SUBELLIPTICITY OF THE $\bar{\partial}$-NEUMANN PROBLEM ON PSEUDO-CONVEX DOMAINS: SUFFICIENT CONDITIONS

BY

J. J. KOHN(1)

Princeton University, Princeton, N.J., U.S.A.

Contents

§ 1. Introduction

The main idea of this work is to analyze a-priori estimates for partial differential operators using the theory of ideals of functions. Here I deal only with the $\bar{\partial}$-Neumann problem; however, it is my belief that this type of analysis will be useful in deriving estimates by algebraic methods in diverse situations (see for example Chapter 3 of [20a]). In particular, by means of the Spencer sequence, a wide class of differential operators can be reduced to the D-Neumann problem (see [30] and [31a]) which in turn seems to be amenable to these methods.

The principal results proved here are Theorems 1.19 and 1.21, they were announced in [20b]. To introduce this paper I give a brief review of those aspects of the $\bar{\partial}$-problem and the $\bar{\partial}$-Neumann problem which motivated my work.

---

(1) This work was done in part while the author was a Guggenheim Fellow. This research was also supported by a National Science Foundation project at Princeton University.

The $\bar{\partial}$-problem. Consider the inhomogeneous Cauchy–Riemann equations on a domain Ω in $\mathbf{C}^n$. To be explicit, let $z_1, \ldots, z_n$ be holomorphic coordinates in $\mathbf{C}^n$ and let $x_j = \operatorname{Re}(z_j)$, $y_j = \operatorname{Im}(z_j)$, we set

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - \sqrt{-1}\,\frac{\partial}{\partial y_j}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + \sqrt{-1}\,\frac{\partial}{\partial y_j}\right)$$

as usual. Now, given functions $\alpha_1, \ldots, \alpha_n$ on Ω, the problem is to solve the equations

$$\frac{\partial u}{\partial \bar{z}_j} = \alpha_j, \quad j = 1, \ldots, n \tag{1.1}$$

and to study the regularity of the solution. Naturally, we must assume that the α_j satisfy the compatibility conditions

$$\frac{\partial \alpha_j}{\partial \bar{z}_k} - \frac{\partial \alpha_k}{\partial \bar{z}_j} = 0. \tag{1.2}$$

Using the notation of differential forms we let $\alpha = \sum \alpha_j d\bar{z}_j$; the equations (1.1) are then expressed by $\bar{\partial} u = \alpha$ and the compatibility conditions (1.2) by $\bar{\partial}\alpha = 0$.

We will assume that Ω is pseudo-convex and has a smooth boundary (see § 2). Since the system (1.1) is elliptic, the regularity properties of u in the interior of Ω are well known. Roughly speaking, on an open set $U \subset\subset \Omega$ a solution u restricted to U is "smoother by one derivative" then α restricted to U. Regularity of u on the boundary is more delicate. Notice that if h is a holomorphic function on Ω then $u + h$ is also a solution of (1.1); thus, "in general" the solutions of (1.1) will not be smooth on the boundary. The problem then is to find some particular solution with good regularity properties at the boundary. In [20d] and [20c] the following result is proved.

THEOREM 1.3. *If $\Omega \subset \mathbf{C}^n$ is pseudo-convex with a C^∞ boundary and if $\alpha_j \in C^\infty(\bar{\Omega})$ and satisfy (1.2) then there exists $u \in C^\infty(\bar{\Omega})$ which satisfies (1.1).*

This result gives global regularity of solutions. The problem of local regularity is the following: given an open set U such that the restriction of α to $U \cap \bar{\Omega}$ is smooth can we find a solution u whose restriction to $U \cap \bar{\Omega}$ is also smooth. The answer to this question, in general, is negative. In [20e] and also in § 9 of this paper, we show that singularities of u can propagate along complex-analytic varieties contained in the boundary of Ω. More precisely, for certain domains Ω we can find an α so that local regularity fails for every solution u. Our construction depends on the fact that the boundary of Ω contains a complex-analytic variety and it is this phenomenon that led us to the main results of this paper.

D. Catlin, in [5], gives an example of a pseudo-convex domain in $\mathbf{C}^3$ for which local regularity fails and whose boundary does not contain any non-trivial complex-analytic varieties.

In recent years many results have been obtained concerning the regularity of solutions of (1.1), on strongly pseudo-convex domains (see [16] for a survey of this field). These results are concerned with estimates of Hölder and L_p norms. In the present work we study pseudo-convex domains which are not strongly pseudo-convex and our results concern estimates of Sobolev norms.

The $\bar{\partial}$-Neumann problem. This problem was formulated by D. C. Spencer to study the $\bar{\partial}$-problem and other properties of the operator $\bar{\partial}$. Here we give a brief description of the problem, for a detailed account see [13].

Let $L_2^{p,q}(\Omega)$ denote the space of square-integrable (p, q)-forms on Ω. The inner product and norm are defined as usual by

$$(1.4)\qquad (\alpha, \beta) = \sum_{I,J} \int_\Omega \alpha_{IJ} \bar{\beta}_{IJ}\, dV, \quad \text{and} \quad \|\alpha\|^2 = (\alpha, \alpha),$$

where $\alpha = \sum \alpha_{IJ} dz_I \wedge d\bar{z}_J$, $\beta = \sum \beta_{IJ} dz_I \wedge d\bar{z}_J$, $I = (i_1, ..., i_p)$, $J = (j_1, ..., j_q)$, $1 \leqslant i_1 < ... < i_p \leqslant n$, $1 \leqslant j_1 < ... < j_q \leqslant n$, $dz_I = dz_{i_1} \wedge ... \wedge dz_{i_p}$ and $d\bar{z}_J = d\bar{z}_{j_1} \wedge ... \wedge d\bar{z}_{j_q}$. Then we have

$$(1.5)\qquad L_2^{p,q-1}(\Omega) \underset{\bar{\partial}^*}{\overset{\bar{\partial}}{\rightleftarrows}} L_2^{p,q}(\Gamma) \underset{\bar{\partial}^*}{\overset{\bar{\partial}}{\rightleftarrows}} L_2^{p,p+1}(\Omega),$$

by $\bar{\partial}$ we mean the closed operator which is the maximal extension of the differential operator and by $\bar{\partial}^*$ we mean the L_2-adjoint of $\bar{\partial}$. We define $\mathcal{H}^{p,q} \subset L_2^{p,q}(\Omega)$ by

$$(1.6)\qquad \mathcal{H}^{p,q} = \{\varphi \in \operatorname{Dom}(\bar{\partial}) \cap \operatorname{Dom}(\bar{\partial}^*) \mid \bar{\partial}\varphi = 0 \quad \text{and } \bar{\partial}^*\varphi = 0\}.$$

Observe that $\mathcal{H}^{0,0}$ is the space of holomorphic functions in $L_2(\Omega)$. The $\bar{\partial}$-Neumann problem for (p, q)-forms can then be stated as follows: given $\alpha \in L_2^{p,q}(\Omega)$ with $\alpha \perp \mathcal{H}^{p,q}$, does there exist $\varphi \in \operatorname{Dom}(\bar{\partial}) \cap \operatorname{Dom}(\bar{\partial}^*)$ with $\bar{\partial}\varphi \in \operatorname{Dom}(\bar{\partial}^*)$ and $\bar{\partial}^*\varphi \in \operatorname{Dom}(\bar{\partial}\varphi)$, such that

$$(1.7)\qquad \bar{\partial}\bar{\partial}^*\varphi + \bar{\partial}^*\bar{\partial}\varphi = \alpha.$$

Observe that if a solution of (1.7) exists then there is a unique solution φ of (1.7) such that $\varphi \perp \mathcal{H}^{p,q}$. We will denote this unique solution by $N\alpha$. If a solution to (1.7) exists for all $\alpha \perp \mathcal{H}^{p,q}$, then we extend the operator N to a linear operator on $L_2^{p,q}(\Omega)$ by setting it equal to 0 on $\mathcal{H}^{p,q}$. Then N is bounded and self-adjoint. Furthermore, if $\bar{\partial}\alpha = 0$, then from (1.7) we obtain $\bar\partial\bar\partial^*\bar\partial N\alpha=0$, taking inner products with $\bar\partial N\alpha$ we get $\|\bar\partial^*\bar\partial N\alpha\|^2=0$ and hence $\bar\partial^*\bar\partial N\alpha=0$. Thus we see from (1.7) that if $\bar\partial\alpha=0$ and $\alpha\perp\mathcal{H}^{p,q}$ then

$$\alpha=\bar\partial\bar\partial^*N\alpha. \tag{1.8}$$

It then follows that $u=\bar\partial^*N\alpha$ is the unique solution to the equation $\bar\partial u=\alpha$ which is orthogonal to the null space of $\bar\partial$.

If the $\bar\partial$-Neumann problem is solvable on (0, 1)-forms and if $f\in L_2(\Omega)\cap\mathrm{Dom}\,(\bar\partial)$ then, applying (1.8) to $\alpha=\bar\partial f$ we can easily deduce that the Bergman orthogonal projection $B: L_2(\Omega)\to\mathcal{H}^{0,0}$ is given by

$$Bf=f-\bar\partial^*N\bar\partial f. \tag{1.9}$$

Then the following result holds (see [17a] and [13]).

Theorem 1.10. *If $\Omega\in\mathbf{C}^n$ is pseudo-convex and if $\bar\Omega$ is compact then the $\bar\partial$-Neumann problem is solvable on (p, q)-forms for all (p, q) and $\mathcal{H}^{p,q}=0$ when $q>0$.*

Subelliptic estimates. These estimates are defined as follows.

Definition 1.11. If $x_0\in\bar\Omega$ we say that the $\bar\partial$-Neumann problem for (p, q)-forms satisfies a subelliptic estimate at x_0 if there exists a neighborhood U of x_0 and constants $\varepsilon>0$ and $C>0$ such that:

$$\|\varphi\|_\varepsilon^2\leqslant C(\|\bar\partial\varphi\|^2+\|\bar\partial^*\varphi\|^2+\|\varphi\|^2) \tag{1.12}$$

for all $\varphi\in\mathcal{D}_U^{p,q}$. Here $\mathcal{D}_U^{p,q}$ denotes the space of (p, q)-forms $\varphi\in\mathrm{Dom}\,(\bar\partial^*)$ such that $\varphi_{IJ}\in C_0^\infty(U\cap\bar\Omega)$, for all components φ_{IJ} of φ. The norm $\|\varphi\|_\varepsilon^2=\sum\|\varphi_{IJ}\|_\varepsilon^2$, denotes the Sobolev ε-norm.

The following theorem (see [21 b] and [13]), shows what implications this estimate has for local regularity of the $\bar\partial$-Neumann problem, the $\bar\partial$-problem and the Bergman operator.

Theorem 1.13. *Suppose that $\Omega\subset\mathbf{C}^n$ is pseudo-convex, the boundary of Ω is C^∞ and that (1.12) holds at $x_0\in\bar\Omega$. Then if $\alpha\in L_2^{p,q}(\Omega)$ and if α is smooth in a neighborhood of x_0, (i.e. a neighborhood in $\bar\Omega$) then $N\alpha$ is also C^∞ in a neighborhood of x_0. Also if (1.12) holds for (0, 1)-forms, if $f\in L_2(\Omega)$ and if f is C^∞ in a neighborhood of x_0 then so is Bf. More precisely, if α and f are in H^s in a neighborhood of x_0 then $N\alpha$ is in $H^{s+2\varepsilon}$, $\bar\partial^*N\alpha$ is in $H^{s+\varepsilon}$ and Bf is in H^s in a neighborhood of x_0.*

In [19a], Kerzman showed how the above theorem can be used to study the regularity of the Bergman kernel function.

In case $\Omega \subset X$ and X is a complex analytic manifold with a hermitian metric the definitions given above extend in a natural way and subellipticity has several important consequences. It should be noted that, according to a result of W. Sweeney (see [31 b]), the validity of (1.12), is independent of the choice of hermitian metric (even though the space $\mathcal{D}_U^{p,q}$ does depend on the choice of metric). We refer again to [21 b] and [13] for a proof of the following.

THEOREM 1.14. *Suppose that $\Omega \subset X$, where X is a complex analytic manifold with a hermitian metric, suppose also that Ω has a C^∞ boundary and that every point in $\bar{\Omega}$ has a neighborhood such that (1.12) holds. Then the space $\mathcal{H}^{p,q}$ is finite dimensional and all of its elements are C^∞ on $\bar{\Omega}$. Furthermore, the operators N, $\bar{\partial}^* N$ and B have the same regularity properties as in Theorem 1.13.*

We will consider the estimate (1.12) on (p, q)-forms for domains which are pseudo-convex and when $q > 0$. It will be shown in § 2 that the validity of (1.12) is independent of p. The estimate (1.12) is always satisfied when $\varepsilon \leqslant 0$ and it cannot be satisfied for any $\varepsilon > 1$. Denote by $\mathcal{E}^q(\varepsilon)$ the subset of $\bar{\Omega}$ such that there exists a neighborhood U of x_0 for which (1.12) holds whenever $\varphi \in \mathcal{D}_U^{p,q}$. Then we have

$$\mathcal{E}^q(\varepsilon) \subset \mathcal{E}^q(\varepsilon') \quad \text{when } \varepsilon \geqslant \varepsilon'.$$

For $\varepsilon = 1$ the estimate (1.12) is an elliptic estimate and we have

$$\mathcal{E}^q(1) = \begin{cases} \Omega & \text{if } q < n \\ \bar{\Omega} & \text{if } q = n, \end{cases} \tag{1.15}$$

the reason for this is that (1.12) is elliptic in the interior for all q and for $q = n$ the space $\mathcal{D}_U^{p,n}$ consists of (p, n)-forms all of whose components vanish on the boundary of Ω. It follows from the general theory of subelliptic estimates that if $x_0 \notin \mathcal{E}^q(1)$ then $x_0 \notin \mathcal{E}^q(\varepsilon)$ for $\varepsilon > \frac{1}{2}$, see [17 b].

The next case is when $\varepsilon = \frac{1}{2}$ and we have the following result (see [17 a] and [13]).

THEOREM 1.16. *If Ω is pseudo-convex and if $x_0 \notin \mathcal{E}^q(1)$ then the following are equivalent*

(a) $x_0 \in \mathcal{E}^q(\frac{1}{2})$

(b) $x_0 \in b\Omega$ (*$b\Omega$ denotes the boundary of Ω*), *$q < n$ and the Levi-form at x_0 has at least $n - q$ positive eigen-values.*

The definition of the Levi-form will be recalled in § 2. The case $\varepsilon = \frac{1}{2}$ has received a great deal of attention in the last few years. In this case there are very precise estimates in terms of Hölder and L_p norms (see, for example [16], [15b], [19a], [23] and [22]), also real-analytic hypoellipticity has been established (see [28], [8] and [29]). Furthermore, asymptotic expansions of the Bergman kernel function have been obtained (see [12], [3] and [18]). When $\varepsilon < \frac{1}{2}$ such results are not known yet except in some special cases (see [6a], [15], [22] and [26]).

The next case for which (1.12) can be completely analyzed is when $q = n-1$. The result is the following

THEOREM 1.17. *If Ω is a pseudo-convex domain contained in an n-dimensional complex analytic manifold X then the following are equivalent.*

(a) $x_0 \in \mathcal{E}^{n-1}(1/m)$, *m an integer.*

(b) *If $V \subset X$ is a complex analytic manifold of dimension $n-1$ and if $x_0 \in V$ then the order of contact of V to $b\Omega$ at x_0 is at most m.*

The proof that (b) implies (a) is given in § 8 (Theorem 8.1). In the case $n=2$ a somewhat weaker result is given in [20e]. Greiner in [14] showed that (a) implies (b) when $n=2$, a proof along the same lines establishes the general case (we do not include this proof in the present paper, it will be part of a more general treatment of necessary conditions).

When $q < n-1$ the determination of when (1.12) holds for a given ε seems to be extremely complicated. What we do here is to give up the attempt to analyze (1.12) for a fixed ε given a-priori, but instead we find conditions for (1.12) to hold for some $\varepsilon > 0$. When our conditions are satisfied we only have a very rough estimate on the size of ε. Setting

$$\mathcal{E}^q = \bigcup_{\varepsilon>0} \mathcal{E}^q(\varepsilon), \tag{1.18}$$

we state one of our principal results in the following theorem.

THEOREM 1.19. *Suppose that Ω is pseudo-convex, that $x_0 \in b\Omega$, that in a neighborhood of x_0 the boundary is real-analytic and that there exists no complex-analytic variety V of dimension greater than or equal to q such that $x_0 \in V \subset b\Omega$. Then $x_0 \in \mathcal{E}^q$, i.e. the estimate (1.12) holds.*

The above theorem is proven in § 6, here we will indicate the method of proof. In § 4 we introduce the notion of a "subelliptic multiplier", this is a C^∞ function f defined on a neighborhood U of x_0 such that there exist positive ε and C so that

$$\|f\varphi\|_\varepsilon^2 \leqslant C(\|\bar\partial\varphi\|^2 + \|\bar\partial^*\varphi\|^2 + \|\varphi\|^2) \tag{1.20}$$

for all $\varphi \in \mathcal{D}_U^{p,q}$. We denote by $I^q(x_0)$ the set of germs of multipliers satisfying (1.20). It is then clear that $x_0 \in \mathcal{E}^q$ if and only if $1 \in I^q(x_0)$ and that if $x \in \bar{\Omega}$, $f \in I^q(x_0)$ and $f(x) \neq 0$ then $x \in \mathcal{E}^q$. We then prove, in § 4, that $I^q(x_0)$ has the following properties:

THEOREM 1.21. *If Ω is pseudo-convex, with a C^∞ boundary and if $x_0 \in \bar{\Omega}$ then we have*

(a) *$I^q(x_0)$ is an ideal.*

(b) *$I^q(x_0) = \sqrt[\mathbf{R}]{I^q(x_0)}$, where $\sqrt[\mathbf{R}]{I^q(x_0)} = \{f \mid$ there exists $g \in I^q(x_0)$ and m such that $|f|^m \leqslant |g|\}$.*

(c) *If $r = 0$ on $b\Omega$ then $r \in I^q(x_0)$ and the coefficients of $\partial r \wedge \bar{\partial} r \wedge (\partial \bar{\partial} r)^{n-q}$ are in $I^q(x_0)$.*

(d) *If $f_1, \ldots, f_{n-q} \in I^q(x_0)$ then the coefficients of $\partial f_1 \wedge \ldots \wedge \partial f_j \wedge \partial r \wedge \bar{\partial} r \wedge (\partial \bar{\partial} r)^{n-q-j}$, with $j \leqslant n-q$, are in $I^q(x_0)$.*

It is then natural to define the ideals $I_k^q(x_0)$ inductively as follows

$$(1.22) \qquad I_1^q(x_0) = \sqrt[\mathbf{R}]{(r, \operatorname{coeff}\{\partial r \wedge \bar{\partial} r \wedge (\partial \bar{\partial} r)^{n-q}\})}$$

$$I_{k+1}^q(x_0) = \sqrt[\mathbf{R}]{(I_k^q(x_0), A_k^q(x_0))},$$

where

$$A_k^q(x_0) = \operatorname{coeff}\{\partial f_1 \wedge \ldots \wedge \partial f_j \wedge \partial r \wedge \bar{\partial} r \wedge (\partial \bar{\partial} r)^{n-q-j}\}.$$

Here $f_1, \ldots, f_{n-q} \in I_k^q(x_0)$ and $j \leqslant n-q$, coeff. $\{\ \}$ stands for the germs of the coefficients of the set of forms $\{\ \}$ and $(\ )$ stands for ideal generated by the sets appearing inside the parenthesis.

