THE MICROBUNDLE REPRESENTATION THEOREM

BY

PER HOLM

The University of Oslo, Oslo, Norway

The present paper contains a generalization of the Kister–Mazur theorem which says that any microbundle over a finite dimensional simplicial complex contains a (up to bundle isomorphism) unique fibre bundle. Precisely, we prove this theorem (or a relativized version of it) for microbundles over arbitrary topological spaces, provided the microbundle admits a trivializing partition of unity on the base. In particular the theorem applies to any microbundle over a paracompact space. At present this is a work that aims at generality and completeness rather than applicability, since so far the Kister–Mazur result covers most of the interesting cases. From a purely esthetical point of view, however, the latter has certain defects. The natural objects to study among the microbundles over a simplicial complex are the piecewise linear microbundles. For such one should of course expect sharper results. Recently Hirsch, Mazur and others have shown that a piecewise linear microbundle contains subcomplexes which are piecewise linear bundles and that any two such are piecewise linearly isomorphic [5]. On the other hand, in the category of topological microbundles it seems unnatural to put any restrictions at all on the base space.

The condition about the existence of a trivializing partition of unity has already been introduced on bundles by Dold, who calls such bundles *numerable*, cf. [3]. Any (micro-) bundle over a normal base space covered by a locally finite family of trivializing open sets is numerable. Products, sums and "pull-back's" of numerable (micro-)bundles are numerable. Dold also shows that the numerable bundles have the good properties shared by bundles over paracompact spaces. In view of his work it almost seems desirable to redefine (micro-)bundles as numerable (micro-)bundles. In any case it has been convenient to do so here. By definition (micro-)bundles in this paper are always numerable.

Besides the techniques purified in [3], an inductive process of Mazur for extending homeomorphisms on open sets in $\mathbf{R}^q$ plays a fundamental role in the sequel.

A preliminary report on this work has already appeared in [6]. The present paper contains the proofs and generalizations of the results announced there. A final word about the generalization from the paracompact to the numerable case. This generalization is not trivial. One reason is that while paracompact spaces are rich on continuous real valued functions, such functions have to be explicitly constructed by means of a single trivializing partition of unity in the numerable case. Another is that a fundamental result such as the germ extension theorem for trivial bundles does not seem to be valid in the general case. It goes through for "numerable" germs, however, which is all one needs. On the other hand, the advantages of working in the numerable case will be obvious in the proofs.

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1. Preliminaries

In the following X denotes an arbitrary topological space. A *partition of unity* on X is a family of continuous functions π_j: $X \to [0, 1]$, $j \in J$, whose *supports* $W_j = \pi_j^{-1}(0, 1]$ form a locally finite cover of X, and whose sum $\Sigma \pi_i$ is everywhere equal to 1. (Note that the supports are *open* subsets of X.) Partitions of unity will be written $(\pi_j, W_j)_{j \in J}$.

If A is a subset of X, a *halo* of A is a set containing the support of some function $\pi: X \to [0, 1]$ which is 1 on A. Thus a halo of A is a neighbourhood of A. Conversely, if X is normal and A is closed, every neighbourhood of A is a halo.

Given a partition of unity $(\pi_j, W_j)_{j \in J}$ on X there is a *derived partition of unity* $(\pi_j', W_j')_{j \in J}$ defined as follows: Form the function $\pi = \sup_{j \in J} \pi_j$. Then π is continuous (since it equals a finite supremum of π_j's over any sufficiently small open set) and positive. Let $\bar{\pi}_j = \max(\pi_j - \pi/2, 0)$ and define $\pi_j' = \bar{\pi}_j / \Sigma \bar{\pi}_i$, $j \in J$. The functions π_i' have supports

$$W_j' = \{x \mid \pi_j(x) > \tfrac{1}{2}\pi(x)\},\ j \in J.$$

If x is any element of X, only finitely many π_j are different from 0 at x, hence there is a particular j such that $\pi(x) = \pi_j(x)$. Then $\pi_j(x) > \frac{1}{2}\pi(x)$ so that $x \in W_j'$. Thus $(W_j')_{j \in J}$ is a cover of X. Since for any j we have $W_j' \subset W_j$, in fact $\overline{W}_j' \subset \{x \mid \pi_j(x) \geqslant \frac{1}{2}\pi(x)\} \subset W_j$, $(W_j')_{j \in J}$ is certainly locally finite. This gives

1.1. LEMMA. *To any partition of unity $(\pi_j, W_j)_{j \in J}$ on X there is a derived partition of unity $(\pi_j', W_j')_{j \in J}$ which is a shrinking of $(\pi_j, W_j)_{j \in J}$,* i.e. *$\overline{W}_j' \subset W_j$ for all j in J.*

If $(\pi_j, W_j)_{j\in J}$ is a partition of unity on X and K is a non-empty subset of J, write $\pi_K = \Sigma_{i\in K}\pi_i$ and $W_K = \bigcup_{i\in K} W_i$. Then W_K is the support of π_K.

Sometimes the term partition of unity will be used also on a family of functions π_j: $X \to [0,\infty)$, $j \in J$, whose supports form a locally finite cover of X and whose sum is everywhere positive, while the families above are referred to as normalized partitions of unity. Of course, every partition of unity (in the wide sense) can be normalized by dividing with its sum, as in the proof of 1.1.

An $\mathbf{R}^q$-*bundle*, $0 \leq q$, is a diagram of maps and spaces

$$X \xrightarrow{s} E \xrightarrow{p} X$$

such that $ps = id_X$, for which the following is true.

1. There exists a collection $\mathcal{A}$ of homeomorphisms Φ: $p^{-1}U \approx U \times \mathbf{R}^q$ (called *local trivializations*), where $U = U_\Phi$ is an open set in X, such that the composite maps

$$U \times \mathbf{R}^q \xrightarrow{\Phi^{-1}} p^{-1}U \xrightarrow{p} U$$

$$U \xrightarrow{s|U} p^{-1}U \xrightarrow{\Phi} U \times \mathbf{R}^q$$

are, respectively, the projection to the first factor and the injection to zero slice. The family $(U_\Phi)_{\Phi\in\mathcal{A}}$ is a cover of X.

2. There exists a partition of unity on X subordinate to $(U_\Phi)_{\Phi\in\mathcal{A}}$ (i.e. whose supports form a cover refining $(U_\Phi)_{\Phi\in\mathcal{A}}$).

Given two local trivializations Φ_1, Φ_2 over open sets U_1, U_2, their composite $\Phi_2\Phi_1^{-1}$ (whenever defined) is a homeomorphism on $(U_1 \cap U_2) \times \mathbf{R}^q$ whose restriction to any fibre maps origin to origin. Thus, if $G(\mathbf{R}^q, 0)$ is the group of homeomorphisms on $\mathbf{R}^q$ keeping the origin 0 fixed, then there is a map $f_{21}: U_1 \cap U_2 \to G(\mathbf{R}^q, 0)$ defined by $\Phi_2\Phi_1^{-1}(x, v) = (x, f_{21}(x)v)$. If $G(\mathbf{R}^q, 0)$ is equipped with the compact open topology, then f_{21} is continuous. An $(\mathbf{R}^q, \mathbf{R}^n)$-*bundle*, $0 \leq n \leq q$, is an $\mathbf{R}^q$-bundle such that all the maps f_{21} actually take values in $G(\mathbf{R}^q, \mathbf{R}^n, 0)$, the subgroup of homeomorphisms that map $\mathbf{R}^n$ onto $\mathbf{R}^n$ (where $\mathbf{R}^n$ is identified with the set of elements $(x_1, x_2, \ldots, x_n, 0, \ldots, 0)$ in $\mathbf{R}^q$). Any $(\mathbf{R}^q, \mathbf{R}^n)$-bundle contains a canonical subbundle with standard fibre $\mathbf{R}^n$. The two form a bundle pair in the definition of [9]. Bundles will be labelled by small greek letters $\xi, \eta, \ldots$.

If $\mathcal{A}$ is the collection of local trivializations of an $\mathbf{R}^q$- or $(\mathbf{R}^q, \mathbf{R}^n)$-bundle, then there is a unique extension $\bar{\mathcal{A}}$ of $\mathcal{A}$ which is maximal with respect to conditions 1 (or its refined version) and 2. As usual we agree to identify two bundle structures which are defined by collections $\mathcal{A}$, $\mathcal{A}'$ with common maximal extension.

Let D^q, $q \geqslant 0$, be the closed unit disk centered at the origin in $\mathbf{R}^q$, and let S^{q-1} be the $(q-1)$-sphere which is its boundary. A *D^q-bundle* is defined like an $\mathbf{R}^q$-bundle except its local trivializations are of the form

$$\Phi : p^{-1}U \approx U \times D^q.$$

An *S^q-bundle* is defined similarly, but is not required to have a zero section as part of its structure. More generally a *bundle with fiber* Y is defined similarly to an S^q-bundle. All bundles in this paper will be $\mathbf{R}^q$-bundles, D^q-bundles or S^q-bundles, though. The standard trivial $\mathbf{R}^q$-bundle (or $(\mathbf{R}^q, \mathbf{R}^n)$-bundle) over X is denoted $\varepsilon^q(X)$ or simply ε^q

$$\varepsilon^q : X \xrightarrow{x \times 0} X \times \mathbf{R}^q \xrightarrow{pr_1} X.$$

The standard trivial D^q-bundle over X is denoted $\delta^q(X)$ or δ^q. (In general though we shorten ε^1 and δ^1 to ε and δ.)

The structure group of a bundle (cf. [9] Ch. 2, sect. 7) will play a modest role in our considerations. They will always be subgroups of $G(\mathbf{R}^q, 0)$, $G(D^q, 0)$ or $G(S^q)$. In section 4 we make some use of the following two lemmas. The notation is the obvious extension of the one introduced above.

1.2. LEMMA. *Let Z and $C \subset Z$ be compact or locally compact and locally connected. Then $G(Z)$ (and $(G(C))$ is a topological group in the compact open topology, and the restriction map $r : G(Z, C) \to G(C)$ is a continuous representation.*

The first part is verified in [2] (Theorems 3 and 4). As for the second it is an easy consequence of the definition of the compact open topology.

1.3. LEMMA. *Let Z be locally compact and locally connected with one point compactification Z', and let $\{\infty\}$ be the complement of Z in Z'. The restriction map $r : G(Z', Z) \to G(Z)$ is an isomorphism of topological groups whose inverse is the canonical extension $j : G(Z) \to G(Z', \infty)$.*

This follows from Theorems 1 and 4 in [2].

