ON THE COMPLETENESS OF SOME SETS OF FUNCTIONS.

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1. Introduction.

A set of functions $\{\psi_n(x)\}$ is said to be complete in a space $L^p(a, b)$ $(1 \leq p \leq \infty)$, if $\int_a^b f(x)\,\overline{\psi}_n(x)\,dx = 0$ $(n = 1, 2, 3, \ldots)$ implies $f(x) \equiv 0$ when $f(x) < L^p(a, b)$. Let the differential equation

$$L y = \lambda y, \quad L \equiv -\frac{d^2}{d x^2} + q(x) \qquad (L)$$

together with linear homogeneous boundary conditions at the end-points of an interval (a, b) $(-\infty < a < b \leq +\infty)$ define a regular or singular boundary-value problem of a Sturm-Liouville type[1], whose eigenfunctions form a set, complete in $L^2(a, b)$. Then, in general, the set of squares on the eigenfunctions cannot be complete in $L^2(a, b)$ (for instance the set $\{\sin^2 n x\}$, belonging to (L) for $q(x) = 0$ and boundary conditions $y(0) = y(\pi) = 0$, has the completeness properties of the set $\{\cos 2 n x\}$). In this paper some completeness properties of sets of eigenfunction-squares will be studied. The problems arose at the study of so-called inverse boundary-value problems, i. e. problems where the differential equation is to be determined from the knowledge of the spectrum and boundary conditions.[2]

The main results are, roughly speaking, the following.

[1] In the sequel we use S-L as an abbreviation of Sturm-Liouville.

[2] G. BORG, Eine Umkehrung der Sturm-Liouvilleschen Eigenwertaufgabe, Acta math. 78 (1945). ——, Inverse Problems in the theory of Characteristic Values of Differential Systems, Dixième Congrès des Mathématiciens Scandinaves, Copenhague 1946. In these papers some results concerning eigenfunction-squares of regular S-L problems are contained.

If to one set of eigenfunction-squares there is added another, belonging to other boundary conditions, then the combined set is complete in the space $M(a, b)$ of functions, effectively bounded on (a, b). The precise wording is given in No. 3 (a trivial instance: $\{\sin^2 nx\}$ and $\left\{\sin^2 \frac{2n+1}{2} x\right\}$, the latter set equivalent to $\{\cos(2n+1)x\}$ and belonging to boundary conditions $y(0) = y'(\pi) = 0$).

The combined set is not normalized, nor in general minimal. Yet holds that at most 2 of the eigenfunction-squares are superfluous — at least in the case of regular or regularly singular *S-L* problems. The proof is given in No. 9.

The examples and results above indicate that one set of eigenfunction squares is complete in a space of functions, defined only on half the original interval — if this is finite. This property is at last, more precisely formulated, proved in No. 10 for regular and regularly singular *S-L* problems.

2. Notations and Known Properties of Singular *S-L* Problems.

Some of the fundamental methods and results concerning singular *S-L* problems will be needed.[1] The differential eq. is as above

$$Ly = \lambda y \tag{L}$$

where

$$L \equiv q(x) - \frac{d^2}{dx^2}.$$

$q(x)$ will be supposed to be continuous within an interval $0 \leq x < b_0$. b_0 may be $+\infty$. If $b_0 < \infty$, $q(x)$ may have a singularity for $x = b_0$.

The boundary conditions are

$$\begin{aligned} &\text{for } x = 0: && y(0)\cos\alpha + y'(0)\sin\alpha = 0, \qquad 0 \leq \alpha < \pi, \\ &\text{for } x = b_0: && y(x) < L^2(0, b_0) \end{aligned} \tag{R}$$

with the addition of another condition in the limit circle case.[2]

We quote this boundary value problem as (L, R). Further we put, following TITCHMARSH,

[1] H. WEYL, Über gewöhnliche Differentialgleichungen mit Singularitäten ..., Math. Ann. 68 (1910).

E. C. TITCHMARSH, Eigenfunction Expansions, Oxford 1946. This book is in the sequel quoted TITCHMARSH. We refer in this No. especially to TITCHMARSH, ch. 2nd.

[2] Cf. H. WEYL, loc. cit.

$\theta(x, \lambda) =$ the solution of (L), which takes the initial values

$$\theta(0, \lambda) = \cos\alpha,\ \theta'(0, \lambda) = \sin\alpha$$

$\phi(x, \lambda) =$ the solution of (L), which takes the initial values

$$\phi(0, \lambda) = \sin\alpha,\ \phi'(0, \lambda) = -\cos\alpha.$$

Then $\phi(x, \lambda)$ satisfies the boundary condition (R) for $x = 0$.

