HARMONIC ESTIMATION IN CERTAIN SLIT REGIONS AND A THEOREM OF BEURLING AND MALLIAVIN

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Introduction

Suppose we form a domain $\mathcal{D}$ consisting of the union of the upper and lower half planes and a finite number of bounded open intervals from the real axis. The remainder, Γ, of the real axis is the boundary of $\mathcal{D}$. If $u(z)$ is subharmonic in $\mathcal{D}$, behaves reasonably well near Γ, and is dominated by $a|\operatorname{Im} z|$ for $|z|$ large, and if, on Γ, the boundary values $u(x \pm i0)$ are known to be less than some given majorant $M(x)$, the possible size of u at any point of $\mathcal{D}$ is governed by two factors:

(i) The allowed rate of growth (or required rate of decrease, as the case may be) of $u(z)$ at ∞.

(ii) The magnitude of the majorant $M(x)$.

The interplay between these two factors is studied in Part I of the present paper. It is remarkable that in many cases their effects are comparable, and depend on the domain $\mathcal{D}$ solely through a quantity, called here the Selberg number, having a quite simple function-theoretic definition.

The results obtained in Part I are specific enough to yield a fairly straightforward duality proof of a theorem of Beurling and Malliavin [2] on the existence of certain kinds of multipliers for entire functions of exponential type. This application is given in Part II.

Part III contains an elementary derivation of another multiplier theorem of Beurling and Malliavin from the one proved in Part II. It can be read independently of the rest of the paper.

Work on the material presented here took several years, and was completed at the end of 1977 while I was staying at the Mittag-Leffler Institute in Sweden. I am very grateful to Professor L. Carleson of that institute for having let me discuss the ideas of

my investigation with him, and especially for encouraging me to not abandom them when it seemed hopelessly bogged down. I also thank him for advice and criticism which helped me to improve the written presentation.

I. Harmonic Estimation in Slit Regions

We are interested in obtaining bounds for subharmonic functions defined in regions $\mathcal{D}$ like the one in Fig. 1, obtained by cutting out a finite number of bounded open intervals from the real axis $\mathbf{R}$. Such regions include the open upper and lower half planes. We will denote $\mathbf{R}\cap\mathcal{D}$ by $\mathcal{O}$ and the boundary $\mathbf{R}\sim\mathcal{O}$ of $\mathcal{D}$ by Γ. It will always be assumed that $0\in\mathcal{O}$.

Each segment of Γ is of course assumed to have *two sides*. If x is on such a segment and not an endpoint of it, and if $U(z)$ is defined for $z\in\mathcal{D}$, we generally have to distinguish between the *two* boundary values

$$U(x+i0)=\lim_{y\to 0+} U(x+iy), \quad U(x-i0)=\lim_{y\to 0+} U(x-iy),$$

assuming that the limits exist. If they are equal, we denote their common value by $U(x)$.

1.

In order to take account of the behaviour, for large values of $|z|$, of functions subharmonic in $\mathcal{D}$, we require a Phragmén-Lindelöf function $Y_{\mathcal{D}}(z)$ with the following properties:

(i) $Y_{\mathcal{D}}(z)$ is harmonic and positive in $\mathcal{D}$

(ii) $Y_{\mathcal{D}}(z)$ is continuous up to Γ and *vanishes* there

(iii) $Y_{\mathcal{D}}(z)=|\operatorname{Im} z|+O(1)$.

Clearly, there can be only one such function.

The first problem we take up is that of finding $Y_{\mathcal{D}}(z)$. Our solution is in terms of the Green's function $G_{\mathcal{D}}(z,w)$ for $\mathcal{D}$. Recall that for fixed $w\in\mathcal{D}$, $G_{\mathcal{D}}(z,w)$ is positive and harmonic in z for $z\in\mathcal{D}$, save near w where it equals $\log(1/|z-w|)$ plus a harmonic function of z. For fixed $w\in\mathcal{D}$, $G_{\mathcal{D}}(z,w)$ is continuous up to the boundary of $\mathcal{D}$ where it *vanishes*—for our domains $\mathcal{D}$ we can thus assume without further ado that $G_{\mathcal{D}}(x,w)$ is defined for all real x, and *zero* if $x\notin\mathcal{O}$. We have the symmetry $G_{\mathcal{D}}(z,w)=G_{\mathcal{D}}(w,z)$ (see [18], p. 17) and, for our domains $\mathcal{D}$, the obvious relation $G_{\mathcal{D}}(z,w)=G_{\mathcal{D}}(\bar{z},\bar{w})$.

Now we have

$$Y_{\mathcal{D}}(z)=\frac{1}{\pi}\lim_{R\to\infty}\int_{-R}^{R}\left[\log\left|1-\frac{z}{t}\right|+G_{\mathcal{D}}(z,t)\right]dt. \tag{1.1}$$

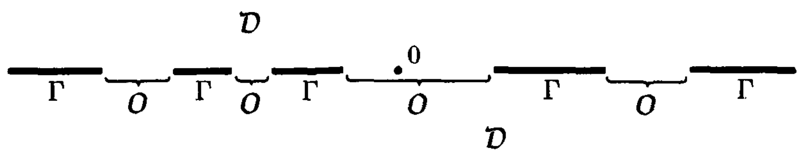

Fig. 1.

Indeed, by contour integration, $\lim_{R\to\infty}\int_{-R}^{R}\log|1-(z/t)|\,dt=\pi|\operatorname{Im} z|$, so that (1.1) can be rewritten

$$Y_{\mathcal{D}}(z)=|\operatorname{Im} z|+\frac{1}{\pi}\int_{\mathcal{O}} G_{\mathcal{D}}(z,t)\,dt. \tag{1.2}$$

From this, properties (ii) and (iii) of $Y_{\mathcal{D}}(z)$ are manifest, and so is (i) except for the harmonicity of $Y_{\mathcal{D}}(z)$ at points of $\mathcal{O}$. To verify that, take a large A with $\mathcal{O}\subseteq(-A,A)$, put $\Gamma_A=\Gamma\cap[-A,A]$, and rewrite (1.1) thus:

$$Y_{\mathcal{D}}(z)=\frac{1}{\pi}\int_{A}^{\infty}\log\left|1-\frac{z^2}{t^2}\right|dt+\frac{1}{\pi}\int_{\Gamma_A}\log\left|1-\frac{z}{t}\right|dt+\frac{1}{\pi}\int_{\mathcal{O}}\left[G_{\mathcal{D}}(z,t)+\log|z-t|+\log\frac{1}{|t|}\right]dt.$$

The harmonicity of this expression at points of $\mathcal{O}$ is clear.

From (1.2) we have, in particular,

$$Y_{\mathcal{D}}(x)=\frac{1}{\pi}\int_{\mathcal{O}} G_{\mathcal{D}}(x,t)\,dt,\quad x\in\mathcal{O}.$$

The integral $\int_{\mathcal{O}} G_{\mathcal{D}}(x,t)\,dt=\int_{-\infty}^{\infty} G_{\mathcal{D}}(x,t)\,dt$ plays an important role in the present study. We call it the Selberg number for $\mathcal{D}$ at x and denote it by $\Lambda_{\mathcal{D}}(x)$. Thus, $Y_{\mathcal{D}}(x)=\pi^{-1}\Lambda_{\mathcal{D}}(x)$.

H. Selberg first studied $\int_{\mathcal{O}} G_{\mathcal{D}}(x,t)\,dt$ and obtained a precise upper bound for it in [16]. See also [18], p. 24.

2.

We now study harmonic measure on the boundary Γ of $\mathcal{D}$. If $w\in\mathcal{D}$ and x runs along Γ, we denote by $d\omega_{\mathcal{D}}(x,w)$ the differential element of harmonic measure for $\mathcal{D}$, as seen from w. Because Γ has two sides, we specify that $\omega_{\mathcal{D}}([x,x+\Delta x],w)$ is to mean the *combined harmonic measure of both sides of Γ along the segment $[x,x+\Delta x]$ lying thereon*. It is also convenient to make $d\omega_{\mathcal{D}}(x,w)$ a differential on all of $\mathbf{R}$ by taking it to be identically zero when x goes through $\mathcal{O}$.

It is possible to give a remarkable upper bound for $\omega_{\mathcal{D}}((-\infty,-x]\cup[x,\infty),0)$, *depending on the domain $\mathcal{D}$ solely through the Selberg number $\Lambda_{\mathcal{D}}(0)$.*

In what follows, we frequently drop the subscript $\mathcal{D}$ to simplify the notation, writing $Y(z)$ for $Y_{\mathcal{D}}(z)$, $G(z,w)$ for $G_{\mathcal{D}}(z,w)$, and so forth. For real t, $G(t,0)$ is continuous save near 0 where it behaves like $\log(1/|t|)$, and is identically zero outside the bounded set $\mathcal{O}$. We can therefore talk about its Hilbert transform. All the material we use about Hilbert transforms is in Chapter 5 of [17]. The symbol p.v. $\int$ denotes a Cauchy principal value of the integral.

LEMMA.
$$\frac{1}{\pi}\,\text{p.v.}\int_{-\infty}^{\infty}\frac{G(t,0)}{x-t}\,dt=\begin{cases}\pi\omega([x,\infty),0) & \text{if } x>0\\ -\pi\omega((-\infty,x],0) & \text{if } x<0.\end{cases} \tag{2.1}$$

Proof. For $\operatorname{Im} z>0$, consider the analytic function
$$F(z)=\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{G(t,0)\,dt}{z-t}; \tag{2.2}$$
since $G(z,0)$ is harmonic for $\operatorname{Im} z>0$ and tends to 0 as $z\to\infty$ in that half plane, we have by Poisson's formula
$$F(z)=G(z,0)+iV(z),\quad \operatorname{Im} z>0, \tag{2.3}$$
where, at points x on the real axis $V(z)$ has a non-tangential boundary value $V(x)$ equal to the left hand side of (2.1) (see [17], Chapter 5).

At each non-zero $x\in\mathcal{O}$, $V'(x)$ exists and equals *zero*. Indeed, such an x lies in a little interval $I\subseteq\mathcal{O}$ having a neighborhood in which $G(z,0)$ is harmonic, so that $F(z)$ can be analytically continued into all of that neighborhood. This makes $V(x)$ infinitely differentiable on I, and by the Cauchy-Riemann equations, $V'(x)=-G_y(x,0)$ which is zero since $G(z,0)=G(\bar z,0)$.