Given two bundles ξ, η over base spaces X, Y (not necessarily of the same fibre type), a *bundle map* of ξ into η is a map of total spaces $\Theta : E \to F$ sending fibres over X into fibres over Y. If ξ and η have zero sections as part of their structure, Θ is required to respect these in the obvious way. Finally, if ξ and η are $(\mathbf{R}^p, \mathbf{R}^m)$- and $(\mathbf{R}^q, \mathbf{R}^n)$-bundles, Θ is in addition required to map subbundle into subbundle. Every bundle map Θ covers a map $\Theta' : X \to Y$ of base spaces. If X equals Y, then Θ' will usually be the identity map. In this case Θ is called an *embedding*, respectively an *isomorphism*, of bundles if it imbeds E in F, respectively maps E homeomorphically onto F. It is then clear what an *automor-*

phism is. We write $\Theta:\xi\to\eta$, respectively $\Theta:\xi\approx\eta$ if Θ is a bundle map, respectively an isomorphism, of ξ to η. If ξ is a bundle over X, any map $f:Z\to X$ induces functorially a bundle $f^*\xi$ of the same type over Z and a bundle map $\bar{f}:f^*\xi\to\xi$. If $A\subset X$, the bundle induced by the inclusion map is written $\xi|A$, and if $\Theta:\xi\to\eta$ is a bundle map covering the identity, we write $\Theta|A:\xi|A\to\eta|A$ for the bundle map defined by Θ.

Later we shall need the following

1.4. Lemma. *Let $F:X\times\mathbf{R}^q\to\mathbf{R}$ be a positive continuous function such that $\|v\|\leqslant\|v'\|$ implies $F(x,v)\leqslant F(x,v')$ for any $x\in X$. Then the map $\Psi:X\times\mathbf{R}^q\to X\times\mathbf{R}^q$ defined by $\Psi(x,v)=(x,F(x,v)v)$ is a bundle automorphism.*

Proof. We show that Ψ is bijective. Since Ψ is fibre preserving, it suffices to show that Ψ maps $x\times\mathbf{R}^q$ bijectively onto $x\times\mathbf{R}^q$ for arbitrary x in X. Suppose v, v' are such that $F(x,v)v=F(x,v')v'$. Then v and v' are linearly dependent. By the order preserving property of F v equals v'. Hence Ψ is injective. To conclude that Ψ is surjective, we have to show that given v' there is a v such that $F(x,v)v=v'$. Since this requires $v=tv'$ for some scalar t, we have to show there is a t with $tF(x,tv')=1$. Since the continuous function $t\mapsto tF(x,tv')$ takes the value 0 and values larger than 1 (for t sufficiently large), by the connectedness of $[0,\infty)$ it takes the value 1 for some t. Thus Ψ is surjective as well.

Clearly Ψ is continuous. We show that Ψ^{-1} is continuous. Let $((x_\gamma, v_\gamma))$ be an ultrasequence in $X\times\mathbf{R}^q$ such that $(\Psi(x_\gamma,v_\gamma))$ converges: $\lim_\gamma\Psi(x_\gamma,v_\gamma)=\Psi(x_0,v_0)$ say. We have to verify that $((x_\gamma,v_\gamma))$ converges to (x_0,v_0). By assumption $\lim_\gamma x_\gamma=x_0$ and

$$\lim_\gamma F(x_\gamma,v_\gamma)v_\gamma=F(x_0,v_0)v_0.$$

Let $t_0=F(x_0,0)>0$. By the continuity of F there is a neighbourhood W_0 of x_0 such that $|F(x,0)-F(x_0,0)|<t_0/2$ for $x\in W_0$. Thus $x\in W_0$ implies $F(x,0)>t_0/2$ and therefore also $F(x,v)>t_0/2$ for any $v\in\mathbf{R}^q$. In other words $F|W_0\times\mathbf{R}^q$ is bounded away from 0. Since ultimately $((x_\gamma,v_\gamma))$ is in $W_0\times\mathbf{R}^q$ and since $(F(x_\gamma,v_\gamma)v_\gamma)$ converges, this implies that ultimately (v_γ) is in some compact subset of $\mathbf{R}^q$ and so converges, $\lim_\gamma v_\gamma=v_0'$, say. But $\Psi(x_0,v_0)=\lim\Psi(x_\gamma,v_\gamma)=\Psi(x_0,v_0')$, thus $v_0=v_0'$ and so $\lim(x_\gamma,v_\gamma)=(x_0,v_0)$.

In this proof an *ultrasequence* simply means a generalized sequence which for any given subset of the space is eventually either in that set or in its complement. It is easily checked that a map $f:X\to Y$ is continuous at $x\in X$ if and only if for each ultrasequence (x_γ) converging to x $(f(x_\gamma))$ converges to $f(x)$.

1.5. Lemma. *Let ξ be a bundle over a space $X\times I$ and let ξ_0,ξ_1 be the bundles over X induced from the inclusions $X\times 0\subset X\times I$, $X\times 1\subset X\times I$. Let V be a halo of a set A in X*

and assume that $\xi\,|\,(V\times I)=(\xi_0\,|\,V)\times I$. *There is an isomorphism* $\xi_0\approx\xi_1$ *which is the identity over some halo* $V'\subset V$ *of* A.

Except for the relativization with respect to V, Lemma 1.5 is a consequence of Theorem 7.8 in [3]. However, the proof of Theorem 7.8 actually works for the relativized version appropriate for the conclusion of Lemma 1.5. (*Warning.* A bundle map in [3] is a stricter concept than in this paper.)

An $\mathbf{R}^q$-*microbundle*, $0\leqslant q$, is a diagram of maps and spaces

$$X\xrightarrow{s}E\xrightarrow{p}X$$

such that $ps=id_X$, for which the following is true

1. There exists a collection $\mathcal{A}$ of homeomorphisms $\Phi: V\approx U\times\mathbf{R}^q$ (called local *trivializations*), where $U=U_\Phi$ and $V=V_\Phi$ are open sets in X and E, such that $sU\subset V$ and $pV\subset U$ and such that the composite maps

$$U\times\mathbf{R}^q\xrightarrow{\Phi^{-1}}V\xrightarrow{p}U$$

$$U\xrightarrow{s}V\xrightarrow{\Phi}U\times\mathbf{R}^q$$

are, respectively, the projection to the first factor and the injection to the zero slice. The family $(U_\Phi)_{\Phi\in\mathcal{A}}$ is a cover of X.

2. There exists a partition of unity on X subordinate to $(U_\Phi)_{\Phi\in\mathcal{A}}$.

Again the collection $\mathcal{A}$ of local trivializations can be extended to a unique maximal collection $\bar{\mathcal{A}}$, and we agree to identify microbundle structures which are defined by collections $\mathcal{A}$, $\mathcal{A}'$ with common maximal extension.

Corresponding to the concept of $(\mathbf{R}^q, \mathbf{R}^n)$-bundles there is a refinement of $\mathbf{R}^q$-microbundles which will be called $(\mathbf{R}^q, \mathbf{R}^n)$-microbundles and which we now define. Given two local trivializations $\Phi_1: V_1\approx U_1\times\mathbf{R}^q$, $\Phi_2: V_2\approx U_2\times\mathbf{R}^q$, their composite $\Phi_2\Phi_1^{-1}$ (whenever defined) is homeomorphism of some open neighbourhood of $(U_1\cap U_2)\times 0$ in $(U_1\cap U_2)\times\mathbf{R}^q$ onto another, whose restriction to any fibre maps origin to origin. Consider the slices of these neighbourhoods that lie in $(U_1\cap U_2)\times\mathbf{R}^n$. An $(\mathbf{R}^q, \mathbf{R}^n)$-*microbundle*, $0\leqslant n\leqslant q$, is an $\mathbf{R}^q$-microbundle all of whose maps $\Phi_2\Phi_1^{-1}$ sends $\mathbf{R}^n$-slice onto $\mathbf{R}^n$-slice. Any $(\mathbf{R}^q, \mathbf{R}^n)$-microbundle contains a canonical submicrobundle with standard fibre $\mathbf{R}^n$. It is constructed as follows. For every local trivialization $\Phi: V\approx U\times\mathbf{R}^q$ of $\bar{\mathcal{A}}$ (the maximal extension of $\mathcal{A}$ respecting $\mathbf{R}^n$-slices) form the restriction $\Phi': V'\approx U\times\mathbf{R}^n$ and the space $E'=\bigcup V'$. If $X\xrightarrow{s}E\xrightarrow{p}X$ is the given microbundle, E' is a subspace of E containing the image of s. The maps induced from s and p then defines an $\mathbf{R}^n$-microbundle $X \xrightarrow{s'} E' \xrightarrow{p'} X$ with local trivializations Φ', Φ running through $\mathcal{A}$ (say). Microbundles will be labelled by small greek letters $\mu, \nu, \ldots$. $\mathbf{R}^q$-bundles and D^q-bundles are obviously microbundles, and so are S^q-bundles provided they have sections.

Given a diagram $X \xrightarrow{s} E \xrightarrow{p} X$ with $ps = id_X$, a *microbundle neighbourhood* of sX in E is a neighbourhood E' such that $X \xrightarrow{s'} E' \xrightarrow{p'} X$ is a microbundle, s' and p' being induced from s and p. We consider some examples.

1.6. For any X there is the standard trivial microbundle $\varepsilon^q: X \xrightarrow{\times 0} X \times \mathbf{R}^q \xrightarrow{pr} X, q \geqslant 0$. A partition of unity for this bundle is defined by the single constant function $\pi: X \to [0, 1]$ with value 1.

1.7. Let X be a denumerable set and let $x_0 \in X$ be a fixed element. Define a topology on X by requiring $U \subset X$ to be open if U is empty or contains x_0. Then X is connected, and so any continuous function $f: X \to \mathbf{R}$ is constant. Consider the standard trivial microbundle $\varepsilon: X \to X \times \mathbf{R} \to X$. Let $U_1 \subset U_2 \subset \ldots$ be a strictly increasing infinite sequence of open sets exhausting X and let $E = \bigcup U_i \times (-1/i, 1/i)$. Then E is an open neighbourhood but not a microbundle neighbourhood of $X \times 0$ in $X \times \mathbf{R}^q$. In fact if E was a microbundle neighbourhood for $X \times 0$, then there should exist a partition of unity (π_j, W_j) of X and homeomorphisms $\Phi_j: V_j \approx W_j \times \mathbf{R}$ as prescribed by the local triviality condition. For all j define $f_j: W_j \to \mathbf{R}$ by $f_j(x) = pr_2 \Phi_j^{-1}(x, 1)$. Then f_j is a positive continuous function and hence so is $f = \Sigma \pi_j f_j: X \to \mathbf{R}$, where $\pi_j f_j$ is defined to be 0 outside W_j. Then, on one hand, f is constant, on the other hand, the set $\{(x, t) \mid (x, t) \in X \times \mathbf{R} \text{ and } t \leqslant f(x)\}$ must be contained in E. This is clearly impossible since U cannot contain proper product sets over X.

1.8. Let X be paracompact. Then any neighbourhood E of $X \times 0$ in $X \times \mathbf{R}^q$ is a microbundle neighbourhood.

