nu = R_\nu(\mathfrak{x}, \mathfrak{h}, \vartheta) = \sum_{i=1}^{n} \int_{D_0} \varphi_{\nu_i}(t)\, a_i(t)\, dt \qquad (0 < \vartheta < 1)$$

where

$$(7.17)\quad a_i(t) = \sum_{j=1}^{n} k_j(t)\left\{\frac{\partial f_i}{\partial u_j}(t, y_1(t) + \vartheta k_1(t), \ldots y_n(t) + \vartheta k_n(t)) - \right.$$
$$\left. - \frac{\partial f_i}{\partial u_j}(t, y_1(t), \ldots y_n(t))\right\}.$$

To prove (i) it will be sufficient to show: if

$$\mathfrak{R} = \mathfrak{R}(\mathfrak{x}, \mathfrak{h}, \vartheta) = (R_1, R_2, \ldots),$$

then there exists to given $\varepsilon > 0$ a positive h_0 such that

$$(7.18)\qquad \|\mathfrak{R}(\mathfrak{x}, \mathfrak{h}, \vartheta)\| < \varepsilon \|\mathfrak{h}\| \text{ for } \|\mathfrak{h}\| < h_0 \text{ and } 0 \leqq \vartheta \leqq 1.$$

We recall that by [15], lemma 3.1 $y_i(t)$ and $k_i(t)$ are continuous functions whose absolute values are bounded by $C\|\mathfrak{x}\|$ and $C\|\mathfrak{h}\|$ respectively where C is the constant of the inequality (7.10). Since the $\partial f_i / \partial u_j$ are continuous it follows immediately that there exists an h_0 to the given ε such that

$$(7.19)\quad a_i^2(t) \leqq \frac{\varepsilon^2}{\lambda_1^2 n^2}\left(\sum_{j=1}^{n} |k_j(t)|\right)^2 \leqq \frac{\varepsilon^2}{\lambda_1^2 n}\sum_{j=1}^{n} k_j^2(t) \text{ for } \|\mathfrak{h}\| < h_0 \text{ and } 0 \leqq \vartheta \leqq 1.$$

If we set $a(t) = (a_1(t), \ldots a_n(t))$, then (7.16) shows that R_ν is the Fourier coefficient of $a(t)$ with respect to $\varphi_\nu(t)$, and we obtain by the use of Bessel's inequality and (7.19)

$$\|\mathfrak{R}\|^2 = \sum_\nu \lambda_\nu R_\nu^2 \leqq \lambda_1 \sum_\nu R_\nu^2 \leqq \lambda_1 \int_D a^2(t)\,dt = \lambda_1 \sum_{i=1}^{n} \int_{D_0} a_\nu^2(t)\,dt$$
$$\leqq \frac{\varepsilon^2}{\lambda_1 n}\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{D_0} k_j^2(t)\,dt = \frac{\varepsilon^2}{\lambda_1}\int_{D_0} k^2(t)\,dt \leqq \varepsilon^2 \|h\|^2.$$

This proves (i). To prove (ii) we have only to show that the expression given by (7.14) is the scalar product $(\mathfrak{L}(\mathfrak{x}, \mathfrak{h}), \mathfrak{h}')$. We omit this verification which is similar to the one used in the proof of lemma 7.1.

Finally, to prove (iii) we notice that the mapping $\Phi(\mathfrak{h}) = k(t)$ is completely continuous.[1] Therefore, it will be sufficient to show that the mapping $k(t) \to \mathfrak{L}$ is continuous. In other words we have to show that $\mathfrak{L}$ as operator on $k(t)$ is bounded. To do this we note first that, as already mentioned, the $y_i(t)$ are continuous. Therefore, the $\frac{\partial f_i}{\partial u_j}(t, y_1(t), \ldots y_n(t))$ are likewise continuous functions of

[1] [15], lemma 2.2.

t and consequently bounded in D_0. If M is an upper bound for their absolute values in D_0, and if we set

$$b_i(t) = \sum_{j=1}^{n} \frac{\partial f_i}{\partial u_i} k_j(t), \qquad b(t) = (b_1(t), \ldots b_n(t)),$$

we have

$$|b_i(t)|^2 \leqq \left(M \sum_{j=1}^{n} |k_j(t)|\right)^2 \leqq M^2 n \sum_{j=1}^{n} k_j^2(t). \tag{7.20}$$