If $x\in\Gamma$ is not an endpoint of one of its components,
$$V'(x)=-\pi\frac{d\omega(x,0)}{dx}. \tag{2.4}$$

For, by the well-known relation between the normal derivative of Green's function and harmonic measure ([18], p. 20),
$$\frac{d\omega(x,0)}{dx}=\frac{1}{2\pi}[G_y(x+i0,0)-G_y(x-i0,0)]=\frac{1}{\pi}G_y(x+i0,0),$$
taking the two sides of Γ and the relation $G(\bar z,0)=G(z,0)$ into account. But $G(t,0)$ vanishes outside $\mathcal{O}$, so (2.2) can be differentiated at $z=x$ here to yield
$$V'(x)=-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{G(t,0)\,dt}{(x-t)^2}.$$

Since $G(x, 0)=0$, (2.3) and (2.2) show that this last expression equals $-G_y(x+i0, 0)$ on taking the limit of a difference quotient. We thus have (2.4).

Near 0, $G(t, 0)=\log(1/|t|)$ plus an infinitely differentiable function of t. Thence, by direct calculation, $V(x)$ has a simple jump discontinuity at 0, increasing by π there. Finally, $V(x)\to 0$ as $x\to\pm\infty$, for $G(t, 0)$ has compact support.

The information about $V(x)$ and $V'(x)$ already obtained implies that $V(x)$ equals the right side of (2.1), provided that it is continuous at the endpoints of the components of $\mathcal{O}$. But it is, for $G(t, 0)$ is easily seen to be Lip $\frac{1}{2}$ at the endpoints and infinitely differentiable elsewhere, save at 0. So $V(x)$ is also Lip $\frac{1}{2}$ at those endpoints ([17], pp. 145–146), and we are done.

Notation. For $x>0$,

$$\Omega_{\mathcal{D}}(x)=\Omega(x)=\omega_{\mathcal{D}}((-\infty, -x]\cup[x, \infty), 0). \tag{2.5}$$

In terms of this notation, (2.1) reads:

COROLLARY. For $x>0$,

$$\Omega_{\mathcal{D}}(x)=\frac{2}{\pi^2}\,\text{p.v.}\int_{-\infty}^{\infty}\frac{x}{x^2-t^2}\,G_{\mathcal{D}}(t, 0)\,dt. \tag{2.6}$$

Since $G(t, 0)$ has compact support,

$$\Omega(x)\sim\frac{2\Lambda(0)}{\pi^2 x}\quad\text{for } x\to\infty.$$

More is true.

THEOREM. *For* $x>0$,

$$\Omega_{\mathcal{D}}(x)\leqslant\frac{\Lambda_{\mathcal{D}}(0)}{\pi x}. \tag{2.7}$$

Proof. Given any $x_0>0$, let $\Gamma_0=\Gamma\cup(-\infty, -x_0]\cup[x_0, \infty)$, $\mathcal{O}_0=\mathbf{R}\sim\Gamma_0$, and let $\mathcal{D}_0$ be the complement of Γ_0 in the complex plane $\mathbf{C}$ (see Fig. 2, top of next page).

Let $G_0(z, w)$ be the Green's function for $\mathcal{D}_0$, and, for $x>0$, let $\Omega_0(x)$ be the harmonic measure of $\Gamma_0\cap[(-\infty, -x]\cup[x, \infty)]$ for $\mathcal{D}_0$, as seen from 0. Since $\mathcal{D}_0\subseteq\mathcal{D}$,

$$\Omega(x_0)\leqslant\Omega_0(x_0). \tag{2.8}$$

For the same reason, $G_0(z, 0)\leqslant G(z, 0)$, so, if $\varrho>1$, by (2.6) applied to Ω_0 and G_0,

$$\Omega_0(\varrho x_0)=\frac{2}{\pi^2}\int_{-x_0}^{x_0}\frac{\varrho x_0 G_0(t, 0)}{\varrho^2 x_0^2-t^2}\,dt\leqslant\frac{2\varrho}{\pi^2(\varrho^2-1)\,x_0}\int_{-x_0}^{x_0}G(t, 0)\,dt\leqslant\frac{2\varrho\Lambda(0)}{\pi^2(\varrho^2-1)\,x_0}, \tag{2.9}$$

for $G_0(t, 0)$ vanishes outside $\mathcal{O}_0\subseteq(-x_0, x_0)$.

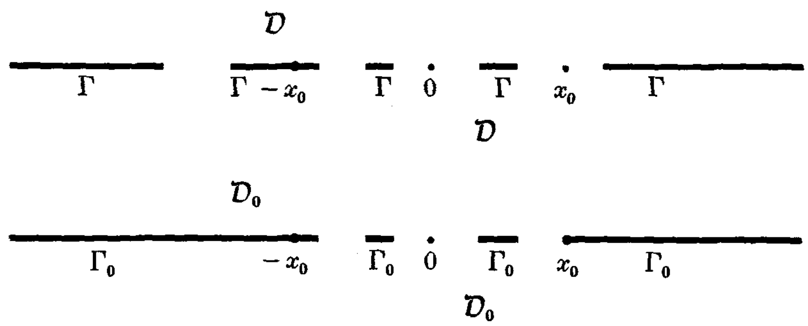

Fig. 2.

The idea now is to show that $\Omega_0(\varrho x_0)/\Omega_0(x_0)$ has a positive lower bound depending only on the parameter $\varrho > 1$. For this purpose, take a third domain

$$\mathcal{E} = \mathbf{C} \sim ((-\infty, -x_0] \cup [x_0, \infty))$$

we have $\mathcal{D}_0 \subseteq \mathcal{E}$ and we put

$$\gamma = \Gamma \cap (-x_0, x_0), \quad \text{then } \mathcal{D}_0 = \mathcal{E} \sim \gamma.$$

Let $d\omega_0(x, 0)$ be the differential element of harmonic measure on Γ_0 for $\mathcal{D}_0$, as seen from 0, and take the function $W_\varrho(z)$ bounded and harmonic in $\mathcal{E}$, determined by the boundary conditions

$$W_\varrho(x \pm i0) = 0, \quad x_0 \leqslant |x| < \varrho x_0,$$
$$W_\varrho(x \pm i0) = 1, \quad \varrho x_0 < |x| < \infty.$$

Then, by Poisson's formula for $\mathcal{D}_0$,

$$\Omega_0(\varrho x_0) = W_\varrho(0) - \int_\gamma W_\varrho(x)\, d\omega_0(x, 0) \tag{2.10}$$

$$\Omega_0(x_0) = 1 - \int_\gamma d\omega_0(x, 0). \tag{2.11}$$

However,

$$W_\varrho(x) \leqslant W_\varrho(0) \quad \text{for } -x_0 < x < x_0. \tag{2.12}$$

To see this, let φ be a conformal mapping of $\{|w| < 1\}$ onto $\mathcal{E}$ which takes the diameter $(-1, 1)$ onto $(-x_0, x_0)$, with $\varphi(0) = 0$. $W_\varrho(\varphi(w))$ is then the *combined harmonic measure of the arcs* $\sigma = \{e^{i\theta};\ \alpha < \theta < \pi - \alpha\}$ *and* $\tilde{\sigma} = -\sigma$, for the unit disk, as seen from w therein. Here α, $0 < \alpha < \pi/2$, is a certain number depending only on ϱ. If $-1 < w < 1$, $W_\varrho(\varphi(w))$ is by symmetry equal to *twice* the harmonic measure of the upper arc, σ. The level lines for this harmonic measure are, however, just the circles through the endpoints of σ. It is now obvious from Fig. 3 that, for $-1 \leqslant w \leqslant 1$, $W_\varrho(\varphi(w))$ is at its maximum when $w = 0$, and (2.12) is proved.

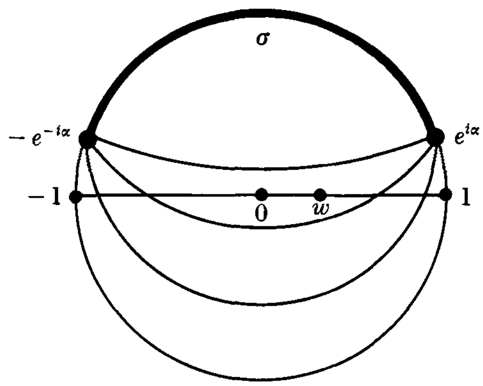

Fig. 3.

From (2.12), (2.11) and (2.10) we have $\Omega_0(\varrho x_0) \geqslant W_\varrho(0)\Omega_0(x_0)$, with $W_\varrho(0)$ depending on ϱ alone by homogeneity. Substitution of this last relation into (2.9) followed by use of (2.8) yields

$$\Omega(x_0) \leqslant \frac{2\varrho}{\pi^2 W_\varrho(0)(\varrho^2-1)} \frac{\Lambda(0)}{x_0}.$$

To obtain (2.7) from this, note that $W_\varrho(0) = 2\pi^{-1}(\frac{1}{2}\pi - \alpha)$, which works out to $(2/\pi)$ arcsin $(1/\varrho)$ by explicit computation with $\varphi(w)$. Put this value into the previous relation and make $\varrho \to \infty$; one gets (2.7) with x_0 instead of x. Q.E.D.

3.

As substitute for a regularity in $\Omega(x)$ which is lacking here ($\Omega'(x) = -\infty$ whenever x or $-x$ is an endpoint of a component of O, and these components may be very numerous), we derive a quadratic inequality involving $d(x\Omega(x))$.

LEMMA. *For* $x \neq 0$,

$$G(x,0) + G(-x,0) = \frac{1}{x}\int_0^\infty \log\left|\frac{x+t}{x-t}\right| d(t\Omega(t)). \tag{3.1}$$

Proof. $G(x,0) \in L^2$, so, continuing, as in the proof of the lemma, § 2, to denote the left side of (2.1) by $V(x)$, we have ([17], Chapter 5)

$$G(x,0) = -\frac{1}{\pi} \text{ p.v.} \int_{-\infty}^\infty \frac{V(t)\,dt}{x-t}.$$

By (2.1) and (2.5), $V(t) - V(-t) = \pi\Omega(t)$ for $t > 0$ which, with the preceding, gives

$$G(x,0) + G(-x,0) = -\text{p.v.} \int_0^\infty \frac{2t\Omega(t)\,dt}{x^2-t^2}. \tag{3.2}$$

The Cauchy principal value on the right is evaluated by integrating by parts, first from 0 to $|x| - \varepsilon$ and from $|x| + \varepsilon$ to ∞, so as to obtain some integrated terms together with a new integral involving $d(t\Omega(t))$. The fact, following from (2.5), that $\Omega(t)$ is Lip $\frac{1}{2}$ makes the sum of the integrated terms go to zero as $\varepsilon \to 0$, and we end up with (3.1).

From 2.6 and the fact that $G(t, 0)$ vanishes outside $\mathcal{O}$ bounded, we get the convergent expansion

$$\Omega(x) = \frac{2\Lambda(0)}{\pi^2 x} + \frac{C}{x^3} + \frac{C'}{x^5} + \dots,$$

valid for large x. The function $\Omega(x)$ decreases from 1 to 0 on $[0, \infty)$ and has the constant value 1 near 0. Using these facts we easily verify that

$$\int_0^\infty \int_0^\infty \log\left|\frac{x+t}{x-t}\right| d(t\Omega(t))\, d(x\Omega(x))$$

converges absolutely.