Recall the definition (and notation) of a germ of continuous maps between topological pairs (cf. [8] p. 65). We now adapt the definition of bundle mapgerms in [8] to our case. Given $\mathbf{R}^q$-microbundles $\mu: X \xrightarrow{s} E \xrightarrow{p} X$, $\nu: Y \xrightarrow{t} F \xrightarrow{q} Y$ a germ $\phi: (E, sY) = (F, tY)$ is a *bundle mapgerm* or simply a *mapgerm*, if the following is true: There is a microbundle neighbourhood V of sX in E and a representative Φ of ϕ on V such that Φ maps each fibre in V injectively into some fibre in F. For $(\mathbf{R}^q, \mathbf{R}^n)$-microbundles we require in addition that Φ map the $\mathbf{R}^n$-submicrobundle of μ in V into the submicrobundle of ν. If $X = Y$ and ϕ covers the identity map, then ϕ is called an *isogerm*. If moreover $\mu = \nu$, then ϕ is called an *autogerm*. For every non-negative integer q there is a category of $\mathbf{R}^q$-microbundles and mapgerms and, for fixed X, a subcategory of $\mathbf{R}^q$-microbundles and isogerms over X. In the latter all morphisms are isomorphisms as will follow from the results of section 3. If μ is a microbundle over X, any map $f:Z\to X$ induces functorially a microbundle $f^*\mu$ of the same type over Z and a mapgerm $\bar{f}:f^*\mu\Rightarrow\mu$. If $A\subset X$, then the microbundle induced by the inclusion map is written $\mu|A$, and if $\nu:\mu\Rightarrow\nu$ is a mapgerm covering the identity, we write $\nu|A$: $\mu|A\Rightarrow\nu|A$ for the mapgerm defined by ν. A subset A of X is *trivializing for* μ if there exists a (global) trivialization of $\mu|A$, $V\approx A\times\mathbf{R}^q$ (assuming that μ is an $\mathbf{R}^q$-microbundle). The trivialization defines an isogerm $\mu|A\Rightarrow\varepsilon^q(A)$ so that $\mu|A$ gets isomorphic to the standard trivial microbundle.

Next we give an important example of autogerms on the standard trivial bundle.

1.9. Let $X\xrightarrow{s}E\xrightarrow{p}X$ be a trivial $\mathbf{R}^q$-microbundle with two global trivializations $\Phi_k:E_k\approx X\times\mathbf{R}^q$, $k=1,2$, fixed throughout this section. We want to show that the maps $\Phi_2\Phi_1^{-1}$, $\Phi_1\Phi_2^{-1}$ define autogerms on $\varepsilon^q(X)$. Since X is not assumed paracompact, this is not at all obvious. In fact we will show that there exist positive continuous functions $\varrho_{12},\varrho_{21}$ on X whose associated neighbourhoods $N(\varrho_{12}), N(\varrho_{21})$ (defined below) are contained in the domains of $\Phi_2\Phi_1^{-1}$, $\Phi_1\Phi_2^{-1}$, resp. The interior of any functional neighbourhood of $X\times 0$, however, is homeomorphic to $X\times\mathbf{R}^q$ by a fibre preserving homeomorphism which is the identity on some smaller functional neighbourhood. Such a homeomorphism is certainly a microbundle trivialization of $\varepsilon^q(X)$. It follows that the maps $\Phi_i\Phi_j^{-1}$ define autogerms.

If $f:X\to\mathbf{R}$ is a positive continuous function, define $N(f)$, $\mathring{N}(f)$ and $\dot{N}(f)$ by

$$(x,v)\in N(f)\Leftrightarrow(x,v)\in X\times\mathbf{R}^q \text{ and } \|v\|\leqslant f(x),$$
$$(x,v)\in \mathring{N}(f)\Leftrightarrow(x,v)\in X\times\mathbf{R}^q \text{ and } \|v\|< f(x),$$
$$(x,v)\in \dot{N}(f)\Leftrightarrow(x,v)\in X\times\mathbf{R}^q \text{ and } \|v\|= f(x).$$

Then $N(f)$ is a closed neighbourhood of $X\times 0$ in $X\times\mathbf{R}^q$ with interior $\mathring{N}(f)$ and boundary $\dot{N}(f)$. If f is constant equal to 1, write $N(f)=N(1)=N$. Thus $N=X\times D^q$, $D^q=D^q(1)$ being the closed unit disk centered at the origin in $\mathbf{R}^q$. In general $N(t)=X\times D^q(t)$ for any real number $t>0$. (Later we shall also make use of these definitions in the case where f takes values in the extended real halfline $[0,\infty]$.)

Define continuous realvalued functions Θ_{ij} on the domain of $\Phi_i\Phi_j^{-1}$, $i,j=1,2$, $i\neq j$, by

$$\Theta_{ij}(x,v)=\|pr_2\Phi_i\Phi_j^{-1}(x,v)\|.$$

Moreover, let $N_j=\Phi_j(\Phi_1^{-1}N\cap\Phi_2^{-1}N)$ and $\dot{N}_j=\Phi_j((\Phi_1^{-1}N\cap\Phi_2^{-1}\dot{N})\cup(\Phi_1^{-1}\dot{N}\cap\Phi_2^{-1}N))$, $j=1,2$. Then N_j is a neighbourhood of $X\times 0$ in $X\times\mathbf{R}^q$ contained in $N\cap$ domain $\Phi_i\Phi_j^{-1}$, and $\dot{N}_j$ is contained in the boundary of N_j in N. Also, for any $x\in X$, $N_j|x$ is a compact neighbourhood of $(x,0)$ in $N|x$ with boundary $\dot{N}_j|x$.

Finally define the realvalued functions ϱ_{ij} on X, $i \neq j$, by

$$\varrho_{ij}(x) = \inf \{\Theta_{ij}(x, v) \mid (x, v) \in \dot{N}_j,\ x \text{ fixed}\}.$$

Geometrically, $\varrho_{ij}(x)$ is measuring the minimal radius in the cross-section of N_j over x. We claim that ϱ_{ij} is a positive continuous function and that $N(\varrho_{ij}) \subset \text{domain } \Phi_j \Phi_i^{-1}$.

Since Θ_{ij} is continuous and positive on the compact set $\dot{N}_j|x$ for any x, $\varrho_{ij}(x) > 0$. Hence ϱ_{ij} is positive. We show that ϱ_{ij} is continuous.

First notice that if W is a neighbourhood of a point x in X, then $N_j|W$ is a neighbourhood of $N_j|x$ in N_j. Moreover, the collection $\{N_j|W\}$ where W runs over the neighbourhoods of x forms a fundamental system for $N_j|x$ in N_j. To see this let $M = X \times B$, where B is some open ball of radius >1 at the origin in $\mathbf{R}^q$. Then, varying B and W, $\{M|W\}$ forms a fundamental system of $N|x$ in $X \times \mathbf{R}^q$. Thus $\{\Phi_i^{-1}(M|W)\}$ forms a fundamental system of neighbourhoods of $\Phi_i^{-1}(N|x)$ in E, $i = 1, 2$. But then $\{\Phi_1^{-1}(M|W) \cap \Phi_2^{-1}(M|W)\}$ forms a fundamental system of $\Phi_1^{-1}(N|x) \cap \Phi_2^{-1}(N|x)$ in E ($\Phi_1^{-1}(N|x) \cap \Phi_2^{-1}(N|x)$ being compact) whose trace on $\Phi_1^{-1}N \cap \Phi_2^{-1}N$ is $\{\Phi_1^{-1}(N|W) \cap \Phi_2^{-1}(N|W)\}$. Thus

$$\{\Phi_1^{-1}(N|W) \cap \Phi_2^{-1}(N|W)\},$$

W varying, forms a fundamental system of $\Phi_1^{-1}(N|x) \cap \Phi_2^{-1}(N|x)$ in $\Phi_1^{-1}N \cap \Phi_2^{-1}N$. Since $\Phi_j(\Phi_1^{-1}(N|W) \cap \Phi_2^{-1}(N|W)) = N_j|W$, the claim follows. The statement ramains true if N_j is replaced with $\dot{N}_j$ everywhere. In fact the latter follows from the former or from a similar direct reasoning. Since $pr_1 \dot{N}_j = X$, this means that $pr_1 : \dot{N}_j \to X$ is an open map.

Let x_0 be any element of X. By the compactness of $\dot{N}_j|x_0$, given $\varepsilon > 0$ there is a finite collection of open balls $B_1, B_2, \ldots, B_r$ in $\mathbf{R}^q$ centered at some $v_1, v_2, \ldots, v_r$ with $(x_0, v_k) \in \dot{N}_j|x_0$ and a neighbourhood W_0 of x_0 in X, such that $\dot{N}_j|x_0 \subset W_0 \times \bigcup B_k \subset \text{domain } \Phi_i \Phi_j^{-1}$ and such that for $(x, v) \in W_0 \times B_k$ $|\Theta(x, v) - \Theta(x_0, v_k)| < \varepsilon$, $k = 1, 2, \ldots, r$. Then $\dot{N}_j \cap (W_0 \times \bigcup B_k)$ is a neighbourhood of $\dot{N}_j|x_0$ in $\dot{N}_j$ (not necessarily filling $\dot{N}_j|W_0$) and so there is a neighbourhood W_0' of x_0 with $\dot{N}_j|W_0' \subset \dot{N}_j \cap (W_0 \times \bigcup B_k)$. It follows that for $(x, v) \in \dot{N}_j|W_0'$ there is an integer s with $|\Theta_{ij}(x, v) - \Theta_{ij}(x_0, v_s)| < \varepsilon$, i.e. $\varrho_{ij}(x_0) - \varepsilon \leqslant \Theta_{ij}(x_0, v_s) - \varepsilon < \Theta_{ij}(x, v) < \Theta_{ij}(x_0, v_s) + \varepsilon$. Therefore $x \in W_0'$ implies $\varrho_{ij}(x_0) - \varepsilon < \varrho_{ij}(x)$. On the other hand, any $(x_0, v_0) \in \dot{N}_j|x_0$ has a neighbourhood U_0 in $\dot{N}_j$ such that $(x, v) \in U_0$ implies $\Theta_{ij}(x, v) < \Theta_{ij}(x_0, v_0) + \varepsilon$. Since $pr_1 : \dot{N}_j \to X$ is an open map, $W'' = pr_1 U_0$ is a neighbourhood of x_0 in X. It follows that for $x \in W_0''$ $\varrho_{ij}(x) < \Theta_{ij}(x_0, v_0) + \varepsilon$. In particular if (x_0, v_0) was chosen a point of minimum of $\Theta_{ij}|(\dot{N}_j|x_0)$, $\varrho_{ij}(x) < \varrho_{ij}(x_0) + \varepsilon$. Thus, given $\varepsilon > 0$ there exist neighbourhoods $W_0' \cap W_0''$ of x_0 such that $x \in W_0' \cap W_0''$ implies $|\varrho_{ij}(x) - \varrho_{ij}(x_0)| < \varepsilon$. Hence ϱ_{ij} is continuous.

By definition $N(\varrho_{ij}) \subset N_i \subset \text{domain } \Phi_j \Phi_i^{-1}$.

Finally, if $f:X \to \mathbf{R}$ is a positive continuous function, let us show that there exists a fibre preserving homeomorphism $\Psi: \mathring{N}(f) \approx X \times \mathbf{R}^q$ which is the identity on $N(f/2)$. We may assume that f is bounded by $1/2$, say. Form the map $F: X \times \mathbf{R}^q \to \mathbf{R}$

$$F(x, v) = \begin{cases} 1, & \|v\| \leqslant f(x)/2 \\ (2/f(x))\,(f(x) - \|v\|) + (2/f(x)^2)\,(\|v\| - f(x)/2), & f(x)/2 \leqslant \|v\| \leqslant f(x) \\ 1/f(x) & f(x) \leqslant \|v\|. \end{cases}$$

Then $F(x, v)$ satisfies the conditions of Lemma 1.4. Therefore the map $(x, v) \mapsto (x, F(x, v)v)$ is a bundle automorphism of $\varepsilon^q(X)$ which sends $\mathring{N}(f)$ onto $\mathring{N}$ and is the identity on $N(f/2)$. Compose the latter with a bundle isomorphism $\mathring{N} \approx X \times \mathbf{R}^q$ which is the identity on $N(1/4)$. This composite gives Ψ.