Let

$$\vartheta(x, \lambda) = \theta(x, \lambda) + l(\lambda)\phi(x, \lambda)$$

and $l(\lambda) = l_b(\lambda)$ be chosen so that

$$\vartheta(b, \lambda)\cos\beta + \vartheta'(b, \lambda)\sin\beta = 0 \tag{2.1}$$

holds, i. e.

$$l(\lambda) = -\frac{\theta(b, \lambda)\cos\beta + \theta'(b, \lambda)\sin\beta}{\phi(b, \lambda)\cos\beta + \phi'(b, \lambda)\sin\beta}. \tag{2.2}$$

When λ and b are fix numbers ($\mathrm{Im}\,(\lambda) \neq 0$), and $z = \cot\beta$ describes the real z-axis, the complex number $l(\lambda)$ describes a circle C_b in the complex l-plane. The interior points of the circle are characterized by the ineq.

$$\int_0^b |\theta(x, \lambda) + l\,\phi(x, \lambda)|^2\,dx < -\frac{\mathrm{Im}\,(l)}{\mathrm{Im}\,(\lambda)}. \tag{2.3}$$

From this follows that if $b' < b$, then the circle $C_{b'}$, includes the circle C_b, and hence that C_b converges to a limit point or a limit circle as $b \to b_0$.

Let $m(\lambda)$ be the limit point or a point on the limit circle (for the definition of which one additional condition at $x = b_0$ is needed, cf. (R) above), then

$$\psi(x, \lambda) = \theta(x, \lambda) + m(\lambda)\phi(x, \lambda) < L^2(0, b_0) \tag{2.4}$$

and

$$\int_0^{b_0} |\psi(x, \lambda)|^2\,dx = -\frac{\mathrm{Im}\,(m(\lambda))}{\mathrm{Im}\,(\lambda)}. \tag{2.5}$$

Regarded as a function of λ, $l_b(\lambda)$ is meromorphic ($|\lambda| < \infty$) and converges boundedly in the upper (and lower) half of the λ-plane to the function $m(\lambda)$.

3. The Theorem of Completeness.

We shall assume that $m(\lambda)$ as $l(\lambda)$ is a meromorphic function ($|\lambda| < \infty$). Then the spectrum $\{\lambda_n\}$ of $(L.\,R)$ is the set of poles, thus a discrete point set.

We shall further assume that there exists a finite number ϱ so that

$$\sum \frac{1}{|\lambda_n|^{\varrho+\varepsilon}} < \infty \qquad (\varepsilon > 0). \tag{3.1}$$

(We shall in the sequel always assume all $\lambda_n \neq 0$; this is no restriction, for the addition of a constant c to $q(x)$ transforms the spectrum $\{\lambda_n\}$ into $\{\lambda_n - c\}$).

If these two conditions are satisfied, we will say that (L, R) *has a point spectrum with a finite convergence exponent* (ϱ).[1]

Let (R^*) be the boundary conditions (R) with α exchanged for α^* (i. e. the conditions at $x = b_0$ unchanged). To all functions, numbers, and relations defined in connection with the problem (L, R), then correspond functions, etc., belonging to the problem (L, R^*). We shall only use a * to distinguish them from the former ones. For instance the eigenvalues of (L, R) will be denoted by $\{\lambda_n\}$, those of (L, R^*) by $\{\lambda_n^*\}$, the normalized eigenfunctions by $\{\psi_n(x)\}$ and $\{\psi_n^*(x)\}$ respectively, the meromorphic functions by $m(\lambda)$ and $m^*(\lambda)$ respectively.

We shall prove the following theorem.

If the spectra of the boundary-value problems (L, R) *and* (L, R^*) $(\alpha^* \not\equiv \alpha \bmod \pi)$ *both are point-spectra with a finite convergence exponent, then the set of all the squares of eigenfunctions* $\{\psi_n^2(x),\ \psi_m^{*2}(x)\}$ $(n, m = 1, 2, 3, \ldots)$ *is complete in class* $M(0, b_0)$ *of functions effectively bounded within the interval* $0 \leq x < b_0$.

Remark. The classical S-L problems, regular or regularly singular at $x = b_0$ satisfy the conditions of the theorem.