On the other hand, we see from (7.13) that L_ν is the Fouriercoefficient of $b(t)$ with respect to $\varphi_\nu(t)$. Therefore, we obtain from the Bessel inequality and (7.20)

$$\|\mathfrak{L}\|^2 = \sum_\nu \lambda_\nu L_\nu^2 \leqq \lambda_1 \sum_\nu L_\nu^2 \leqq \lambda_1 \int_D b^2(t)\,dt =$$

$$= \lambda_1 \sum_{i=1}^{n} \int_{D_0} b_i^2(t)\,dt \leqq \lambda_1 M^2 n \int_{D_0} \sum_{j=1}^{n} k_j^2(t)\,dt = \lambda_1 M^2 n \int_{D_0} k^2(t)\,dt$$

which proves the required boundedness of L since the last integral is the square of the norm of $k(t)$ in L^2.

Theorem 7.1. *Let $\mu = \mu(t, u)$ be the smallest eigenvalue of the matrix $(\partial f_i / \partial u_j)$*[1], *and, as always, λ_1 the greatest of the eigenvalues of the* (*not necessarily symmetric*) *kernel $K(s, t)$ defined by* (7.2). *In addition to the hypotheses of lemma* 7.2 *we assume that there exists a positive constant $\bar{c}$ such that for all $t, u_1, \ldots u_n$ in the product space $D_0 \times E^n$*

$$\mu \geqq -\bar{c} > -1/\lambda_1. \tag{7.21}$$

Then: (i) *the Hammerstein scalar $i(\mathfrak{x})$ has exactly one critical point $\mathfrak{a}$,* (ii) *for the norm $\|\mathfrak{a}\|$ of this critical point the estimate*

$$\|\mathfrak{a}\|^2 < \frac{\lambda_1 \sum_{i=1}^{n} \int_{D_0} f_i^2(t, 0, \ldots 0)\,dt}{(1 - \bar{c}\lambda_1)^2} \tag{7.22}$$

holds, and (iii) *$i(\mathfrak{a})$ is an absolute minimum in H.*

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[1] This matrix is symmetric since $\sum_{i=1}^{n} f_i\,du_i$ is a total differential, namely the differential of the line integral (7.1).

Proof. Since it is easily seen that the gradient of $\|\mathfrak{x}\|^2/2 = \sum_\nu \lambda_\nu x_\nu^2/2$ is given by $(x_1, x_2, \ldots)$ and the second differential by $\sum_\nu \lambda_\nu h_\nu h'_\nu$ it follows from lemmas 7.1 and 7.2 that the gradient $\mathfrak{g}(\mathfrak{x})$ of $i(\mathfrak{x}) = \|\mathfrak{x}\|^2/2 + I(\mathfrak{x})$, the differential $\mathfrak{l}(\mathfrak{x}, \mathfrak{h})$ of $\mathfrak{g}(\mathfrak{x})$, and the second differential $d_2(\mathfrak{x}, \mathfrak{h}, \mathfrak{h}')$ of $i(\mathfrak{x})$ exist, and that $\mathfrak{g}(\mathfrak{x})$ and d_2 are given by

$$\mathfrak{g}(\mathfrak{x}) = (x_1 + G_1(\mathfrak{x}),\ x_2 + G_2(\mathfrak{x}), \ldots) \tag{7.23}$$

$$d_2(\mathfrak{x}, \mathfrak{h}, \mathfrak{h}') = \sum_\nu \lambda_\nu h_\nu h'_\nu + \sum_{i=1}^{n} \sum_{j=1}^{n} \int_{D_0} k_i(t)\, k'_j(t)\, \frac{\partial f_i}{\partial u_j}(t, y_1(t), \ldots y_n(t))\, dt \tag{7.24}$$

where the $G_\nu(\mathfrak{x})$ are defined by (7.12).