The discussion above could easily have been generalized by starting with a tubular neighbourhood $N(t) = X \times D^q(t)$ of radius $t > 0$. This would have given a positive continuous function $\varrho(x, t)$ of two variables such that for t fixed $N(\varrho(\cdot, t)) \subset N(t) \subset \text{domain}\ \Phi_2 \Phi_1^{-1}$. The following is really a special case of this situation. The proof is easier than that in 1.9 and will be omitted.

1.10. Lemma. *Let Φ be a bundle embedding of a trivial bundle $\varepsilon^q(X)$ in itself and define the realvalued functions ϱ_Φ, σ_Φ on $X \times (0, \infty)$ by*

$$\varrho_\Phi(x, t) = \inf \{\|pr_2 \Phi(x, v)\| \mid v \in \dot{D}^q(t), \quad x, t \text{ fixed}\}$$

$$\sigma_\Phi(x, t) = \sup \{\|pr_2 \Phi(x, v)\| \mid v \in \dot{D}^q(t), \ x, t \text{ fixed}\}$$

Then ϱ_Φ, σ_Φ are positive continuous functions.

Note that if Φ is only defined on a neighbourhood of $X \times D^q(t)$ for some $t > 0$, then, of course, the lemma remains valid when ϱ_Φ, σ_Φ are considered as functions on $X \times (0, t]$.

2. The germ extension theorem

This section is concerned about representing autogerms on the standard trivial $\mathbf{R}^q$-bundle by automorphisms.

2.1. Lemma. *Let ϕ be an autogerm on a trivial bundle ε^q. There is a bundle embedding of ε^q in itself whose germ is ϕ.*

Proof. Let X be the base of ε^q and let Φ be a representative of ϕ defined on a microbundle neighbourhood E' of $X \times 0$. Let $(\pi_j, W_j)_{j \in J}$ be a trivializing partition of unity for E'. Then there are local trivializations $\Phi_j: V_j \approx W_j \times \mathbf{R}^q$, $j \in J$, with $V_j \subset E'$. Since Φ_1^{-1} is a bundle embedding of $\varepsilon^q(W_j)$ in itself, by Lemma 1.10 there is a positive continuous function $f_j = \varrho_{\Phi_j^{-1}}(\cdot, 1)$ on W_j such that $N(f_j) \subset V_j \subset E'$. Form the function $f = \Sigma \pi_j \cdot f_j$, where $\pi_j \cdot f_j$ is defined to be zero outside W_j. Then f is positive and continuous on X, and $N(f)$ is a neighbourhood of $X \times 0$ contained in E'. We may assume f bounded by $\frac{1}{2}$. Let F: $X \times \mathbf{R}^q \to \mathbf{R}$ be the function constructed in the end of example 1.9 and let $\Psi_1 : \mathring{N}(f) \approx \mathring{N}$ be the homeomorphism $(x, v) \mapsto (x, F(x, v)v)$. Finally, let $\Psi_2 : \mathring{N} \approx X \times \mathbf{R}^q$ be some bundle isomorphism which is the identity on $N(\frac{1}{4})$. Then $\Phi \Psi_1^{-1} \Psi_2^{-1}$ is a bundle embedding of ε^q that coincides with Φ on $N(f/2)$.

Notice that the proof relies heavily on the existence of a trivializing partition of unity for E' and fails to cover the case where ϕ is just an autogerm in the sense of Milnor. We next complete the above result by showing that any bundle embedding of ε^q has the germ of an automorphism. The proof is based on the Hirsch–Mazur induction technique.

2.2. Lemma. *Let Φ be a bundle embedding of a trivial bundle ε^q in itself. There is a bundle automorphism Φ' on ε^q such that germ $\Phi' =$ germ Φ.*

Proof. Let X be the base of ε^q. Using the map $\varrho_\Phi(\cdot, 1) : X \to \mathbf{R}$ as we did with f in 1.9, we construct a map $F' : X \times \mathbf{R}^q \to \mathbf{R}$ such that Ψ',

$$\Psi'(x, v) = (x, F'(x, v)v),$$

is a bundle automorphism of ε^q with germ the identity, which maps $N(\varrho_\Phi(\cdot, 1)$ onto $N(2)$. Then $\Psi' \Phi = \Phi_1$ is a bundle embedding with germ $\Phi_1 =$ germ Φ and such that $\Phi_1 N(1) \supset N(2)$. We now proceed by induction. With Φ_1 as described and $\Phi_0 = \Phi$, suppose we have constructed embeddings $\Phi_1, \Phi_2, \dots, \Phi_n : \varepsilon^q \to \varepsilon^q$, $n \geqslant 1$, such that

I_n: $\quad \Phi_i N(i) \supset N(i+1), \qquad i = 1, 2, \dots, n,$

II_n: $\quad \Phi_i | N(i-1) = \Phi_{i-1} | N(i-1), \qquad i = 1, 2, \dots, n.$

Since Φ_n^{-1} is defined on $N(n+1)$, by Lemma 1.10 there is a positive continuous function $\varrho_{\Phi_n^{-1}}(\cdot, n) : X \to \mathbf{R}$ measuring the minimal radii of $\Phi_n^{-1} N(n)$. Let $g = \min(\varrho_{\Phi_n^{-1}}(\cdot, n), n - \frac{1}{2})$. Then $\Phi_n N(g) \subset N(n)$.

Similarly, Φ_n defines a positive continuous function $\sigma_{\Phi_n}(\cdot, n+1) : X \to \mathbf{R}$. Let $h = \max(\sigma_{\Phi_n}(\cdot, n+1), n+2)$. Then $\Phi_n N(n+1) \subset N(h)$.

Now, let $G, H : X \times \mathbf{R}^q \to \mathbf{R}$ be the positive continuous functions

$$G(x, v) = \begin{cases} g(x)/n, & \|v\| \leqslant n \\ (\|v\| - n) + (g(x)/n)(n + 1 - \|v\|), & n \leqslant \|v\| \leqslant n+1 \\ 1, & n+1 \leqslant \|v\|, \end{cases}$$

$$H(x,v)=\begin{cases}1, & \|v\|\leqslant n\\ (h(x)/n+1)(\|v\|-n)+(n+1-\|v\|), & n\leqslant\|v\|\leqslant n+1\\ h(x)/n+1, & n+1\leqslant\|v\|.\end{cases}$$

Then the maps

$$\Psi(x, v) = (x, G(x, v)v), \qquad \Theta(x, v) = (x, H(x, v)v)$$

are bundle automorphisms (Lemma 1.4) such that $\Psi N(n)=N(g)$, $\Psi N(n+1)=N(n+1)$ and Ψ equals the identity outside $N(n+1)$, and such that $\Theta N(n+1)=N(h)$ and Θ is the identity on $N(n)$. Finally let $\Gamma: X\times\mathbf{R}^q\to X\times\mathbf{R}^q$ be the map which equals $\Phi_n\Psi^{-1}\Phi_n^{-1}$ on $\Phi_n N(n+1)$ and the identity outside. Then Γ is clearly a bundle automorphism. Moreover, since $N(h)=\Phi_n N(n+1)\cup(N(h)-\Phi_n N(n+1))$ and $\Gamma\Phi_n N(n+1)=\Phi_n N(n+1)$, we have $\Gamma N(h)=N(h)$. Define Φ_{n+1} to equal the composite embedding $\Gamma\Theta\Phi_n\Psi$. Then Φ_{n+1} satisfies the induction conditions:

$$I_{n+1}:\quad \Phi_{n+1}N(n+1)=\Gamma\Theta\Phi_n\Psi N(n+1)=\Gamma\Theta\Phi_n N(n+1)\supset\Gamma\Theta\Phi_n N(n)\supset\Gamma\Theta N(n+1)$$
$$=\Gamma N(h)=N(h)\supset N(n+2).$$

II_{n+1}: If $(x,v)\in N(n)$, then $\Psi(x,v)\in N(g)$ and so $\Phi_n\Psi(x,v)\in\Phi_n N(g)\subset N(n)$. Therefore $\Theta\,\Phi_n\Psi(x,v)=\Phi_n\Psi(x,v)$ and $\Gamma\,\Theta\,\Phi_n\Psi(x,v)=\Gamma\,\Phi_n\Psi(x,v)$. Since $\Phi_n\Psi(x,v)\in N(n)\subset\Phi_n N(n)\subset\Phi_n N(n+1)$, $\Gamma\Phi_n\Psi(x,v)=\Phi_n\Psi^{-1}\Phi_n^{-1}\Phi_n\Psi(x,v)=\Phi_n(x,v)$. Thus $\Phi_{n+1}(x,v)=\Phi_n(x,v)$.

Hence, by induction there exists a sequence of bundle embeddings $\Phi_1, \Phi_2, \ldots : X\times\mathbf{R}^q\to X\times\mathbf{R}^q$ with germ $\Phi_1=$ germ Φ and such that Φ_i satisfies I_i and II_i, $i=1, 2, \ldots$. By the conditions II_i there is a limit bundle map $\Phi': X\times\mathbf{R}^q\to X\times\mathbf{R}^q$ defined by $\Phi'|N(i)=\Phi_i|N(i)$, $i=1, 2, \ldots$, which is clearly an embedding, and by the conditions I_i Φ is epic and therefore actually an automorphism. Finally germ $\Phi'=$ germ $\Phi_1=$ germ Φ.

Combining 2.1 and 2.2 we get the following important result.

2.3. Germ extension theorem. *Let ϕ be an autogerm on a trivial bundle ε^q. There is a bundle automorphism on ε^q whose germ is ϕ.*

In particular the germ extension theorem applies to the transition autogerms of example 1.9. In fact the proofs show that if $\Phi_i\Phi_j^{-1}$ is a representative of such a germ on ε^q, then there is an automorphism Φ on ε^q which coincides with $\Phi_i\Phi_j^{-1}$ on some functional neighbourhood $N(f)$ of the zero section.

3. The microbundle representation theorem

The result below is an important step in the inductive arguments to follow in this section.