4. A Boundary-Value Problem with Eigenfunctions $\{\psi_n^2(x)\}$ and $\{\psi_n^{*2}(x)\}$.

Let $y_1(x)$ and $y_2(x)$ be solutions of (L), then $u(x) = y_1 \cdot y_2$ is a solution of

$$D u + 4 \lambda u' \equiv u''' + 4(\lambda - q(x)) u' - 2 q'(x) u = 0, \tag{D}$$

where for the sake of simplicity we assume $q'(x)$ continous $(0 \leq x < b_0)$. Then $\psi_n^2(x)$ and $\psi_n^{*2}(x)$ are solutions of (D) for $\lambda = \lambda_n$ and $\lambda = \lambda_n^*$ respectively. They may bo looked upon as the eigenfunctions of a boundary-value problem, consisting of the eq. (D) and boundary conditions, corresponding to the conditions (R) and (R^*) above.[2] We will prove our theorem by constructing a function $\Gamma(x, t, \lambda)$ which will serve as a Green's function of this problem.

[1] This is of course especially the case, if $m(\lambda)$ is assumed to be of finite order.

[2] Cf. G. Borg, Inverse Problems ... Dixième Congrès des Mathématiciens Scandinaves.

5. Green's Function of the Boundary-Value Problem in No. 4.

Under the assumptions in No. 3, $m(\lambda)$ must be real on the real axis of λ, for we have $l(\bar{\lambda}) = \overline{l(\lambda)}$. Hence $\lim\limits_{\text{Im}(\lambda)=0} \dfrac{\text{Im}(m(\lambda))}{\text{Im}(\lambda)} =$ finite number except possibly for $\lambda = \{\lambda_n\}$. Thus by (2.5)

$$\psi(x, \lambda) < L^2(0, b_0) \qquad\qquad \text{for } \lambda \neq \{\lambda_n\}. \tag{5.1}$$

To the boundary value problem (L, R) belongs the following Green's function

$$\gamma(x, t, \lambda) = \begin{cases} \psi(x, \lambda)\,\phi(t, \lambda), & x \geq t \\ \phi(x, \lambda)\,\psi(t, \lambda), & x < t, \end{cases} \tag{5.2}$$

which according to (5.1) satisfies

$$\int_0^{b_0} |\gamma(x, t, \lambda)|^2\, dx < \infty, \quad \lambda \neq \{\lambda_n\}, \quad t \text{ fix.} \tag{5.3}$$

We recall the following two properties of Green's function, which are of importance in proving completeness theorems.

a. $\gamma(x, t, \lambda)$ and $\dfrac{\partial \gamma(x, t, \lambda)}{\partial t}$ are continuous within $0 \leq x, t < b_0$ except at $t = x$, where

$$\frac{\partial \gamma(x, t, \lambda)}{\partial t}\Bigg|_{t=x+0}^{t=x-0} = -1. \tag{5.4}$$

b. $$-L(\gamma) + \lambda\gamma \equiv \frac{\partial^2 \gamma(x, t, \lambda)}{\partial t^2} + (\lambda - q(t))\,\gamma(x, t, \lambda) = 0, \quad \begin{matrix} t \neq x \\ x \text{ fix.} \end{matrix} \tag{5.5}$$

From the function $\Gamma(x, t, \lambda)$ mentioned above, we shall require two corresponding properties

1° Γ and $\dfrac{\partial \Gamma}{\partial t}$ continuous within $0 \leq x, t < b_0$

$\dfrac{\partial^2 \Gamma}{\partial t^2}$ » $\dfrac{\partial^3 \Gamma}{\partial t^3}$ » » $0 \leq x, t < b_0, \ t \neq x$

$$\frac{\partial^2 \Gamma(x, t, \lambda)}{\partial t^2}\Bigg|_{t=x+0}^{t=x-0} = +4 \tag{5.6}$$

2° $$D\Gamma(x, t, \lambda) + 4\lambda \Gamma_t'(x, t, \lambda) = 0 \ (x \text{ fix}) \text{ for } 0 \leq t < b_0, \ t \neq x. \tag{5.7}$$