THEOREM. *We have*

$$\int_0^\infty \int_0^\infty \log\left|\frac{x+t}{x-t}\right| d(t\Omega(t))\, d(x\Omega(x)) \leqslant \frac{[\Lambda(0)]^2}{\pi}. \tag{3.3}$$

Proof. By (3.1), $\int_0^\infty \log(|x+t|/|x-t|) d(t\Omega(t)) \geqslant 0$ for $x > 0$, while $d(x\Omega(x)) \leqslant \Omega(x)dx$ there since $\Omega(x)$ is decreasing. So by (3.1), the left side of (3.3) is

$$\leqslant \int_0^\infty x[G(x,0) + G(-x,0)]\,\Omega(x)\,dx,$$

which in turn is $\leqslant \pi^{-1}\Lambda(0) \int_0^\infty [G(x,0) + G(-x,0)]dx$ by (2.7). This last equals $[\Lambda(0)]^2/\pi$ by definition of the Selberg number (§ 1).

4.

We apply the results of §§ 1–3 in order to obtain estimates for certain functions subharmonic in $\mathcal{D}$.

LEMMA. *Let $v(z)$, subharmonic in $\mathcal{D}$, have the following properties:*

(i) *At each $x \in \Gamma$ not an endpoint of one of its components, v has boundary values satisfying $v(x \pm i0) \leqslant 0$,*

(ii) *There is an $\alpha < \frac{1}{2}$ such that $v(z) \leqslant O(|z-c|^{-\alpha})$ near any endpoint c of any component of Γ,*

(iii) *$v(z) \leqslant a|\operatorname{Im} z| + O(1)$ for $z \in \mathcal{D}$ with $|z|$ large.*

Then $v(z) \leqslant aY_{\mathcal{D}}(z)$ for $z \in \mathcal{D}$. In particular, for $x \in \mathcal{O}$, $v(x) \leqslant a\Lambda_{\mathcal{D}}(x)/\pi$.

Proof. Suppose first that we know how to construct a Phragmén-Lindelöf function $p(z)$ behaving thus: $p(z)$ is harmonic and positive in $\mathcal{D}$ and tends to ∞ like $|z-c|^{-\gamma}$ when z tends to any endpoint c of any component of Γ; here, γ is a number between α and $\frac{1}{2}$. Granted this, the lemma follows easily. Indeed, using the properties of $Y_{\mathcal{D}}(z)$ established in § 1, we see by the principle of maximum that $v(z)-aY_{\mathcal{D}}(z)-\varepsilon p(z)<0$ in $\mathcal{D}$ for each $\varepsilon>0$.

Here is a construction of $p(z)$. Let the infinite components of Γ be $(-\infty, a_0]$ and $[b_0, \infty)$, and the finite ones (if there are any) be $[a_1, b_1], \ldots, [a_n, b_n]$. Then

$$p(z)=\operatorname{Re}\left\{\left(\frac{b_0-z}{z-a_0}\right)^{\gamma}+\left(\frac{z-a_0}{b_0-z}\right)^{\gamma}\right\}+\operatorname{Re}\sum_{k=1}^{n}\left\{\left(\frac{z-b_k}{z-a_k}\right)^{\gamma}+\left(\frac{z-a_k}{z-b_k}\right)^{\gamma}\right\}$$

does the job.

COROLLARY. *Let $M(t)$ be positive and continuous on Γ, save perhaps at the endpoints of its components, and suppose that*

$$\int_{\Gamma} M(t)\, d\omega_{\mathcal{D}}(t, 0)<\infty. \tag{4.1}$$

Let $u(z)$ be subharmonic in $\mathcal{D}$ and have, at each $x\in\Gamma$ not an endpoint of one of its components, boundary values satisfying $u(x\pm i0)\leqslant M(x)$. Let $u(z)$ also have properties (ii) *and* (iii) *required of $v(z)$ in the above lemma.*

Then, for $z\in\mathcal{D}$,

$$u(z)\leqslant aY_{\mathcal{D}}(z)+\int_{\Gamma} M(t)\, d\omega_{\mathcal{D}}(t, z). \tag{4.2}$$

Proof. Since $M(t)\geqslant 0$, (4.1) implies by Harnack's theorem that $\int_{\Gamma} M(t)d\omega_{\mathcal{D}}(t, z)<\infty$ for every $z\in\mathcal{D}$. The corollary thence follows on applying the above lemma to

$$v(z)=u(z)-\int_{\Gamma} M(t)\, d\omega_{\mathcal{D}}(t, z).$$

Remark 1. $M(t)$ is allowed to become infinite at the endpoints of the components of Γ (and at ∞), but as long as (4.1) holds, (4.2) furnishes a useable bound for $u(z)$.

Remark 2. Relation (4.2) still holds (and we shall in Part II, have occasion to use it) in certain situations where $M(t)$ is *not* $\geqslant 0$ on Γ. *Suppose, for instance, that* $M(t)=M_1(t)+M_2(t)$, *where* $M_1(t)\geqslant 0$ *satisfies* (4.1) *and* $M_2(t)$, *of variable sign, is such that*

$$\int_{\Gamma} |M_2(t)|\, d\omega_{\mathcal{D}}(t, 0)<\infty.$$

If the harmonic function

$$u_2(z) = \int_\Gamma M_2(t)\, d\omega_{\mathcal{D}}(t, z)$$

is

(i) $\geqslant -O(|z-c|^{-\alpha})$ *near any endpoint c of any component of* Γ, *with an* $\alpha < \frac{1}{2}$;

(ii) *bounded below for all z in* $\mathcal{D}$ *of sufficiently large modulus;*

then (4.2) *holds for subharmonic functions* $u(z)$ *fulfilling the other conditions of the corollary.*

This extension is immediate. Simply write $u_1(z) = u(z) - u_2(z)$, then the corollary applies as it stands with $u_1(z)$ in place of $u(z)$ and $M_1(t)$ in place of $M(t)$. Add $u_2(z)$ to both sides of the inequality corresponding to (4.2).

If $M(t) \geqslant 0$, we would like to use the theorem of § 2 so as to estimate the integral on the right side of (4.2) in terms of $\int_\Gamma (M(x)/x^2)\,dx$. This, however, is not possible in general, because $d\omega(t, 0)/dt$ is infinite at the endpoints of the components of Γ, and there is no limitation on the number of these components. In order to be able to make such a comparison, we must impose a *certain smoothness*, of a nature determined by the results in § 3, on the majorant $M(t)$.

Definition. Let $M(x)$ be even. We call $M(x)/x$ a *Green potential* on $(0, \infty)$ if

$$\frac{M(x)}{x} = \int_0^\infty \log\left|\frac{x+t}{x-t}\right| d\varrho(t) \quad \text{for } x \neq 0 \tag{4.3}$$

with a real signed measure ϱ making the integral absolutely convergent for all $x \in \mathbf{R}$.

Definition. If $M(x)/x$ is a Green potential on $(0, \infty)$ and if

$$\int_0^\infty \int_0^\infty \log\left|\frac{x+t}{x-t}\right| |d\varrho(t)|\,|d\varrho(x)| < \infty$$

with ϱ the real signed measure from (4.3), we call

$$E\left\langle\frac{M(x)}{x}\right\rangle = \int_0^\infty \int_0^\infty \log\left|\frac{x+t}{x-t}\right| d\varrho(t)\, d\varrho(x) \tag{4.4}$$

the *energy* of $M(x)/x$.

This energy appears in the work of Beurling and Malliavin [2]. I think one of the reasons for its appearance there is that it is a natural measure for the kind of smoothness $M(t)$ must have.

THEOREM. *If $M(x)$ is positive and even, $M(x)/x$ a Green potential on $(0, \infty)$, and the integral in* (4.4) *absolutely convergent,*

$$\int_\Gamma M(x)\,d\omega_D(x,0) \leqslant \frac{\Lambda_D(0)}{\pi}\left[\int_0^\infty \frac{M(x)}{x^2}\,dx + \sqrt{\pi E\left\langle\frac{M(x)}{x}\right\rangle}\right]. \tag{4.5}$$

Proof. Since $M(x)$ is even, by (2.5),

$$\int_\Gamma M(x)\,d\omega(x,0) = -\int_0^\infty M(x)\,d\Omega(x) = \int_0^\infty \Omega(x)\frac{M(x)}{x}\,dx - \int_0^\infty \frac{M(x)}{x}\,d(x\Omega(x)). \tag{4.6}$$

Because $M(x)\geqslant 0$, the first integral on the right is $\leqslant (\Lambda(0)/\pi)\int_0^\infty (M(x)/x^2)\,dx$ by (2.7). According to the hypothesis, we can use (4.3) in the second integral on the right hand side of (4.6), obtaining

$$\int_0^\infty\int_0^\infty \log\left|\frac{x+t}{x-t}\right| d\varrho(t)\,d(x\Omega(x)). \tag{4.7}$$

Now, as long as the double integral in (4.4) and a corresponding one involving $d\sigma$ *converge absolutely,* the real bilinear form

$$\int_0^\infty\int_0^\infty \log\left|\frac{x+t}{x-t}\right| d\varrho(t)\,d\sigma(x)$$

in $d\varrho$ and $d\sigma$ is *positive definite.* (See [13], p. 92 and pp. 215–219; $\log(|z+\bar{w}|/|z-w|)$ is the Green's function for the right half plane. A more elementary discussion is given in [9], pp. 255–256.) *Therefore Schwarz' inequality holds for it.* This, applied to the expression in (4.7), shows that the latter is in modulus $\leqslant \pi^{-1/2}\Lambda(0)\,(E\langle M(x)/x\rangle)^{1/2}$, by (4.4) and the theorem of § 3. Substituting this estimate back into (4.6), we get (4.5), Q.E.D.

Remark. If $u(z)$ is subharmonic in $\mathcal{D}$ and satisfies the hypothesis of the above corollary with an $M(t)$ fulfilling the conditions of the theorem, the relation $\pi Y(0) = \Lambda(0)$ yields, with (4.2) and (4.5),

$$u(0) \leqslant \frac{\Lambda_D(0)}{\pi}\left[a + \int_0^\infty \frac{M(t)}{t^2}\,dt + \sqrt{\pi E\left\langle\frac{M(t)}{t}\right\rangle}\right]. \tag{4.8}$$

Thus, at the cost of making rather special assumptions about the majorant $M(x)$ for $u(x\pm i0)$ on Γ, we have obtained an estimate for $u(0)$ which depends on $\mathcal{D}$ only through the Selberg number $\Lambda_D(0)$.

5.