3.1. LEMMA. *Let ξ be an $\mathbf{R}^q$-bundle with a trivializing partition of unity $(\pi_i, W_i)_{i=1,2}$, and let $\phi:\xi \Rightarrow \varepsilon^q$ be an isogerm to the trivial bundle. Let $\Phi_1:\xi\,|\,W_1 \approx \varepsilon^q\,|\,W_1$ be a trivialization of ξ over W_1 whose germ is $\phi\,|\,W_1$. There is a trivialization $\Phi:\xi\approx\varepsilon^q$ of ξ whose germ is ϕ such that $\Phi\,|\,W_1 - W_2 = \Phi_1\,|\,W_1 - W_2$.*

Proof. Consider the isogerm $\phi\,|\,W_2:\xi\,|\,W_2 \Rightarrow \varepsilon^q\,|\,W_2$. Since $\xi\,|\,W_2$ is trivial, by the germ extension theorem there is a trivialization $\Phi_2:\xi\,|\,W_2\approx\varepsilon^q\,|\,W_2$ whose germ is $\phi\,|\,W_2$. With $(\pi_i', W_i')_{i=1,2}$ the derived partition of unity of $(\pi_i, W_i)_{i=1,2}$, let $A = W_2 - W_2'$, $U = W_1 \cap W_2$. Then A and U are subsets of W_2, and with $\pi = \pi_1'\,|\,W_2$, we have $A = \pi^{-1}(1)$ and $A \subset \pi^{-1}(0,1] \subset U$. Since $\{W_1 - \overline{W}_2', W_2\}$ forms an open covering of the base, it suffices to construct a trivialization Ψ_2 of $\xi\,|\,W_2$ such that germ $\Psi_2 =$ germ Φ_2 and $\Psi_2\,|\,A = \Phi_1\,|\,A$. Translate first the whole problem to $W_2 \times \mathbf{R}^q$ by means of Φ_2. Thus we have to construct an automorphism $\Psi_2' = \Psi_2\Phi_2^{-1}$ of $\varepsilon^q(W_2)$ such that germ $\Psi_2' =$ germ identity and $\Psi_2'\,|\,A = \Phi_1\Phi_2^{-1}\,|\,A$. Write Φ' for $\Phi_1\Phi_2^{-1}\,|\,U$. There is a map $\Pi: W_2 \times \mathbf{R}^q \to W_2 \times \mathbf{R}^q$ defined by $\Pi(x, v) = (x, \pi(x)v)$ which is a bundle automorphism over $\pi^{-1}(0, 1]$. Define Ψ_2' by

$$\Psi_2' = \begin{cases} \Pi^{-1}\Phi'\Pi & \text{on } \pi^{-1}(0,1] \times \mathbf{R}^q \\ \text{identity} & \text{outside.} \end{cases}$$

Then Ψ_2' is bijective, and it is bicontinuous except possibly over the boundary of $\pi^{-1}(0, 1]$ in W_2. Suppose $((x_\gamma, v_\gamma))$ is a generalized sequence in $\pi^{-1}(0, 1] \times \mathbf{R}^q$ converging to some point (x, v) in the boundary in $W_2 \times \mathbf{R}^q$ (thus $(x, v) \in U \times \mathbf{R}^q$). Then ultimately $\Pi(x_\gamma, v_\gamma)$ belongs to the neighbourhood of $W_2 \times 0$ (and of $(x, 0)$) where Φ' equals the identity. But then $\Psi_2'(x_\gamma, v_\gamma) = (x_\gamma, v_\gamma)$. Thus $\lim \Psi_2'(x_\gamma, v_\gamma) = (x, v) = \Psi_2'(x, v)$ showing that Ψ_2' is continuous also at (x, v).

Suppose that $((x_\gamma, v_\gamma))$ is a generalized sequence of elements from $\pi^{-1}(0, 1] \times \mathbf{R}^q$ such that $(\Psi_2'(x_\gamma, v_\gamma))$ converges to some element $\Psi_2'(x, v) = (x, v)$ in the boundary. We have $\Psi_2'(x_\gamma, v_\gamma) = (x_\gamma, 1/\pi(x_\gamma)\cdot pr_2\Phi'(x_\gamma, \pi(x_\gamma)\cdot v_\gamma))$. Since $1/\pi(x_\gamma) \to \infty$, $pr_2\Phi'(x_\gamma, \pi(x_\gamma)\cdot v_\gamma) \to 0$ and so $\Phi'(x_\gamma, \pi(x_\gamma)\cdot v_\gamma) \to (x, 0)$. Since Φ' is a trivialization defined on all of $U \times \mathbf{R}^q$, this implies that $\pi(x_\gamma)v_\gamma \to 0$. But then ultimately $\Phi'(x_\gamma, \pi(x_\gamma)\cdot v_\gamma) = (x_\gamma, \pi(x_\gamma)\cdot v_\gamma)$ and therefore ultimately $\Psi_2'(x_\gamma, v_\gamma) = (x_\gamma, v_\gamma)$. Thus $\lim\,(x_\gamma, v_\gamma) = \lim \Psi_2'(x_\gamma, v_\gamma) = \Psi_2'(x, v) = (x, v)$ showing that $\Psi_2'^{-1}$ is continuous at $\Psi_2'(x, v)$.

3.2. COROLLARY. *Let ξ be an $\mathbf{R}^q$-bundle, and let $\phi:\xi\Rightarrow\varepsilon^q$ be an isogerm to the trivial bundle. There is a trivialization $\Phi:\xi\approx\varepsilon^q$ whose germ is ϕ.*

Proof. Let $(\pi_j, W_j)_{j\in J}$ be a trivializing partition of unity for ξ. By the germ extension theorem (or by the lemma above) for any $j \in J$ there is a trivialization of ξ over W_j

$$\xi \mid W_j \approx \varepsilon^q(W_j)$$

whose germ is $\phi \mid W_j$. Consider then the collection of pairs (K, Φ) where $K \subset J$ is non-empty and $\Phi: \xi \mid W_K \approx \varepsilon^q(W_K)$ is a trivialization of ξ over W_K with germ $\phi \mid W_K$. Order this collection by defining $(K, \Phi) \leqslant (K', \Phi')$ if the following two conditions are satisfied:

(a) $K \subset K'$.

(b) If $x \in W_K$ and $\pi_K(x) = \pi_{K'}(x)$, then $\Phi \mid x = \Phi' \mid x$.

It is clear that this relation is an ordering i.e. that it is reflexive and transitive. Suppose (K_r, Φ_r), $r \in R$, are the members of a linearly ordered family of pairs. Let $K = \bigcup K_r$ and let $x \in W_K$. There is a neighbourhood $V(x)$ of x in W_K meeting only a finite number of W_i with $i \in K$, say $i_1, i_2, \dots, i_n$. Let r be such that all $i_1, i_2, \dots, i_n$ is contained in K_r. Then for $(K_r, \Phi_r) \leqslant (K_s, \Phi_s)$, $\pi_{K_s} \mid V(x) = \pi_{K_r} \mid V(x)$ and so $\Phi_s \mid V(x) = \Phi_r \mid V(x)$. In other words, if V is a sufficiently small open subset of W_K, then ultimately all Φ_r coincide over V. It follows that there is a well-defined isomorphism $\Phi = \lim \Phi_r : \xi \mid W_K \approx \varepsilon^q(W_K)$. Thus the pair (K, Φ) belongs to our collection, and it is moreover an upper bound for the family (K_r, Φ_r), $r \in R$. It follows that the collection of pairs is in fact inductively ordered and so contains maximal elements. Let (K, Φ) be such an element. If $K \neq J$, let $j \in J - K$ and write $K' = K \cup \{j\}$. Form the partition of unity $(\pi_i, W_i)_{i=1,2}$ on $W_{K'}$ with $\pi_1 = \pi_K/\pi_{K'}$, $\pi_2 = \pi_j/\pi_{K'}$. Then $W_1 = W_K$ and $W_2 = W_j$ and so the bundle $\xi \mid W_{K'}$ satisfies the conditions of Lemma 3.1 with Φ_1 equal Φ in (K, Φ) above and ϕ equal $\phi \mid W_{K'}$. It follows from 3.1 then that there is a trivialization Φ' of $\xi \mid W_{K'}$ whose germ is $\phi \mid W_{K'}$ such that $(K, \Phi) \leqslant (K', \Phi')$. Since $(K', \Phi') \neq (K, \Phi)$ this contradicts the maximality of (K, Φ). Therefore we must have $K = J$, and so $\Phi: \xi \approx \varepsilon^q$ is the trivialization claimed.

We are now ready to prove the result towards which we have been heading.

3.3. Microbundle representation theorem. (a) *Let μ be an $\mathbf{R}^q$-microbundle over some space X. Let V be a halo of a set A in X, and let η be an $\mathbf{R}^q$-bundle over V contained in $\mu \mid V$. There is an $\mathbf{R}^q$-bundle ξ over X contained in μ such that $\xi \mid V' = \eta \mid V'$ for some halo $V' \subset V$ of A.*

(b) *Let ξ_1, ξ_2 be $\mathbf{R}^q$-bundles over X contained in the microbundle μ such that $\xi_1 \mid V' = \xi_2 \mid V'$ for some halo V' of A in X. There is a bundle isomorphism $\xi_1 \approx \xi_2$ which is the identity over some halo $V'' \subset V'$ of A.*

Proof. Let $(\pi_i, W_i)_{i\in J}$ be a trivializing partition of unity for μ. Consider the collection of all triples $(K, \xi, (\Phi_k))$, where K is a non-empty subset of J, ξ is an $\mathbf{R}^q$-bundle over W_K

contained in $\mu | W_K$ and (Φ_k) is a family of trivializations $\Phi_k : \xi | W_K \approx \varepsilon^q(W_k)$, $k \in K$. Order this collection by defining $(K, \xi, (\Phi_k)) \leqslant (K', \xi', (\Phi'_k))$ if the following two conditions are satisfied:

(a) $K \subset K'$.

(b) If $x \in W_K$ and $\pi_K(x) = \pi_{K'}(x)$, then
$\xi | x = \xi' | x$ and $\Phi_k | x = \Phi'_k | x$ for any $k \in K$ with $x \in W_k$.

Clearly this relation is an ordering. We show that the ordering is in fact inductive. Let $(K_r, \xi_r, (\Phi^r_i))$, $r \in R$, be the members of some linearly ordered family of triples. Let $K = \bigcup K_r$ and let $x \in W_K$. There is a neighbourhood $V(x)$ of x in W_K meeting a finite number of W_K with $k \in K$, say $k_1, k_2, \ldots, k_n$. Let r be such that all $k_1, k_2, \ldots, k_n$ belong to K_r. Then, for $(K_s, \xi_s, (\Phi^s_j))$ succeeding $(K_r, \xi_r, (\Phi^r_i))$, $\pi_{K_s} | V(x) = \pi_{K_r} | V(x)$ and so $\xi_s | V(x) = \xi_r | V(x)$. If moreover $V(x)$ is contained in W_k for some $k \in K$, then $\Phi^s_k | V(x) = \Phi^r_k | V(x)$. It follows that there is a bundle $\xi = \lim_r \xi_r$ over W_K contained in $\mu | W_K$ with trivializations $\Phi_k = \lim_r \Phi^r_k$ over W_k, $k \in K$. Obviously $(K, \xi, (\Phi_k))$ is an upper bound for the family $(K_r, \xi_r, (\Phi^r_i))$, $r \in R$. Thus, by Zorn's lemma the collection of triples admits maximal elements. Let $(K, \xi, (\Phi_k))$ be such an element. We show that $K = J$ which will prove that ξ is a bundle over X contained in μ.