It then follows that $I_k^q(x_0) \subset I^q(x_0)$ and hence $1 \in I_k^q(x_0)$ implies $x_0 \in \mathcal{E}^q$. In § 5 we study the geometric meaning of these ideals, they appear to measure the maximum order of contact that a complex analytic variety of dimension q through x_0 can have with the boundary of Ω. One must distinguish here between the order of contact that can be achieved by complex analytic manifolds and by complex-analytic varieties. Consider, for example, a pseudo-convex domain in $\mathbf{C}^3$ whose boundary, near the origin, is given by the function r, defined by:

$$(1.23) \qquad r(z_1, z_2, z_3) = \operatorname{Re}(z_3) + |z_1^2 - z_2^3|^2 + \exp[-(|z_1|^2 + |z_2|^2 + |z_3|^2)^{-1}].$$

The order of contact, with $r = 0$, of simple complex analytic curves at the origin is at most 6; but the curve defined by $z_3 = 0$, $z_1^2 = z_2^3$ has infinite order of contact. Such behaviour has been studied in [2]. In this case, for $x \in b\Omega$ and $x \neq 0$ the maximum order of contact of all complex analytic curves is at most 2. In a forthcoming publication we will show that in the domain defined by $r \leqslant 0$ there is no subelliptic estimate for $(0, 1)$-forms at the origin, i.e. $0 \notin \mathcal{E}^1$.

Returning to Theorem 1.19, when the boundary is analytic near x_0 we restrict ourselves to germs of real-analytic functions in the definition of the ideals $I_k^q(x_0)$. We then use the theory of ideals of real analytic functions to show that $1 \in I_k^q(x_0)$ for some k is equivalent to the non-existence of real-analytic varieties of "holomorphic dimension" (see Definition 6.16) greater or equal to q contained in the boundary near x_0. We then apply a theorem of Diederich and Fornaess (see [9]) to show that this is equivalent to the non-existence of complex-analytic varieties of dimension greater than or equal to q. Finally, we apply a theorem due to Fornaess (see Theorem 6.23) which shows that not having q-dimensional complex analytic varieties in the boundary arbitrarily close to x_0 is equivalent to not having a q-dimensional complex analytic variety through x_0 in the boundary.

In § 7 we consider the special case of domains whose boundary is given by

$$r(z_1, \ldots, z_n) = \operatorname{Re}(z_n) + \sum_{j=1}^{m} |h_j(z_1, \ldots, z_n)|^2 + a = 0 \tag{1.24}$$

where $h_1, \ldots, h_m$ are holomorphic functions and $a \in \mathbf{C}$. For these domains, if $r(z^0) = 0$ we construct a sequence of ideals of germs of holomorphic functions $J_k^q(z^0)$ such that $1 \in J_k^q(z^0)$ if and only if there is no complex analytic variety of dimension greater or equal to q through z^0 which lies in $r=0$. This is also equivalent to the condition that the dimension of the variety $\{z \mid z_n = z_n^0, h_j(z) = h_j(z^0) \text{ for } j = 1, \ldots, m\}$ is less than q. Our construction leads us to a formula for the dimension of a complex analytic variety (see Theorem 7.10).

In this article we do not take up the question of necessity. The problem is to prove that if Ω is pseudo-convex then $x_0 \in \mathcal{E}^q$ implies $1 \in I_k^q(x_0)$ for some k. We can prove this for very large classes of domains, but as yet we do not have the proof in general. In [10], Egorov announces a result which implies that if there is a non-singular complex-analytic curve through $x_0 \in b\Omega$, with contact m then $x_0 \notin \mathcal{E}^1(\varepsilon)$ when $\varepsilon > 1/m$. This result implies the converse of Theorem 1.19 in the case $q=1$; for if a complex-analytic curve is contained in the boundary then at every regular point x in the curve we have $x \notin \mathcal{E}^1(\varepsilon)$ for $\varepsilon > 1/m$ for all m, thus $x \notin \mathcal{E}^1$ for all regular points and hence for all points of the curve. In [22], Krantz shows that the type of condition considered by Egorov is necessary for subellipticity in the sense of Hölder estimates when $q = n-1$.

I am greatly indebted to J. E. Fornaess, R. C. Gunning and J. Mather for several discussions which were very helpful, especially in the study of ideals of functions. I also wish to express my thanks to L. Hörmander and H.-M. Maire who read the original version of this manuscript and suggested several revisions, corrections and clarifications which have been incorporated in the present text.

§ 2. The basic estimate on pseudo-convex domains

In this section we recall the basic estimate for the $\bar\partial$-Neumann problem on pseudo-convex domains (for a detailed exposition of this material see [13]).

Let X be an n-dimensional complex-analytic manifold with a hermitian metric. Let $\Omega \subset X$ be an open subset of X and let $b\Omega$ denote the boundary of Ω. Throughout this paper we will restrict ourselves to domains Ω such that $b\Omega$ is smooth in the following sense. We assume that in a neighborhood U of $b\Omega$ there exists a C^∞ real-valued function r such that $dr \neq 0$ in U and $r(x)=0$ if and only if $x \in b\Omega$. Without loss of generality, we shall assume that $r>0$ outside of $\bar\Omega$ and that $r<0$ in Ω. For $x \in X$, we denote by $\mathbf{C}T_x$ the complex-valued tangent vectors to X at x and we have the direct sum decomposition $\mathbf{C}T_x = T_x^{1,0} \oplus T_x^{0,1}$, where $T_x^{1,0}$ and $T_x^{0,1}$ denote the holomorphic and anti-holomorphic vectors at x respectively.

We denote by $A_x^{p,q}$ the space of (p, q)-forms at x and by $\langle\ ,\ \rangle_x$ the pairing of $A_x^{p,q}$ with its dual space, we will also denote by $\langle\ ,\ \rangle_x$ the inner product induced on $A_x^{p,q}$ by the hermitian metric and by $|\ \ |_x$ the associated norm. We will denote: by $T^{1,0}$, $T^{0,1}$ and $A^{p,q}$ the bundles with fibers $T_x^{1,0}$, $T_x^{0,1}$ and $A_x^{p,q}$ respectively; by $\Gamma(T^{1,0}, U)$, $\Gamma(T^{0,1}, U)$ and $\Gamma(A^{p,q}, U)$ the spaces of C^∞ sections of these bundles; and by $\boldsymbol{T}_x^{1,0}$, $\boldsymbol{T}_x^{0,1}$, $\boldsymbol{A}_x^{p,q}$ the set of germs at x of local C^∞ sections of these bundles. Finally we will set $\mathcal{A}^{p,q} = \Gamma(A^{p,q}, \bar\Omega)$, that is (p, q)-forms which are C^∞ up to and including the boundary.

Definition 2.1. If $\theta \in A_x^{0,1}$, we define the map int (θ): $A_x^{p,q} \to A_x^{p,q-1}$ as follows, given $\varphi \in A_x^{p,q}$ then int $(\theta)\varphi$ is the element of $A_x^{p,q-1}$ which satisfies

$$\langle \operatorname{int}(\theta)\varphi, \omega\rangle_x = \langle \varphi, \theta \wedge \omega\rangle_x \tag{2.2}$$

for all $\omega \in A_x^{p,q-1}$. Thus the map int (θ) is the adjoint of the map given by $\omega \mapsto \theta \wedge \omega$.

For each $x \in X$ we denote by $(dV)_x$ the unique positive (n, n)-form such that: $|(dV)_x| = 1$. We call dV the volume element. If $x \in b\Omega$ we define $(dS)_x$ to be the unique real $(2n-1)$-form on $b\Omega$ such that $(dr)_x \wedge (dS)_x = |dr|_x (dV)_x$. If $\varphi, \psi \in \mathcal{A}^{p,q}$ we define the inner products:

$$(\varphi, \psi) = \int_\Omega \langle \varphi, \psi\rangle_x (dV)_x, \tag{2.3}$$

$${}^b(\varphi, \psi) = \int_{b\Omega} \langle \varphi, \psi\rangle_x (dS)_x \tag{2.4}$$

and the corresponding norms:

$$\|\varphi\|^2 = (\varphi, \varphi) \quad \text{and} \quad {}^b\|\varphi\|^2 = {}^b(\varphi, \varphi). \tag{2.5}$$

The subspace $\mathcal{D}^{p,q}=\mathcal{D}^{p,q}(\Omega)$ of $\mathcal{A}^{p,q}$ is defined by:

(2.6) $$\mathcal{D}^{p,q}=\{\varphi\in\mathcal{A}^{p,q}\,|\,(\text{int}\,(\bar\partial r)\varphi)_x=0 \quad \text{for } x\in b\Omega\},$$

The operators $\bar\partial$: $\mathcal{A}^{p,q}\to\mathcal{A}^{p,q+1}$ and $\bar\partial^*$: $\mathcal{D}^{p,q+1}\to\mathcal{A}^{p,q}$, then satisfy

(2.7) $$(\varphi,\bar\partial^*\psi)=(\bar\partial\varphi,\psi),$$

for all $\varphi\in\mathcal{A}^{p,q}$ and $\psi\in\mathcal{D}^{p,q+1}$. It can be shown that $\mathcal{D}^{p,q}=\mathcal{A}^{p,q}\cap\text{Dom}\,(\bar\partial^*)$, see [13].

The quadratic form Q is defined on $\mathcal{D}^{p,q}$ by:

(2.8) $$Q(\varphi,\psi)=(\bar\partial\varphi,\bar\partial\psi)+(\bar\partial^*\varphi,\bar\partial^*\psi)+(\varphi,\psi),$$

for $\varphi,\psi\in\mathcal{D}^{p,q}$.

Definition 2.9. If $x\in b\Omega$ we denote by $\mathbf{C}T_x(b\Omega)$ the space of complex-valued tangent vectors to $b\Omega$, i.e. $\mathbf{C}T_x(b\Omega)$ is the subspace of $\mathbf{C}T_x$ consisting of all S such that $S(r)=0$. We set $T^{1,0}_x(b\Omega)=\mathbf{C}T_x(b\Omega)\cap T^{1,0}_x$ and $T^{0,1}_x(b\Omega)=\mathbf{C}T_x(b\Omega)\cap T^{0,1}_x$.

Definition 2.10. The *Levi-form* is the quadratic form on $T^{1,0}_x(b\Omega)$ denoted by $\mathcal{L}_x(L,L')$ and defined by:

(2.11) $$\mathcal{L}_x(L,L')=\langle\partial\bar\partial r, L\wedge\bar L'\rangle_x, \quad \text{where } L,L'\in T^{1,0}_x(b\Omega).$$

We say that Ω is *pseudo-convex* if for each $x\in b\Omega$ the form $\mathcal{L}_x$ is non-negative.

If $x_0\in b\Omega$ then there exists a neighborhood U of x_0 such that on $U\cap\bar\Omega$ we can choose C^∞ vector fields with values in $T^{1,0}$, which at each point $x\in U\cap\bar\Omega$ are an orthonormal basis of $T^{1,0}$. Let $L_1,\dots,L_n$ be such a basis, then for each $x\in U\cap\bar\Omega$ we have $\langle(L_i)_x,(L_j)_x\rangle_x=\delta_{ij}$. We wish to write the operators $\bar\partial$ and $\bar\partial^*$ in terms of this basis. Let $\omega_1,\dots,\omega_n$ be the dual basis of (1, 0)-forms on $U\cap\bar\Omega$, so for each $x\in U\cap\bar\Omega$ we have $\langle(\omega_i)_x,(L_j)_x\rangle_x=\delta_{ij}$. We denote by $\bar L_1,\dots,\bar L_n$ the conjugates of the L_i (i.e. $\bar L_i(f)=\overline{L_i(\bar f)}$), these form an orthonormal basis of $T^{0,1}$ on $U\cap\bar\Omega$ and $\bar\omega_1,\dots,\bar\omega_n$, the conjugates of the ω_i, are the local basis of $\Gamma(A^{0,1}, U\cap\bar\Omega)$ which is dual to $\bar L_1,\dots,\bar L_n$. If φ is in $\mathcal{A}^{p,q}$ then on $U\cap\bar\Omega$ φ can be written as follows:

(2.12) $$\varphi=\sum{}'\varphi_{IJ}\,\omega_I\wedge\bar\omega_J,$$

where $I=(i_1,\dots,i_p)$, $J=(j_1,\dots,j_q)$, the i_k and j_k are integers between 1 and n. The symbol $\sum'$ signifies that the summation is restricted to strictly increasing p-tuples I and q-tuples J. The forms ω_I and $\bar\omega_J$ are given by

(2.13) $$\omega_I=\omega_{i_1}\wedge\dots\wedge\omega_{i_p} \quad \text{and } \bar\omega_J=\bar\omega_{j_1}\wedge\dots\wedge\bar\omega_{j_q}.$$

We then have

(2.14) $$\bar\partial\varphi=(-1)^p\sum{}'\sum_j\bar L_j(\varphi_{IJ})\,\omega_I\wedge\bar\omega_j\wedge\bar\omega_J+\sum{}^2 f^{IJ}_{HL}\varphi_{IJ}\,\omega_H\wedge\bar\omega_L,$$

where H and L run through increasing p-tuples and $(q+1)$-tuples respectively. We also have:

$$\bar{\partial}^*\varphi = (-1)^{p+1} \sum{}' \sum_j L_j(\varphi_{I,jk})\, \omega_I \wedge \bar{\omega}_K + \sum{}' g^{IJ}_{HK} \varphi_{IJ}\, \omega_H \wedge \bar{\omega}_K, \tag{2.15}$$

where the summations are over increasing ruples (I and H run through p-tuples, J through q-tubles and K through $(q-1)$-tuples) and

$$\varphi_{I,jk} = \begin{cases} 0 & \text{if } j \in K \\ \operatorname{sgn}\begin{pmatrix} jK \\ \langle jK \rangle \end{pmatrix} \varphi_{I\langle jK\rangle} & \text{if } j \notin K, \end{cases} \tag{2.16}$$

here $\langle jK \rangle$ denotes the increasingly ordered q-tuple with elements $(j, k_1, \ldots, k_{q-1})$ and $\operatorname{sgn}\begin{pmatrix} jK \\ \langle jK \rangle \end{pmatrix}$ is the sign of the permutation taking jK to $\langle jK \rangle$. The coefficients f^{IJ}_{HL} and g^{IJ}_{HK} are C^∞ functions on $U \cap \bar{\Omega}$.

We fix r so that $|\partial r|_x = 1$ in a neighborhood of $b\Omega$. For $x_0 \in b\Omega$, in a small neighborhood U of x_0, we choose $\omega_1, \ldots, \omega_n$ to be $(1, 0)$-forms on U such that $\omega_n = \partial r$ and such that $\langle \omega_i, \omega_j \rangle = \delta_{ij}$ for $x \in U$. We then define $L_1, \ldots, L_n, \bar{L}_1, \ldots, \bar{L}_n, \bar{\omega}_1, \ldots, \bar{\omega}_n$ as above. Note that on $U \cap b\Omega$, we have

$$L_j(r) = \bar{L}_j(r) = \delta_{jn}. \tag{2.17}$$

Thus $L_1, \ldots, L_{n-1}$ and $\bar{L}_1, \ldots, \bar{L}_{n-1}$ are local bases of $T^{1,0}(U \cap b\Omega)$ and $T^{0,1}(U \cap b\Omega)$ respectively. We define a vector field T on $U \cap b\Omega$ with values in $\mathbf{C}T(U \cap b\Omega)$ by:

$$T = L_n - \bar{L}_n. \tag{2.18}$$

Observe that $L_1, \ldots, L_{n-1}, \bar{L}_1, \ldots, \bar{L}_{n-1}, T$ are a local basis for $\Gamma(\mathbf{C}T(U \cap b\Omega)$. We denote the Levi form in terms of these bases by:

$$c_{ij}(x) = \langle \partial\bar{\partial} r, L_i \wedge \bar{L}_j \rangle_x, \tag{2.19}$$

for $i, j = 1, \ldots, n$ and $x \in U \cap b\Omega$. On $b\Omega$, for $i, j < n$ we have

$$[L_i, \bar{L}_j] = c_{ij} T + \sum_1^{n-1} a^k_{ij} L_k + \sum_1^{n-1} b^k_{ij} \bar{L}_k, \tag{2.20}$$

where $[L_i, \bar{L}_j] = L_i \bar{L}_j - \bar{L}_j L_i$, as usual.

If $\varphi \in \mathcal{A}^{p,q}$; then, in terms of the local basis, the condition (2.6) is expressed as follows: $\varphi \in \mathcal{D}^{p,q}$ whenever

$$\varphi_{IJ}(x) = 0, \quad \text{when } n \in J \quad \text{and } x \in b\Omega. \tag{2.21}$$

Here φ_{IJ} denotes components of φ in (2.12) relative to the local basis defined above.