The last property indicates that $\Gamma(x, t, \lambda)$ must be a sum of products of solutions of (L) (cf. No. 4). The property 1° together with 4° later in this No. require the functions $\gamma(x, t, \lambda)$ and $\gamma^*(x, t, \lambda)$ for the construction of $\Gamma(x, t, \lambda)$. If we put

$$\begin{aligned} g(x, t, \lambda) = \theta(x, \lambda)\phi(t, \lambda) - \theta(t, \lambda)\phi(x, \lambda) &\equiv \psi(x, \lambda)\phi(t, \lambda) - \psi(t, \lambda)\phi(x, \lambda) \\ &\equiv \psi^*(x, \lambda)\phi^*(t, \lambda) - \psi^*(t, \lambda)\phi^*(x, \lambda) \end{aligned} \tag{5.8}$$

it is easy to prove that the function

$$\Gamma(x, t, \lambda) = \begin{cases} 2\gamma(x, t, \lambda)\cdot\gamma^*(x, t, \lambda), & x \geq t \\ 2\gamma(x, t, \lambda)\cdot\gamma^*(x, t, \lambda) + 2g(x, t, \lambda)(\gamma + \gamma^*), & x \leq t \end{cases} \tag{Γ}$$

has the required properties. Since $I(x, t, \lambda)$ is a sum of products of solutions of (L) — for $t \neq x$ — 2° follows. The property 1° is easily proved through straightforward computations, using the relations (5.4) and (5.4*) and

$$g(t, t, \lambda) = 0, \quad \frac{\partial g(x, t, \lambda)}{\partial t}\Bigg/_{t=x} = -1. \tag{5.9}$$

Now let $f(x) < M(0, b_0)$. The integral $\int_0^{b_0} \Gamma(x, t, \lambda) f(x)\,dx$ (t fix, $\lambda \neq \{\lambda_n\}, \{\lambda_n^*\}$) exists according to (5.3), (5.3*). Put

$$\Phi(t, \lambda) = \int_0^{b_0} \Gamma(x, t, \lambda) f(x)\,dx, \quad f(x) < M(0, b_0). \tag{5.10}$$

Then $\Phi(t, \lambda)$ and the first two derivatives are absolutely continuous within $0 \leq t < b_0$. Using 1° we find

$$\Phi'(t, \lambda) = \int_0^{b_0} \Gamma_t'(x, t, \lambda) f(x)\,dx, \quad \Phi''(t, \lambda) = \int_0^{b_0} \Gamma_t''(x, t, \lambda) f(x)\,dx$$

$$\Phi'''(t, \lambda) = \int_0^{b_0} \Gamma_t'''(x, t, \lambda) f(x)\,dx - 4f(t) \quad \text{p. p. (presque partout)}$$

and hence by (5.7)

$$D\,\Phi(t, \lambda) + 4\lambda\,\Phi_t'(t, \lambda) = -4f(t) \quad \text{p. p. for } 0 \leq t < b_0. \tag{5.11}$$

The function $\Gamma(x, t, \lambda)$ has some further properties, which will be needed:

3° If the spectra $\{\lambda_n\}$ and $\{\lambda_n^*\}$ are point-spectra with a finite convergence exponent, then $\Phi(t, \lambda)$ is a meromorphic function of a finite order ($|x| < \infty$) for every fix t-value. $\{\lambda_n\}$ and $\{\lambda_n^*\}$ are the only poles.

$\Phi(t, \lambda)$ converges to o, $|\lambda| \to \infty$, except possibly in an arbitrarily small angle $|\arg \lambda| < \varepsilon$, $|\arg \lambda - \pi| < \varepsilon$.

4° The principal part of $\Phi(t, \lambda)$ at a pole $\lambda = \lambda_n$ is of the form

$$(\lambda - \lambda_n)^{-1} \cdot v_n(t) \int_0^{b_0} \psi_n^2(x) f(x)\, dx. \tag{5.12}$$

The proof of the theorem is then brought to an end in the following way. Assume

$$\int_0^{b_0} \psi_n^2(x) f(x)\, dx = \int_0^{b_0} \psi_n^{*2}(x) f(x)\, dx = 0 \quad (n = 1, 2, 3, \ldots). \tag{5.13}$$

Hence, owing to (5.12), (5.12*) $\Phi(t, \lambda)$ is an integral function (t fix), which according to 3° is of finite order. Since it converges to o in the manner mentioned in 3° it follows by the well-known theorem of Phragmén-Lindelöf that $\Phi(t, \lambda)$ is bounded in all the λ-plane, i. e. constant and so $= 0$ according to 3°. Thus

$$\Phi(t, \lambda) \equiv 0 \qquad \text{for all } t < b_0 \text{ and all } \lambda.$$

Hence, on account of (5.11)

$$f(t) = 0 \qquad \text{p. p.},$$

which means that the set $\{\psi_n^2(x), \psi_m^{*2}(x)\}$ is complete in $M(0, b_0)$.