We want to show first that $d_2(\mathfrak{x}, \mathfrak{h}, \mathfrak{h}')$ is uniformly positive definite in H in the sense of definition 6.2. By a well known property of the smallest eigenvalue of a symmetric matrix[1] it follows from (7.21) that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} k_i(t)\, k_j(t)\, \frac{\partial f_i}{\partial u_j}(t, y_1(t), \ldots y_n(t)) \geqq -\bar{c} \sum_{i=1}^{n} k_i^2(t). \tag{7.25}$$

Observing that

$$\sum_{i=1}^{n} \int_{D_0} k_i^2(t)\, dt = \int_{D_0} k^2(t)\, dt = \sum_\nu (\lambda_\nu h_\nu)^2 \leqq \lambda_1 \|\mathfrak{h}\|^2$$

and that the first sum in (7.24) for $\mathfrak{h} = \mathfrak{h}'$ equals $\|\mathfrak{h}\|^2$, we see from (7.24), (7.25) that

$$d_2(\mathfrak{x}, \mathfrak{h}, \mathfrak{h}) \geqq \|\mathfrak{h}\|^2 - \bar{c}\lambda_1 \|\mathfrak{h}\|^2 = \|\mathfrak{h}\|^2 (1 - \bar{c}\lambda_1).$$

This shows that (6.5) (for $q = d_2$) is satisfied with

$$c = 1 - \bar{c}\lambda_1. \tag{7.26}$$

Therefore, d_2 is uniformly positive definite in H since $c > 0$ by (7.21).

It follows from lemma 2.4 that d_2 is not degenerate and, therefore, from lemma 2.5 that the differential $\mathfrak{l}(\mathfrak{x}, \mathfrak{h})$ of $\mathfrak{g}(\mathfrak{x})$ is non-singular. By lemmas 7.1, 7.2 it is now obvious that the hypotheses 3.1—3.4 and hypothesis 4.1 and, consequently, all assumptions of theorem 6.2 and its corollary are satisfied. It follows, therefore, from this theorem and the corollary that $i(\mathfrak{x})$ has exactly one critical point $\mathfrak{a}$, that $i(\mathfrak{x})$ is an absolute minimum and that

[1] See e.g. COURANT—HILBERT, Methoden der mathematischen Physik, 2nd ed., 1930, vol. 1, Chapter 1, § 3.

$$(7.27) \qquad \|\mathfrak{a}\| < \|\mathfrak{g}(\mathfrak{o})\|/c$$

where c is defined by (7.26).

It remains to prove the estimate (7.22). Using (7.23), (7.12) and Bessel's inequality we see that

$$\|\mathfrak{g}(\mathfrak{o})\|^2 = \|\mathfrak{G}(\mathfrak{o})\|^2 = \sum_\nu \lambda_\nu G_\nu^2(\mathfrak{o}) \leqq \lambda_1 \sum_\nu G_\nu^2(\mathfrak{o})$$

$$\leqq \lambda_1 \int_D f^2(t, 0)\, dt = \lambda_1 \sum_{i=1}^n \int_{D_0} f_i^2(t, 0, \dots 0)\, dt.$$

Combining this inequality with (7.27) we obtain (7.22).

Theorem 7.2. *Under the same assumptions as in theorem 7.1, there exists one and only one pair of conjugate n-tuples* $y(s) = (y_1(s), \dots y_n(s))$, $y^*(s) = (y_1^*(s), \dots y_n^*(s))$ *(def. 7.1) satisfying the system (1.11).*[1] *For this solution the following estimates hold*

$$(7.28) \qquad \sum_{i=1}^n \int_{D_0} y_i^2(t)\, dt = \sum_{i=1}^n \int_{D_0} y_i^{*2}(t)\, dt \leqq \frac{\lambda_1^2 \sum_{i=1}^n \int_{D_0} f_i^2(t, 0, \dots 0)\, dt}{(1 - \bar{c}\lambda_1)^2}$$

$$(7.29) \qquad \left.\begin{matrix} |y(t)| \\ |y^*(t)| \end{matrix}\right\} \leqq C \frac{\sqrt{\lambda_1 \sum_{i=1}^n \int_{D_0} f_i^2(t, 0, \dots 0)\, dt}}{1 - \bar{c}\lambda_1}$$

where C is the constant of Assumption A (see (7.10)) and $\bar{c}$ the constant appearing in the inequality (7.21).[2]