If U is an open subset of X then the space $\mathcal{D}_U^{p,q}$, which is defined in connection with (1.12) is also given by:

$$\mathcal{D}_U^{p,q} = \{\varphi \in \mathcal{D}^{p,q} \mid \operatorname{supp}(\varphi) \subset U \cap \bar{\Omega}\}. \tag{2.22}$$

THEOREM 2.23. (Basic estimate.) *If $x_0 \in b\Omega$ and Ω is pseudo-convex then there exists a neighborhood U of x_0 and a constant $C>0$ such that*

$$\|\varphi\|_{\bar{z}}^2 + \sum{}' \sum_{i,j} \int_{b\Omega} c_{ij}\, \varphi_{I,iK}\, \bar{\varphi}_{I,jK}\, dS \leq C Q(\varphi, \varphi) \tag{2.24}$$

for all $\varphi \in \mathcal{D}_U^{p,q}$, with $q \geq 1$. Here $\|\varphi\|_{\bar{z}}$ denotes the norm given by:

$$\|\varphi\|_{\bar{z}}^2 = \sum \|\bar{L}_j \varphi_{IJ}\|^2 + \|\varphi\|^2. \tag{2.25}$$

Observe that if $u \in C_0^\infty(U \cap \bar{\Omega})$ with $u(x)=0$ on $U \cap b\Omega$, then

$$\sum \|L_j u\|^2 \leq \text{const.}\ (\sum \|\bar{L}_j u\|^2 + \|u\|^2), \tag{2.26}$$

where the constant is independent of u. Hence we have

$$\|u\|_1^2 \leq \text{const.}\ \|u\|_{\bar{z}}^2, \tag{2.27}$$

for all u satisfying the above. Here $\|u\|_1$ denotes the Sobolev 1-norm, i.e. the sum of the L_2-norms of the first derivatives of u. Combining this observation with (2.21) and (2.25) we obtain

$$\|\varphi\|_{\bar{z}}^2 + \sum \|\varphi_{I,nK}\|_1^2 + \sum{}' \left\| \sum_{j=1}^{n-1} L_j(\varphi_{I,jK}) \right\|^2 + \sum{}' \sum_{i,j} \int_{b\Omega} c_{ij}\, \varphi_{I,iK}\, \bar{\varphi}_{I,jK}\, dS \leq \text{const.}\ Q(\varphi, \varphi), \tag{2.28}$$

for all $\varphi \in \mathcal{D}_U^{p,q}$ with $q \geq 1$, since the third term on the left is bounded by

$$\text{const.}\ (\|\bar{\partial}^* \varphi\|^2 + \sum \|\varphi_{I,nK}\|_1^2 + \|\varphi\|^2) \tag{2.29}$$

and hence by const. $Q(\varphi, \varphi)$.

Notice that conversely we have

$$Q(\varphi, \varphi) \leq \text{const.} \left(\|\varphi\|_{\bar{z}}^2 + \left| \sum{}' \sum_{i,j} \int_{b\Omega} c_{ij}\, \varphi_{I,iK}\, \bar{\varphi}_{J,jK}\, dS \right| \right). \tag{2.30}$$

for all $\varphi \in \mathcal{D}_U^{p,q}$. This inequality is a consequence of the definitions and holds without the assumption of pseudo-convexity.

The estimates that we will derive will be valid for (p, q)-forms if and only if they are valid for $(0, q)$-forms, by virtue of the following.

LEMMA 2.31. *Let E be a norm on $C_0^\infty(U\cap\bar{\Omega})$ and denote also by E the norm on $\mathcal{D}_U^{p,q}$ defined by:*

$$E(\varphi)^2=\sum E(\varphi_{IJ})^2.$$

Then the following are equivalent. There exists $C>0$ such that

$$E(\varphi)^2\leqslant CQ(\varphi,\varphi),\quad \text{for all } \varphi\in\mathcal{D}_U^{p,q}; \tag{2.32}$$

and there exists $C>0$ such that

$$E(\psi)^2\leqslant CQ(\psi,\psi),\quad \text{for all } \psi\in\mathcal{D}_U^{0,q}. \tag{2.33}$$

Proof. The inequalities (2.28) and (2.30) show that $Q(\varphi,\varphi)$ is equivalent to

$$\sum_I{}'\left\{\sum_J{}'\|\varphi_{IJ}\|_Z^2+\sum\|\varphi_{I,nK}\|^2+\sum_K{}'\left\|\sum_{j=1}^{n-1}L_j(\varphi_{I,jK})\right\|^2+\sum_K{}'\sum_{i,j}\int_{b\Omega}c_{ij}\varphi_{I,iK}\bar{\varphi}_{I,jK}\,dS\right\}, \tag{2.34}$$

thus (2.32) is equivalent to the sum (over I) of the inequality (2.33) applied to $\psi_I\in\mathcal{D}_U^{0,q}$ with $\psi_I=\sum_J'\varphi_{IJ}\bar{\omega}_J$.

Remark 2.35. In the case of (0, 1)-forms on pseudo-convex domains the third term in (2.34) is $\|\sum_1^{n-1}L_j\varphi_j\|^2$ which is dominated by $Q(\varphi,\varphi)$. It is important to note that $\sum_1^{n-1}\|L_j\varphi_j\|^2$ is in general *not* bounded by $Q(\varphi,\varphi)$: (relative to any basis L_i,ω_i) as can be seen in the case of $\Omega\subset\mathbf{C}^4$, where r near the origin is given by

$$r(z)=\operatorname{Re}(z_4)+|z_1|^6+|z_1^2+z_2^2|^2+|z_3|^4. \tag{2.36}$$

These types of bounds are studied by Derridj in [7].

§ 3. Tangential Sobolev norms

In our study of (1.12) we will use tangential pseudo-differential operators on $U\cap\bar{\Omega}$, with U a neighborhood of $x_0\in b\Omega$. These will be expressed in terms of boundary coordinates which are defined as follows.

Definition 3.1. If $x_0\in b\Omega$ we will call a system of real C^∞ coordinates, defined in a neighborhood U of x_0, *boundary coordinates* if one of the coordinate functions is r. We will denote such a system by $(t_1,\ldots,t_{2n-1},r)$ and we call the t_j *tangential* coordinates and r the *normal* coordinate.

For $u\in C_0^\infty(U\cap\bar{\Omega})$ we define $\tilde{u}$, the *tangential Fourier transform* of u, by

$$\tilde{u}(\tau,r)=\int_{\mathbf{R}^{2n-1}}e^{-it\cdot\tau}u(t,r)\,dt, \tag{2.3}$$

where

$$\tau = (\tau_1, \ldots, \tau_{2n-1}), \quad t = (t_1, \ldots, t_{2n-1}),$$
$$t \cdot \tau = \sum t_j \tau_j \quad \text{and } dt = dt_1, \ldots, dt_{2n-1}.$$

For each $s \in \mathbf{R}$ we define $\Lambda^s u$ by:

$$\widetilde{\Lambda^s u}(\tau, r) = (1 + |\tau|^2)^{s/2} \tilde{u}(\tau, r), \tag{3.3}$$

where $|\tau|^2 = \sum \tau_j^2$.

Further, we define $|||u|||_s$, the *tangential s-norm* of u, by

$$|||u|||_s^2 = \int_{-\infty}^{0} \int_{\mathbf{R}^{2n-1}} |\Lambda^s u(t, r)|^2 dt\, dr. \tag{3.4}$$

Of course, if s is a non-negative integer, then $|||u|||_s^2$ is equivalent to $\sum_{|\alpha| \leq s} \|D_t^\alpha u\|^2$, where $\alpha = (\alpha_1, \ldots, \alpha_{2n-1})$ and the subscript t denotes differentiation with respect to the tangential variables.

Definition 3.5. P is a *tangential pseudo-differential operator* of order m on $C_0^\infty(U \cap \bar{\Omega})$ if it can be expressed by:

$$Pu(t, r) = \int_{\mathbf{R}^{2n-1}} e^{-it \cdot \tau} p(t, r, \tau)\, \tilde{u}(\tau, r)\, d\tau. \tag{3.6}$$

Here $p \in C^\infty(\mathbf{R}^{2n}_- \times \mathbf{R}^{2n-1})$, where $\mathbf{R}^{2n}_-$ consist of $(t, r) \in \mathbf{R}^{2n}$ with $r \leq 0$. The function p is called the symbol of P and satisfies the following inequalities, for multiindices $\alpha = (\alpha_1, \ldots, \alpha_{2n})$, $\beta = (\beta_1, \ldots, \beta_{2n-1})$ there exists a constant $C = C(\alpha, \beta)$ such that:

$$|D^\alpha D_\tau^\beta p(t, r, \tau)| \leq C(1 + |\tau|)^{m - |\beta|}. \tag{3.7}$$

Both, tangential s-norms and tangential pseudo-differential operators have natural extensions to the space $\mathcal{S}(\mathbf{R}^{2n}_-)$, i.e. the space of C^∞ functions all of whose derivatives are rapidly decreasing.

PROPOSITION 3.8. *If P is a tangential pseudo-differential operator of order m then for each $s \in \mathbf{R}$ there exists $C_s > 0$ such that:*

$$|||Pu|||_s \leq C_s |||u|||_{s+m} \quad \text{for all } u \in \mathcal{S}(\mathbf{R}^{2n}_-). \tag{3.9}$$

Furthermore, if P^* is the adjoint of P then P^* is a tangential pseudo-differential operator of order m and if p and p^* are the symbols of P and P^* then $\bar{p} - p^*$ is the symbol of an operator of order $m-1$. If P' is a tangential pseudo-differential operator of order m' with symbol p', *then PP' is a tangential pseudo-differential operator of order $m+m'$; if q is the symbol of PP' then $pp'-q$ is the symbol of an operator of order $m+m'-1$. Hence, the commutator $[P, P'] = PP'-P'P$ has order $m+m'-1$.*

The proof of the above is exactly the same as the proof of the corresponding properties of pseudo-differential operators. The only tangential pseudo-differential operators which are used in this paper are the elements of the algebra generated, under composition and taking adjoints, by the Λ^s and the tangential differential operators (i.e. operators of the form $\sum a_\alpha(t, r) D_t^\alpha$, where the a_α and all their derivatives are bounded). These will arise because the subelliptic estimate (1.12) can be expressed entirely in terms of the tangential ε-norm. More precisely, we have the following proposition.

PROPOSITION 3.10. *If $x_0 \in b\Omega$ then $x_0 \in \mathcal{E}^q(\varepsilon)$ if and only if there exists a neighborhood U' of x_0 and constant $C'>0$ such that*

$$|||\varphi|||_\varepsilon^2 \leqslant C'Q(\varphi,\varphi), \quad \textit{for all } \varphi \in \mathcal{D}_{U'}^{p,q}. \tag{3.11}$$

This proposition is an easy consequence of the fact that $b\Omega$ is non-characteristic with respect to Q, see [21 b].

§ 4. Ideals and modules of subelliptic multipliers

For $x_0 \in \bar{\Omega}$ and U a neighborhood of x_0 we wish to study functions $f \in C^\infty(U \cap \bar{\Omega})$ which satisfy (1.20). For $x_0 \in b\Omega$ (or near $b\Omega$) the inequality (1.20) is equivalent to the following:

$$|||f\varphi|||_\varepsilon^2 \leqslant CQ(\varphi,\varphi), \quad \text{for all } \varphi \in \mathcal{D}_U^{p,q}, \tag{4.1}$$

which is a consequence of Proposition 3.10.

Observe that if f' is a function defined on a neighborhood U' of x_0, such that $f=f'$ on $U \cap U'$ then f' satisfies (1.20) or (4.1), for all $\varphi \in \mathcal{D}_{U\cap U'}^{p,q}$. Thus, denoting the set of germs of C^∞ functions at x_0 by $C^\infty(x_0)$, we are led to the following definition.

Definition 4.2. For $x_0 \in \bar{\Omega}$ we define $I^q(x_0) \subset C^\infty(x_0)$, the *subelliptic multipliers* at x_0, as follows. $f \in I^q(x_0)$ if and only if there exists a neighborhood U of x_0 and constants $\varepsilon > 0$ and $C>0$ such that (1.20) holds. Here we denote by f both the germ at x_0 and a representative of this germ defined on a sufficiently small U.

It is a consequence of Lemma 2.31 that the sets $I^q(x_0)$ are independent of p.

Definition 4.3. To each $x_0 \in \bar{\Omega}$ and $q \geqslant 1$, we associate the module $M^q(x_0) \subset A^{1,0}(x_0)$, which is defined as follows. $\sigma \in M^q(x_0)$ if and only if there exists a neighborhood U of x_0 and constants $C>0$, $\varepsilon>0$ such that:

$$\|\text{int}(\bar{\sigma})\varphi\|_\varepsilon^2 \leqslant CQ(\varphi,\varphi), \quad \text{for all } \varphi \in \mathcal{D}_U^{p,q}. \tag{4.4}$$

As above, if x_0 is near $b\Omega$ we can replace $\| \ \|_\varepsilon$ by $||| \ |||_\varepsilon$ in (4.4). Here again, σ stands both for the germ at x_0 and a $(1, 0)$-form on a sufficiently small U representing the germ.

Definition 4.5. If $J \subset C^\infty(x_0)$, then the *real radical* of J, denoted by $\sqrt[\mathbf{R}]{J}$, is the set of all $g \in C^\infty(x_0)$ such that there exists and integer m and an $f \in J$ so that

$$|g|^m \leqslant |f|$$

on some neighborhood of x_0.

Definition 4.6. If $S \subset A^{1,0}(x_0)$ then $\det_k S$ is the subset of $C^\infty(x_0)$ consisting of all f that, for x near x_0, can be expressed by:

$$f(x) = \langle \sigma^1(x) \wedge \ldots \wedge \sigma^k(x), \theta(x) \rangle_x,$$

where $\sigma^1, \ldots, \sigma^k \in S$ and $\theta \in A^{k,0}(x_0)$.

The following proposition gives information about $I^q(x_0)$ and $M^q(x_0)$ which will enable us to prove Theorem 1.21.

PROPOSITION 4.7. *If Ω is pseudo-convex and if $x_0 \in \bar{\Omega}$, then $I^q(x_0)$ and $M^q(x_0)$ have the following properties.*

(A) *$1 \in I^n(x_0)$ and for all q, whenever $x_0 \in \Omega$, then $1 \in I^q(x_0)$.*

(B) *If $x_0 \in b\Omega$, then $r \in I^q(x_0)$.*

(C) *If $x_0 \in b\Omega$, then* int $(\theta)\partial\bar{\partial}r \in M^q(x_0)$, *for all $\theta \in A^{0,1}(x_0)$ such that $\langle \theta, \bar{\partial} r \rangle = 0$ on $b\Omega$.*

(D) *$I^q(x_0)$ is an ideal.*

(E) *If $f \in I^q(x_0)$ and if $g \in C^\infty(x_0)$ with $|g| \leqslant |f|$ in a neighborhood of x_0, then $g \in I^q(x_0)$.*

(F) $I^q(x_0) = \sqrt[\mathbf{R}]{I^q(x_0)}$.

(G) *$\partial I^q(x_0) \subset M^q(x_0)$, where $\partial I^q(x_0)$ denotes the set of $\partial f \in A^{1,0}(x_0)$ with $f \in I^q(x_0)$.*

(H) $\det_{n-q+1} M^q(x_0) \subset I^q(x_0)$.

Observe that, due to (A), the properties (B) to (H) are non-trivial only when $x_0 \in b\Omega$.

Proof of (A). If $\varphi \in \mathcal{D}^{p,n}$ then $\varphi = 0$ on $b\Omega$ and hence (1.20) holds with $\varepsilon = 1$. If $x_0 \in \Omega$ choose U so that $\bar{U} \cap b\Omega = \emptyset$, then (1.20) again holds with $\varepsilon = 1$ since supp $(\varphi) \subset U$.

Proof of (B). We choose U so that r is defined on U, and we have

$$\|r\varphi\|_1^2 \leqslant \text{const. } \|r\varphi\|_{\frac{1}{2}}^2 \leqslant \text{const. } \|\varphi\|_{\frac{1}{2}}^2 \leqslant \text{const. } Q(\varphi, \varphi). \tag{4.8}$$

The following lemma will be used in the proofs of (C) and (G).

LEMMA 4.9. *Let $L_1, \ldots, L_n$ be the special local basis defined in a neighborhood U of $x_0 \in b\Omega$ and characterized by (2.17). Let $u, v \in C_0^\infty(U \cap \bar{\Omega})$, then we have*

$$(4.10) \qquad (L_i u, v) = -(u, \bar{L}_i v) + \delta_{in} \int_{b\Omega} u\bar{v}\, dS + (u, g_i v),$$

where $g_i \in C^\infty(\bar{U} \cap \bar{\Omega})$.

Proof. In terms of a boundary coordinate system we have

$$L_i u = \sum a_i^k \frac{\partial u}{\partial t_k} + b_i \frac{\partial u}{\partial r},$$

where $b_i = \delta_{in}$ on $b\Omega$, hence

$$(L_i u, v) = (u, L_i^* v) + \delta_{in} \int_{b\Omega} u\bar{v}\, dS,$$

where

$$L_i^* v = -\bar{L}_i v - \left(\sum_k \frac{\partial \bar{a}_i^k}{\partial t_k} + \frac{a\bar{b}_i}{\partial r} \right) v$$

so (4.10) follows.