The theorem at the end of the preceding § applies to certain positive majorants $M(x)$ of the form (4.3) for which the double integral in (4.4) is *not* absolutely convergent. Indeed, positive definiteness of the real bilinear form used in the proof of that theorem makes $\sqrt{E\langle\ \rangle}$ a Hilbert space norm on the collection of Green potentials (4.3) with absolutely convergent integrals (4.4), and it is manifest that (4.5) and (4.8) continue to hold, with $E\langle M(t)/t\rangle$ defined by continuity, as long as $M(x)/x$ is in the *closure* of that collection under the norm $\sqrt{E\langle\ \rangle}$.

Under this head falls the important case when $M(x)$ is the *logarithm of an entire function of exponential type. Then the second term on the right in* (4.5) *can be expressed in terms of the first.* In dealing with such majorants, one can reduce the situation (see §§ 1 and 3 of Part II) to one where

$$M(x) = \log T(x), \tag{5.1}$$

with $T(z)$ an entire function of exponential type $2B$ say, such that

$$T(0) = 1,\ T(x) = T(-x) \geqslant 1 \quad \text{for } x \in \mathbf{R}, \tag{5.2}$$

and

$$\int_0^\infty \frac{\log T(x)}{x^2}\, dx < \infty. \tag{5.3}$$

By a well-known extension of a factorization theorem due to Riesz ([3], p. 125), condition (5.3) is enough ([3], p. 86) to guarantee the existence of an entire function $G(z)$ of exponential type B, *having all its zeros in* $\operatorname{Im} z < 0$, *such that* $T(x) = |G(x)|^2$ *for real* x. In the present case, (5.2) implies $T(-z) = T(z)$, so, since also $T(\bar{z}) = \overline{T(z)}$, the zeros of $T(z)$ lying in $\operatorname{Im} z < 0$ can be enumerated thus:

$$\{\lambda_n,\ -\bar{\lambda}_n;\ n = 1, 2, 3, \ldots\},$$

with $\operatorname{Re} \lambda_n \geqslant 0$. Under the circumstances, the construction on page 125 of [3] shows that we can take(¹)

$$G(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{\lambda_n}\right)\left(1 + \frac{z}{\bar{\lambda}_n}\right). \tag{5.4}$$

Now, for $\operatorname{Im} \lambda < 0$ and $\operatorname{Im} z \geqslant 0$, by Poisson's formula,

$$\log \left|\left(1 - \frac{z}{\lambda}\right)\left(1 + \frac{z}{\bar{\lambda}}\right)\right| = \frac{1}{\pi}\int_0^\infty \log\left|1 - \frac{z^2}{t^2}\right| \left(\frac{\operatorname{Im}\lambda}{|\lambda - t|^2} + \frac{\operatorname{Im}\lambda}{|\lambda + t|^2}\right) dt. \tag{5.5}$$

Taking the logarithm of the modulus on each side of (5.4), substituting (5.5), and changing the order of integration and summation, we get

$$\log |G(z)| = \int_0^\infty \log\left|1 - \frac{z^2}{t^2}\right| d\nu(t) \quad \text{for } \operatorname{Im} z \geqslant 0, \tag{5.6}$$

(¹) The right side of (5.4) perhaps also contains single factors of the form $(1 + z/i\mu_k)$ with $\mu_k > 0$. The right side of (5.7) must then be modified accordingly.

where

$$\frac{d\nu(t)}{dt}=\frac{1}{\pi}\sum_{1}^{\infty}\left(\frac{|\operatorname{Im}\lambda_n|}{|\lambda_n-t|^2}+\frac{|\operatorname{Im}\lambda_n|}{|\lambda_n+t|^2}\right). \tag{5.7}$$

The change in order of integration and summation is surely justified when $\pi/4 \leqslant \arg z \leqslant 3\pi/4$, for then $\log|1-z^2/t^2| \geqslant 0$ for $t \in \mathbf{R}$. This means in particular that the right hand integral in (5.6) converges whenever $z=iy$, $y>0$. But it is then easy to see that the integral converges uniformly on compact subsets of $\operatorname{Im} z>0$, clearly yielding a function harmonic in $\operatorname{Im} z \geqslant 0$. Since the left side of (5.6) is also harmonic there, the two sides agree for $\operatorname{Im} z \geqslant 0$.

For $z=x$ real, the right side of (5.6) may be integrated by parts twice, the second partial integration resembling the one applied to (3.2). Taking $\nu(0)=0$, one finds ([11], pp. 136–137),

$$\log|G(x)| = -x\int_0^{\infty}\log\left|\frac{x+t}{x-t}\right| d\left(\frac{\nu(t)}{t}\right) \quad \text{for } x \in \mathbf{R}. \tag{5.8}$$

Thus, $\log T(x)/x = 2\log|G(x)|/x$ *is a Green potential on* $(0, \infty)$ *according to the definition of* § 4.

Now, under conditions (5.2) *and* (5.3), $E\langle \log T(x)/x\rangle$ *has meaning and is finite.* This was first seen by Beurling and Malliavin [2]. Quantitatively,

$$E\left\langle\frac{\log T(x)}{x}\right\rangle \leqslant 2eJ(J+B) \tag{5.9}$$

with $J=\int_0^{\infty}(\log T(x)/x^2)\,dx$, and we have the

THEOREM. *If* $T(x)$ *is entire of exponential type* $2B$ *with* $T(x)=T(-x)\geqslant 1$ *and* $T(0)=1$, *then*

$$\int_{\Gamma}\log T(x)\,d\omega_{\mathcal{D}}(x,0) \leqslant \frac{\Lambda_{\mathcal{D}}(0)}{\pi}\left[J+\sqrt{2\pi eJ(J+B)}\right],$$

where

$$J=\int_0^{\infty}\frac{\log T(x)}{x^2}\,dx.$$

Proof. Firstly, in the special situation where

$$\int_0^{\infty}\int_0^{\infty}\log\left|\frac{x+t}{x-t}\right|\left|d\left(\frac{\nu(t)}{t}\right)\right|\left|d\left(\frac{\nu(x)}{x}\right)\right|<\infty, \tag{5.10}$$

the theorem follows directly from (4.5) and (5.9). In order to *obtain* (5.9), we may here apply (4.4) directly to (5.8), getting

$$E\left\langle\frac{\log T(x)}{x}\right\rangle = -4\int_0^{\infty}\frac{\log|G(x)|}{x}\,d\left(\frac{\nu(x)}{x}\right) \leqslant 2\int_0^{\infty}\frac{\nu(x)}{x}\,\frac{\log T(x)}{x^2}\,dx,$$

since $\log T(x) = 2\log|G(x)| \geqslant 0$ and $\nu'(x) \geqslant 0$ by (5.7). Formula (5.9) will follow as soon as we show that

$$\frac{\nu(x)}{x} \leqslant e(B+J). \tag{5.11}$$

To this end, observe that, by direct calculation with (5.6), we have the Jensen formula

$$\int_0^r \frac{\nu(t)}{t}\,dt = \frac{1}{2\pi}\int_0^\pi \log|G(re^{i\theta})|\,d\theta. \tag{5.12}$$

On the other hand, for $\operatorname{Im} z \geqslant 0$, $\log|G(z)|$ is continuous and $\geqslant \log|G(\operatorname{Re} z)| \geqslant 0$ by (5.4), since $\operatorname{Im}\lambda_n < 0$. The Poisson representation for positive harmonic functions here yields

$$\log|G(z)| = B\operatorname{Im} z + \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\operatorname{Im} z\,\log|G(t)|}{|z-t|^2}\,dt \quad \text{for } \operatorname{Im} z > 0, \tag{5.13}$$

$G(z)$ being of exponential type B.

Substitute (5.13) into (5.12) and perform an integration on the variable r (idea of B. Nyman, [15], pp. 14–16). We find

$$\int_0^R\left(\int_0^r \frac{\nu(t)}{t}\,dt\right)\frac{dr}{r} = \frac{BR}{\pi} + \frac{1}{2\pi^2}\int_{-\infty}^{\infty}\left(\int_0^R \log\left|\frac{r+t}{r-t}\right|\frac{dr}{r}\right)\frac{\log|G(t)|}{t}\,dt. \tag{5.14}$$

Now

$$\left|\int_0^R \log\left|\frac{r+t}{r-t}\right|\frac{dr}{r}\right| \leqslant \frac{\pi^2}{2}\frac{R}{|t|}$$

and, since $\nu(t)$ is increasing, the left side of (5.14) is $\geqslant \nu(R/e^2))$. Putting $x = R/e^2$, we obtain a better inequality than (5.11). Thus (5.9), and the theorem, hold under the condition (5.10).

Suppose that (5.10) is not fulfilled. According to the remarks at the beginning of this §, (4.5) will *still* hold—by an evident weak convergence argument applied in the appropriate Hilbert space—if we can construct a sequence of Green potentials

$$Q_n(x) = -\int_0^\infty \log\left|\frac{x+t}{x-t}\right| d\left(\frac{\nu_n(t)}{t}\right) \tag{5.15}$$

with

$$\int_0^\infty\int_0^\infty \log\left|\frac{x+t}{x-t}\right|\left|d\left(\frac{\nu_n(t)}{t}\right)\right|\left|d\left(\frac{\nu_n(x)}{x}\right)\right| < \infty, \tag{5.16}$$

$$E\langle Q_n(x)\rangle \leqslant \text{const.}, \tag{5.17}$$

and

$$\int_0^\infty \frac{\log|G(x)|}{x}\,d\sigma(x) = \lim_{n\to\infty}\int_0^\infty Q_n(x)\,d\sigma(x) \tag{5.18}$$

for every signed measure σ with $\int_0^\infty |d\sigma(x)| < \infty$. (Note that $d\sigma(x) = d(x\Omega(x))$ has this property.)

We take

$$\nu_n(x) = \begin{cases} \nu(x), & 0 \leqslant x \leqslant n \\ \nu(n), & x > n. \end{cases}$$

Using (5.15) and applying (5.6), (5.8) to ν_n, we get

$$xQ_n(x) = \int_0^n \log\left|1 - \frac{x^2}{t^2}\right| d\nu(t), \tag{5.19}$$

from which we see that $Q_n(x) \geqslant 0$ for $x \geqslant \sqrt{2}\,n$ since $d\nu(t) \geqslant 0$. For the same reason, if $0 \leqslant x \leqslant \sqrt{2}\,n$, by (5.6), $xQ_n(x) = \log|G(x)| - \int_n^\infty \log|1-(x^2/t^2)|\,d\nu(t) \geqslant \log|G(x)|$, which is $\geqslant 0$ by (5.2). We thus have

$$Q_n(x) \geqslant 0 \quad \text{for } x \geqslant 0. \tag{5.20}$$

Also, $Q_n(x) \leqslant (1/x) \int_0^n \log(1 + (x^2/t^2))\,d\nu(t) \leqslant A$, a constant independent of n, when $x \geqslant 0$, since $\nu(t)$ is $O(t)$ on $[0, \infty)$. Because (5.19) and (5.6) clearly imply $Q_n(x) \to \log|G(x)|/x$ pointwise on $\mathbf{R}$ as $n \to \infty$, (5.18) now follows from the bounded convergence theorem.