Proof. Since by definition the eigenfunctions $\varphi_\nu(s)$, $\varphi_\nu^*(s)$ satisfy the linear integral equation

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[1] We draw attention to the definition 7.1 of conjugate n-tuples. If one asks for solutions of (1.11) satisfying condition (i) but not necessarily (ii) of this definition, then the uniqueness assertion of theorem 7.2 is no more true. If e.g. (1.11) consists of one single linear equation whose kernel K admits an eigenfunction $n(t)$ to the eigenvalue 0, it is easily seen that with $y(t)$, $y^*(t)$ also $y(t) + n(t)$, $y^*(t)$ is a pair of solutions of (1.11) satisfying condition (i) of definition (7.1). If, however, the kernel of (1.11) is closed, i.e., $M = M^* = L^2$, condition (ii) of definition 7.1 is automatically satisfied, and the uniqueness assertion is true.

We finally remark that the y^*-part of a solution is always uniquely determined since the Schmidt expansion theorem shows that for a solution $y(t)$, $y^*(t)$, the function $y^*(t)$ is always in M^*.

[2] We note that if the so-called bilinear series for the kernel converges, we may take $\mathrm{Max}\,|K(s,s)|$ for C, and that in case of one single integral equation c may be taken as $\min \partial f(t, u)/\partial u$.

$$\lambda_\nu \varphi_\nu(t) = \int_D K(t, s) \varphi_\nu^*(s)\, ds$$

we obtain from (7.12)

$$\lambda_\nu G_\nu = \int_D \varphi_\nu^*(s) \int_D K(t, s) f(t, y(t))\, dt\, ds,$$

and since

$$\lambda_\nu x_\nu = \int_D \varphi_\nu^*(s) y^*(s)\, ds,$$

we have for the components g_ν of the gradient $\mathfrak{g}(\mathfrak{x})$ of $i(\mathfrak{x})$

$$\lambda_\nu g_\nu = \lambda_\nu x_\nu + \lambda_\nu G_\nu = \int_D \varphi_\nu^*(s) \left[y^*(s) + \int_D K(t, s) f(t, y(t))\, dt\right] ds. \tag{7.30}$$

Now for the critical point $\mathfrak{x} = \mathfrak{a}$ whose existence is assured by theorem 7.1, we have $\mathfrak{g}(\mathfrak{a}) = 0$. Therefore, (7.30) shows that for $y(t) = \Phi(\mathfrak{a})$, $y^*(t) = \Phi^*(\mathfrak{a})$ all Fourier coefficients of the quantity contained in the bracket of equation (7.30) are zero, and this quantity itself will be shown to be zero once it is proved to be an element of the space M^* spanned by the $\varphi_\nu^*(s)$. But this is true since $y^*(s) < M^*$ by the definition of $y^*(s)$ and since by the Schmidt expansion theorem

$$\int_D K(t, s) f(t, y(t))\, dt$$

can be expanded according to the $\varphi_\nu^*(s)$. This proves the existence of a solution of (1.11) since the n-tuple formed by the left members of these equations is just the n-tuple associated with the quantity in the bracket of (7.30) by the rule given in (7.3), (7.4) and by the definition of $f(t, y(t))$ given in lemma 7.1.

The uniqueness follows easily from the fact that because of (7.30) the conjugate pair $y(t) = \Phi(\mathfrak{x})$, $y^*(t) = \Phi^*(\mathfrak{x})$ is a solution of (1.11) only if $\mathfrak{g}(\mathfrak{x}) = \mathfrak{o}$, i.e., if $\mathfrak{x}$ is a critical point of $i(\mathfrak{x})$ since by theorem 7.1 there exists only one critical point.