Proof of (C). We will use the special local basis in $U \cap \bar{\Omega}$ described in section 2. It suffices to prove (C) in the case $\theta = \bar{\omega}_k$ for $k = 1, \ldots, n-1$. We have:

$$(4.11) \qquad \partial\bar{\partial} r = \sum_{i,j} c_{ij}\, \omega_i \wedge \bar{\omega}_j,$$

where the c_{ij} are given by (2.19) for $i, j = 1, \ldots, n$, hence for $i, j = 1, \ldots, n-1$ they satisfy (2.20). Then

$$(4.12) \qquad \operatorname{int}(\bar{\omega}_k)\, \partial\bar{\partial} r = \sum_i c_{ik}\, \omega_i.$$

If $\varphi \in \mathcal{D}_U^{0,q}$, then

$$(4.13) \qquad \varphi = \sum_J{}' \varphi_J\, \omega_J$$

and

$$(4.14) \qquad \varphi_J = 0 \quad \text{on } b\Omega \text{ whenever } n \in J.$$

Now setting

$$(4.15) \qquad \sigma^k = \sum_i c_{ik}\, \omega_i, \quad \text{for } k = 1, \ldots, n-1; \quad \text{we have}$$

$$(4.16) \qquad \operatorname{int}(\bar{\sigma}^k)\, \varphi = \sum_K{}' \sum_i c_{ik}\, \varphi_{iK}\, \bar{\omega}_K.$$

To prove (C) we will show that

$$(4.17)\qquad |||\operatorname{int}(\bar\sigma^k)\varphi|||_{1/2}^2 \leqslant CQ(\varphi,\varphi),\quad \text{for all } \varphi\in\mathcal{D}_U^{0,q} \quad\text{and}\quad k=1,\dots,n-1.$$

We will first show that there exists $C>0$ such that

$$(4.18)\qquad \Big|\sum_{i,k}(c_{ik}\varphi_{iK}, Du_k)\Big| \leqslant C\left(Q(\varphi,\varphi)+\sum_{k<n}\|u_k\|_z^2+\sum_{i,k<n}\int_{b\Omega}c_{ik}u_i\bar u_k\,dS\right),$$

for all $\varphi\in\mathcal{D}_U^{0,q}$ and $u_k\in C_0^\infty(U'\cap\bar\Omega)$, $k=1,\dots,n-1$; where, U' is a neighborhood of $\bar U$ and D is any first order differential operator. It will suffice to prove (4.18) in the cases when $D=L_i$ and $D=\bar L_i$, $i=1,\dots,n$. For $D=\bar L_i$, (4.18) follows by applying the Schwartz inequality. Similarly, if $D=L_i$ with $i<n$ we first apply (4.10), then the Schwartz inequality and (2.28). Finally, if $D=L_n$ we obtain by use of (4.10):

$$(4.19)\qquad \sum_{i,k}(c_{ik}\varphi_{iK}, L_n u_k)=\sum_{i,k<n}\int_{b\Omega}c_{ik}\varphi_{iK}\bar u_k\,dS+O(\|\varphi\|_z(\sum\|u_k\|)),$$

here the term $i=n$ does not appear in the boundary integral since $\varphi_{nK}=0$ on $b\Omega$. Since the Levi-form is non-negative, we have

$$(4.20)\qquad \Big|\sum_{i,k<n}c_{ik}\varphi_{iK}\bar u_k\Big|\leqslant\Big(\sum_{i,k<n}c_{ik}\varphi_{iK}\bar\varphi_{kK}\Big)^{1/2}\Big(\sum_{i,k<n}c_{ik}u_i\bar u_k\Big)^{1/2},$$

on $b\Omega$. Then (4.18) follows by integrating the above over the boundary and invoking (2.28).

We will use (4.18) with u_k defined by

$$(4.21)\qquad u_k=\sum_{j<n}c_{jk}\,\zeta S^0\varphi_{jK},$$

where $\zeta\in C_0^\infty(U')$, $\zeta=1$ on U and S^0 is a tangential pseudo-differential operator of order 0. First, we show that

$$(4.22)\qquad \sum_k\|u_k\|_z^2+\sum_{i,k<n}\int_{b\Omega}c_{ik}u_i\bar u_k\,dS\leqslant \text{const. } Q(\varphi,\varphi).$$

The first term is estimated by:

$$(4.23)\qquad \|u_k\|_z^2\leqslant\text{const. }\|\zeta S^0\varphi\|_z^2\leqslant\text{const. }Q(\zeta S^0\varphi,\zeta S^0\varphi)\leqslant\text{const. }Q(\varphi,\varphi).$$

To estimate the boundary integral, we have on $b\Omega$:

$$(4.24)\qquad \sum_{i,k<n}c_{ik}u_i\bar u_k\leqslant\text{const. }\sum_{k<n}|u_k|^2$$

and, using the Schwartz inequality, we obtain

$$(4.25)\qquad \sum_{k<n} |u_k|^2 = \sum_{k,j<n} c_{jk}\zeta S^0\varphi_{jK} u_k \leqslant \text{const.} \Big(\sum_{k,j<n} c_{jk}\zeta S^0\varphi_{jK}\overline{\zeta S^0\varphi_{kK}}\Big)^{1/2}\Big(\sum_{k<n}|u_k|^2\Big)^{1/2},$$

hence

$$(4.26)\qquad \sum_{i,k<n} c_{ik} u_i \bar{u}_k \leqslant \text{const.} \sum_{k,j<n} c_{jk}\zeta S^0\varphi_{jK}\overline{\zeta S^0\varphi_{kK}}.$$

Thus, by (2.28), the integral over the boundary (4.26) is bounded by const. $Q(\zeta S^0\varphi, \zeta S^0\varphi)$ and hence by const. $Q(\varphi, \varphi)$; which concludes the proof of (4.22).

Putting all this together, and replacing D by $\partial/\partial t_m$ in (4.18) (with $m<2n$), we obtain

$$(4.27)\qquad \left|\sum_{i,k,j}\left(c_{ik}\varphi_{iK}, \frac{\partial}{\partial t_m} S^0(c_{jk}\varphi_{jK})\right)\right| \leqslant \text{const.}\ Q(\varphi, \varphi),$$

where we have replaced $(\partial/\partial t_m)c_{jk}\zeta S^0\varphi_{jK}$ by $\zeta(\partial/\partial t_m)S^0(c_{jk}\varphi_{jK})$ (ζ does not appear in (4.27) since it is one on the support of φ); the difference between these terms is $O(\|\varphi\|)$ and hence dominated by the right hand side.

We will now conclude the proof of (C) by showing how (4.17) follows from (4.27). Set $S^0 = -(\partial/\partial t_m)\Lambda^{-1}$ in (4.27) and sum over m. Observe that

$$(4.28)\qquad -\sum_{1}^{2n-1}\frac{\partial^2}{\partial t_m^2}\Lambda^{-1} = \Lambda^1 - \Lambda^{-1},$$

and hence

$$(4.29)\qquad \sum_k \Big|\Big|\Big|\sum_i c_{ik}\varphi_{iK}\Big|\Big|\Big|_{1/2}^2 = \Big|\sum_{i,k,j}(c_{ik}\varphi_{ik}, \Lambda^1(c_{jk}\varphi_{jK}))\Big| \leqslant \text{const.}\ Q(\varphi, \varphi);$$

which establishes (4.17).

Proof of (D). Property (D) follows immediately from the following inequality. For any $g\in C^\infty(\bar{U})$ there exists $C>0$ so that:

$$(4.30)\qquad |||gu|||_\varepsilon \leqslant C|||u|||_\varepsilon$$

for all $u\in C_0^\infty(U\cap\bar{\Omega})$. Thus if $f\in I^q(x_0)$ and $g\in C^\infty(x_0)$ we can conclude that $fg\in I^q(x_0)$ by replacing u with $f\varphi$ in (4.30), with $\varphi\in \mathcal{D}_U^{0,q}$ and U suitably small.

Property (E) is a consequence of the following lemma.

LEMMA 4.31. *If* $\varepsilon\leqslant 1$, $f, g\in C^\infty(\bar{U})$ *and if* $|g|\leqslant|f|$, *then*

$$(4.32)\qquad |||gu|||_\varepsilon \leqslant |||fu|||_\varepsilon + \text{const.}\ \|u\|$$

for all $u\in C_0^\infty(U\cap\bar{\Omega})$.

Proof. The operators $[\Lambda^\varepsilon, g]$ and $[f, \Lambda^\varepsilon]$ are of order $\varepsilon-1$ and hence bounded in L_2 so that we have

$$|||gu|||_\varepsilon = \|\Lambda^\varepsilon(gu)\| = \|g\Lambda^\varepsilon u\| + O(\|u\|) \tag{4.33}$$

and

$$\|g\Lambda^\varepsilon u\| \leqslant \|f\Lambda^\varepsilon u\| = |||fu|||_\varepsilon + O(\|u\|),$$

which gives (4.32).

For the proof of (F) we need the following lemma.

LEMMA 4.34. *If* $0<\delta\leqslant 1/m$, *then there exists* $C>0$ *such that*

$$|||gu|||_\delta \leqslant |||g^m u|||_{m\delta} + C\|u\| \tag{4.35}$$

for all $u \in C_0^\infty(U \cap \bar{\Omega})$.

Proof. Proceeding by induction we assume that the left hand side of (4.35) is bounded by $|||g^k u|||_{k\delta} + \text{const.}\ \|u\|$ for $k<m$. Then for any j, with $0\leqslant j\leqslant k$ and $(k+j)\delta\leqslant 1$, we have

$$\begin{aligned}|||g^k u|||_{k\delta}^2 &= (g^k\bar{g}^j\Lambda^{(k+j)\delta}u, g^{k-j}\Lambda^{(k-j)\delta}u) + O(\|u\|^2)\\ &\leqslant |||g^{k+j}u|||_{(k+j)\delta}|||g^{k-j}u|||_{(k-j)\delta} + O(\|u\|^2).\end{aligned}$$

If m is even we obtain the desired estimate by setting $k=j=m/2$.

If m is odd, set $k=(m+1)/2$ and $j=(m-1)/2$, we then have

$$|||gu|||_\delta^2 \leqslant |||g^m u|||_{m\delta}|||gu|||_\delta + \text{const.}\ \|u\|^2,$$

which proves the desired inequality (4.35).

Proof of (F). If $g \in \sqrt[\mathbf{R}]{I(x_0)}$ then on some neighborhood U of x_0 we have $|g|^m \leqslant |f|$, where f satisfies (4.1). Hence, combining (4.1) with 4.31 and 4.34 we obtain

$$|||g\varphi|||_{\varepsilon/m}^2 \leqslant \text{const.}\ Q(\varphi, \varphi), \tag{4.36}$$

for all $\varphi \in \mathcal{D}_U^{0,q}$. Therefore, $g \in I^q(x_0)$ which proves (F).

Proof of (G). By Lemma 2.31 it suffices to consider $\varphi \in \mathcal{D}_U^{0,q}$. Then, if $f \in I^q(x_0)$ and satisfies (4.1) we have

$$\operatorname{int}(\overline{\partial f})\,\varphi = \sum_K{}' \sum_j (L_j f)\,\varphi_{jK}\,\bar{\omega}_K, \tag{4.36}$$

where φ is given by (4.13). Thus,

$$|||\operatorname{int}(\overline{\partial f})\varphi|||_\delta^2 = \sum_K{}' |||\sum_j (L_j f)\varphi_{jK}|||_\delta^2. \tag{4.37}$$

Setting

$$\psi_K = \sum_j (L_j f)\varphi_{jK},$$

we have

$$\begin{aligned} |||\sum_j (L_j f)\varphi_{jK}|||_\delta^2 &= \sum_j (\Lambda^\delta((L_j f)\varphi_{jK}), \Lambda^\delta \psi_K) \\ &= \sum_j ((L_j f)\Lambda^\delta\varphi_{jK}, \Lambda^\delta\psi_K) + O(|||\varphi|||_{2\delta-1}\|\psi_K\|) \\ &= -\sum_j (fL_j\Lambda^\delta\varphi_{jK}, \Lambda^\delta\psi_K) - \sum_j (f\Lambda^\delta\varphi_{jK}, \bar L_j\Lambda^\delta\psi_K) \\ &\quad + O(|||f\varphi|||_{2\delta}\|\psi_K\| + |||\varphi|||_{2\delta-1}\|\psi_K\|) \\ &= (-\sum_j L_j\varphi_{jK}, \Lambda^{2\delta}(f\psi_K)) - \sum_j (\Lambda^{2\delta}(f\varphi_{jK}), \bar L_j\psi_K) \\ &\quad + O(|||f\varphi|||_{2\delta}^2 + |||\varphi|||_{2\delta-1}^2 + \|\varphi\|^2), \end{aligned} \tag{4.38}$$

where the term $|||\varphi|||_{2\delta-1}$ in the second line arises in estimating $\|\Lambda^\delta[\Lambda^\delta, L_j f]\varphi_{jK}\|$; the third line is obtained by an application of Lemma 4.9, the boundary term does not appear since $\Lambda^\delta\varphi_{nK}=0$ on $b\Omega$. The new error terms on the fourth line come from the last term in (4.10), that is

$$(f\Lambda^\delta\varphi_{jK}, g_j\Lambda^\delta\psi_K) = (\Lambda^{2\delta}(f\varphi_{jK}), g_j\psi_K) + O(|||\varphi|||_{2\delta-1}\|\psi_K\|)$$

here the term $|||\varphi|||_{2\delta-1}$ comes from commutators as above. In the last line of (4.38) we have used $\psi_K = O(\|\varphi\|)$. Now, from (2.15), we have

$$\|\sum_j L_j\varphi_{jK}\| \leqslant \|\bar\partial^*\varphi\| + \text{const.}\,\|\varphi\|. \tag{4.39}$$

From the definition of ψ_K we obtain

$$\|\Lambda^{2\delta}(f\psi_K)\| \leqslant \text{const.}\,(|||f\varphi|||_{2\delta} + |||\varphi|||_{2\delta-1}), \tag{4.40}$$

and

$$\|\bar L_j\psi_K\| \leqslant \text{const.}\,\|\varphi\|_z. \tag{4.41}$$

Setting $\varepsilon = 2\delta$ and combining the above, we obtain

$$|||\operatorname{int}(\overline{\partial f})\varphi|||_{\varepsilon/2}^2 \leqslant \text{const.}\,(|||f\varphi|||_\varepsilon^2 + Q(\varphi,\varphi)); \tag{4.42}$$

hence property (G) follows from (4.1).

For the proof of property (H) we will need the proposition given below. The case $q=1$ is somewhat simpler then the case of other q.

Definition 4.43. If (a_{ij}) is a matrix with $i, j=1, \ldots, n$ and if I and J are two m-tuples of integers between 1 and n; we define the $m \times m$ matrix (a_{jk}^{IJ}) by

$$(a_{ij}^{IJ}) = \begin{pmatrix} a_{i_1 j_1} \cdots a_{i_1 j_m} \\ \vdots \quad\quad \vdots \\ a_{i_m j_1} \cdots a_{i_m j_m} \end{pmatrix}, \tag{4.44}$$

where $I=(i_1, \ldots, i_m)$ and $J=(j_1, \ldots, j_m)$. We then define a norm $\delta^m(a_{ij})$ on the mth exterior powers by

$$\delta^m(a_{ij}) = (\sum_{I,J}{}' \,|\det a_{ij}^{IJ})|^2)^{1/2}, \tag{4.45}$$

where the sum runs over all m-tuples I and J; and "det" denotes the determinant.

PROPOSITION 4.46. *Suppose for each $x \in \overline{U \cap \Omega}$, that $(a_{ij}(x))$ is a semi-definite matrix, and that $\omega_1, \ldots, \omega_n$ form a basis of the $(1, 0)$-forms on $\overline{U \cap \Omega}$; then there exists $C>0$ such that:*

$$\delta^{n-q+1}(a_{ij}(x)) \sum{}' |\varphi_J(x)|^2 \leqslant C \sum_K{}' \sum_{i,j} a_{ij}(x)\, \varphi_{iK}(x)\, \overline{\varphi_{jK}(x)} \tag{4.47}$$

for all $x \in \overline{U \cap \Omega}$ and all $\varphi \in \mathcal{A}^{0,q}(\overline{U \cap \Omega})$.

Proof. At each x we define the inner product $\langle\ ,\ \rangle_x$ by $\langle \omega_i(x), \omega_j(x)\rangle_x = \delta_{ij}$. Let $(s_{ki}(x))$ be a unitary matrix such that

$$a_{ij}(x) = \sum_h \lambda_h(x)\, s_{hi}(x)\, \overline{s_{hj}(x)}, \tag{4.48}$$

where $\lambda_1(x), \ldots, \lambda_n(x)$ are the eigenvalues of $a_{ij}(x)$. Then we obtain

$$\sum_K{}' \sum_{i,j} a_{ij}(x)\, \varphi_{iK}(x)\, \overline{\varphi_{jK}(x)} = \sum_J{}' \,(\sum_{h \in J} \lambda_h(x))\, |\varphi'_J(x)|^2), \tag{4.49}$$

where

$$\varphi'_J(x) = \sum \operatorname{sgn} \begin{pmatrix} i_1 \ldots i_q \\ J \end{pmatrix} s_{i_1 h_1} \cdots s_{i_q h_q} \varphi_{h_1 \ldots h_q}, \tag{4.50}$$

and

$$\operatorname{sgn} \begin{pmatrix} i_1 \ldots i_q \\ J \end{pmatrix} = 0 \quad \text{if } (i_1, \ldots, i_q) \neq J$$

sign of permutation π of $(1, \ldots, q)$ for which $i_m = j_{\pi(m)}$, where $J=(j_1, \ldots, j_q)$.

It follows from (4.48) that there is a $C_0 > 0$ such that

$$\delta^{n-q+1}(x) \leqslant C_0 \sum_{1 \leqslant h_1 < \ldots < h_{n-q+1} \leqslant n} \lambda_{h_1}(x) \ldots \lambda_{h_{n-q+1}}(x), \tag{4.51}$$

Now, we let

$$C_1 = \max_x \sum_1^n \lambda_j(x). \tag{4.52}$$

Then for any strictly increasing q-tuple J we have:

$$\delta^{n-q+1}(x) \leqslant C_0 C_1^{n-q} \binom{n}{q} \sum_{h \in J} \lambda_h(x), \tag{4.53}$$

since each term in the sum in (4.51) must have at least one factor whose subscript is in J.

Since (s_{ik}) is unitary, we have $(s_{ik})^{-1} = (\overline{s_{ki}})$ and hence

$$\varphi_J(x) = \sum \operatorname{sgn} \begin{pmatrix} i_1 \ldots i_q \\ J \end{pmatrix} \bar{s}_{i_1 h_1} \ldots \bar{s}_{i_q h_q} \varphi'_{h_1 \ldots h_q}. \tag{4.54}$$

Therefore, there is a C_2 such that

$$\sum' |\varphi_J(x)|^2 \leqslant C_2 \sum' |\varphi'_J(x)|^2 \tag{4.55}$$

for all $x \in \overline{U \cap \Omega}$. The estimate (4.47) then follows by combining (4.55), (4.53) and (4.49); thus concluding the proof of the proposition.