To prove (5.17) we use (5.15), (4.4), the fact that $d\nu_n(x) \geqslant 0$ and (5.20) to deduce

$$E\langle Q_n(x)\rangle \leqslant \int_0^\infty \frac{\nu_n(x)}{x^2} Q_n(x)\,dx. \tag{5.21}$$

Write, for the moment, $\varrho(t) = \nu_n(t)/t$; we have $\varrho(t) = \nu(n)/t$ for $t \geqslant n$, also $\varrho(0) = \nu'(0)$ is finite by (5.7). Substituting (5.15) into the *right side* of (5.21) and changing the order of integration in the resulting double integral, we obtain

$$\begin{aligned}
-\int_0^\infty \int_0^\infty \log\left|\frac{x+t}{x-t}\right| \frac{\varrho(x)}{x}\,dx\,d\varrho(t) &= -\varrho(t) \int_0^\infty \log\left|\frac{t+x}{t-x}\right| \frac{\varrho(x)}{x}\,dx \Big|_0^\infty \\
&\quad + \int_0^\infty \left(\frac{d}{dt}\int_0^\infty \log\left|\frac{t+x}{t-x}\right| \frac{\varrho(x)}{x}\,dx\right) \varrho(t)\,dt \\
&= c[\varrho(0)]^2 + \int_0^\infty \left(\frac{d}{dt}\int_0^\infty \log\left|\frac{1+\xi}{1-\xi}\right| \frac{\varrho(t\xi)}{\xi}\,d\xi\right) \varrho(t)\,dt \\
&= c[\nu'(0)]^2 + \int_0^\infty \left(\int_0^\infty \log\left|\frac{1+\xi}{1-\xi}\right| \varrho'(t\xi)\,d\xi\right) \varrho(t)\,dt \\
&= c[\nu'(0)]^2 - \int_0^\infty \frac{Q_n(t)}{t} \frac{\nu_n(t)}{t}\,dt,
\end{aligned}$$

where

$$c = \int_0^\infty \log\left|\frac{1+\xi}{1-\xi}\right|\frac{d\xi}{\xi},$$

all the steps being easily justified. We see that the *right side* of (5.21) equals $\frac{1}{2}c[\nu'(0)]^2$, and (5.17) holds.

The verification of (5.9) remains. Using weak convergence and applying (5.18) with $d\sigma(x) = d(\nu_m(x)/x)$, we see that

$$E\left\langle\frac{\log|G(x)|}{x}\right\rangle = \lim_{m\to\infty}\left[\lim_{n\to\infty}\int_0^\infty Q_n(x)\,d\left(-\frac{\nu_m(x)}{x}\right)\right] = -\lim_{m\to\infty}\int\frac{\log|G(x)|}{x}\,d\left(\frac{\nu_m(x)}{x}\right).$$

Since $\log|G(x)| \geqslant 0$,

$$-\int_0^\infty \frac{\log|G(x)|}{x}\,d\left(\frac{\nu_m(x)}{x}\right) \leqslant \int_0^\infty \frac{\nu_m(x)}{x^2}\,\frac{\log|G(x)|}{x}\,dx \leqslant \int_0^\infty \frac{\nu(x)}{x}\,\frac{\log|G(x)|}{x^2}\,dx,$$

and

$$E\left\langle\frac{\log T(x)}{x}\right\rangle = 4E\left\langle\frac{\log|G(x)|}{x}\right\rangle \leqslant 4\int_0^\infty \frac{\nu(x)}{x}\,\frac{\log|G(x)|}{x^2}\,dx.$$

From here, the computation runs as it did at the beginning.

The theorem is completely proved.

II. The Theorem of Beurling and Malliavin

1.

Suppose that $F(z)$ is an entire function of exponential type, say of type A. The *theorem of Beurling and Malliavin* [2] says that if

$$\int_0^\infty \frac{\log^+|F(x)|}{1+x^2}\,dx < \infty,$$

then, for every $\eta > 0$ there is a non-zero entire function $f(z)$ of exponential type 2η with both $|f(x)|$ and $|f(x)F(x)|$ bounded on the real axis.

We wish to prove this theorem using the results of Part I. It is enough to show the existence of a non-zero $f(z)$ of arbitrary exponential type $2\eta > 0$ such that $|f(x)|^2(1+|F(x)|^2+|F(-x)|^2)$ is bounded on $\mathbf{R}$. In other words, we want such an f with $T(x)|f(x)|^2$ bounded on $\mathbf{R}$, $T(z)$ being the entire function of exponential type $2A$ given by

$$T(z) = 1 + F(z)F^*(z) + F(-z)F^*(-z). \tag{1.1}$$

(Here, and in all that follows, we use systematically the notation $F^*(z) = \overline{F(\bar{z})}$ for entire functions $F(z)$.) The advantage of introducing $T(z)$ is that now $T(x) = T(-x) \geq 1$ for real x, while we still have

$$\int_{-\infty}^{\infty} \frac{\log T(x)}{1+x^2}\, dx < \infty. \tag{1.2}$$

As in Part I, § 5, this condition enables us to apply a generalization of Riesz' factorization theorem ([3], pp. 125 and 86), getting an entire function $G(z)$ of exponential type A having all its zeros in $\operatorname{Im} z < 0$ with

$$T(z) = G(z)\, G^*(z). \tag{1.3}$$

Evidently, $|G(-x)| = |G(x)|$.

A Phragmén-Lindelöf theorem ([3], p. 82) says that $|f(x)G(x)|$ is bounded for real x if and only if $|f(x+3i)\,G(x+3i)|$ is. We can obtain a non-zero entire f of exponential type 2η, making the latter expression bounded, *provided that there is a non-zero g of exponential type η with*

$$\int_{-\infty}^{\infty} \frac{|g(x)\,G(x+3i)|}{1+x^2}\, dx < \infty.$$

For then the f given by $f(z+3i) = z^{-2} g(z) \sin^2 (\eta z/2)$ will work ([3], p. 82).

The reason for using $G(x+3i)$ instead of $G(x)$ is that its behaviour has a certain regularity. *Henceforth, we work with the entire function*

$$C(z) = e^{-3A}\, G(z+3i). \tag{1.4}$$

LEMMA. *For real x, $|C(x)| = |C(-x)| \geq 1$, and*

$$\int_{-\infty}^{\infty} \frac{\log |C(x)|}{1+x^2}\, dx < \infty.$$

If x and x' are both real,

$$|C(x')| \leq |C(x)|^{\exp(|x'-x|/3)}. \tag{1.5}$$

Proof. As we saw in proving the theorem of Part I, § 5, $\log |G(z)|$ has a Poisson representation in $\operatorname{Im} z > 0$, which, for $z = x+3i$ can here be written

$$\log |C(x)| = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{3 \log |G(t)|\, dt}{(x-t)^2 + 9}. \tag{1.6}$$

Since $|G(t)| = |G(-t)| \geq 1$, the first statement of the lemma is immediate, and (1.6) yields

$$\frac{d \log |C(x)|}{dx} \leq \tfrac{1}{3} \log |C(x)|$$

on differentiation. This last relation gives us (1.5).

In the rest of Part II, L^2 means $L^2(-\infty, \infty)$ and the H^p spaces involved always refer to the upper half plane. Property (1.5) of $C(x)$ plays no role in the following result.

THEOREM. *Let $\eta > 0$. If the cone*

$$K = \left\{ p(x) + \frac{e^{2i\eta x}}{|C(x)|^2} f(x); \quad f \in H^2, p \geqslant 0 \text{ and } p \in L^2 \right\}$$

is not dense in L^2, there is a non-zero entire function $g(z)$ of exponential type η with

$$\int_{-\infty}^{\infty} \frac{|C(x)\, g(x)|}{1+x^2} dx < \infty. \tag{1.7}$$

Proof. Non-density of K implies the existence of a non-zero $Q \in L^2$ with

$$\operatorname{Re} \int_{-\infty}^{\infty} Q(x)\, k(x)\, dx \geqslant 0 \quad \text{for all } k \in K.$$

From this we see immediately that

$$\operatorname{Re} Q(x) \geqslant 0 \quad \text{a.e., } x \in \mathbf{R} \tag{1.8}$$

and that $e^{2i\eta x}\varphi(x) \in H^2$, where $\varphi(x) = Q(x)/|C(x)|^2$. (See [5], p. 195. Note that $\varphi \in L^2$ because $|C(x)| \geqslant 1$.)

By (1.8) there is a function $\psi(x)$, $-\pi/2 \leqslant \psi(x) \leqslant \pi/2$, with

$$\varphi(x)\, e^{-i\psi(x)} \geqslant 0 \quad \text{a.e., } x \in \mathbf{R}. \tag{1.9}$$

Take now the harmonic conjugate function

$$\tilde{\psi}(x) = \frac{1}{\pi} \text{ p.v.} \int_{-\infty}^{\infty} \left(\frac{1}{x-t} + \frac{t}{t^2+1} \right) \psi(t)\, dt;$$

the bounds on ψ ensure that $(x+i)^{-2/p} \exp(\tilde{\psi}(x) - i\psi(x))$ is in H^p for every $p < 1$. (See, for instance, [8], p. 70 together with [7], p. 130.) Using this statement with $p = 2/3$, the fact that $\varphi \in L^2$ and Hölder's inequality with exponents 4 and 4/3, one sees that

$$(x+i)^{-4}\, e^{2i\eta x} \varphi(x)\, e^{\tilde{\psi}(x) - i\psi(x)} \in H^{1/2} \tag{1.10}$$

There is thus a function $H(z)$ regular in $\operatorname{Im} z > 0$, having almost everywhere on $\mathbf{R}$ a non-tangential boundary value $H(x)$ equal to $\varphi(x) e^{\tilde{\psi}(x) - i\psi(x)}$. By (1.9), $H(x) \geqslant 0$ a.e., and (1.10) implies that $H(z)$ is locally in $H^{1/2}$ near $\mathbf{R}$. A simple extension ([10], p. 1203) of a theorem due independently to Helson and Sarason ([6], pp. 10–11) and to Neuwirth and Newman ([14]) now says that the Schwarz reflection

$$H(\bar{z}) = \overline{H(z)} \tag{1.11}$$

furnishes an analytic continuation of $H(z)$ across $\mathbf{R}$.

$H(z)$ is thus *entire*, and for $\operatorname{Im} z > 0$, $(z+i)^{-4} e^{2i\eta z} H(z) \in H^{1/2}$. This, together with (1.11), implies that $H(z)$ is of exponential type 2η by a straightforward argument ([10], pp. 1203–1204).