It remains to prove the estimates (7.28), (7.29). Since

$$\sum_{i=1}^{n} \int_{D_0} y_i^2(t)\, dt = \int_D y^2(t)\, dt = \sum_\nu \Big(\int_D y(t) \varphi_\nu(t)\, dt\Big)^2 =$$
$$= \sum_\nu (\lambda_\nu x_\nu)^2 \leqq \lambda_1 \sum_\nu \lambda_\nu x_\nu^2 = \lambda_1 \|\mathfrak{x}\|^2$$

and since the corresponding inequality holds if $y(t)$ is replaced by $y^*(t)$, (7.28) follows from (7.22) since as already observed, $\mathfrak{x}$ is the critical point $\mathfrak{a}$ of $i(\mathfrak{x})$ if

$\Phi(\mathfrak{x}) = y(t)$, $\Phi^*(\mathfrak{x}) = y^*(t)$ is the solution of (1.11). To derive also (7.29) from (7.22) we have only to recall that $C\|\mathfrak{x}\|$ is an upper bound for $|y(t)|$ and $|y^*(t)|$ ([15], lemma 3.1).

The following theorem was first obtained by M. Golomb.[1]

Theorem 7.3. *If in addition to the assumptions of theorem 7.2 the kernel $K(s, t)$ defined by (7.2) is symmetric and positive definite, then the system (1.12) has one and only one solution $y(t) = (y_1(t), \dots y_n(t))$, and this solution is continuous.*

Proof. Under the present conditions we may assume $\varphi_\nu(s) = \varphi_\nu^*(s)$. Therefore, $y_i(s) = y_i^*(s)$ and the theorem follows immediately from the preceding theorem 7.2 and the final remark made in the footnote to theorem 7.2 concerning the uniqueness.

Bibliography.

1. ALEXANDROFF and HOPF, *Topologie*, Berlin, 1935.
2. FRÉCHET, *La notion de différentielles dans l'analyse générale*, Annales de l'École Normale Supérieure, (3) vol. 42 (1925), 293—323.
3. M. GOLOMB, *Zur Theorie der nichtlinearen Integralgleichungen, Integralgleichungssysteme und allgemeinen Funktionalgleichungen*, Mathematische Zeitschrift, vol. 39 (1934), 45—75.
4. ——, *Über Systeme von nichtlinearen Integralgleichungen*, Publications Mathématiques de l'université de Belgrade, vol. V (1936), 52—83.
5. L. M. Graves, *Riemann integration and Taylor's theorem in general analysis*, Transactions of the American Mathematical Society, vol. 29 (1927), 163—177.
6. L. M. GRAVES and T. H. HILDEBRANDT, *Implicit functions and their differentials in general analysis*, Transactions of the American Mathematical Society, vol. 29 (1927), 126—153.
7. A. HAMMERSTEIN, *Nichtlineare Integralgleichungen nebst Anwendungen*, Acta mathematica, vol. 54 (1930), 118—176.
8. M. KERNER, *Die Differentiale in der allgemeinen Analysis*, Annals of Mathematics, (2) vol. 34 (1933), 546—572.
9. J. LERAY and M. SCHAUDER, *Topologie et équations fonctionnelles*, Annales de l'École Normale Supérieure, vol. 51 (1934), 45—78.
10. E. H. ROTHE, *Zur Topologie der topologischen Ordnung und der Vektorfelder in Banachschen Raumen*, Compositio Matematica 5 (1937), 177—197.
11. ——, *The theory of topological order in some linear topological spaces*, Iowa State College Journal of Science, vol. 13 (1939), 373—390.

[1] [4], Satz 1. The conditions of the present theorem 7.3 are slightly less restrictive (regarding the kernels) than in [4] as follows from [15], lemma 5.1.

12. E. H. ROTHE, *Topological proofs of uniqueness theorems in the theory of differential and integral equations,* Bulletin of the American Mathematical Society, vol. 45 (1939), 606—613.

13. ——, *Gradient mappings in Hilbert space,* Annals of Mathematics, vol. 47 (1946), 580—592.

14. ——, *Gradient mappings and extrema in Banach spaces,* Duke Mathematical Journal, vol. 15 (1948), 421—431.

15. ——, *Weak topology and non-linear integral equations,* Transactions of the American Mathematical Society, vol. 66 (1949), 75—92.

16. SEIFERT und THRELFALL, *Variationsrechnung im Grossen (Theorie von Marston Morse),* Teubner, Leipzig und Berlin, 1938.