Proof of (H). Suppose $\sigma^k \in M^q(x_0)$, with $k = 1, \ldots, n-q+1$, satisfying (4.4); that is, if

$$\sigma^k = \sum_j \sigma_j^k \omega_j, \tag{4.56}$$

then, for each $(q-1)$-tuple $K = (k_1, \ldots, k_{q-1})$, we have

$$|||\sum_i \sigma_i^k \varphi_{iK}|||_\varepsilon^2 \leqslant CQ(\varphi, \varphi) \tag{4.57}$$

for all $\varphi \in \mathcal{D}_U^{0,q}$. We will show that the function that takes $x \in U \cap \bar{\Omega}$ to $\langle \sigma^1 \wedge \ldots \wedge \sigma^{n-q+1}, \theta \rangle_x$ is in $I^q(x_0)$ for all $\theta \in \mathcal{A}^{n-q+1,0}(U \cap \bar{\Omega})$. It will suffice to show this in the case $\theta = \omega_H$, for all $H = (h_1, \ldots, h_{n-q+1})$ with $1 \leqslant h_1 < \ldots < h_{n-q+1} \leqslant n$. We then have

$$(\sigma^1 \wedge \ldots \wedge \sigma^{n-q+1}, \omega_H) = \det(\sigma_{h_j}^k). \tag{4.58}$$

Let K be the ordered $(q-1)$-tuple consisting of all integers between 1 and n which are not in H. Since $\varphi_{iK} = 0$, whenever $i \in K$, the sum in (4.57) runs over all $i \in H$. Then, we have

$$|||\sum_i \sigma_i^k \varphi_{iK}|||_\varepsilon^2 = \sum_{i,j \in H} (\sigma_i^k \bar{\sigma}_j^k \Lambda^\varepsilon \varphi_{iK}, \Lambda^\varepsilon \varphi_{jK}) + O(|||\varphi|||_{\varepsilon-\frac{1}{2}}^2), \tag{4.59}$$

where the error term estimates

$$\sum_{i,j\in H} \{([\Lambda^\varepsilon, \sigma_i^k]\varphi_{iK}, \Lambda^\varepsilon(\sigma_j^k\varphi_{jK})) + (\sigma_i^k\Lambda^\varepsilon\varphi_{iK}, [\Lambda^\varepsilon, \sigma_j^k]\varphi_{jK})\}. \tag{4.60}$$

Let

$$a_{ij}(x) = \sum_{k=1}^{n-q+1} \sigma_i^k(x)\,\bar{\sigma}_j^k(x). \tag{4.61}$$

Applying Proposition 4.46 with φ_{iK} replaced by $\Lambda^\varepsilon\varphi_{iK}$, we obtain

$$\delta^{n-q+1}(a_{ij}(x)) \sum{}' |\Lambda^\varepsilon\varphi_J(x)|^2 \leqslant C \sum_K{}' \sum_{i,j,k} \sigma_i^k(x)\,\bar{\sigma}_j^k(x)\,\Lambda^\varepsilon\varphi_{iK}(x)\,\overline{\Lambda^\varepsilon\varphi_{jK}(x)}. \tag{4.62}$$

Furthermore, we have

$$\delta^{n-q+1}(a_{ij}(x)) \geqslant \sum_H |\det \sigma_{h_j}^k(x)|^2. \tag{4.63}$$

Integrating the above and estimating commutators as in (4.60), we obtain

$$|||\det(\sigma_{h_j}^k)\,\varphi|||_\varepsilon \leqslant \sum_K{}' |||\sum_i \sigma_i^k\varphi_{iK}|||_\varepsilon^2 + \text{const.}\, |||\varphi|||_{\varepsilon-\frac{1}{2}}^2. \tag{4.64}$$

Therefore, we conclude from (4.57), that det $(\sigma_{h_j}^k) \in I^q(x_0)$, thus completing the proof of (H).

The proof of one of our principal results now follows immediately from Proposition 4.7.

Proof of Theorem 1.21. The only properties of $I^q(x_0)$ which are not explicitly stated in Proposition 4.7 are (c) and (d). These are obtained by combining (C) with (H) and (G) with (H), respectively.

§ 5. Subelliptic stratifications and orders of contact

We define the ideals $I_k^q(x_0)$ below and then show that this definition coincides with the one given by (1.22). If $x_0 \in b\Omega$ we define the sequence of ideals $I_1^q(x_0) \subset \ldots \subset I_k^q(x_0) \subset I^q(x_0)$ and the sequence of modules $M_1^q(x_0) \subset \ldots \subset M_k^q(x_0) \subset M^q(x_0)$ by:

$$M_1^q(x_0) = \{\partial r, \operatorname{int}(\theta)\,\partial\bar{\partial} r \quad \text{for all } \theta \in A_{x_0}^{0,1} \quad \text{with } \theta \perp \bar{\partial} r\} \tag{5.1}$$

$$I_1^q(x_0) = \sqrt[\mathbf{R}]{(r, \det{}_{n-q+1} M_1^q(x_0))} \tag{5.2}$$

and inductively

$$M_k^q(x_0) = \{M_{k-1}^q(x_0), \partial I_{k-1}^q(x_0)\} \tag{5.3}$$

$$I_k^q(x_0 = \sqrt[\mathbf{R}]{(I_{k-1}^q(x_0), \det{}_{n-q+1} M_k^q(x_0))}. \tag{5.4}$$

PROPOSITION 5.5. *The ideals $I_k^q(x_0)$ are also given by (1.22).*

Proof. If we choose (as usual) $\omega_1, ..., \omega_n$ to be an orthonormal basis of $A_{x_0}^{1,0}$ with $\omega_n = \partial r$ and if

$$\partial\bar{\partial}r = \sum_{i,j} c_{ij}\,\omega_i \wedge \bar{\omega}_j.$$

then define $\tau_1, ..., \tau_{n-1}$ by:

$$\tau_j = \operatorname{int}(\bar{\omega}_j)\,\partial\bar{\partial}r = \sum c_{ij}\,\omega_i.$$

Then $M_1(x_0)$ is generated by: $\tau_1, ..., \tau_{n-1}$ and ω_n. Hence $I_1^q(x_0)$ is the real radical of the ideal generated by the determinants of the $(n-q)\times(n-q)$ minors of (c_{ij}) with $i, j < n$, and the function r. This establishes (1.22) for $k=1$. The general case then follows by induction.

If V is a complex-analytic variety defined in a neighborhood U of x_0 we denote by $\mathfrak{I}_{x_0}(V)$ the ideal of germs of holomorphic functions that vanish on V and by $\mathfrak{J}_{x_0}(V)$ the ideal of germs of complex-valued real-analytic functions that vanish on V. We will make use of a result of R. Ephraim (see [11]) which asserts that when V is irreducible then $\mathfrak{J}_{x_0}(V)$ is generated by $\mathfrak{I}_{x_0}(V)$ and $\overline{\mathfrak{I}_{x_0}(V)}$, where $\overline{\mathfrak{I}_{x_0}(V)} = \{\bar{f} \mid f \in \mathfrak{I}_{x_0}(V)\}$.

Definition 5.6. If V is a germ of a complex-analytic variety at $x_0 \in b\Omega$ then we define the *order of contact* of V to $b\Omega$ at x_0, denoted by $O(x_0, V)$, by

$$O(x_0, V) = O_{x_0}(r/\mathfrak{J}_{x_0}(V)) = \max_{g \in \mathfrak{J}_{x_0}(V)} O_{x_0}(r-g), \tag{5.7}$$

where $O_{x_0}(f)$ denotes the order of vanishing of f at x_0. Let $\mathfrak{V}^q(x_0)$ denote the set of germs of q-dimensional, irreducible varieties containing x_0. Then we define $O^q(x_0)$, the *q-order* of x_0, by:

$$O^q(x_0) = \max_{V \in \mathfrak{V}^q(x_0)} O(x_0, V). \tag{5,8}$$

Let $\mathfrak{W}^q(x_0)$ be the set of all germs of q-dimensional complex manifolds containing x_0. Then we define the *regular q-order* of x_0, denoted by reg $O^q(x_0)$, by

$$\operatorname{reg} O^q(x_0) = \max_{V \in \mathfrak{W}^q(x_0)} O(x_0, V). \tag{5.9}$$

Observe that

$$\operatorname{reg} O^q(x_0) \leqslant O^q(x_0). \tag{5.10}$$

In fact, for r in $\mathbf{C}^3$ given by

$$r(z) = \operatorname{Re}(z_3) + |z_1^2 - z_2^3|^2,$$

we have reg $O^1(x_0)=6$ and $O^1(x_0)=\infty$. This type of phenomenon has been studied in [2] and [6a]. In [6a], D'Angelo shows that, if reg $O^1(x_0)\leqslant 4$ then $O^1(x_0)=\text{reg } O^1(x_0)$.

Lemma 5.11. *If $O^q(x_0)=m$ then there exist germs of holomorphic functions $h_1, \dots, h_k$ at x_0 and polynomials A_i, B_i such that*

$$r=\sum A_i h_i+\sum B_i \bar{h}_i+O(|z|^m).$$

Proof. By definition of $O^q(x_0)$ there exists a germ of a q-dimensional irreducible variety $V\in \mathcal{V}^q(x_0)$ such that $O(x_0, V)=m$. Let $h_1, \dots, h_k$ be the generators of $\mathcal{I}_{x_0}(V)$ then, by the above cited theorem of Ephraim, we conclude that $h_1, \dots, h_k, \bar{h}_1, \dots, \bar{h}_k$ generate $\mathcal{J}_{x_0}(V)$. Thus the function g which attains the maximum in (5.7) can be expressed in terms of these generators, which concludes the proof.

Lemma 5.12. *Given $N>0$ there exists a holomorphic coordinate system $z_1, \dots, z_n$ with origin at x_0 such that*

$$r=2\text{Re}\,(z_n)+\sum_{\substack{|\alpha|>0,\,|\beta|>0\\ |\alpha+\beta|<N}} a_{\alpha\beta}\, z^\alpha \bar{z}^\beta+O(|z|^N). \tag{5.13}$$

Proof. Choose any holomorphic coordinate system $w_1, \dots, w_n$ with origin x_0. Then by expanding in Taylor series we have

$$r=\text{Re}\Big(\sum_{|\alpha|<N} c_\alpha w^\alpha\Big)+\sum_{\substack{|\alpha|>0,\,|\beta|>0\\ |\alpha+\beta|<N}} b_{\alpha\gamma}\, w^\alpha \bar{w}^\beta+O(|w|^N).$$

Let $z_1, \dots, z_n$ be any holomorphic coordinate systems with origin at x_0 and with $z_n=\frac{1}{2}\sum_{|\alpha|<N} c_\alpha w^\alpha$, then substituting in the above, we obtain (5.13).

Lemma 5.14. *Given $N>O(x_0, V)$, where V is a germ of an irredubible complex-analytic variety through x_0; then*

$$O_{x_0}(z_n/\mathcal{I}_{x_0}(V))\geqslant O(x_0, V).$$

Proof. Let $h_1, \dots, h_k$ be generators of $\mathcal{I}_{x_0}(V)$ then, by 5.11,

$$r=\sum A_i h_i+\sum B_i \bar{h}_i+O(|z|^m), \quad \text{with } m=O(x_0, V). \tag{5.15}$$

From (5.13) we have:

$$z_n-\sum A_i h_i=\sum B_i \bar{h}_i-\bar{z}_n+\sum{}'' a_{\alpha\beta} z^\alpha \bar{z}^\beta+O(|z|^m).$$

We can write $A_i=F_i+G_i$, when F_i is holomorphic and G_i has a power series expansion all of whose terms have one of the $\bar{z}_j$ as a factor. Then

$$z_n-\sum_i F_i h_i=G+O(|z|^m),$$

where G is a polynomial each of whose terms contains at least one $\bar{z}_j$. Hence all partial derivatives of the left hand side up to order m vanish at x_0, which completes the proof.

In the following proposition the main assertion that (b) is equivalent to (d) is proven, in the case $q=1$ by J. D'Angelo in [6b].

PROPOSITION 5.16. *If Ω is pseudo-convex then the following are equivalent*

(a) $1 \in I_1^q(x_0)$

(b) *The Levi-form at x_0 has at least $n-q$ positive eigen-values*

(c) reg $O^q(x_0)=2$

(d) $O^q(x_0)=2$

Proof. That (a) is equivalent to (b) is an immediate consequence of the definition of $I_1^q(x_0)$. It is also clear that (d) implies (c), by (5.10) and since from (5.13) we see that reg $O^q(x_0) \geqslant 2$. We will first prove that (c) implies (b). Choosing the coordinate system $z_1, \ldots, z_n$ so that (5.13) holds with $N=3$ we have

$$r(z) = 2\mathrm{Re}\,(z_n) + \sum r_{z_i \bar{z}_j}(0)\, z_i \bar{z}_j + O(|z|^3). \tag{5.17}$$

We will assume (c) holds and that (b) does not hold. Thus

$$\dim\left\{ z \,|\, z_n = 0,\ \sum_{i=1}^{n-1} r_{z_i \bar{z}_j}(0)\, z_i = 0,\ j = 1, \ldots, n-1 \right\} \geqslant q. \tag{5.18}$$

So by (5.17) the order of contact of the linear space defined in (5.18) is greater or equal to 3, which contradicts (c).

Now assuming (a) we will prove (d). Let $V \in \mathcal{V}^q(x_0)$ and let $h_1, \ldots, h_k$ be generators of $\mathcal{I}_{x_0}(V)$. Suppose $O(x_0, V) > 2$, then, by 5.11:

$$\begin{aligned} r &= \sum A_i h_i + \sum B_i \bar{h}_i + O(|z|^3) \\ \partial\bar{\partial} r &= -\sum \bar{\partial} A_i \wedge dh_i + \sum \partial B_i \wedge d\bar{h}_i + \theta + O(|z|), \end{aligned} \tag{5.19}$$

where $\theta = 0$ on V. From (5.17) we have,

$$\partial r = dz_n + O(|z|). \tag{5.20}$$

By virtue of (5.14) we know that $z_n|_V = O(|z|^3)$, hence

$$(\partial r)_{x_0} = (dz_n)_{x_0} = \sum_1^k c_i (dh_i)_{x_0}. \tag{5.21}$$

Let α and β be (1, 0)-forms with constant coefficients such that

$$(\bar{\partial}A\ )_{x_0} = a_i d\bar{z}_n + \bar{\alpha}_i \quad \text{and} \quad (\partial B_i)_{x_0} = b_i dz_n + \beta_i \tag{5.22}$$

with

$$a_i = \left(\frac{\partial A_i}{\partial \bar{z}_n}\right)_{x_0} \quad \text{and} \quad b_i = \left(\frac{\partial B_i}{\partial z_n}\right)_{x_0}.$$

Then, from (5.19), we have

$$(\partial\bar{\partial}r)_{x_0} = \sum (dh_i)_{x_0} \wedge \bar{\alpha}_i + \sum \beta_i \wedge (d\bar{h}_i)_{x_0} + \sum a_i (dh_i)_{x_0} \wedge d\bar{z}_n + \sum b_i dz_n \wedge (dh_i)_{x_0}. \tag{5.23}$$

The restriction of $(\partial\bar{\partial}r)_{x_0}$ to $T^{1,0}_{x_0}(b\Omega)$ is given by the first two terms on the right of (5.23). This is a semi-definite hermitian form which vanishes on the intersection of the annihilators of the $(dh_j)_{x_0}$. Hence we have, using (5.21)

$$(\partial\bar{\partial}r)_{x_0} = \sum a_{ij} (dh_i)_{x_0} \wedge (d\bar{h}_j)_{x_0}. \tag{5.24}$$

Then

$$\partial r \wedge \bar{\partial} r \wedge (\partial\bar{\partial}r)^{n-q} = \sum dh_{i_1} \wedge \dots \wedge dh_{i_{n-q+1}} \wedge \psi_{i_1 \dots i_{n-q+1}} + O(|z|). \tag{5.25}$$

Since V is q-dimensional at most $n-q$ of the dh_j are linearly independent at regular points of V. Hence $dh_{i_1} \wedge \dots \wedge dh_{i_{n-q+1}} = 0$ on V and hence $[\partial r \wedge \bar{\partial} r \wedge (\partial\bar{\partial}r)^{n-q}]_{x_0} = 0$ so that $1 \notin I^q_1(x_0)$ which is a contradiction and concludes the proof.

Definition 5.26. A is an *admissible* vector-field in a neighborhood U of x_0 if $\langle A, \partial r\rangle = 0$ and $\langle A, \bar{\partial} r\rangle = 0$. In particular for $x \in b\Omega$, $A_x \in T^{1,0}_x(b\Omega) + T^{0,1}_x(b\Omega)$.

LEMMA 5.27. *If c_{ij} is a component of the Levi-form and if $A_1, \dots, A_m$ are admissible vector fields then $A_1, \dots, A_m(c_{ij}) \in I^{n-1}_{m+1}(x_0)$.*

Proof. Since

$$\partial r \wedge \bar{\partial} r \wedge \partial\bar{\partial} r = \sum_{i,j<n} c_{ij}\, \omega_i \wedge \bar{\omega}_j \wedge \omega_n \wedge \bar{\omega}_n$$

each $c_{\ j} \in I^{n-1}_1(x_0)$ when $i, j < n$. Further

$$\partial c_{ij} \wedge \partial r = \sum_{k<n} (L_k c_{ij})\, \omega_k \wedge \omega_n.$$

Hence $L_k c_{ij} \in I^{n-1}_2(x_0)$ and also $L_k c_{ji} \in I^{n-1}_2(x_0)$; but $L_k c_{ji} = L_k \bar{c}_{ij}$ and $|L_k \bar{c}_{ij}| = |\bar{L}_k c_{ij}|$, hence $\bar{L}_k c_{ij} \in I^{n-1}_2(x_0)$. Since the admissible vectors are combinations of the L_k and $\bar{L}_k$ the lemma follows for $m = 1$. For $m = 2$ we apply the same argument to $\partial A c_{ij}$, when A is an admissible vcetor field and similarly conclude the proof by induction.

LEMMA 5.28. *Let F be a real-valued C^∞ function defined in a neighborhood of the origin in $\mathbf{R}^n$. Suppose that $F \geqslant 0$ and $F(0)=0$. Let X be a real C^∞ vector field defined in a neighborhood of $0 \in \mathbf{R}^n$. Then either $X^j F(0)=0$ for all j or there exists some integer k such that $X^j F(0)=0$ if $j<2k$ and $X^{2k}F(0)>0$.*

Proof. It suffices to consider X such that $X \neq 0$ in a neighborhood of 0. Choose a coordinate system $x_1, \ldots, x_n$ so that $X=\partial/\partial x_1$, then

$$F(x)=\sum_{j=1}^{m} a_j(0, x_2, \ldots, x_n)\, x_1^j + O(|x|^{n+1}).$$

Choosing m to be the smallest number such that $a_m(0, \ldots, 0) \neq 0$ we see that $F(x_1, 0, \ldots, 0) \geqslant 0$ implies that m is even and $a_m(0, \ldots, 0)>0$ which concludes the proof.

LEMMA 5.29. *If $f, g_1, \ldots, g_m$ are complex-valued C^∞ functions in a neighborhood of $0 \in \mathbf{R}^n$ such that*

$$|f|^{2p} \leqslant \text{const.} \sum_{1}^{m} |g_j|^2; \tag{5.30}$$

furthermore, if X is a real C^∞ vector field and $X^j f(0)=0$ for $j<k$ and $X^k f(0) \neq 0$ then for some j and some $q \leqslant pk$ we have $X^q g_j(0) \neq 0$.