We have $H(x) \geq 0$ and $\int_{-\infty}^{\infty} \sqrt{H(x)}/(1+x^2)\,dx < \infty$. The extension of Riesz' factorization theorem already used ([3], p. 125) thus applies ([3], p. 86), and we have $H(z) = g(z) g^*(z)$ for some evidently non-zero entire function $g(z)$ of exponential type η.

We have, on the real axis, $|C(x)|^2 H(x) = |C(x)|^2 |\varphi(x)| e^{\tilde{\varphi}(x)} = |Q(x)| e^{\tilde{\varphi}(x)}$ where $Q \in L^2$. Repeating the Hölder-inequality argument used to prove (1.10) with $Q(x)$ in place of $\varphi(x)$, we get

$$\int_{-\infty}^{\infty} \frac{|C(x)| \sqrt{H(x)}}{1+x^2}\,dx < \infty.$$

which is the same as (1.7). Q.E.D.

2.

To prove the theorem of Beurling and Malliavin it is enough, by the discussion and theorem in the preceding §, to show that for arbitrary $\eta > 0$ the cone

$$K = \left\{ p(x) + \frac{e^{2i\eta x} f(x)}{|C(x)|^2};\quad f \in H^2,\ p \in L^2 \text{ and } p \geq 0 \right\}$$

is *not dense* in $\mathbf{L}^2$. Our procedure in what follows is *to assume that K is dense and thereby derive a contradiction.*

We begin by carrying out a construction under this assumption, described in steps (a) and (b) below.

(a) If K *is* dense in L^2, for each N the function $Q_N(x)$ equal to $-|C(x)|^2$ for $|x| \leq N$ and to *zero elsewhere* belongs to the *closure* of K, so we can certainly find $P_N(x) \geq -Q_N(x)|C(x)|^2 \geq 0$ and $f_N \in H^2$ such that

$$\left\| \frac{P_N(x) - e^{2i\eta x} f_N(x)}{|C(x)|^2} \right\|_2 < \frac{1}{2}. \tag{2.1}$$

Since $|C(x)| = |C(-x)|$, this is clearly satisfied with $\frac{1}{2}[P_N(x) + P_N(-x)]$ instead of $P_N(x)$ and $\frac{1}{2}[f_N(x) + \overline{f_N(-x)}]$ instead of $f_N(x)$; in other words, we can and do assume in what follows that

$$P_N(x) = P_N(-x), \quad f_N(x) = \overline{f_N(-x)}.$$

We may clearly *also* assume that $P_N(x)$ is of *compact support.*

We now fix a quantity L equal to max $(24\pi/\eta, \frac{1}{2})$ and put

$$R_N(x) = \frac{1}{2L}\int_{-L}^{L} P_N(x+t)\,dt,$$

$$\varphi_N(x) = \frac{1}{2L}\int_{-L}^{L} e^{2i\eta t} f_N(x+t)\,dt, \tag{2.2}$$

$$W(z) = \frac{1}{2L}\int_{-L}^{L} [C(z+t)\,C^*(z+t)]^2\,dt. \tag{2.3}$$

Then $R_N(x)$ is $\geqslant 0$ and of compact support, φ_N belongs to $\mathcal{A}$ (the space of functions analytic in $\operatorname{Im} z > 0$, continuous on $\operatorname{Im} z \geqslant 0$, and going to *zero* as $z \to \infty$), and $W(z)$ is entire of exponential type $4A$, with $W(x) \geqslant 1$ on the real axis. We also have: $R_N(-x) = R_N(x)$, $\varphi_N(-x) = \overline{\varphi_N(x)}$, $W(-x) = W(x)$.

LEMMA. *If $N > 2L$, there is a bounded open set O_N on* $\mathbf{R}$ *such that*

(i) $O_N = -O_N$, $(-N+L,\ N-L) \subseteq O_N$, *and each component of O_N has length $> L$;*

(ii) *For* $x \in O_N$, $\operatorname{Re}[e^{2i\eta x}\varphi_N(x)] \geqslant 0$,

(iii) *For real* $x \notin O_N$, $|\varphi_N(x)|^2 \leqslant 9[W(x)]^{\exp L}$.

Also:

(iv) *For* $-N+L \leqslant x \leqslant N-L$, $\operatorname{Re}[e^{2i\eta x}\varphi_N(x)] \geqslant \frac{1}{2}W(x)$.

Proof. First of all, by Schwarz' inequality and (2.1),

$$R_N(x) - e^{2i\eta x}\varphi_N(x) = \frac{1}{2L}\int_{x-L}^{x+L} (P_N(s) - e^{2i\eta s} f_N(s))\,ds$$

is in modulus

$$\leqslant \frac{1}{2}\sqrt{\frac{1}{4L^2}\int_{-L}^{L} |C(x+t)|^4\,dt},$$

i.e., since $2L \geqslant 1$,

$$|R_N(x) - e^{2i\eta x}\varphi_N(x)| \leqslant \tfrac{1}{2}\sqrt{W(x)}. \tag{2.4}$$

Suppose $x_0 \in \mathbf{R}$ and

$$R_N(x_0) > 2[W(x_0)]^{(\exp L)/2}. \tag{2.5}$$

Since $P_N \geqslant 0$, by (2.2),

$$R_N(x) > [W(x_0)]^{(\exp L)/2} \tag{2.6}$$

throughout *at least one* of the intervals $(x_0 - L, x_0]$, $[x_0, x_0 + L)$. Take one on which (2.5) holds and call it I_0.

By (2.3) and (1.5), with help of the mean value theorem, $W(x) \leqslant [W(x_0)]^{\exp L}$ for $|x-x_0| \leqslant L$ so, from (2.5) and (2.6),

$$R_N(x) > \sqrt{W(x)}, \quad x \in I_0. \tag{2.7}$$

From (2.4) and (2.7), $\operatorname{Re}[e^{2i\eta x}\varphi_N(x)] > 0$ on I_0, so, since $\varphi_N \in \mathcal{A}$ is continuous, $\operatorname{Re}[e^{2i\eta x}\varphi_N(x)] > 0$ on some *larger* open interval $I(x_0) \supset I_0$ with $x_0 \in I(x_0)$. $I(x_0)$ has length $> L$.

For each $x_0 > 0$ satisfying (2.5) *choose* such an $I(x_0)$, and let Ω_+ be the union of all of them. Ω_+ is *bounded* because the set of $x_0 > 0$ satisfying (2.5) is bounded, R_N being of compact support. Take $\mathcal{O}_N$ to be $(-N+L, N-L) \cup \Omega_+ \cup (-\Omega_+)$ together with any one-point components of the complement of that set in $\mathbf{R}$.

We now have (i) by construction of $\mathcal{O}_N$, and (ii) holds because $\varphi_N(x) = \overline{\varphi_N(-x)}$. If $x \notin \mathcal{O}_N$, (2.5) *fails* for $x_0 = x$ by evenness of R_N, so (2.4) yields $|\varphi_N(x)| \leqslant \frac{5}{2}[W(x)]^{(\exp L)/2}$ since $W(x) \geqslant 1$. This is stronger than (iii). Finally, (iv) follows from (2.2), (2.3), (2.4), and the fact that $P_N(x) \geqslant -Q_N(x)|C(x)|^2$ which equals $|C(x)|^4$ for $-N \leqslant x \leqslant N$.

(b) We now drop the subscript N and write $\mathcal{O}$ instead of $\mathcal{O}_N$ and φ instead of φ_N. We henceforth denote the component of $\mathcal{O}$ containing 0 by $(-l, l)$; we saw in the lemma of step (a) that $l \geqslant N-L$. *We think of l as a parameter which we can take as large as we like; it is not to be confounded with the fixed quantity* $L = \max(24\pi/\eta, \frac{1}{2})$.

We use the notation of Part I, writing Γ for $\mathbf{R} \sim \mathcal{O}$ and taking $\mathcal{D} = \mathbf{C} \sim \Gamma$.

By the lemma of step (a) there is a function $\psi(x)$ defined on $\mathcal{O}$ with $-\pi/2 \leqslant \psi(x) \leqslant \pi/2$ there and

$$e^{2i\eta x}\varphi(x)e^{-i\psi(x)} \geqslant 0, \quad x \in \mathcal{O}. \tag{2.8}$$

We take $\psi(x)$ to be *zero* on $\Gamma = \mathbf{R} \sim \mathcal{O}$. Since $\varphi(-x) = \overline{\varphi(x)}$ and $\mathcal{O} = -\mathcal{O}$,

$$\psi(x) = -\psi(-x). \tag{2.9}$$

Write

$$\tilde{\psi}_1(x) = \frac{1}{\pi}\,\text{p.v.}\int_{-2}^{2}\frac{\psi(t)\,dt}{x-t}. \tag{2.10}$$

The Hilbert transform is an isometry in L^2 ([17], Chapter 5), whence $\int_{-1}^{1}(\tilde{\psi}_1(x))^2dx \leqslant \pi^2$, so by (2.9), which implies $\tilde{\psi}_1(-x) = \tilde{\psi}_1(x)$, we can find a b, $0 \leqslant b \leqslant 1$, with $\tilde{\psi}_1(b) > -3$.

We now fix such a $b \in [0, 1]$ and write

$$\tilde{\psi}_2(x) = \frac{1}{\pi}\,\text{p.v.}\int_{|t|>2}\left(\frac{1}{x-t} + \frac{t}{t^2-b^2}\right)\psi(t)\,dt. \tag{2.11}$$

By (2.9), $\tilde{\psi}_2(-x) = \tilde{\psi}_2(x)$ so in particular $\tilde{\psi}_2(b) = 0$.

The function $\tilde{\psi}(x)=\tilde{\psi}_1(x)+\tilde{\psi}_2(x)$ *will be taken as the harmonic conjugate of* ψ. Although this definition of $\tilde{\psi}$ is different from the one used in proving the theorem of § 1, we still have, for each $p<1$

$$(x+i)^{-2/p}\exp[\tilde{\psi}(x)-i\psi(x)]\in H^p, \tag{2.12}$$

as is easily verified. Note that by the choice of b,

$$\tilde{\psi}(b)=\tilde{\psi}_1(b)>-3. \tag{2.13}$$

Let us now put

$$\Phi(x)=e^{2i\eta x}\varphi(x)e^{\tilde{\psi}(x)-i\psi(x)}. \tag{2.14}$$

Since $\varphi\in\mathcal{A}$, $(x+i)^{-2/p}\Phi(x)\in H^p$ for each $p<1$ by (2.12). When $\operatorname{Im} z>0$, the analogue of (2.14) with z instead of x gives the analytic function $\Phi(z)$ having boundary data $\Phi(x)$ on $\mathbf{R}$, provided that the evident suitable definition of $\tilde{\psi}(z)-i\psi(z)$ is used. From (2.8) we have $\Phi(x)\geqslant 0$ for $x\in\mathcal{O}$; therefore an argument like the one used in proving the theorem of § 1 shows that $\Phi(z)$ has an analytic continuation across $\mathcal{O}$ into all of $\mathcal{D}$ obtained by putting

$$\Phi(\bar{z})=\overline{\Phi(z)}. \tag{2.15}$$

Because ψ has compact support, (2.10), (2.11) and (2.14) make

$$|\Phi(z)|\leqslant \text{Const. } e^{-2\eta|\operatorname{Im} z|} \tag{2.16}$$

for $\operatorname{Im} z>0$ and $|z|$ large; we see from (2.15) that (2.16) continues to hold in $\operatorname{Im} z<0$ for large $|z|$.