6. Proof of the Property 3° of No. 5.

If the reverse is not explicitly stated we assume $\lambda \neq \{\lambda_n\}$ and $\neq \{\lambda_n^*\}$. According to (5.10) and (Γ) we have

$$\Phi(t, \lambda) = 2 \int_0^{b_0} \gamma(x, t, \lambda)\, \gamma^*(x, t, \lambda) f(x)\, dx + 2 \int_0^{t} g(x, t, \lambda)(\gamma + \gamma^*) f(x)\, dx. \tag{6.1}$$

The first integral may be expanded in a series. We have[1]

$$\gamma(x, t, \lambda) \sim \sum_{n=1}^{\infty} \frac{\psi_n(x)\, \psi_n(t)}{\lambda - \lambda_n} \tag{6.2}$$

hence by (5.3) and the Parseval relation

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[1] TITCHMARSH, pag. 33. $\{\psi_n(t) \cdot (\lambda - \lambda_n)^{-1}\}$ are the Fourier coefficients of $\gamma(x, t, \lambda)$ with respect to the set $\{\psi_n(x)\}$.

$$\int_0^{b_0} \gamma(x,t,\lambda)\gamma^*(x,t,\lambda)f(x)\,dx = \sum_{n=1}^{\infty} \frac{\psi_n(t)}{\lambda-\lambda_n} \int_0^{b_0} \gamma^*(x,t,\lambda)f(x)\psi_n(x)\,dx$$

$$= \sum_{n=1}^{\infty} \frac{\psi_n(t)}{\lambda-\lambda_n} \sum_{m=1}^{\infty} \frac{\psi_m^*(t)}{\lambda-\lambda_m^*} a_{nm} \tag{6.3}$$

where $a_{nm} = \int_0^{b_0} \psi_n(x)\psi_m^*(x)f(x)\,dx$. The summation order may as well be inverted.

To prove that the left hand member is a meromorphic function of a finite order, we will first estimate the difference

$$d = \int_0^{b_0} \gamma(x,t,\lambda)\gamma^*(x,t,\lambda)f(x)\,dx - \sum_{n=1}^{N}\sum_{m=1}^{N} \frac{\psi_n(t)\psi_m^*(t)}{(\lambda-\lambda_n)(\lambda-\lambda_m^*)} a_{nm} \tag{6.4}$$

for all λ belonging to the region D_R: $|\lambda| < R$ with the exclusion of small circles round the poles λ_n and λ_n^* $(n = 1, 2, 3, \ldots)$. From (6.3) follows

$$d = \sum_{n=N+1}^{\infty} \frac{\psi_n(t)}{\lambda-\lambda_n} \sum_{m=1}^{\infty} \frac{\psi_m^*(t)}{\lambda-\lambda_m^*} a_{nm} + \sum_{m=N+1}^{\infty} \frac{\psi_m^*(t)}{\lambda-\lambda_m^*} \sum_{n=1}^{\infty} \frac{\psi_n(t)}{\lambda-\lambda_n} a_{nm}$$

$$+ \sum_{n,m=N+1}^{\infty} \frac{\psi_n(t)}{\lambda-\lambda_n} \frac{\psi_m^*(t)}{\lambda-\lambda_m^*} a_{nm}. \tag{6.5}$$

Now for the first of the series on the right we get

$$\sum_{n=N+1}^{\infty}\sum_{m=1}^{\infty} \frac{\psi_n\psi_m^*}{(\lambda-\lambda_n)(\lambda-\lambda_m^*)} a_{nm} = \int_0^{b_0} \left(\gamma(x,t,\lambda) - \sum_1^N \frac{\psi_n(x)\psi_n(t)}{\lambda-\lambda_n}\right)\gamma^*(x,t,\lambda)f(x)\,dx \tag{6.6}$$

(the Parseval relation, cf. (6.2)). Hence, if $|f(x)| < M$ p. p.,

$$\left|\sum_{N+1}^{\infty}\sum_1^{\infty}\right| \le M\left\{\int_0^{b_0}\left|\gamma - \sum_1^N\right|^2 dx \int_0^{b_0} |\gamma^*|^2 dx\right\}^{\frac{1}{2}} =$$

$$= M\left\{\sum_{N+1}^{\infty}\left|\frac{\psi_n(t)}{\lambda-\lambda_n}\right|^2 \cdot \sum_1^{\infty}\left|\frac{\psi_n^*(t)}{\lambda-\lambda_n^*}\right|^2\right\}^{\frac{1}{2}} \tag{6.7}$$

The series

$$\sum_1^{\infty}\left|\frac{\psi_n(t)}{\lambda-\lambda_n}\right|^2, \quad \sum_1^{\infty}\left|\frac{\psi_n^*(t)}{\lambda-\lambda_n^*}\right|^2,$$

which are convergent according to (6.2) and (5.3), are also uniformly convergent in D_R, hence uniformly bounded (this fact being easily proved, is taken for granted). It is then always possible to choose N independent of λ so great as to make the right-hand member of (6.7) less than a given ε in D_R.