Proof. Assume that $X^q g_j(0)=0$ for $j=1, \ldots, m$ and all $q<pk$. Let $F=\text{const.} \sum_1^m |g_j|^2 - |f|^{2p}$, then

$$X^{2pk}F(0)=\text{const.} \sum_{1}^{m} |X^{pk} g_j(0)|^2 - |X^k f(0)|^2$$

and the result follows from 5.29.

Definition 5.31. $\mathfrak{L}^k(x_0) \subset \mathbf{C}T_{x_0}$ is defined inductively as follows:

$\mathfrak{L}^1(x_0)=$ germs of admissible vector fields. $\mathfrak{L}^k(x_0)=\mathfrak{L}^{k-1}(x_0)+[\mathfrak{L}^1(x_0), \mathfrak{L}^{k-1}(x_0)]$. $\dot{\mathfrak{L}}^k(x_0) \subset \mathbf{C}T_{x_0}$ is the subspace obtained by evaluating all elements of $\mathfrak{L}^k(x_0)$ at x_0. Note that $\dot{\mathfrak{L}}^1(x_0)=T^{1,0}_{x_0}(b\Omega)+T^{0,1}_{x_0}(b\Omega)$. We say that x_0 is of *finite type* if for some integer m we have $\dot{\mathfrak{L}}^m(x_0)=\mathbf{C}T_{x_0}(b\Omega)$ and if m is the least such integer we say that x_0 is of *type* m.

Observe that if $f \in C^\infty(x_0)$ and $A \in \mathfrak{L}^k(x_0)$ then $fA \in \mathfrak{L}^k(x_0)$ since

$$f[B_1, B_2]=B_2(f)\, B_1+[fB_1, B_2],$$

so that if $B_1 \in \mathfrak{L}^1(x_0)$ and $B_2 \in \mathfrak{L}^{k-1}(x_0)$ then $f[B_1, B_2] \in \mathfrak{L}^k(x_0)$.

LEMMA 5.32. *$x_0 \in b\Omega$ is of type greater than or equal to m, with $m \geqslant 3$, if and only if whenever $A_1, \ldots, A_k \in \mathfrak{L}^1(x_0)$, with $k<m-2$, then $A_1 \ldots A_k c_{ij}(x_0)=0$ if $i, j<n$. Furthermore, x_0 is of type 2 if and only if $c_{ij}(x_0) \neq 0$ for some i, j with $i, j<n$.*

Proof. With our usual notation (see 2.20) we have

$$[L_i, \bar{L}_j] = c_{ij} T \bmod \mathcal{L}^1(x_0),$$

if $i, j < n$. For any $S \in \mathbf{C}T_{x_0}$ we have $S = aT \bmod \mathcal{L}^1(x_0)$. Thus by induction we obtain, when $i, j < n$

$$[A_1, [A_2, \ldots, [A_k, [L_i, \bar{L}_j] \ldots] = (A_1 \ldots A_k(c_{ij}) + R_{kij}(c_{ij}))\, T \bmod \mathcal{L}^{k+1}(x_0), \tag{5.33}$$

where R_{kij} is a polynomial in the $A_1, \ldots, A_k$ of degree less than k. The desired conclusion then follows by evaluating (5.33) at x_0.

LEMMA 5.34. *Suppose that $x_0 \in b\Omega$ is of type greater than p and that $f \in C^\infty(x_0)$ has the properties that $f(x_0) = 0$ and that $A_1 \ldots A_k f(x_0) = 0$ whenever $k < p$ and $A_j \in \mathcal{L}^1(x_0)$. Then if $A_1, \ldots, A_p \in \mathcal{L}^1(x_0)$ and if π is a permutation of $\{1, \ldots, p\}$ we have*

$$A_1 \ldots A_p f(x_0) = A_{\pi(1)} \ldots A_{\pi(p)} f(x_0). \tag{5.35}$$

Furthermore, if for some choice of $A_1, \ldots, A_p \in \mathcal{L}^1(x_0)$, we have $A_1 \ldots A_p f(x_0) \neq 0$ then there exists $A \in \mathcal{L}^1(x_0)$ such that $A^p f(x_0) \neq 0$. Finally, if in the last statement the $A_1, \ldots, A_p$ are real then there exists a real $A \in \mathcal{L}^1(x_0)$ such that $A^p f(x_0) \neq 0$.

Proof. From (5.33) it follows that

$$A_1 \ldots A_p = A_{\pi(1)} \ldots A_{\pi(p)} + \sum_{i,j<n} P_{ij}(c_{ij})\, T + P_p, \tag{5.36}$$

where the P_{ij} and P_p are polynomials in $A_1, \ldots, A_p$ of degree less than p. Hence (5.35) follows by applying (5.36) to f and evaluating at x_0.

If $A_1 \ldots A_p f(x_0) \neq 0$, let $A = \sum s_j A_j$. Then $A^p f(x_0)$ is a homogeneous polynomial in the s_j's and the coefficient of $s_1 \ldots s_p$ equals $p!\, A_1 \ldots A_p f(x_0) \neq 0$. Hence the polynomial is not identically zero and so for some choice of the s_j we have $A^p f(x_0) \neq 0$. If the A_j are real we can choose the s_j real and obtain a real A as required.

PROPOSITION 5.37. *$1 \in I_m^{n-1}(x_0)$ if and only if x_0 is of finite type.*

Proof. By Lemmas 5.32 and 5.34 it will suffice to prove that $1 \in I_m^{n-1}(x_0)$ is equivalent to the existence of $A \in \mathcal{L}^1(x_0)$ such that $A^p(c_{ij}) \neq 0$ with $p \geqslant 0$ and some $i, j < n$. From Lemma 5.27 it follows that if $A^p(c_{ij}) \neq 0$, with $A \in \mathcal{L}^1(x_0)$ $i, j < n$ and $p \geqslant 0$ then $1 \in I_{p+1}^{n-1}(x_0)$.

Suppose that $1 \in I_m^{n-1}(x_0)$ then there exists a function $f^{(1)} \in I_{m-1}^{n-1}(x_0)$ such that, for some $i < n$, $L_i f^{(1)} \neq 0$. Then there exist functions $f_1^{(2)}, \ldots, f_{k_2}^{(2)} \in I_{m-2}^{n-1}(x_0)$ and p_1, such that

$$|f^{(1)}|^{2p_1} \leqslant \sum_{s=1}^{n-1} \sum_{j=1}^{k_2} |L_s f_j^{(2)}|^2.$$

Let A be either Re (L_i) or Im (L_i) so that $Af^{(1)} \neq 0$. Then, by 5.9, there exist s, j and q so that $A^q L_s f_j^{(2)} \neq 0$. Let $f^{(2)} = f_j^{(2)}$ and let B be either Re (L_s) or Im (L_s) so that $A^q B f^{(2)} \neq 0$. Now suppose that x_0 is of type greater than $q_2 = q+1$ then, from Lemma 5.34, we conclude that there exists a real $A_2 \in \mathcal{L}^1(x_0)$ and an integer q_2 so that $A_2^{q_2} f^{(2)} \neq 0$. Similarly we obtain $f^{(3)} \in I_{m-3}^{n-1}(x_0)$ and a real $A_3 \in \mathcal{L}^1(x_0)$ such that $A_3^{q_3} f^{(3)} \neq 0$. After repeating this procedure $m-1$ times we obtain $f^{(m-1)} \in I_1^{n-1}(x_0)$ and a real $A_{m-1} \in \mathcal{L}^1(x_0)$ so that $A_{m-1}^{q_{m-1}} f^{(m-1)} \neq 0$, further

$$|f^{(m-1)}|^{2p_{m-1}} \leqslant \sum_{i,j<n} |c_{ij}|^2$$

hence $A_{m-1}^k c_{ij} \neq 0$, for some k, i, j; which, by 5.32, concludes the proof.

PROPOSITION 5.38. *x_0 is of type m if and only if* reg $O^{n-1}(x_0) = m$.

Proof. Choose coordinates $z_1, \ldots, z_n$ with origin at x_0 so that $r = \operatorname{Re}(z_n) + F + O(|z|^{m+1})$, where F is a mixed polynomial vanishing at 0. Let

$$L_j = \frac{\partial}{\partial z_j} - \frac{r_{z_j}}{r_{z_n}} \frac{\partial}{\partial z_n}, \quad j = 1, \ldots, n-1,$$
$$L_n = \frac{1}{r_{z_n}} \frac{\partial}{\partial z_n},$$
$$T = L_n - \bar{L}_n.$$

Let $V = \{z_n \mid z_n = 0\}$. Then, by Lemma 5.14, $O(x_0, V) = m$ if and only if reg $O^{n-1}(x_0) = m$. We also have $O(x_0, V) = m$ if and only if there is some $i, j < n$ and $\alpha_1, \ldots, \alpha_{2n-1}$ with $\alpha_1 + \ldots + \alpha_{2n-2} = m-2$ such that

$$B_1^{\alpha_1} \ldots B_{2n-2}^{\alpha_{2n-2}} F_{z_i \bar{z}_j}(0) \neq 0 \tag{5.39}$$

and this expression equals zero whenever $i, j < n$ and $\alpha_1 + \ldots + \alpha_{2n-2} < m-2$, where

$$B_i = \frac{\partial}{\partial z_i}, \; B_{i+n-1} = \frac{\partial}{\partial \bar{z}_i}, \quad i = 1, \ldots, n-1.$$

We will show that if $O(x_0, V) \geqslant m$ then

$$B_2^{\alpha_1} \ldots B_{2n-2}^{\alpha_{2n-2}} F_{z_i \bar{z}_j}(0) = A_1^{\alpha_1} \ldots {}_{2n-2}^{\alpha_{2n-2}} c_{ij}(0), \tag{5.40}$$

whenever $i, j < n$ and $\alpha_1 + \ldots + \alpha_{2n-1} \leqslant m-2$, where

$$A_i = L_i, \; A_{i+n-1} = \bar{L}_i, \quad i = 1, \ldots, n-1.$$

The desired conclusion follows from (5.40) by applying Lemma 5.32. To prove (5.40) we first note that for $i, j < n$ we have

$$c_{ij} = F_{z_i\bar z_j} - \frac{r_{\bar z_j}}{r_{\bar z_n}} F_{z_i\bar z_n} - \frac{r_{z_i}}{r_{z_n}} F_{z_n\bar z_j} + \frac{r_{z_i}\bar r_{z_j}}{|r_{z_n}|^2} F_{z_n\bar z_n} + O(|z|^{m-1}) = F_{z_i\bar z_j} + h_{ij}F_{z_i} + g_{ij}F_{\bar z_j} + O(|z|^{m-1}),$$

where $h_{ij}, g_{ij} \in C^\infty(x_0)$. Furthermore

$$A_i = \begin{cases} B_i + h_i F_{z_i}\dfrac{\partial}{\partial z_n}, & i = 1, \dots, n-1 \\[2ex] B_i + g_i F_{\bar z_{i-n+1}}\dfrac{\partial}{\partial z_n}, & i = n, \dots, 2n-2, \end{cases}$$

where $h_i, g_i \in C^\infty(x_0)$. Now (5.40) is easily established by induction on $k = \alpha_1 + \dots + \alpha_{2n-2}$.

§ 6. The real-analytic case

In this section we will suppose that r is real-analytic in a neighborhood of $x_0 \in b\Omega$. We will deal only with real-analytic functions. We will denote by $\mathscr{A}(x_0)$ the set of germs of real-analytic functions at x_0. If $S \subset \mathscr{A}(x_0)$ then (S) denotes the ideal of germs of real-analytic functions generated by S and $\sqrt[\mathbf{R}]{S}$ denotes the set of all $f \in \mathscr{A}(x_0)$ such that there exists an m and a $g \in S$ with $|f|^m \leqslant |g|$. In this section $I_k^q(x_0)$ will denote the ideal of germs of real-analytic functions defined by (1.22) where ( ) and $\sqrt[\mathbf{R}]{\ }$ are interpreted as above. Before we enter into an examination of the ideals $I_k^q(x_0)$ we will state some properties of ideals of germs of real-analytic functions.

Let I be an ideal of germs of real-analytic functions at $0 \in \mathbf{R}^p$. Let $\mathfrak{V}(I)$ denote the germ of the real-analytic variety defined by I; that is, if $f_1, \dots, f_k$ are generators of I which are defined on a neighborhood U of 0, then $U \cap \mathfrak{V}(I) = \{x \in U \mid f_j(x) = 0, j = 1, \dots, k\}$. If $x \in \mathfrak{V}(I)$ we denote by $\mathfrak{J}_x\mathfrak{V}(I)$ the ideal of germs of real-analytic functions at x which vanish on $\mathfrak{V}(I)$. The following is proved in [24].

THEOREM 6.1. (Lojasiewicz). *If I is an ideal of germs of real-analytic functions at $0 \in \mathbf{R}^p$, then* $\mathfrak{J}_0\mathfrak{V}(I) = \sqrt[\mathbf{R}]{I}$.

As usual we will complexify $\mathbf{R}^p$ by the embedding of $\mathbf{R}^p$ into $\mathbf{C}^p$ given by $z_j = x_j$, where $x_1, \dots, x_p$ are coordinates in $\mathbf{R}^p$ and $z_1, \dots, z_p$ are coordinates in $\mathbf{C}^p$. If f is a real-analytic function on an open set $U \subset \mathbf{R}^p$ then there exists an open set $\tilde U \subset \mathbf{C}^p$ such that $\tilde U \cap \mathbf{R}^p = U$ and a holomorphic function $\tilde f$ on $\tilde U$ such that $\tilde f = f$ on U. We call $\tilde f$ the *complexification* of f, and if I is an ideal of germs of real-analytic functions then we denote by $I^{\mathbf{C}}$ the ideal of germs of holomorphic functions generated by the complexifications of the elements of I. We will also denote by $\mathfrak{V}(I^{\mathbf{C}})$ the germ of the complex-analytic variety defined by $I^{\mathbf{C}}$ and if $z \in \mathfrak{V}(I^{\mathbf{C}})$ we will denote by $\mathfrak{I}_z \mathfrak{V}(I^{\mathbf{C}})$ the ideal of germs of holomorphic functions at z that vanish on $\mathfrak{V}(I^{\mathbf{C}})$.

PROPOSITION 6.2. *Let I be an ideal of germs of real-analytic functions at $0 \in \mathbf{R}^p$ such that $I = \sqrt[\mathbf{R}]{I}$. Then we have*

$$\dim_{\mathbf{R}} \mathfrak{V}(I) = \dim_{\mathbf{C}} \mathfrak{V}(I^{\mathbf{C}}). \tag{6.3}$$

Proof. In Narasimhan [25], Proposition 1, page 91, it is shown that

$$(\mathfrak{I}_0 \mathfrak{V}(I))^{\mathbf{C}} = \mathfrak{I}_0 \mathfrak{V}((\mathfrak{I}_0 \mathfrak{V}(I))^{\mathbf{C}}).$$

Applying 6.1 we have

$$\mathfrak{I}_0 \mathfrak{V}(I^{\mathbf{C}}) = I^{\mathbf{C}}. \tag{6.4}$$

Then (6.3) follows by Proposition 3 of [25], p. 93.

H. Cartan in [4] shows that in $\mathbf{R}^3$ if $I = (z(x^2+y^2) - x^3)$ then, for any $z \neq 0$, the ideal $\mathfrak{I}_{(0,0,z)} \mathfrak{V}(I)$ is not generated by $\mathfrak{I}_{(0,0,0)} \mathfrak{V}(I)$. For our purposes this difficulty can be overcome by means of the following result.

PROPOSITION 6.5. *If I is an ideal of germs of real-analytic functions at $0 \in \mathbf{R}^p$ and if $I = \sqrt[\mathbf{R}]{I}$, then there exists a sequence of points $x^{(\nu)} \in \mathfrak{V}(I)$ such that $x^{(\nu)}$ converges to 0 and such that each $x^{(\nu)}$ has a neighborhood U_ν with the property that if $y \in U_\nu \cap \mathfrak{V}(I)$ then $\mathfrak{I}_y \mathfrak{V}(I)$ is generated by the elements of I.*

Proof. Let $m = \dim_{\mathbf{R}} \mathfrak{V}(I)$, then we can choose a sequence $x^{(\nu)} \in \mathfrak{V}(I)$ with $\lim_{\nu \to \infty} x^{(\nu)} = 0$ such that $\mathfrak{V}(I)$ is regular and of dimension m at $x^{(\nu)}$ (see Theorem 1, page 41 of [25]). Let U'_ν be a neighborhood of $x^{(\nu)}$ such that every $y \in U'_\nu \cap \mathfrak{V}(I)$ is a regular point of $\mathfrak{V}(I)$ and $\mathfrak{V}(I)$ has dimension m at y. Let $\tilde{U} \subset \mathbf{C}^p$ be a neighborhood of 0 such that for every $z \in \tilde{U} \cap \mathfrak{V}(I^{\mathbf{C}})$ the ideal $\mathfrak{I}_z \mathfrak{V}(I^{\mathbf{C}})$ is generated by elements of $I^{\mathbf{C}}$ (such a $\tilde{U}$ exists by Oka's theorem). If $y \in \tilde{U} \cap U'_\nu \cap \mathfrak{V}(I)$ then y is a regular point of $\mathfrak{V}(I^{\mathbf{C}})$ and so there exists $h_1, \ldots, h_{p-m} \in I^{\mathbf{C}}$ so that $(dh_1)_y \wedge \ldots \wedge (dh_{p-m})_y \neq 0$. The restrictions of $h_1, \ldots, h_{p-m}$ to $\mathbf{R}^p$ are elements of I which generate $\mathfrak{I}_y \mathfrak{V}(I)$. Hence the neighborhoods $U_\nu = \tilde{U} \cap U'_\nu$ have the desired property.

Returning to our ideals $I_k^q(x_0)$ we let $\mathfrak{V}_k^q$ be the germ of a real-analytic variety at x_0 given by

$$\mathfrak{V}_k^q(x_0) = \mathfrak{V}(I_k^q(x_0)). \tag{6.6}$$

Definition 6.7. If I is an ideal of germs of analytic function at x_0 and if $x \in \mathcal{V}(I)$ then we define $Z_x^{1,0}(I)$ the *Zariski tangent space* of I at x as follows

$$Z_x^{1,0}(I) = \{L \in T_x^{1,0} \mid L(f) = 0 \quad \text{if } f \in I\}. \tag{6.8}$$

If V is a germ of a real-analytic variety at x_0 then we define

$$Z_x^{1,0}(V) = Z_x^{1,0}(\mathcal{J}_x V). \tag{6.9}$$

The following is then immediate.