Now in fact, $\psi(t)$ vanishes outside $\mathcal{O}$. Therefore $\Phi(z)$ is, in $\operatorname{Im} z\geqslant 0$, continuous up to $\Gamma=\mathbf{R}\sim\mathcal{O}$ *except, perhaps, at the endpoints of the components of* $\mathcal{O}$. By (2.15), the same is true in $\operatorname{Im} z\leqslant 0$, and we see from (2.14) and the lemma in step (a) that, if $x\in\Gamma$ is *not* such an endpoint,

$$|\Phi(x\pm i0)|\leqslant 3[W(x)]^{(\exp L)/2}e^{\tilde{\psi}(x)}. \tag{2.17}$$

Concerning the behaviour of $\Phi(z)$ near the endpoints of the components of $\mathcal{O}$, we now have the

LEMMA. *If* c *is an endpoint of a component of* $\mathcal{O}$ *and* $z\in\mathcal{D}$ *is close to* c,

$$|\Phi(z)|\leqslant O(|z-c|^{-2}). \tag{2.18}$$

Proof. Without loss of generality, let c be a *left* endpoint of such a component.

If $-\pi/6<\arg(z-c)<\pi/6$ and z is sufficiently close to c, $|\Phi|^{1/4}$ is subharmonic in the circle of radius $\frac{2}{3}|c-z|$ about z and there satisfies $|\Phi(\zeta)|^{1/4}\leqslant\text{const.}\,|\operatorname{Im}\zeta|^{-1/2}$ by (2.15) and the fact that $(x+i)^{-4}\Phi(x) \in H^{1/2}$. Integration around this circle yields $|\Phi(z)|^{1/4} \leqslant$ const. $|z-c|^{-1/2}$.

If $z \in \mathcal{D}$ satisfies $\pi/6 < \arg(z-c) < 11\pi/6$, use the facts that ψ vanishes outside $\mathcal{O}$ and that $\varphi \in \mathcal{A}$ together with (2.11), (2.14) and (2.15) to obtain

$$\log|\Phi(z)| \leqslant \tfrac{1}{2} \log \frac{1}{|z-c|} + O(1)$$

for z close enough to c.

We thus have (2.18), or better, in both sectors.

3.

The function $\Phi(z)$ obtained in § 2 has one more property. By the lemma in step (a) of the construction in § 2 together with the fact that $W(x) \geqslant 1$ and (2.13), (2.14):

$$|\Phi(b)| \geqslant \tfrac{1}{2} e^{-3}. \tag{3.1}$$

Of course, $b \in \mathcal{O}$ since $0 \leqslant b \leqslant 1$ and l is large. *Our idea now is to use the results of Part I to show that* (2.16) *and* (2.17) *contradict* (3.1). $\log|\Phi(z)|$ is subharmonic in $\mathcal{D}$ and by (2.16), (2.17) and (2.18), fulfills the conditions required by the corollary in Part I, § 4 with $a = -2\eta$ and

$$M(x) = \log 3 + \tfrac{1}{2} e^{L} \log W(x) + \tilde{\psi}(x). \tag{3.2}$$

We would like to use the corollary to conclude that

$$\log|\Phi(b)| \leqslant -2\eta Y(b) + \int_{\Gamma} M(x)\, d\omega(x, b) \tag{3.3}$$

in the notation of Part I.

The majorant $M(x)$ given by (3.2) is not necessarily $\geqslant 0$ on Γ; we are therefore obliged to fall back on Remark 2 to the corollary of Part I § 4. Writing $M_1(x) = \tfrac{1}{2} e^{L} \log W(x) + \log 3$, $M_2(x) = \tilde{\psi}(x)$, we have $M_1(x) \geqslant 0$, and must examine the behaviour of $U_2(z) = \int_{\Gamma} \tilde{\psi}(t)\, d\omega(t, z)$.

Since ψ is of compact support, $|\tilde{\psi}(t)|$ is bounded for large $|t|$, hence $|U_2(z)|$ is bounded for large $|z|$. Suppose z tends to an endpoint, c—without loss of generality, a *left* endpoint—of a component of $\mathcal{O}$. Put $z = c + \zeta^2$ with $\operatorname{Re} \zeta > 0$, then $U_2(z)$ may be estimated by using Poisson's integral for the right ζ-plane. For $t \in \Gamma$ just to the *left* of c we have $|\tilde{\psi}(t)| \leqslant O(\log 1/|t-c|)$, from which we easily find $|U_2(z)| \leqslant O(\log 1/|\zeta|) = O(\log 1/|z-c|)$.

The behaviour of $U_2(z)$ is, by Remark 2 to the corollary in question, more than sufficient for the justification of (3.3). Using the definition of the Selberg number $\Lambda_D(b)=\Lambda(b)$ given in Part I, § 1, (3.2) and (3.3) yield

$$\log|\Phi(b)| \leqslant -\frac{2\eta}{\pi}\Lambda(b)+\int_\Gamma \tilde{\psi}(x)\,d\omega(x,b)+\tfrac{1}{2}e^L\int_\Gamma \log W(x)\,d\omega(x,b)+\log 3. \tag{3.4}$$

We shall estimate each of the integrals on the right in (3.4) *as multiples of* $\Lambda(b)$.

(a) *Estimate of* $\int_\Gamma \tilde{\psi}(x)\,d\omega(x,b)$.

Since l, the half-width of the component of O containing 0, is *large*, we have, say, $|\tilde{\psi}_1(x)|\leqslant 1$ on Γ so $\int_\Gamma \tilde{\psi}_1(x)\,d\omega(x,b)\leqslant 1$, and the main problem is to estimate $\int_\Gamma \tilde{\psi}_2(x)\,d\omega(x,b)$. By (2.9) and (2.11),

$$\int_\Gamma \tilde{\psi}_2(x)\,d\omega(x,b)=\frac{1}{\pi}\int_\Gamma\int_2^\infty\left(\frac{2t}{x^2-t^2}+\frac{2t}{t^2-b^2}\right)\psi(t)\,dt\,d\omega(x,b). \tag{3.5}$$

As in Part I, let us denote the Green's function for $\mathcal{D}$ by $G_{\mathcal{D}}(z,w)$ or just $G(z,w)$. A well-known formula (the derivation given in [18], p. 87 holds for the kind of infinite domains considered here) says that

$$G(t,b)=\log\frac{1}{|t-b|}+\int_\Gamma \log|t-x|\,d\omega(x,b). \tag{3.6}$$

For the time being, write

$$G(t)=G(t,b)+G(-t,b). \tag{3.7}$$

Since $|\psi(t)|$ vanishes outside $O=\mathbf{R}\sim\Gamma$ and is bounded, it is easy to verify absolute convergence of the double integral on the right in (3.5). After changing the order of integration therein, we find, with the help of (3.6),

$$\int_\Gamma \tilde{\psi}_2(x)\,d\omega(x,b)=-\frac{1}{\pi}\int_2^\infty \psi(t)\,G'(t)\,dt \tag{3.8}$$

For *positive* $t\in O$, by (3.6) and (3.7),

$$\frac{d}{dt}\left[\frac{dG(t)}{d\log(t^2-b^2)}\right]=-\int_\Gamma\frac{2t(x^2-b^2)}{(t^2-x^2)^2}\,d\omega(x,b)<0, \tag{3.9}$$

since $0\leqslant b\leqslant 1$ and l is large.

Let (α, β), with $l<\alpha<\beta$, be a component of $\mathcal{O}$. Note that $G(t)$ vanishes on Γ since $\mathcal{O}=-\mathcal{O}$. It is also easy to see that $G'(\alpha+)=\infty$, $G'(\beta-)=-\infty$. From (3.9) we see that $G'(t)$ has precisely *one* zero in (α, β), say at m, $\alpha<m<\beta$. Since $G(\alpha)=G(\beta)=0$, $G(m)>0$, and $|\psi(t)|\leqslant\pi/2$, we find

$$-\frac{1}{\pi}\int_\alpha^\beta \psi(t)\,G'(t)\,dt\leqslant G(m). \tag{3.10}$$

According to the lemma in step (a) of the construction in § 2, $\beta-\alpha>L$. So *one* of the differences $m-\alpha$, $\beta-m$ is $>L/2$; without loss of generality, say it is the *first* one. By (3.9), $G(t)$ is a concave function of $\log(t^2-b^2)$ for $m-L/2\leqslant t\leqslant m$. Since $G(m-L/2)>0$, this concavity yields

$$G(t)\geqslant \tfrac{1}{3}G(m) \quad \text{for } m-L/4\leqslant t\leqslant m \tag{3.11}$$

provided that m/L is large. From (3.10) and (3.11), by positivity of $G(t)$,

$$-\frac{1}{\pi}\int_\alpha^\beta \psi(t)\,G'(t)\,dt\leqslant \frac{12}{L}\int_\alpha^\beta G(t)\,dt, \tag{3.12}$$

which certainly holds whenever l/L is large enough, since $m>l$.

In like manner, provided that l/L is sufficiently large,

$$-\frac{1}{\pi}\int_2^l \psi(t)\,G'(t)\,dt\leqslant \frac{12}{L}\int_2^l G(t)\,dt. \tag{3.13}$$

Use (3.12) and (3.13) to sum the right-hand integral in (3.8) over the separate components of $\mathcal{O}\cap(2,\infty)$. We get

$$\int_\Gamma \tilde{\psi}_2(x)\,d\omega(x,b)\leqslant\frac{12}{L}\int_2^\infty G(t)\,dt.$$

Since $\int_\Gamma \tilde{\psi}_1(x)d\omega(x,b)\leqslant 1$ we have, by (3.7) and the definition of $\Lambda_{\mathcal{D}}(b)$ given in Part I § 1,

$$\int_\Gamma \tilde{\psi}(x)\,d\omega(x,b)\leqslant 1+\frac{12\Lambda_{\mathcal{D}}(b)}{L}, \tag{3.14}$$

valid whenever l/L is large.

(b) *Estimate of* $\int_\Gamma \log W(x)\,d\omega(x,b)$.