Analogous arguments apply to the other two series of (6.5), whence the difference d (6.4) is less than 3ε for all $\lambda < D_R$ if N is chosen great enough. Hence $\int_0^{b_0} \gamma(x, t, \lambda)\gamma^*(x, t, \lambda) f(x)\, dx$ is meromorphic in $|\lambda| < \infty$ and $\{\lambda_n\}$, $\{\lambda_n^*\}$ are the only poles.

The order of this meromorphic function is finite, for by (3.1) one can choose the polynomials $Q_\nu(\lambda) = \frac{\lambda}{\lambda_\nu} + \cdots + \frac{1}{q}\left(\frac{\lambda}{\lambda_\nu}\right)^q$ of a degree $q \leq \varrho$, so that

$$P(\lambda) = \prod_{\nu=1}^{\infty}\left(1 - \frac{\lambda}{\lambda_\nu}\right) e^{Q_\nu(\lambda)}, \quad P^*(\lambda) = \prod_{\nu=1}^{\infty}\left(1 - \frac{\lambda}{\lambda_\nu^*}\right) e^{Q_\nu^*(\lambda)}$$

are integral functions of finite order with the zeros λ_ν and λ_ν^* respectively $(\nu = 1, 2, 3, \ldots)$. Then

$$P(\lambda) \cdot P^*(\lambda) \cdot \int_0^{b_0} \gamma(x, t, \lambda)\gamma^*(x, t, \lambda) f(x)\, dx$$

is an integral function of a finite order. For, putting $P_n = e^{Q_n(\lambda)} \prod_{\nu \neq n} \left(1 - \frac{\lambda}{\lambda_\nu}\right) e^{Q_\nu(\lambda)}$, $P_n^* = \ldots$, we get from (6.7) for $N = 0$

$$\left| P(\lambda) P^*(\lambda) \int_0^{b_0} \gamma\gamma^* f\, dx \right| \leq M \left\{ \sum_1^\infty \left| \frac{\psi_n(t) P_n(\lambda)}{\lambda_n} \right|^2 \sum_1^\infty \left| \frac{\psi_n^*(t) P_n^*(\lambda)}{\lambda_n^*} \right|^2 \right\}^{\frac{1}{2}} \leq$$

$$\leq M \left\{ \sum_1^\infty \left| \frac{\psi_n}{\lambda_n} \right|^2 \sum_1^\infty \left| \frac{\psi_n^*}{\lambda_n^*} \right|^2 \right\}^{\frac{1}{2}} \operatorname*{Max}_{n, m} \operatorname*{Max}_{|\lambda| = r} |P_n(\lambda) P_m^*(\lambda)|.$$

Now, if $|\lambda| = r$ is great enough, the right hand member is $\leq e^{r^{\varrho+\varepsilon}}$, for such an inequality holds manifestly, uniformly in n, for all $P_n(\lambda)$ and $P_n^*(\lambda)$. Thus $\int_0^{b_0} \gamma\gamma^* f\, dx$ is the quotient of two integral functions of a finite order, which proves the statement above.

At last we prove that $\int_0^{b_0} \gamma\gamma^* f\, dx$ converges to zero, when $|\lambda| \to \infty$ in the region

$$\left.\begin{array}{r}\varepsilon \leq \arg \lambda \leq \pi - \varepsilon \\ \pi + \varepsilon \leq \arg \lambda \leq 2\pi - \varepsilon\end{array}\right\} \tag{6.8}$$

(ε being arbitrarily small, as above we assume $\lambda_n, \lambda_n^* \neq 0$).