LEMMA 6.10. *If I is an ideal of germs of real-analytic functions at x_0 and if $x \in \mathcal{V}(I)$ then*

$$Z_x^{1,0}(\mathcal{V}(I)) \subset Z_x^{1,0}(I). \tag{6.11}$$

If $\mathcal{J}_x \mathcal{V}(I)$ is generated by elements of I then equality holds in (6.11).

PROPOSITION 6.12. *If $x \in \mathcal{V}_k^q(x_0)$ then $x \in \mathcal{V}_{k+1}^q(x_0)$ if and only if*

$$\dim\,(Z_x^{1,0}(I_k^q(x_0)) \cap \mathcal{N}_x) \geq q, \tag{6.13}$$

where $\mathcal{N}_x$ is defined by:

$$\mathcal{N}_x = \{L \in T_x^{1,0}(b\Omega) \mid \langle (\partial\bar{\partial}r)_x, L \wedge \bar{L}\rangle = 0\}. \tag{6.14}$$

Proof. If $L_1, \ldots, L_n$ is the usual local basis of $T^{1,0}$ with $\langle L_i, \partial r\rangle = \delta_{in}$ and $c_{ij} = \langle \partial\bar{\partial}r, L_i \wedge \bar{L}_j\rangle$ so that (c_{ij}) with $i, j < n$ on $b\Omega$ is the Levi form; then $x \in \mathcal{V}_{k+1}^q(x_0)$ if and only if the following system has at least q linearly independent solutions.

$$\begin{aligned} &\sum_{i=1}^{n-1} c_{ij}(x)\,\zeta_i = 0, \quad j = 1, \ldots, n-1 \\ &\sum_{i=1}^{n} [L_i(f)]_x\,\zeta_i = 0, \quad f \in I_k^q(x_0). \end{aligned} \tag{6.15}$$

For $x \in \mathcal{V}_k^q(x_0)$ and $L = \sum_{i=1}^n \zeta_i L_i$ the above system characterizes those L such that $L_x \in Z_x^{1,0}(I_k^q(x_0)) \cap \mathcal{N}_x$, which concludes the proof.

Definition 6.16. If V is a real-analytic variety contained in $b\Omega$ we define the *holomorphic dimension* of V by

$$\text{hol. dim}\,(V) = \min_{x \in V} \dim Z_x^{1,0}(V) \cap \mathcal{N}_x. \tag{6.17}$$

PROPOSITION 6.18. *If* $V \subset U \cap b\Omega$ *is a real-analytic variety and if* hol. dim $(V) \geq q$ *then* $V \subset \mathcal{V}_m^q(x_0)$ *for all* m.

Proof. If $x \in V$ then dim $\mathcal{N}_x \geq q$ hence $x \in \mathcal{V}_1^q(x_0)$, so that $V \subset \mathcal{V}_1^q(x_0)$. Assume that $V \subset \mathcal{V}_k^q(x_0)$, then, applying (6.10), we obtain for $x \in V$:

$$Z_x^{1,0}(V) \subset Z_x^{1,0}(\mathcal{V}_k^q(x_0)) \subset Z_x^{1,0}(I_k^q(x_0)). \tag{6.19}$$

Then, intersecting the above with $\mathcal{N}_x$ and applying Proposition 6.12 we conclude that $x \in \mathcal{V}_{k+1}^q(x_0)$ hence $V \subset \mathcal{V}_{k+1}^q(x_0)$ so that $V \subset \mathcal{V}_m^q(x_0)$ for all m.

PROPOSITION 6.20. *If for every real-analytic variety* $V \subset U \cap b\Omega$ *we have* hol. dim $(V) < q$ *then* $\mathcal{V}_{2n}^q(x_0) = \emptyset$.

Proof. We will show that, if $\mathcal{V}_k^q(x_0) \neq \emptyset$, then

$$\dim \mathcal{V}_k^q(x_0) > \dim \mathcal{V}_{k+1}^q(x_0). \tag{6.21}$$

Suppose (6.21) does not hold. Then, these dimensions are equal and hence in an open set W with the property that every $y \in W \cap \mathcal{V}_{k+1}^q(x_0)$ is a regular point at which the dimension of $\mathcal{V}_{k+1}^q(x_0)$ is maximal, we have $W \cap \mathcal{V}_k^q(x_0) = W \cap \mathcal{V}_{k+1}^q(x_0)$. Now by 6.5 we can choose such a $W \subset U$ so that for each $y \in W \cap \mathcal{V}_k^q(x_0)$ the ideal $\mathcal{J}_y \mathcal{V}_k^q(x_0) = \mathcal{J}_y \mathcal{V}_{k+1}^q(x_0)$ is generated by the elements of $I_k^q(x_0)$. Hence by Proposition 6.12 we conclude that hol. dim $W \cap \mathcal{V}_k^q(x_0) \geq q$, which is a contradiction. Hence (6.21) holds and the conclusion follows since dim $\mathcal{V}_1^q(x_0) \leq$ dim $b\Omega = 2n-1$.

It then follows that if in some neighborhood U of x_0 there is no $V \subset U \cap b\Omega$ with hol. dim $V \geq q$ then $1 \in I_{2n}^q(x_0)$ and hence a subelliptic estimate holds at x_0 for (p, q)-forms. Observe that if W is a complex-analytic variety with $W \subset b\Omega$ then hol. dim $W =$ dim W since then $Z_x^{1,0}(W) \subset \mathcal{N}_x$ for all $x \in W$. The converse of this is the following deep result of Diederich and Fornaess (see [9]).

THEOREM 6.22. (Diederich and Fornaess). *If* Ω *is pseudo-convex, if* r *is analytic in a neighborhood* U *of* $x_0 \in b\Omega$ *and if there exists a real analytic variety* $V \subset U \cap b\Omega$ *with* hol. dim $(V) = q$ *then there exists a complex-analytic variety* $W \subset U \cap b\Omega$ *with* dim $W = q$.

Using this theorem we see that a subelliptic estimate holds if there are no complex-analytic varieties of dimension greater or equal to q in some neighborhood of x_0. Actually, this is equivalent to the condition that there is no variety in $b\Omega$ of dimension q which contains x_0, by a result that was obtained by J. Fornaess and which is given below. The proof given here is also due to Fornaess; it uses the methods developped in [9].

THEOREM 6.23. (Fornaess.) *If W_k is a sequence of complex varieties with* $\dim W_k \geqslant q$, $W_k \subset b\Omega$ *and x_0 a cluster point of this sequence them there exists a complex variety W such that* $\dim W \geqslant q$, $W \subset b\Omega$ *and* $x_0 \in W$.

Choose a neighborhood U of x_0 such that the Taylor series of r about x_0 converges in U. Let $\tilde{r}$ be the complexification of r. Now we need the following result which is proved in section 6 of [9].

PROPOSITION 6.24. *There exists a neighborhood U' of x_0 such that $U' \subset U$ and such that if W is an irreducible complex-analytic variety in $U' \cap b\Omega$ then there exists a complex analytic variety W' such that $W \subset W' \subset U' \cap b\Omega$ and such that W is closed in U'; that is: $\overline{W}' \cap U' = W'$. Furthermore, for any complex analytic variety $W \subset U' \cap b\Omega$ we have $\tilde{r}(z, \bar{w}) = 0$ whenever $z, w \in W$.*

Proof of Theorem 6.23. We may suppose that the W_k are closed irreducible varieties contained in $U' \cap b\Omega$. Let $p^{(1)}$ be a cluster point of the W_k, then we can find a subsequence, which we also denote by $\{W_k\}$ such that $p_k^{(1)} \in W_k$ and $p^{(1)} = \lim_{k\to\infty} p_k^{(1)}$. Now, let $p^{(2)}$ be a cluster point of this subsequence whose distance from $p^{(1)}$ is maximal. We then choose a further subsequence $\{W_k\}$ such that $p_k^{(2)} \in W_k$ and $\lim_{k\to\infty} p_k^{(2)} = p^{(2)}$. Proceeding inductively and using diagonalization we finally obtain a sequence $\{W_k\}$ and for each m we have $p_k^{(m)} \in W_k$ and $\lim_{k\to\infty} p_k^{(m)} = p^{(m)}$. If C denotes the set of cluster points of $\{W_k\}$; then the sequence $\{p^{(m)}\}$ is dense in C. For every k we have $p_k^{(i)}, p_k^{(j)} \in W_k$ hence $\tilde{r}(p_k^{(i)}, \bar{p}_k^{(j)}) = 0$ and hence if p and $p' \in C$ we have $\tilde{r}(p, \bar{p}') = 0$. Let W' be defined by

$$W' = \bigcap_{p' \in C} \{p \in U' \mid \tilde{r}(p, \bar{p}') = 0\}. \tag{6.24}$$

Thus W' is a closed complex-analytic variety contained in U' and $W' \supset C$; furthermore, if $w' \in W$ and $c \in C$ then we have

$$\tilde{r}(w', \bar{c}) = \tilde{r}(c, \bar{w}') = 0. \tag{6.25}$$

Let W be defined by

$$W = \bigcap_{w' \in W'} \{w \in U' \mid \tilde{r}(w, \bar{w}') = 0\}. \tag{6.26}$$

Then $C \subset W \subset W'$ and if $w \in W$ we have $r(w) = \tilde{r}(w, \bar{w}) = 0$. Hence W is a closed complex-analytic subvariety of $U' \cap b\Omega$, it remains to show that $\dim W \geqslant q$. Consider the stratification $\hat{W}_0 \subset \hat{W}_1 \subset \dots \subset \hat{W}_l = W$, where the $\hat{W}_j$ are the singular points of $\hat{W}_{j+1}$ for $j = 1, \dots, l-1$.

Let d be the smallest integer such that $C - \hat{W}_d$ does not cluster at x_0. Let $W^1, \dots, W^s$ be the irreducible germs of $\hat{W}_d$. Then $(W^i - \hat{W}_{d-1}) \cap C$ clusters at x_0 for some $i \in \{1, \dots, s\}$.

Fix such an i. It suffices to show that dim $W^i \geq q$. Choose $p^{(m)} \in W^i - \tilde{W}_{d-1}$ and a neighborhood U'' of $p^{(m)}$ such that $U'' \cap W^i$ consists of regular points of W^i and $U'' \cap C$ in contained in W^i. Let $\eta_1, \ldots, \eta_n$ be holomorphic coordinates with origin at $p^{(m)}$ such that $\Delta \subset U''$ and

$$W^i \cap \Delta = \{(\eta_1, \ldots, \eta_n) \in \Delta \,|\, \eta_{t+1} = \ldots = \eta_n = 0\},$$

where $\Delta = \{(\eta_1, \ldots, \eta_n) \in \mathbf{C}^n \,|\, |\eta_j| < 1, j = 1, \ldots, n\}$ and $t = \dim W^i$. Then on points of $C \cap \Delta$ we have $\eta_{t+1} = \ldots = \eta_n = 0$. Let $\tilde{\Delta} = \{\eta \,|\, |\eta_j| < \frac{1}{2}, j = 1, \ldots, n\}$. It then follows, if k is sufficiently large, that

$$W_k \cap \tilde{\Delta} \subset \{\eta \in \tilde{\Delta} \,|\, |\eta_j| < \tfrac{1}{4}, j = t+1, \ldots, n\}.$$

Hence the map

$$\pi_k \colon W_k \cap \tilde{\Delta} \to \{(\eta_1, \ldots, \eta_t) \,|\, |\eta_j| < \tfrac{1}{2}, j = 1, \ldots, t\}$$

is proper. This is only possible if dim $W_k \leq t$. Hence dim $W \geq q$ since dim $W_k \geq q$ and $t =$ dim $W^i \leq$ dim W.

The above results are then summarized by the following theorem.

THEOREM 6.27. *Assume that Ω is pseudo-convex, $x_0 \in b\Omega$ and r is real-analytic in a neighborhood of x_0. Then the following conditions are equivalent:*

(a) $1 \in I_k^q(x_0)$ *for some k.*

(b) *There exists a neighborhood U of x_0 such that $U \cap b\Omega$ does not contain any complex analytic varieties of dimension q.*

(c) *If W is a germ of a complex-analytic variety at x_0 such that $W \subset b\Omega$ then* dim $W < q$.

Theorem 1.19 then follows since (a) implies that $x_0 \in \mathcal{E}^q$.

§ 7. Some special domains

In this section we consider domains $\Omega \subset \mathbf{C}^n$ whose defining function r is given, near the origin, by:

$$r(z_1, \ldots, z_n) = \operatorname{Re}(z_n) + \sum_{j=1}^{m} |h_j(z_1 \ldots, z_n)|^2 + a, \tag{7.1}$$

where $h_1, \ldots, h_m$ are holomorphic functions, $a \in \mathbf{R}$ and $r(0, \ldots, 0) = 0$; so that

$$a = -\sum_{j=1}^{m} |h_j(0, \ldots, 0, 0)|^2.$$

Then we have

$$r_{z_k \bar{z}_l} = \sum_{j=1}^{m} h_{jz_k} \bar{h}_{j\bar{z}_l} \tag{7.2}$$

and hence

$$\sum_{k,i=1}^{n} r_{z_k \bar{z}_i} \zeta_k \bar{\zeta}_i = \sum_{j=1}^{m} \left| \sum_{k=1}^{n} h_{jz_k} \zeta_k \right|^2. \tag{7.3}$$

Thus, the domain is pseudo-convex.

PROPOSITION 7.4. *If W is a germ of a complex-analytic variety such that $W \subset b\Omega$ then the functions z_n and h_j are constant on W. In particular, if W contains the origin then W is contained in the variety V given by $V = \{z_n = 0, h_j(z_1, \ldots, z_n) = h_j(0, \ldots, 0)$ for $j = 1, \ldots, m\}$. Note that $V \subset b\Omega$.*

Proof. Let $z^0 = (z_1^0, \ldots, z_n^0)$ be a regular point of W. Let $z_n' = z_n - z_n^0$, then by Lemma 5.12 we have $z_n' = 0$ on W. We choose coordinates $z_1', \ldots, z_n'$ with origin at z^0 so that, in a neighborhood of z^0 the variety W is given by $z_{p+1}' = \ldots = z_n' = 0$. Let h_j' be the function given by $h_j'(z') = h_j(z(z'))$, then we have

$$r(z') = \operatorname{Re}(z_n') + \sum_{j=1}^{m} |h_j'(z')|^2 + a + \operatorname{Re}(z_n^0). \tag{7.5}$$

Evaluating r on W we obtain

$$\sum_{j=1}^{m} |h_j'(z_1', \ldots, z_p', 0, \ldots, 0)|^2 = -a - \operatorname{Re}(z_n^0). \tag{7.6}$$

Applying $\partial^2/\partial z_k' \partial \bar{z}_k'$ to (7.6) and summing on k gives

$$\sum_{k=1}^{p-1} \sum_{j=1}^{m} \left| \frac{\partial h_j'}{\partial z_k'} (z_1', \ldots, z_p', 0, \ldots, 0) \right|^2 = 0. \tag{7.7}$$

Hence the h_j are constant on W.

Applying Theorem 6.27 we find that the following conditions are equivalent.

(a) $1 \in I_k^q(0)$ for some k.

(b) $\dim V < q$.

(c) $\dim \{z_n = z_n^0, h_j = h_j(z^0)\} < q$, where z^0 is close to the origin.

Observe that

$$\partial r \wedge (\partial \bar{\partial} r)^{n-q} = \tfrac{1}{2} dz_n \wedge (\sum_j \partial h_j \wedge \overline{\partial h_j})^{n-q} + \ldots, \tag{7.8}$$

where the dots represent forms in which either ∂h_j or $\overline{\partial h_j}$ appears as factor at least $n-q+1$ times.

Definition 7.9. Let $J_k^q(0)$ denote ideals of germs of holomorphic functions at 0 defined by

$$J_1^q(0) = \sqrt{(\text{coeff.}\{dz_n \wedge dh_{j_1} \wedge \dots \wedge dh_{j_{n-q}}\})}$$

and

$$J_k^q(0) = \sqrt{(J_{k-1}^q(0), \text{coeff.}\{df_1 \wedge \dots \wedge df_{n-q+1}\} \quad \text{where } f_k \in J_{k-1}^q(0) \cup \{z_n, h_j\})}.$$

We set

$$J^q(0) = \bigcup_k J_k^q(0).$$

PROPOSITION 7.10. *The conditions* (a), (b) *and* (c) *are equivalent to* $1 \in J^q(0)$. *Furthermore, if* $1 \notin J^q(0)$ *then* $\dim V \geq q$, *where* $V = \{z \mid z_n = 0, h_j(z) = h_j(0), j = 1, ..., m\}$.

Proof. We will show that (b) is equivalent to $1 \in J^q(0)$. The proof is along the same lines as that of Proposition 6.20; it is much simpler because it is based only on properties of ideals of holomorphic functions.

Suppose that $\dim V < q$, define $\mathcal{J}_z$ by

$$\mathcal{J}_z = \{L \in T_z^{1,0} \mid L(z_n) = L(h_j) = 0\}. \tag{7.11}$$

Suppose $1 \notin J_k^q(0)$ and let A be an open subset of reg $\mathcal{V}(J_k^q(0))$ which is so close to the origin that (c) is satisfied for all $z^0 \in A$. If $z \in A$ and $z \in \mathcal{V}(J_{k+1}^q(0))$ then, since (by Oka's theorem) $\mathcal{J}_z(\mathcal{V}(J_k^q(0))$ is generated by $J_k^q(0)$, we have (by Cramer's rule)

$$\dim (Z_z^{1,0}\mathcal{V}(J_k^q(0)) \cap \mathcal{J}_z) \geq q. \tag{7.12}$$

If there were an open subset $A' \subset A$ with $A' \subset \mathcal{V}(J_{k+1}^q(0))$ then A' would contain an open subset A'' on which the left hand side of (7.12) is constant. Hence, by the Frobenius theorem, A'' is a complex manifold of dimension greater or equal to q. On the other hand if $z^0 \in A''$ then for each $z \in \text{reg } A''$ we have $T_z^{1,0}(A'')$ is a subspace of the tangent space to $\{z \mid z_n = z_n^0, h_j(z) = z_n^0\}$, which contradicts (c). Hence A cannot have an open subset contained in $\mathcal{V}(J_{k+1}^q(0))$ and therefore $\dim \mathcal{V}(J_{k+1}^q(0) < \dim \mathcal{V}(J_k^q(0))$. Thus we conclude inductively that $\mathcal{V}(J^q(0)) = \emptyset$, so that $1 \in J^q(0)$.