Suppose $K \neq J$ and let $j \in J - K$. Since W_j is trivializing for μ, there is a bundle η over W_j contained in $\mu | W_j$ and a trivialization $\Psi_j : \eta \approx \varepsilon^q(W_j)$. Then $(\{j\}, \eta, \Psi_j)$ is a triple in our collection. We glue together $(K, \xi, (\Phi_k))$ and $(\{j\}, \eta, \Psi_j)$ to a larger triple, thereby reaching a contradiction: Since $\xi | W_K \cap W_j$ and $\eta | W_K \cap W_j$ are both contained in $\mu | W_K \cap W_j$, their total spaces have a non-empty intersection E. It follows from example 1.9 that E is a microbundle neighbourhood of the zero-section in $\xi | W_K \cap W_j$. Therefore its identity map determines an isogerm $\xi | W_K \cap W_j \Rightarrow \eta | W_K \cap W_j$. Since η is trivial, by Lemma 3.2 there is a bundle isomorphism $\Theta : \xi | W_K \cap W_j \approx \eta | W_K \cap W_j$ with germ the identity. In particular $\xi | W_K \cap W_j$ is trivial and so, like $\eta | W_K \cap W_j$, gives rise to a trivialization of μ over $W_K \cap W_j$. Again by example 1.9 there is then a positive continuous function ϱ on $W_K \cap W_j$ such that $\Psi_j E \supset N(\varrho)$. We may suppose that Θ is the identity on $\Psi_j^{-1} N(\varrho)$. Let $K' = K \cup \{j\}$. Then $\{(\pi_K, W_K), (\pi_j, W_j)\}$ forms a partition of unity on $W_{K'}$ except for normalization. After normalizing (so that, keeping the same notation, $\pi_K + \pi_j = 1$), form the derived partition of unity $\{(\pi'_K, W'_K), (\pi'_j, W'_j)\}$. Let F_K, F_j be the subsets of $W_{K'}$ defined by

$$x \in F_K \Leftrightarrow x \in W_K \quad \text{and} \quad \pi'_K(x) \geqslant \pi_K(x)$$

$$x \in F_j \Leftrightarrow x \in W_j \quad \text{and} \quad \pi'_j(x) \geqslant \pi_j(x)$$

and let $A = F_K \cap F_j$. Since $F_K \subset W'_K \subset \overline{W}'_K$, F_K is a closed subset of $W_{K'}$. Similarly F_j and

A are closed in $W_{K'}$. Clearly $F_K \cup F_j = W_{K'}$. Let $\xi_K = \xi | F_K$, $\eta_j = \eta | F_j$ and $\Theta' = \Theta | A$. Form the bundle $\xi' = \xi_K \cup_{\Theta'} \eta_j$ which is the disjoint union of ξ_K and η_j identified over A by Θ'. That this is a bundle in the definition of section 1 follows from the fact that it has $(\pi_i, W_i)_{i \in K'}$ as trivializing partition of unity. Indeed, consider a trivialization $\Phi_i : \xi | W_i \approx \varepsilon^q(W_i)$ for $i \in K$ and let Φ_{iK}, Φ_{ij} be the restrictions of Φ_i to $W_i \cap F_K$, $W_i \cap F_j$ respectively and let Θ_{ij} be the restriction of Θ to $W_i \cap F_j$. Then the pair $\Phi_{iK}, \Phi_{ij}\Theta_{ij}^{-1}$ defines a trivialization $\Phi_i' = \Phi_{iK} \cup_{\Theta'} \Phi_{ij}\Theta_{ij}^{-1}$ of ξ' over W_i. Moreover, when restricted to $\xi' | W_i - W_j = \xi | W_i - W_j$ this trivialization equals $\Phi_i | W_i - W_j$. Similarly one shows that there is a trivialization Φ_j' of ξ' over W_j. We should like to finish the proof by concluding that the triple $(K', \xi', (\Phi_i'))$ is larger than the maximal triple $(K', \xi, (\Phi_i))$. Unfortunately the former does not belong to our collection since (because of the identification by Θ') ξ' need not be contained in $\mu | W_{K'}$. The part of ξ' corresponding to where Θ is the identity is contained in $\mu | W_{K'}$ however, as is the part of ξ' over $W_{K'} - A$. Let $\pi : W_{K'} \to [0, 1]$ be the continuous function that equals $1 - (\pi_K' \pi_j')/(\pi_K \pi_j)$ on $W_K' \cap W_j'$ and 1 outside. Then $A \subset \pi^{-1}(0)$. Pick an order preserving homeomorphism $[0, 1] \approx [0, \infty]$ of $[0, 1]$ to the extended real half-line and let π_∞ be the composite of π with this. Then π_∞ is 0 on A and ∞ outside $W_K' \cap W_j'$. Let ϱ_∞: $W_{K'} \to [0, \infty]$ be the continuous function that equals $\pi_K \pi_j \varrho$ on $W_K \cap W_j$ and 0 outside. Finally let $\varrho_\infty' = \pi_\infty + \varrho_\infty$. Then ϱ_∞' is a positive continuous function on $W_{K'}$ to $[0, \infty]$ that takes the value ∞ outside W_j' and is less than ϱ on A. It follows that $\Psi^{-1}N(\varrho_\infty' | W_j)$ is the total space of bundle η'' over W_j contained in η and contained in $\xi' | W_j$ as well (since Θ operates as the identity on the common part of ξ and η'' over A). Thus there is a well-defined bundle $\xi'' = (\xi | W_K - W_j') \cup \eta'' = (\xi' | W_K - W_j') \cup \eta''$ over $W_{K'}$ contained in ξ' and in $\mu | W_{K'}$, such that $\xi'' | W_K - W_j = \xi' | W_K - W_j = \xi | W_K - W_j$. This bundle has $(\pi_i, W_i)_{i \in K'}$ as trivializing partition of unity. In fact, an obvious modification will change Φ_i' restricted to $\xi'' | W_i$ to a trivialization $\Phi_i'' : \xi'' | W_i \approx \varepsilon^q(W_i)$, $i \in K'$, such that $\Phi_i'' | W_i - W_j = \Phi_i' | W_i - W_j = \Phi_i | W_i - W_j$ for $i \in K$. Then the triple $(K', \xi'', (\Phi_i''))$ is in fact a member of our ordered collection, which strictly majorizes $(K, \xi, (\Phi_i))$. The contradiction shows that $K = J$ and so ξ is a bundle contained in μ. This proves the first part of the theorem in the case where A is empty.

Suppose finally that over some halo V of a non-empty set A in the base a bundle η is given contained in $\eta | V$. Let ψ be a function on the base to $[0, 1]$ which is 1 on A and 0 outside V. Without loss of generality we may suppose that V actually is the support of ψ. Let W be the open subset of all x for which $\psi(x) < \frac{1}{2}$. Then there is a non-negative real valued function π with support W such that $\psi + \pi = 1$. Let $(\pi_j, W_j)_{j \in J}$ be a trivializing partition of unity for the microbundle μ and let $(\psi_l, V_l)_{l \in L}$ and $(\Psi_l)_{l \in L}$ be a trivializing partition of unity and a family of corresponding local trivializations for the bundle η. Form the subset $K \subset J$ consisting of those indicies k for which W_k meet W and the disjoint union $L \cup K$. Then the two families $(\psi_l, V_l)_{l \in L}$ and $(\pi_k \pi, W_k \cap W)_{k \in K}$ together form a trivializing partition of unity for μ. With respect to this we have the admissible triple $(L, \eta, (\psi_l))$, which is then majorized by some maximal triple $(L \cup K, \xi, (\Phi_i))$. Let V' be the subset of V on which $\psi > \frac{1}{2}$. Then V' is a halo of A. By definition of the ordering $\xi | V' = \eta | V'$ and $\Phi_l | V' = \Psi_l | V'$ for $l \in L$. This completes the proof of (a).

The proof of (b) is now easy. Let ξ_1, ξ_2 and V' be as described under (b) in the theorem. Then we have a microbundle $\bar{\mu} = \mu \times I$ over $X \times I$ and $t_0, t_1 \in I$ with $0 < t_0 < t_1 < 1$ such that $\bar{\mu} | X \times [0, t_0]$ contains $\xi_1 \times [0, t_0]$, $\bar{\mu} | X \times [t_1, 1]$ contains $\xi_2 \times [t_1, 1]$ and $\bar{\mu} | V' \times I$ contains $(\xi_1 | V') \times I = (\xi_2 | V') \times I$. Therefore we have the bundle $\xi_1 \times [0, t_0] \cup (\xi_1 | V') \times I \cup \xi_2 \times [t_1, 1]$ over a halo $X \times [0, t_0] \cup V' \times I \cup X \times [t_1, 1]$ of $X \times 0 \cup A \times I \cup X \times 1$ in $X \times I$, contained in $\bar{\mu} | X \times [0, t_0] \cup V' \times I \cup X \times [t_1, 1]$. By (a) there is a bundle $\bar{\xi}$ over $X \times I$ contained in $\bar{\mu}$ which restricts to this given bundle over any sufficiently small halo of $X \times 0 \cup A \times I \cup X \times 1$. By the compactness of I any halo of $X \times 0 \cup A \times I \cup X \times 1$ contains a subset of the form $X \times 0 \cup V'' \times I \cup A \times 1$ where $V'' \subset V'$ is a halo of A in X. Part (b) follows now from Lemma 1.5.

Remark 1. Note that the proof of the representation theorem actually gives some more information than announced in the theorem. Thus under (a), if $(\psi_l, V_l)_{l \in L}$ is a trivializing partition of unity for η with associated local trivializations $(\Psi_l)_{l \in L}$, then there is a trivializing partition of unity $(\phi_k, U_k)_{k \in K} \cup (\psi_l, V_l)_{l \in L}$ for ξ with associated local trivializations $(\Phi_i)_{i \in K \cup L}$ such that $(\phi_k, U_k)_{k \in K}$ is disjoint with V' and $\Phi_l | V_l \cap V' = \Psi_l | V_l \cap V'$ for all $l \in L$. By (b) any solution ξ admits such nice trivializations provided V' is small enough.

Remark 2. The relation "ξ contained in μ" could everywhere have been replaced with "ξ microbundle isomorphic to μ" in Theorem 3.3 (together with the other obvious changes); the proof is easily modified to take care of this case. In particular (b) can be given the slightly strengthened form: *Let ξ_1, ξ_2 be $\mathbf{R}^q$-bundles over some space X and let $\phi : \xi_1 \Rightarrow \xi_2$ be an isogerm. Let $\Phi' : \xi_1 | V' \approx \xi_2 | V'$ be an isomorphism of ξ_1 to ξ_2 over some halo V' of A in X whose germ is $\phi | V'$. There is an isomorphism $\Phi : \xi_1 \approx \xi_2$ whose germ is ϕ such that $\Phi | V'' = \Phi' | V''$ over some halo $V'' \subset V'$ of A.*

Remark 3. Theorem 3.3 above is stated in its relativized form, relativization taken with respect to subsets in the base. We could however equally well have relativized with respect to subsets in the fibre or carried through both relativizations simultaneously, i.e. stated the representation theorem for $(\mathbf{R}^q, \mathbf{R}^n)$-microbundles and $(\mathbf{R}^q, \mathbf{R}^n)$-bundles. In fact the proof of Theorem 3.3 with the preceeding lemmas go through with nominal changes to give this refined result. Notice also that remarks 1 and 2 hold for the refined representation theorem.