From (6.6) and (6.7) follows for $N = 0$

$$\int_0^{b_0} \gamma\gamma^* f\,dx = O\left(\left\{\sum_1^\infty \left|\frac{\psi_n}{\lambda - \lambda_n}\right|^2 \cdot \sum_1^\infty \left|\frac{\psi_n^*}{\lambda - \lambda_n^*}\right|^2\right\}^{\frac{1}{2}}\right). \tag{6.9}$$

Putting $\psi_n = \delta_n$ we have

$$\sum \left|\frac{\delta_n}{\lambda_n}\right|^2 < \infty \qquad \text{(cf. p. 272, last row, p. 273)}$$

and

$$\left\|\frac{\delta_n}{\lambda - \lambda_n}\right|^2 - \left|\frac{\delta_n}{\lambda_n}\right|^2\right| = \left|\frac{\delta_n}{\lambda_n}\right|^2 \cdot \left|\frac{|\lambda_n|^2 - |\lambda - \lambda_n|^2}{|\lambda - \lambda_n|^2}\right|.$$

Put $\lambda = re^{i\vartheta}$ and let for a moment λ_n be a variable, continuously varying from $-\infty$ to $+\infty$. Then the last factor assumes its max. for $\lambda_n = \frac{r}{\cos\vartheta}$ and the max. is $\cos^2\vartheta(1 - \cos^2\vartheta)^{-1}$. Further the inequality

$$\cos^2\vartheta \cdot (1 - \cos^2\vartheta)^{-1} \leq \cos^2\varepsilon(1 - \cos^2\varepsilon)^{-1}$$

holds for all λ:s belonging to (6.8). From this follows that the series on the right of (6.9) converge uniformly towards 0 in the region (6.8). For if we choose N fix and so great as to make

$$\sum_{N+1}^\infty \left|\frac{\delta_n}{\lambda_n}\right|^2 \frac{1}{1 - \cos^2\varepsilon} \leq \varepsilon'$$

(ε' arbitrarily small) then for every λ in the region (6.8)

$$\sum_{N+1}^\infty \left|\frac{\delta_n}{\lambda - \lambda_n}\right|^2 = \sum_{N+1}^\infty \left(\left|\frac{\delta_n}{\lambda - \lambda_n}\right|^2 - \left|\frac{\delta_n}{\lambda_n}\right|^2\right) + \sum_{N+1}^\infty \left|\frac{\delta_n}{\lambda_n}\right|^2 \leq \frac{1}{1 - \cos^2\varepsilon} \sum_{N+1}^\infty \left|\frac{\delta_n}{\lambda_n}\right|^2 \leq \varepsilon'.$$

Hence, since every term of the series in (6.9) converges to 0, the truth of the statement follows.

Thus the first integral of (6.1) has the property 3° of No. 5. We will now prove that this is also the case of the second. We shall restrict ourselves to one part of the integral:

$$I(t, \lambda) = \int_0^t g(x, t, \lambda)\,\gamma(x, t, \lambda) f(x)\,dx = \int_0^t g(x, t, \lambda)\,\phi(x, \lambda)\,\psi(t, \lambda) f(x)\,dx.$$

The remaining part is to be treated in formally the same way. Since $\phi(t,\lambda)$ and $\theta(t,\lambda)$ for t fix $< b_0$ are integral functions of a finite order, and $\psi(t,\lambda) = \theta(t,\lambda) + m(\lambda)\phi(t,\lambda)$, we immediately find that $I(t,\lambda)$ is a meromorphic function with the poles $\{\lambda_n\}$. Its order is finite, if this holds of $m(\lambda)$. This is really the fact. Using TITCHMARSH (rel. 2.5.7) we obtain the Parseval relation

$$\int_0^{b_0} \psi(t,\lambda)\, g(t)\, dt = \sum_{n=1}^{\infty} \frac{r_n^{\frac{1}{2}} \int_0^{b_0} \psi_n(t)\, g(t)\, dt}{\lambda - \lambda_n}$$

($r_n =$ the residue of $m(\lambda)$ in $\lambda = \lambda_n$, $g(x) < L^2(0, b_0)$), i. e. for

$$g(x) = \begin{cases} 1, & 0 \le x \le b' = \min\left(1, \frac{b_0}{2}\right), \\ 0, & b' < x \le b_0 \end{cases}$$

$$m(\lambda) \int_0^{b'} \phi(t,\lambda)\, dt = \sum_{n=1}^{\infty} (\lambda - \lambda_n)^{-1} r_n^{\frac{1}{2}} \int_0^{b_0} \psi_n(t)\, g(t)\, dt - \int_0^{b'_0} \theta(t,\lambda)\, dt. \tag{6.10}$$

As in the case of the series (6.3) one finds that the series of (6.10) represents a meromorphic function of a finite order (owing to the property (3.1)). Hence from the definition of the order of a meromorphic function we conclude that $m(\lambda)$, defined by (6.10), is so too.