If, conversely, $\dim V \geq q$, then (7.12) holds at all points of $z \in V'$, where V' denotes the union of components V of dimension greater or equal to q. Hence $V' \subset \mathcal{V}(J_k^q(0))$ for all k and thus $1 \notin J_k^q(0)$.

Observe that in the above proposition z_n plays the same role as the h_j; hence, we obtain the following result, whose proof is analogous to the one given above.

THEOREM 7.13. *Let $h_0, \ldots, h_m$ be germs of holomorphic functions at $0 \in \mathbf{C}^n$ and let $V = \{z \mid h_j(z) = 0, j = 0, \ldots, m\}$. Define the ideals of germs J_k^q as follows*

$$J_1^q = \sqrt{(\text{coeff.}\{dh_{j_0} \wedge \ldots \wedge dh_{j_{n-q}}\})},$$

where $(j_0, \ldots, j_{n-q})$ range over all $(n-q+1)$-tuples of integers from 0 to m; inductively we let

$$J_{k+1}^q = \sqrt{(J_k^q, \{\text{coeff.}(df_0 \wedge \ldots \wedge df_{n-q})\}, f_j \in J_k^q \cup \{h_0, \ldots, h_m\})}.$$

We set $J^q = \bigcup_k J_k^q$. Note that $J^0 \subset J^1 \subset \ldots \subset J^n$. Let q_0 be the unique integer such that $1 \in J^q$ if $q > q_0$ and $1 \notin J^q$ if $q \leqslant q_0$. Then $\dim V = q_0$.

§ 8. Estimates of $(p, n-1)$-forms

In this section we prove the following which is an extension of the main result in [20e].

THEOREM 8.1. *If Ω is pseudo-convex $x_0 \in b\Omega$ and if* $\operatorname{reg} O^{n-1}(x_0) = m$ *then $x_0 \in \mathcal{E}^{n-1}(1/m)$.*

Proof. If $\varphi \in \mathcal{D}_U^{0,n-1}$, where U is a neighborhood of x_0, we can write

$$\varphi = u\bar{\omega}^1 \wedge \ldots \wedge \bar{\omega}^{n-1} + \psi \wedge \bar{\omega}^n, \tag{8.2}$$

where $\psi = 0$ on $b\Omega$, then we have

$$Q(\varphi, \varphi) \sim \sum_{i=1}^{n-1} \|L_i u\| + \sum_{i=1}^{n} \|\bar{L}_i u\| + \|u\|^2 + \|\psi\|_1^2. \tag{8.3}$$

Thus to show that $x_0 \in \mathcal{E}^{n-1}(1/m)$ it suffices to prove that

$$|||u|||_{1/m}^2 \leqslant \text{const.}\left(\sum_{i=1}^{n-1} \|L_i u\|^2 + \sum_{i=1}^{n} \|\bar{L}_i u\|^2 + \|u\|^2\right) \tag{8.4}$$

for all $u \in C_0^\infty(U \cap \bar{\Omega})$.

We first reduce the estimate (8.4) to an estimate on the boundary, following a procedure developed by L. Hörmander (see [17b]) and which was applied to the D-Neumann problem by W. Sweeney (see [31a]).

Applying Proposition 5.8 of [31a] we conclude that there exists a pseudo-differential operator P of order one operating on $C_0^\infty(U \cap b\Omega)$, such that (8.4) holds if and only if:

$$'\|u\|_{1/m}^2 \leqslant \text{const.}\left(\sum_{i=1}^{n-1} ('\|L_i u\|^2 + '\|\bar{L}_i u\|^2) + '\|Pu\|^2 + '\|u\|^2\right), \tag{8.5}$$

for all $u \in C_0^\infty(U \cap b\Omega)$, where $'\|\ \|$ denotes norms on $U \cap b\Omega$.

Setting:

$$X_i=\begin{cases} L_i+\bar{L}_i, & i=1,\ldots,n-1\\ \sqrt{-1}\,(L_{i-n+1}-\bar{L}_{i-n+1}), & i=n,\ldots,2n-2,\end{cases} \tag{8.6}$$

we have, by Lemma 5.32, that $[X_{i_1},[X_{i_2},\ldots,[X_{i_{p-1}},X_{i_p}]]\ldots]$ for $p\leqslant m$ span the tangent vector fields on $U\cap b\Omega$, when U is small. In [17c], Hörmander proves that this condition implies that for each $\varepsilon<1/m$ there exists C such that

$$'\|u\|_\varepsilon^2\leqslant C\left(\sum_{i=1}^{2n-2}{}'\|X_iu\|^2+{}'\|u\|^2\right),\quad \text{for all } u\in C_0^\infty(U\cap b\Omega). \tag{8.7}$$

In [27], E. Stein and L. Rothschild, proves that (8.7) holds also for $\varepsilon=1/m$. From this (8.5) follows, since

$$\sum_{i=1}^{n-1}({}'\|L_iu\|^2+{}'\|\bar{L}_iu\|^2)+\|u\|^2\sim\sum_{i=1}^{2n-2}{}'\|X_iu\|^2+{}'\|u\|^2. \tag{8.8}$$

The operator P that appears in (8.5) can be described quite explicitly using the results of [17b] and [31a]. The principal symbol of P, denoted by p is given by

$$p(t,\tau)=-\sigma_t(T,\tau)+\sqrt{|\sigma_t(T,\tau)|^2+\sum_{j=1}^{n-1}|\sigma_t(L_j,\tau)|^2}, \tag{8.9}$$

where $t\in U\cap b\Omega$, $\tau\in\Lambda^1(T_t(b\Omega))$, $T=L_n-\bar{L}_n$ and $\sigma_t(T,\tau)$ denotes the symbol of T evaluated at τ.

§ 9. Propagation of singularities for $\bar{\partial}$

In [20e] we discussed propagation of singularities for $\bar{\partial}$ on Levi-flat domains in $\mathbf{C}^2$, here we will give a natural generalization of this for domains in $\mathbf{C}^n$ whose boundary contains a germ of a complex-analytic curve.

Definition 9.1. If $\alpha\in L_2^{p,q}(\Omega)$ we define the *singular support* of α to be the closed subset of $\bar{\Omega}$, denoted by sing. supp. (α), as follows. If $x\in\bar{\Omega}$ then $x\notin$ sing. supp. (α) if there exists a neighborhood U of x such that the restriction of α to $U\cap\bar{\Omega}$ (denoted by $\alpha|_{U\cap\bar{\Omega}}$) is in C^∞.

An immediate consequence of Theorem 1.13 is the following.

THEOREM 9.2. *If Ω is pseudo-convex and $\alpha\in L_2^{p,q}(\Omega)$ with $\bar{\partial}\alpha=0$ then there exist $u\in L_2^{p,q-1}(\Omega)$ such that $\bar{\partial}u=\alpha$. Furthermore, if $x_0\in\mathcal{E}^q$ then there exists a neighborhood U of x_0 such that*

$$U\cap\text{sing. supp. }(u)\subset\text{sing. supp. }(\alpha), \tag{9.3}$$

where $u\in L_2^{p,q-1}(\Omega)$ is the unique solution of $\bar{\partial}u=\alpha$ which is orthogonal to the null space of $\bar{\partial}$.

Definition 9.4. Let $x_0 \in b\Omega$ we say that Ω admits a *local holomorphic separating function* at x_0 if there exists a neighborhood U of x_0 and a holomorphic function g on U such that $g(x_0)=0$ and whenever $\operatorname{Re} g(x)=0$ then $x \notin U \cap \Omega$.

The example in [21a] shows that this condition is rather restrictive. Recent results of Bedford and Fornaess (see [1]) indicate that peak functions can substitute for separating functions in many applications.

PROPOSITION 9.5. *Suppose Ω is pseudo-convex, that $x_0 \in b\Omega$ and that the following hypotheses are satisfied:*

(a) *Ω admits a local holomorphic separating function g at x_0 such that $dg \neq 0$.*

(b) *There is a complex-analytic curve V such that $x_0 \in V$ and $V \subset b\Omega$.*

(c) *g vanishes on V.*

Then for any neighborhood U of x_0 there exists an open set $U' \subset U$ and a form $\alpha \in L_2^{0,1}(\Omega)$, with $\bar\partial\alpha=0$, such that $U' \cap$ sing. supp. $(\alpha)=\emptyset$ and for every $u \in L_2(\Omega)$ which satisfies $\bar\partial u=\alpha$ we have $U' \cap$ sing. supp. $(u) \neq \emptyset$.

Proof. Let $z_1, ..., z_n$ be holomorphic coordinates with origin at x_0 such that $z_n = \pm g$, where the sign is chosen so that $\operatorname{Re}(z_n) \leqslant 0$ in Ω (near x_0). Let $a \in U \cap \operatorname{reg}(V)$ and let $\varrho \in C_0^\infty(U)$, such that $\varrho(z)=1$ if $|z-a| \leqslant \gamma$ and $\varrho(z)=0$ if $|z-a| \geqslant 2\gamma$; where γ is so small that if z satisfies $|z-a| \leqslant 3\gamma$ then: $z \in U$; $\operatorname{Re}(z_n) \leqslant 0$ if $z \in \bar\Omega$ and also if $z \in V$ then $z \in \operatorname{reg} V$.

We define α by:

$$\alpha = \begin{cases} (-z_n)^{-1/4}\bar\partial\varrho & \text{in } U \cap \bar\Omega \\ 0 & \text{outside of } U \cap \bar\Omega, \end{cases} \tag{9.6}$$

where we choose the principal value of $(-z_n)^{-1/4}$. Observe that $\bar\partial\alpha=0$, that $\alpha \in L_2^{0,1}(\Omega)$ and that

$$\text{sing. supp. }(\alpha) = \{z \in U \cap \Omega \,|\, \gamma \leqslant |z-a| \leqslant 2\gamma \quad \text{and } z_n = 0\}. \tag{9.7}$$

Let K be a small closed neighborhood of the above set and let $U' = U - K$. Then we have $U' \cap$ sing. supp. $(\alpha)=\emptyset$. Suppose there exists a function $u \in L_2(\Omega)$ such that $\bar\partial u = \alpha$ and suppose that $U' \cap$ sing. supp. $(u)=\emptyset$. Let $h = u - (-z_n)^{-1/4}\varrho$. Then h is holomorphic. For small δ we restrict h to the set $\{z \,|\, |z-a| < 4\gamma,\ z_j = a_j \text{ for } j=2, ..., n-1 \text{ and } z_n = -\delta\}$ and we obtain the function of one variable f_δ defined by

$$f_\delta(z_1) = u(z_1, a_2, ..., a_{n-1}, -\delta) - \frac{\varrho(z_1, a_2, ..., a_{n-1}, -\delta)}{\delta^{1/4}}. \tag{9.8}$$

The assumption that $U' \cap \text{sing. supp.}\,(u)=\emptyset$ implies that $u(a_1, \ldots, a_{n-1}, -\delta)$ is bounded independently of δ and that $u(z_1, a_2, \ldots, a_{n-1}, -\delta)$ evaluated on the set $\{z_1 \mid |z_1-a_1| = 3\gamma\}$ is bounded independently of δ, (for $\delta < \gamma$). Hence from (9.8) we conclude that $f_\delta(z_1)$ is bounded independently of δ on the circle $|z_1-a_1| = 3\gamma$ (since $\varrho=0$ there) and that $|f_\delta(a_1)| > 1/\delta^{1/4} - M$, where M is the bound of $|u(a_1, \ldots, a_{n-1}, -\delta)|$. Since f_δ is holomorphic the value $f_\delta(a_1)$ is an average of the values of f_δ on the circle $|z_1-a_1| = 3\gamma$; which, for small δ, is a contradiction. Hence $U' \cap \text{sing. supp.}\,(u) \neq \emptyset$.

References

[1]. BEDFORD, E. & FORNAESS, J. E., A construction of peak functions on weakly pseudoconvex domains. Preprint.

[2]. BLOOM, T. & GRAHAM, I., A geometric characterization of points of type m on real hypersurfaces. *J. Differential Geometry*, to appear.

[3]. BOUTET DE MONVEL, L. & SJÖSTRAND, J., Sur la singularité des noyaux de Bergman et de Szegö. *Soc. Math. de France Astérisque*, 34–35 (1976), 123–164.

[4]. CARTAN, H., Variétés analytiques reélles et variétés analytiques complexes. *Bull. Soc. Math. France*, 85 (1957), 77–99.

[5]. CATLIN, D., *Boundary behaviour of holomorphic functions on weakly pseudo-convex domains.* Thesis, Princeton Univ. 1978.

[6]. D'ANGELO, J., (a) Finite type conditions for real hypersurfaces. *J. Differential Geometry*, to appear.

—— (b) A note on the Bergman kernel. *Duke Math. J.*, to appear.

[7]. DERRIDJ, M., Sur la régularité des solutions du problème de Neumann pour $\bar{\partial}$ dans quelques domains faiblement pseudo-convexes. *J. Differential Geometry*, to appear.

[8]. DERRIDJ, M. & TARTAKOFF, D., On the global real-analyticity of solutions of the $\bar{\partial}$-Neumann problem. *Comm. Partial Differential Equations*, 1 (1976), 401–435.

[9]. DIEDERICH, K. & FORNAESS, J. E., Pseudoconvex domains with real-analytic boundary. *Ann. of Math.*, 107 (1978), 371–384.

[10]. EGOROV, YU. V., Subellipticity of the $\bar{\partial}$-Neumann problem. *Dokl. Akad. Nauk. SSSR*, 235, No. 5 (1977), 1009–1012.

[11]. EPHRAIM, R., C^∞ and analytic equivalence of singularities. *Rice Univ. Studies*, Complex Analysis, Vol. 59, No. 1 (1973), 11–31.

[12]. FEFFERMAN, C., The Bergman kernel and biholomorphic mappings. *Invent. Math.*, 26 (1974), 1–65.

[13]. FOLLAND, G. B. & KOHN, J. J., *The Neumann problem for the Cauchy–Riemann complex.* Ann. of Math. Studies, No. 75, P.U. Press, 1972.

[14]. GREINER, P. C., On subelliptic estimates of the $\bar{\partial}$-Neumann problem in $\mathbf{C}^2$. *J. Differential Geometry*, 9 (1974), 239–250.

[15]. GREINER, P. C. & STEIN, E. M., (a) On the solvability of some differential operators of type $\Box_b$, preprint.

—— (b) *Estimates for the $\bar{\partial}$-Neumann problem.* Math. Notes No. 19, Princeton University Press 1977.

[16]. HENKIN, G. M. & ČIRKA, E. M., *Boundary properties of holomorphic functions of several complex variables.* Problems of Math. Vol. 4, Moscow 1975, 13–142.

[17]. HÖRMANDER, L., (a) L^2 estimates and existence theorems for the $\bar{\partial}$ operator. *Acta Math.*, 113 (1965), 89–152.

—— (b) Hypoelliptic second order differential equations. *Acta Math.*, 119 (1967), 147–171.

—— (c) Pseudo-differential operators and non-elliptic boundary problems. *Ann. of Math.*, 83 (1966), 129–209.

[18]. KASHIWARA, M., Analyse micro-locale du noyau de Bergman. *Sem. Goulaouic–Schwartz* 1976–1977. Exposé N° VIII.

[19]. KERZMAN, N., (a) The Bergman-kernel function: differentiability at the boundary. *Math. Ann.*, 195 (1972), 149–158.

—— (b) Hölder and L^p estimates for solutions of $\bar\partial u = f$ in strongly pseudo-convex domains. *Comm. Pure Appl. Math.*, 24 (1971), 301–379.

[20]. KOHN, J. J., (a) Lectures on degenerate elliptic problems. *Proc. CIME Conf. on Pseudo-differential Operators*, Bressanone (1977), to appear.

—— (b) Sufficient conditions for subellipticity on weakly pseudo-convex domains. *Proc. Nat. Acad. Sci. U.S.A.*, 74 (1977), 2214–2216.

—— (c) Methods of partial differential equations in complex analysis. *Proc. of Symp. in Pure Math.*, vol. 30, part 1 (1977), 215–237.

—— (d) Global regularity for $\bar\partial$ on weakly pseudo-convex manifolds. *Trans. Amer. Math. Soc.*, 181 (1973), 273–292.

—— (e) Boundary behaviour of $\bar\partial$ on weakly pseudo-convex manifolds of dimension two. *J. Differential Geometry*, 6 (1972), 523–542.

[21]. KOHN, J. J. & NIRENBERG, L., (a) A pseudo-convex domain not admitting a holomorphic support function. *Math. Ann.*, 201 (1973), 265–268.

—— (b) Non-coercive boundary value problems. *Comm. Pure Appl. Math.*, 18 (1965), 443–492.

[22]. KRANTZ, S. G., (a) Characterizations of various domains of holomorphy via $\bar\partial$ estimates and applications to a problem of Kohn, preprint.

—— (b) Optimal Lipschitz and L^p regularity for the equation $\bar\partial u = f$ on strongly pseudo-convex domains. *Math. Ann.*, 219 (1976), 223–260.

[23]. LIEB, I., Ein Approximationssatz auf streng pseudo-konvexen Gebieten. *Math. Ann.*, 184 (1969), 55–60.

[24]. LOJASIEWICZ, S., *Ensembles semi-analytiques*. Lecture note (1965) at I.H.E.S.; Reproduit n° A66–765, Ecole polytechnique, Paris.

[25]. NARASIMHAN, R., *Introduction to the theory of analytic spaces*. Lectures notes in Math. No. 25, Springer Verlag 1966.

[26]. RANGE, R. M., On Hölder estimates for $\bar\partial u = f$ on weakly pseudoconvex domains, preprint.

[27]. ROTHSCHILD, L. P. & STEIN, E. M., Hypoelliptic differential operators and nilpotent groups. *Acta Math.*, 137 (1976), 247–320.

[28]. TARTAKOFF, D., The analytic hypoellipticity of $\Box_b$ and related operators on non-degenerate C–R manifolds, preprint.

[29]. TREVES, F., Analytic hypo-ellipticity of a class of pseudodifferential operators with double characteristics and application to the $\bar\partial$-Neumann problem, preprint.

[30]. SPENCER, D. C., Overdetermined systems of linear partial differential equations. *Bull. Amer. Math. Soc.*, 75 (1969), 176–239.

[31]. SWEENEY, W. J., (a) The D-Neumann problem. *Acta Math.*, 120 (1968), 223–277.

—— (b) A condition for subellipticity in Spencer's Neumann problem. *J. Differential Equations*, 21 (1976), 316–362.

Received May 3, 1978