The circle $|z|<l$ lies in $\mathcal{D}$ and $0\leqslant b\leqslant 1$. So, since $\log W(x)\geqslant 0$, by Harnack's theorem,

$$\int_\Gamma \log W(x)\,d\omega(x,b)\leqslant\frac{l+1}{l-1}\int_\Gamma \log W(x)\,d\omega(x,0). \tag{3.15}$$

The function $W(z)$, given by (2.3), is entire, of exponential type $4A$, and satisfies $W(x)=W(-x)\geqslant 1$, $x\in\mathbf{R}$. By the regularity of $|C(x)|$ established in the lemma of § 1 together with (2.3),

$$\int_{-\infty}^{\infty}\frac{\log W(x)}{1+x^2}\,dx<\infty. \tag{3.16}$$

Put

$$W_l(z)=1+\left(\frac{z}{l}\right)^2 W(z). \tag{3.17}$$

For $x\in\Gamma$, $|x|\geqslant l$, so

$$\int_\Gamma \log W(x)\,d\omega(x,0)\leqslant\int_\Gamma \log W_l(x)\,d\omega(x,0). \tag{3.18}$$

According to the theorem of Part I § 5, the integral on the right in (3.18) may be estimated in terms of

$$J_l=\int_0^\infty \frac{\log W_l(x)}{x^2}\,dx \tag{3.19}$$

alone, because $W_l(0)=1$, $W_l(x)=W_l(-x)\geqslant 1$ for $x\in\mathbf{R}$, and $W_l(z)$ is entire, of exponential type $4A$. Using that theorem we find

$$\int_\Gamma \log W_l(x)\,d\omega(x,0)\leqslant\frac{\Lambda(0)}{\pi}\left[J_l+\sqrt{2\pi e J_l(J_l+2A)}\right]. \tag{3.20}$$

By Part I § 1, for $x\in\mathcal{O}$, $\Lambda(x)=\pi Y(x)$ with $Y(z)$ positive and harmonic in $\mathcal{D}$. Another application of Harnack's theorem thus yields

$$\Lambda(0)\leqslant\frac{l+1}{l-1}\Lambda(b). \tag{3.21}$$

Now (3.16), (3.17) and Lebesgue's dominated convergence theorem applied to (3.19) show that $J_l\to 0$ for $l\to\infty$. So (3.15), (3.18), (3.20) and (3.21) together yield

$$\int_\Gamma \log W(x)\,d\omega(x,b)\leqslant\delta(l)\,\Lambda_{\mathcal{D}}(b), \tag{3.22}$$

with an expression $\delta(l)$, depending *only* on $W(z)$ and the parameter l, which tends to zero as $l\to\infty$.

4.

Now we can finish the proof of the theorem of Beurling and Malliavin by showing *that* (3.1) *cannot hold if the parameter* l (the half-width of the component of $\mathcal{O}$ containing

0) *is very large*. Recall that in step (a) of the construction in § 2 we took L equal to $\max(24\pi/\eta, \frac{1}{2})$. Substitution of (3.14) and (3.22) into (3.4) thus yields, for large l,

$$\log|\Phi(b)| \leq \Lambda_{\mathcal{D}}(b)\left\{-\frac{2\eta}{\pi}+\frac{\eta}{2\pi}+\frac{1}{2}e^{L}\delta(l)\right\}+\log 3+1, \tag{4.1}$$

where $\delta(l)\to 0$ as $l\to\infty$. Another application of (3.21) gives us

$$\log|\Phi(b)| \leq -\frac{\eta\Lambda_{\mathcal{D}}(0)}{2\pi}+\log 3+1, \tag{4.2}$$

valid for all sufficiently large l.

Let now $\mathcal{D}_l=\mathbf{C}\sim\{(-\infty, -l]\cup[l, \infty)\}$, and call $G_l(z, w)$ the Green's function for $\mathcal{D}_l$. We have $\mathcal{D}_l\subseteq\mathcal{D}$ so $G_l(t, 0)\leq G_{\mathcal{D}}(t, 0)$ for $t\in\mathbf{R}$, whence (Part I § 1),

$$\Lambda_{\mathcal{D}}(0)=\int_{-\infty}^{\infty}G_{\mathcal{D}}(t, 0)\,dt \geq \int_{-\infty}^{\infty}G_l(t, 0)\,dt. \tag{4.3}$$

The second integral on the right in (4.3) is, *from homogeneity considerations alone*, seen to be a *purely numerical multiple* of l. (In fact, it is equal to πl.) Therefore (4.2) contradicts (3.1) for large enough l, and our proof of the theorem of Beurling and Malliavin is complete.

Remark. Examination of the details in the preceding line of argument would permit us to obtain quantitative information about the function $g(z)$ satisfying (1.7) which is shown to exist by the theorem of § 1. Such information would be expressed in terms of η, A, and the behaviour of J_l (formula (3.19)) for large l—thus, ultimately, in terms of the behaviour of $|C(x)|$.

III. Addendum

In [2], Beurling and Malliavin also proved that if $\log W(x)$ is *positive* and *uniformly continuous* on $\mathbf{R}$ and if

$$\int_{-\infty}^{\infty}\frac{\log W(x)}{1+x^2}\,dx<\infty, \tag{1}$$

then, for each $\eta>0$ there is a non-zero entire function $f(z)$ of exponential type 2η with $f(x)\,W(x)$ bounded on $\mathbf{R}$. As we now show, *this result is in fact a simple consequence of the corresponding one proved in Part II for the case where $W(x)$ is an entire function of exponential type*. The following discussion is elementary, and does not use material from Parts I or II.

By working with $W_*(x)$ instead of $W(x)$, with

$$\log W_*(x) = \sup\{\omega(x);\ 0 \leqslant \omega(t) \leqslant \log W(t) \quad \text{and } |\omega'(t)| \leqslant k,\ t \in \mathbf{R}\},$$

k being a suitable constant, we easily reduce the situation to one where *uniform continuity* is replaced by the property of being *uniformly* Lip 1 on $\mathbf{R}$. Supposing henceforth this reduction *made*, we assume, without loss of generality, that

$$|\log W(x) - \log W(x')| \leqslant |x - x'| \text{ on } \mathbf{R}. \tag{2}$$

If we can find an entire function $T(z)$ of exponential type 2 with

$$\tfrac{1}{4}[W(x)]^{\sqrt{2}} \leqslant T(x), \quad x \in \mathbf{R}, \tag{3}$$

$$\int_{-\infty}^{\infty} \frac{\log T(x)}{1+x^2}\, dx < \infty, \tag{4}$$

we will be done by the result established in Part II.

Take

$$\Omega(x) = \pi(x^2+1)[W(x)]^2 \tag{5}$$

and write

$$\langle f, g \rangle_\Omega = \int_{-\infty}^{\infty} \frac{f(x)\overline{g(x)}}{\Omega(x)}\, dx, \quad \|f\|_\Omega = \sqrt{\langle f, f \rangle_\Omega}.$$

For $z \in \mathbf{C}$, let $M(z)$ be sup $|f(z)|$ *for f ranging over the entire functions of exponential type $\leqslant 1$, bounded on the real axis, with $\|f\|_\Omega \leqslant 1$.*

First of all, for real x,

$$M(x) \geqslant \tfrac{1}{2}[W(x)]^{1/\sqrt{2}}. \tag{6}$$

Indeed, if $x_0 \in \mathbf{R}$, take the test function

$$f_0(z) = \cos\sqrt{(z - x_0)^2 - \tfrac{1}{2}(\log W(x_0))^2}. \tag{7}$$

The idea of using such test functions is on p. 252 of L. de Brange's book [4]; the particular form (7) was once suggested to me by a paper of H. Widom [19]. Since $\cos w$ is even, $f_0(z)$ is entire; it is clearly of exponential type 1 and bounded on $\mathbf{R}$. Since $\log W(x) \geqslant 0$ we see, with the help of (2) and a simple diagram, that $|f_0(x)| \leqslant M(x)$ on $\mathbf{R}$, whence, by (5), $\|f_0\|_\Omega \leqslant 1$. Therefore $M(x_0) \geqslant |f_0(x_0)| = \cosh(2^{-1/2} \log W(x_0))$, proving (6) with $x = x_0$.

Observe that $M(x) \geqslant 1$ since $W(x) \geqslant 1$. Let us show that

$$\int_{-\infty}^{\infty} \frac{\log M(x)}{1+x^2}\, dx < \infty. \tag{8}$$

If $f(z)$ is entire of exponential type $\leqslant 1$ and bounded on the real axis, we have, e.g. from p. 93 of [3],

$$\log|f(z)| \leqslant |y| + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|y| \log|f(t)|\,dt}{|z-t|^2}. \tag{9}$$

By the inequality between geometric and arithmetic means, for $|y| \geqslant 1$ the right side of (9) is easily seen to be

$$\leqslant |y| + \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|y|}{|z-t|^2} \log\left[\frac{\Omega(t)}{\pi}\right] dt + \log\|f\|_\Omega,$$

and this yields

$$\log M(x+i) \leqslant 1 + \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\log(\Omega(t)/\pi)}{(x-t)^2+1}\,dt. \tag{10}$$

The analogue of (9) holds with $f(z+i)$ in place of $f(z)$. If $\|f\|_\Omega \leqslant 1$ we have by definition $|f(x+i)| \leqslant M(x+i)$, so, using (10) together with Fubini's theorem,

$$|f(x)| \leqslant 2 + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log[\Omega(t)/\pi]\,dt}{(x-t)^2+4},$$

and finally, by (5),

$$M(x) \leqslant 2 + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log((t^2+1)[W(t)]^2)}{(x-t)^2+4}\,dt \tag{11}$$

for $x \in \mathbf{R}$, from which (8) follows by (1).

Let $\{p_n(z)\}$ be any sequence of entire functions of exponential type $\leqslant 1$ bounded on the real axis, *complete and orthonormal with respect to the inner product* $\langle\, ,\rangle_\Omega$. In view of the completeness, a straightforward computation with Schwarz' inequality shows

$$\sum_n |p_n(x)|^2 = [M(x)]^2, \quad x \in \mathbf{R}. \tag{12}$$

For each N, put $T_N(z) = \sum_{n \leqslant N} p_n(z) p_n^*(z)$ where $p_n^*(z) = \overline{p_n(\bar{z})}$; each $T_N(z)$ is entire, of exponential type 2, and bounded on the real axis; in addition, for $x \in \mathbf{R}$,

$$0 \leqslant T_N(x) \leqslant (M(x))^2, \tag{13}$$

by (12). We see from (13) and (8), together with the inequality from p. 93 of [3] used earlier, that the $T_N(z)$ form a *normal family in the complex plane*. By (12), $T_N(x) \to_N (M(x))^2$ on $\mathbf{R}$, so the $T_N(z)$ tend to an *entire function* $T(z)$ with $T(x) = (M(x))^2$ on the real axis. An argument of Akhiezer ([1], pp. 285–287; his reasoning is reproduced in [12], pp. 629–631) shows that $T(z)$ is in fact of exponential type 2. By (6) and (8), $T(x)$ satisfies (3) and (4), and we have finished.

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