Theorem 3.3 has the following immediate consequences (by Lemma 1.5), cf. [8] p. 58.

3.4. Corollary. *Let μ be a microbundle and let $f_0 \simeq f_1$ be homotopic maps to the base of μ. Then $f_0^*\mu \approx f_1^*\mu$.*

In fact if for some halo V of a subset A in the domain of f_0, f_1 we have $f_0 \simeq f_1$ rel V, then $f_0^*\mu \approx f_1^*\mu$ by an isogerm which is the identity over A.

3.5. Corollary. (a) *Any microbundle over a contractible space is trivial.* (b) *If μ is a microbundle and f is a map to the base of μ, then μ can be extended to a microbundle over the mapping cone C_f if and only if $f^*\mu$ is trivial.*

We give a final application of the techniques used to prove 3.1 and 3.2. As above the proof and the theorem actually hold for $(\mathbf{R}^q, \mathbf{R}^n)$-bundles, $0 \leqslant n \leqslant q$.

3.6. Theorem. *Let $\xi: X \xrightarrow{s} E \xrightarrow{p} X$ be an $\mathbf{R}^q$-bundle. Let V be a halo of a set A in X, and let $H: E|V \times I \to E|V$ be a fibre homotopy relative to sV from $s|V\,p|V$ to $id_E|V$ such that H_t is an embedding of $\xi|V$ into itself for $t>0$. There is a fibre homotopy $H': E \times I \to E$ relative to sX from sp to id_E such that H'_t is an embedding of ξ into itself for $t>0$, and such that H' coincides with H on $E|V' \times I$ for some halo $V' \subset V$ of A.*

Proof. Let $(\pi_j, W_j)_{j\in J}$ be a trivializing partition of unity for ξ with associated local trivializations $\Phi_j: \xi|W_j \approx \varepsilon(W_j)$. For each $j \in J$ there is a fibre homotopy $H_j: E|W_j \times I \to E|W_j$ corresponding under Φ_j to the homotopy $(W_j \times \mathbf{R}^q) \times I \to W_j \times \mathbf{R}^q$ that maps $(x, v; t)$ to (x, tv). Clearly $H_j: s|W_j\,p|W_j \simeq id_E|W_j$ rel sW_j, and $(H_j)_t$ is a bundle embedding of $\xi|W_j$ into itself for $t>0$. More generally, consider the collection of pairs (K, H) where K is a non-empty subset of J and $H: E|W_k \times I \to E|W_k$ is a fibre homotopy relative to sW_K from $s|W_K\,p|W_K$ to $id_E|W_K$ with H_t an embedding for $t>0$. Then this collection is non-empty. Order it by defining $(K, H) \leqslant (K', H')$ if the following conditions are satisfied

(a) $K \subset K'$

(b) If $e \in E|W_K$ and $\pi_K(p(e)) = \pi_{K'}(p(e))$ then $H(e,t) = H'(e,t)$ for all $t \in I$.

This gives an inductive ordering and so each pair is majorized by a maximal pair. Let (K, H) be a maximal pair. We show that $K = J$. If $K \neq J$, let $K' = K \cup \{j\}$ where $j \in J - K$. From H and H_j construct a homotopy $H': E|W_{K'} \times I \to E|W_{K'}$ as follows. From the partition of unity $\{(\pi_K, W_K), (\pi_j, W_j)\}$ on W_K (normalized if necessary) form the derived partition of unity $\{(\pi'_K, W'_K), (\pi'_j, W'_j)\}$. Define H' on $E|W'_K \cap W'_j$ by

$$H'(e,t)=\begin{cases} H_j(H(e,t/\pi'_K(p(e))),\pi'_K(p(e))) & \text{for } \pi'_K(p(e))\geqslant t\\ H_j(e,t) & \text{for } \pi'_K(p(e))\leqslant t\end{cases}$$

Then extend H' so that it coincides with H on $W'_K - W'_j$ and with H_j on $W'_j - W'_K$. Clearly H' is well-defined on $E \mid W_{K'} \times I$ and everywhere continuous except possibly at the points $(e, 0)$ with $p(e)$ in the boundary of W'_K in $W_{K'}$. However, H and H_j are both defined and continuous in a neighbourhood of $e \times I$ for such a point e. Since $H_j(H(e\times I), 0) = H_j(e, 0) = H(e, 0) = s(p(e))$, for any neighbourhood U' of $s(p(e))$ there is a neighbourhood U of e in $W_K \cap W_j$ and an $\varepsilon > 0$ such that $H(U\times[0,\varepsilon])\subset U'$, $H_j(U\times[0,\varepsilon])\subset U'$ and $H_j(H(U\times I)\times [0,\varepsilon])\subset U'$ (using the compactness of I). It follows that $H'(U\times[0,\varepsilon])\subset U'$ which shows that H' is in fact continuous also at $(e, 0)$. Obviously H' is a fibre homotopy relative to $sW_{K'}$ from $s \mid W_{K'}\, p \mid W_{K'}$ to $id_E \mid W_{K'}$. It is easy to see that H'_t is open, hence a bundle embedding, for $t > 0$. Therefore (K', H') is a pair in our ordered set strictly majorizing (K, H). The contradiction shows that we must have $K = J$. This proves the theorem for $A = \emptyset$. The case where A is non-empty now follows by a trick similar to that in the proof of the representation theorem.

By the Theorems 3.3 and 3.6 every microbundle contains a neighbourhood contractible along the fibres to the zero section. By an argument of Milnor this implies that composite microbundles are isomorphic to Whitney sums and vice versa ([8], p. 12):

3.7. COROLLARY. *Let $\mu: X \xrightarrow{s} E \xrightarrow{p} X$, $\nu: E \xrightarrow{t} E' \xrightarrow{q} E$ be microbundles whose composite $\mu\circ\nu$ is defined. Then $\mu\circ\nu \approx \mu\oplus s^*\nu$. Similarly, if μ, μ' are microbundles over X, then*

$$\mu\oplus\mu' \approx \mu\circ p^*\mu'.$$

Proof. From their definitions the microbundles $\mu\oplus\mu'$ and $\mu\circ p^*\mu'$ are in fact identic. To show the first part of the corollary we may assume that μ (and ν) is a bundle. Then, by 3.6 $sp \simeq id_E$ and so $p^*s^*\nu \approx \nu$. Thus $\mu\circ\nu \approx \mu\circ p^*s^*\nu = \mu\oplus s^*\mu$.

Notice that if μ and ν are actually bundles, their composite, although a microbundle, need not be a bundle. By the representation theorem, however, it does contain an essentially unique bundle which could be called the *composite bundle* and which is bundle isomorphic to the Whitney sum of bundles $\mu\oplus s^*\mu$.

4. Sphere bundles and Thom spaces. The Hirsch–Mazur theorem

For any integer $q \geqslant 0$ let $\mathbf{R}^q \subset S^q$ be a fixed embedding of $\mathbf{R}^q$ into its one-point compactification S^q, the q-sphere, and let ∞ denote the complementary point of $\mathbf{R}^q$ in S^q. Any homeomorphism of $(\mathbf{R}^q, 0)$ extends uniquely to a homeomorphism of $(S^q, 0, \infty)$. By

Lemma 1.3 this correspondence is an isomorphism $G(\mathbf{R}^q, 0) \approx G(S^q, 0, \infty)$ of topological groups. It follows that to any $\mathbf{R}^q$-bundle $\xi: X \xrightarrow{s} E \xrightarrow{p} X$ there is a functorially associated S^q-bundle $\xi_\infty: E_\infty \xrightarrow{p_\infty} X$ over the same base with two sections s_0, s_∞, and that ξ is naturally imbedded in ξ_∞ with zero section s corresponding to s_0 and total space E corresponding to $E_\infty - \operatorname{im} s_\infty$. The bundle ξ_∞ is constructed as follows. Let (f_{VW}) be the family of transition maps of ξ arising from the collection of all local trivializations $\Phi_V: \xi \mid V \approx \varepsilon^q(V)$. Thus $f_{VW}: V \cap W \to G(\mathbf{R}^q, 0)$ are continuous maps defined on non-empty $V \cap W$ by $f_{VW}(x)v = v'$ if and only if $(\Phi_V \Phi_W^{-1})(x, v) = (x, v')$. Then the relations $f_{UV} \cdot f_{VW} = f_{UW}$ hold on $U \cap V \cap W$ for any trivializing sets U, V, W with non-empty intersection. Composing with the isomorphism $G(\mathbf{R}^q, 0) \approx G(S^q, 0, \infty)$ give continuous maps $f_{VW\infty}: V \cap W \to G(S^q, 0, \infty)$ such that the corresponding relations $f_{UV\infty} \cdot f_{VW\infty} = f_{UW\infty}$ hold. Then the usual gluing process applies to give a space E_∞ and a continuous map $p_\infty: E_\infty \to X$ with fibres homeomorphic to S^q and with two sections s_0, s_∞. Clearly there is a natural fibre preserving embedding $E \to E_\infty$ under which the sections correspond as described. This defines $\xi_\infty: E_\infty \xrightarrow{p_\infty} X$ as an S^q-bundle. (Obviously a partition of unity trivializing for ξ is trivializing for ξ_∞ and conversely.) Therefore ξ_∞ is a bundle related to ξ as claimed. By the natural embedding we may and will consider ξ as contained in ξ_∞.

Next consider the standard embedding $Y \subset CY$ of a compact space Y into its (unreduced) cone CY. A homeomorphism $\alpha \in G(Y)$ extends to a homeomorphism $C\alpha \in G(CY)$, and the correspondence $C: G(Y) \to G(CY)$ is a continuous representation. It follows that to any bundle $\zeta: E \xrightarrow{p} X$ with fibre Y there is a functorially associated bundle $\zeta_C: E_C \xrightarrow{p_C} X$ with fibre CY, and since any homeomorphism $C\alpha$ keeps the cone top point fixed, ζ_C has a canonical section $s_C: X \to E_C$. Observe that E_C can be naturally identified with Z_p, the mapping cylinder of p. Under this identification p_C corresponds to the canonical retraction $Z_p \to X$. In the case where $Y = S^{q-1}$, $q \geq 1$, identify CS^{q-1} with D^q by the standard homeomorphism. Then ζ_C (with its zero-section) is a disk bundle. Note that the embedding $S^{q-1} \subset D^q$ induces an embedding $\zeta \subset \zeta_C$.

Let $\mathbf{R}^q \subset D^q$ be a fixed embedding of $\mathbf{R}^q$ onto the interior of the q-dimensional unit disk such that origin corresponds to origin. Then $\mathbf{R}^q$ sits in both D^q and S^q, and the identity map on $\mathbf{R}^q$ extends to a homeomorphism $D^q/S^{q-1} \approx S^q$ by which the two spaces will be identified. Any homeomorphism on D^q maps boundary to boundary and interior to interior according to the theorem of invariance of domains. By 1.2 the restriction homeomorphisms $G(D^q, 0) \to G(S^{q-1})$ and $G(D^q, 0) \to G(\mathbf{R}^q, 0)$ are continuous.

Consider a q-dimensional disk bundle $\eta: X \xrightarrow{s} F \xrightarrow{p} X$. The structure group is $G(D^q, 0)$, and the restriction homeomorphisms define associated bundles $\dot{\eta}: \dot{F} \xrightarrow{\dot{s}} X$ and $\mathring{\eta}: X \xrightarrow{\mathring{s}} \mathring{F} \xrightarrow{\mathring{p}} X$ with fibres S^{q-1} and $\mathbf{R}^q$ respectively, called the *boundary bundle* and the *interior bundle* of η. There are natural embeddings of $\dot{\eta}$ and $\mathring{\eta}$ in η induced from the inclusions $S^{q-1} \subset D^q$ and $(\mathbf{R}^q, 0) \subset (D^q, 0)$ by which $\dot{\eta}$ and $\mathring{\eta}$ are identified with subbundles of η. Finally the canonical homeomorphism $G(D^q, 0) \to G(S^q, 0, \infty)$ which maps $\alpha: D^q \to D^q$ to $\alpha/S^{q-1}: D^q/S^{q-1} \to D^q/S^{q-1}$ is continuous and so defines an associated S^q-bundle $\eta/\dot{p}$ with two sections s_0 and s_∞. This bundle may be considered obtained from η by collapsing the boundary in each fibre. Since this collapsing does not affect $\mathring{\eta}$, $\mathring{\eta}$ sits in $\eta/\dot{p}$.

4.1. Lemma. *There is a natural bundle isomorphism $\eta/\dot{p} \approx \mathring{\eta}_\infty$ which is the identity on $\mathring{\eta}$.*

Let $\xi: X \xrightarrow{s} E \xrightarrow{p} X$ be an $\mathbf{R}^q$-bundle. For any pair of subsets (A, B) in the base X define the *Thom space* $T_\xi(A, B)$ to be the pointed space

$$T_\xi(A, B) = p_\infty^{-1}A/(s_\infty A \cup p_\infty^{-1}B),$$

the collapsed subset $s_\infty A \cup p_\infty^{-1}B$ serving as base point $*$. In the classical case where ξ is an orthogonal bundle the Thom space T_ξ of ξ is usually defined as follows: Pick out a disk bundle $\eta: X \xrightarrow{s'} F \xrightarrow{p'} X$ contained in ξ (by means of some riemannien metric on ξ, say). Then T_ξ is defined to be the pointed space $F/\dot{F}$. This determines T_ξ up to a base point preserving homeomorphism. In fact we may form $F/\dot{F}$ by first collapsing the boundary of each fibre in F, which gives the total space $F/\dot{p}'$ of $\eta/\dot{p}'$, and then collapsing im $\dot{s}_\infty$ in $F/\dot{p}'$. By Lemma 4.1 the result is base point preserving homeomorphic to $\mathring{F}_\infty/\text{im}\ \mathring{s}_\infty = T_{\mathring{\eta}}(X, \emptyset)$. But $\mathring{\eta}$ is $G(\mathbf{R}^q, 0)$-isomorphic to ξ by the microbundle representation theorem (in fact $\mathring{\eta}$ and ξ are O^q-isomorphic), and so $T_{\mathring{\eta}}(X, \emptyset)$ is base point preserving homeomorphic to $T_\xi(X, \emptyset)$. Altogether $T_\xi \approx T_\xi(X, \emptyset)$. It follows that our definition is an extension (and relativization) of the classical one. Notice that the classical definition breaks down because there is no way of constructing disk bundles in a general $\mathbf{R}^q$-bundle. In fact W. Browder has recently shown that there exist $\mathbf{R}^q$-bundles even over finite polyhedrons that do not contain (or are contained in) disk bundles. On the other hand, M. Hirsch and B. Mazur have shown that (piecewise linear) $\mathbf{R}^q$-bundles over polyhedra which split off a (piecewise linear) $\mathbf{R}^1$-bundle do in fact imbed as interiors of (piecewise linear) D^q-bundles, cf. [4]. Moreover, Hirsch shows that two D^q-bundles over a polyhedron having isomorphic interior bundles get isomorphic after having a trivial D^1-bundle added. Disregarding the piecewise linear aspect it is easy to extend the proof to cover the case of general $\mathbf{R}^q$- and D^q-bundles. In fact the proof is much shorter because one need not bother with piecewise linear structures. Since this result will be applied in [7] we will sketch a proof. (Contrary to the proof in [4] this does not make use of any representation theorem for microbundles.) First some preliminary information. Every homeomorphism on D^q restricts to a homeomorphism on the boundary S^{q-1}. Conversely, every homeomorphism on S^{q-1} extends radially to a homeomorphism on D^q keeping the origin fixed. Let $r: G(D^q, 0) \to G(S^{q-1})$, $i: G(S^{q-1}) \to G(D^q, 0)$ be the corresponding continuous representations. Then $ri = id_{S^{q-1}}$ and so i imbeds $G(S^{q-1})$ as a subgroup of $G(D^q, 0)$ onto which $G(D^q, 0)$ retracts by r. In fact the Alexander radialization process (cf. [1] and [4]) shows that $G(S^{q-1})$ is a strong deformation retract of $G(D^q, 0)$.

4.2. Lemma. *There is a strong deformation retraction.*

$$H: (G(D^q), G(D^q, 0)) \times I \to (G(D^q), G(D^q, 0)) \text{ rel } iG(S^{q-1})$$

from id_{D_q} *to* ir *such that* H_t *is a continuous endomorphism on* $(G(D^q), G(D^q, 0))$, $t \in I$.

Proof. Define $H: (G(D^q) G(D^q, 0)) \times I \to (G(D^q), G(D^q, 0))$ by

$$H(\alpha, 1-t)(x) = \begin{cases} 0 & t=0,\ \|x\|=0 \\ t \cdot \alpha(x/t) & t>0,\ \|x\| \leqslant t \\ \|x\| \cdot \alpha(x/\|x\|) & t \geqslant 0,\ \|x\| > t. \end{cases}$$

This map is easily seen to be continuous. Also it is easy to check that H_t is an endomorphism for arbitrary t in I.

It follows that the structure group of a bundle with fibre D^q reduces to (a subgroup of) $G(S^{q-1})$. More precisely

4.3. Theorem. *Let η be a bundle with fibre D^q. There is an isomorphism $\eta \approx \dot{\eta}_C$ which is the identity on $\dot{\eta}$. If η is a D^q-bundle (i.e. has a prescribed zero section) over a halo V' of a set A in the base, the isomorphism $\eta \approx \dot{\eta}_C$ can be chosen so as to give an isomorphism of D^q-bundles over a halo $V'' \subset V'$ of A.*

4.4. Corollary. *Let η_0, η_1 be D^q-bundles or just bundles with fibre D^q. Any isomorphism $\dot{\eta}_0 \approx \dot{\eta}_1$ extends to an isomorphism $\eta_0 \approx \eta_0$.*

Proof of 4.3. Let (g_{VW}) be the family of transition maps of η arising from the collection of all local trivializations $\Psi_V: \eta \mid V \approx \delta^q(V)$. For each $g_{VW}: V \cap W \to G(D^q, 0)$ define G_{VW}: $(V \cap W) \times I \to G(D^q, 0)$ to be the composite $H \circ (g_{VW} \times id_I)$ where H is the homotopy in Lemma 4.2. Then (G_{VW}) is a defining family of transition maps for a D^q-bundle $\bar{\eta}$ over $X \times I$, X being the base of η. In fact

$$G_{UV}(x, t) \cdot G_{VW}(x, t) = H_t(g_{UV}(x) \cdot H_t(g_{VW}(x)) = H_t(g_{UV}(x) \cdot g_{VW}(x)) = H_t(g_{UW}(x)) = G_{UW}(x, t)$$

for any $(x, t) \in (U \cap V \cap W) \times I$, according to Lemma 4.2. Also if $(\pi_j, W_j)_{j \in J}$ is a trivializing partition of unity for η, then $(\pi_j \times id_I, W_j \times I)_{j \in J}$ is a trivializing partition of unity for $\bar{\eta}$.

Clearly $\bar{\eta}|0$ and $\bar{\eta}|1$ are naturally isomorphic to η and $\dot{\eta}_C$ respectively. But then $\eta \approx \dot{\eta}_C$ by Lemma 1.5. The refinements of the relative case are easily taken care of. We omit the details.

Notice that 4.3 implies that an $\mathbf{R}^q$-bundle which is the interior of a D^q-bundle contains D^q-bundles. The converse, of course, follows from the representation theorem. We can now formulate and prove the Hirsch–Mazur theorem in the general topological context.

4.5. THEOREM (*Hirsch–Mazur*). *Let ξ be an $\mathbf{R}^q$-bundle over some space X. Then $\xi \oplus \varepsilon$ is isomorphic to the interior of the D^{q+1}-bundle $\xi_{\infty C}$.*

Let η_0, η_1 be D^q-bundles over X with isomorphic interiors. Then $\eta_0 \oplus \delta$ is isomorphic to $\eta_1 \oplus \delta$.

We only indicate the proof and leave the details to the reader. Notice first that $\xi_{\infty C}$ is isomorphic to $\xi_{\infty C'}$, the "dented" mapping cylinder of ξ_∞ obtained from $\xi_{\infty C}$ by collapsing the radius to the ∞-point in each fibre. In fact $\xi_{\infty C}$ and $\xi_{\infty C'}$ have isomorphic boundary bundles and so by 4.4 are isomorphic (as bundles with fibre D^{q+1}). Therefore $\xi_{\infty C}$ and $\xi_{\infty C'}$ have isomorphic interiors. On the other hand, the interior of $\xi_{\infty C'}$ is isomorphic to $\xi_{\infty C}$ with its boundary and its "sheet" of ∞-radii removed. This bundle, however, is easily seen to be isomorphic to $\xi \oplus \varepsilon$. Moreover, it is easy to pick the different isomorphisms in such a way that the composite int $\xi_{\infty C} \approx \xi \oplus \varepsilon$ maps zero section to zero section. The second part of the theorem follows from the fact that for disk bundles η there is an isomorphism $\eta \oplus \delta \approx \dot{\eta}_{\infty C}$. It is in fact trivial to construct such an isomorphism on the boundary bundles and then again one uses 4.4.

References

[1]. ALEXANDER, J. W., On the deformation of an n-cell. *Proc. Nat. Acad. Sci. U.S.A.*, 9 (1923), 406–407.

[2]. ARENS, R., Topologies for homeomorphism groups. *Amer. J. Math.*, 68 (1946), 593–610.

[3]. DOLD, A., Partitions of unity in the theory of fibrations. *Ann. of Math.*, 78 (1963), 223–225.

[4]. HIRSCH, M. W., On non linear cell bundles. To appear.

[5]. HIRSCH, M. W. & MAZUR, B., Smoothings of piecewise linear manifolds. To appear.

[6]. HOLM, P., Microbundles and bundles. *Bull. Amer. Math. Soc.*, 72 (1966), 545–548.

[7]. HOLM, P., Microbundles and S-duality. To appear. (See also *Bull. Amer. Math. Soc.*, 72 (1966), 549–554.)

[8]. MILNOR, J., Microbundles. Part 1. *Topology*, 3 (1964), 53–79.

[9]. SPANIER, E. H., *Algebraic topology*. McGraw-Hill (1966).

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