It thus remains to be proved that $|I(t,\lambda)| \to 0$, when $|\lambda| \to \infty$ in the region (6.8). For the proof we need the properties of $m(\lambda)$, mentioned at the end of No. 2. In consequence of them

$$\vartheta(t,\lambda) = \theta(t,\lambda) + l_b(\lambda)\,\phi(t,\lambda)$$

describes (for λ, t fix) a circle S_b as $z = \cot\beta$ describes the real axis of z (for l_b describes the circle C_b and $\vartheta(t,\lambda)$ is of the form $a + b\,l_b$). The circle S_b is interior to the circle $S_{b'}$, if $b' < b$. If $m(\lambda)$ is the limit point or a point of the limit circle of the circles $C_b\,(b \to b_0)$, then $\psi(t,\lambda) = \theta(t,\lambda) + m(\lambda)\phi(t,\lambda)$ is the corresponding point, relatively the circles S_b, i. e. a point inside an arbitrary one of them. Let M_b be the centrum, r_b the radius of the circle S_b $(b < b_0)$. Then we have

$$|\psi(t,\lambda)| \le |M_b| + r_b. \tag{6.11}$$

The radius of S_b is manifestly $|\phi(t,\lambda)|$ times the one of C_b, and the centrum is by the formula $M_b = \theta + l\phi$ given in terms of the centrum of C_b. Then, using the well-known expressions for these last quantities[1], we get

$$|M_b| = |W(g(b, t, \lambda), \bar{\phi}(b)| \cdot \{2\operatorname{Im}(\lambda) \cdot \int_0^b |\phi(t, \lambda)|^2 dt\}^{-1}$$

$$r_b = |\phi(t, \lambda)| \cdot \{2\operatorname{Im}(\lambda) \cdot \int_0^b |\phi(t, \lambda)|^2 dt\}^{-1},$$

where $W(u(x), v(x)) = u(x)v'(x) - u'(x)v(x)$ and $g(x, t, \lambda)$ is defined in (5.8).

Now we will choose the fix number $b < b_0$ and $> t, x$. As is well known, we have for all λ:s greater than a positive number R

$$\phi(t, \lambda) = (\cos\sqrt{\lambda}\, t \sin\alpha - \lambda^{-\frac{1}{2}} \sin\sqrt{\lambda}\, t \cos\alpha)(1 + O(\lambda^{-\frac{1}{2}}))$$

$$g(x, t, \lambda) = \lambda^{-\frac{1}{2}} \sin\sqrt{\lambda}(x - t)(1 + O(\lambda^{-\frac{1}{2}})).$$

Putting $\lambda = r^2 e^{2i\gamma}$, we get after simple calculations

$$\left.\begin{matrix} |\phi(t, \lambda)| \\ r^{-1}|\phi'(t, \lambda)| \end{matrix}\right\} \leq \begin{cases} c\, e^{tr|\sin\gamma|}, & \sin\alpha \neq 0 \\ c\, r^{-1} e^{tr|\sin\gamma|}, & \sin\alpha = 0 \end{cases}$$

$$|W(g(b, t, \lambda), \bar{\phi}(b, \lambda))| \leq \begin{cases} c\, e^{(2b-t)r|\sin\gamma|}, & \sin\alpha \neq 0 \\ c\, r^{-1} e^{(2b-t)r|\sin\gamma|}, & \sin\alpha = 0 \end{cases}$$

$$\left|2\operatorname{Im}(\lambda)\int_0^b |\phi(t, \lambda)|^2 dt\right| \geq \begin{cases} c\, r|\cos\gamma| \cdot |\sin\operatorname{hyp}(2br\sin\gamma)|, & \sin\alpha \neq 0 \\ c\, r^{-1}|\cos\gamma| \cdot |\sin\operatorname{hyp}(2br\sin\gamma)|, & \sin\alpha = 0. \end{cases}$$

The relations hold for all λ:s in the region (6.8) and satisfying $|\lambda| > R$. c is here and in the sequel a constant, independent of x, t and λ, which has not necessarily the same numerical value in all cases.

According to (6.8) we have $\frac{\varepsilon}{2} \leq \gamma \leq \frac{\pi}{2} - \frac{\varepsilon}{2}$ or $\frac{\pi}{2} + \frac{\varepsilon}{2} \leq \gamma \leq \pi - \frac{\varepsilon}{2}$, i. e. $|\cos\gamma| \geq \left|\sin\frac{\varepsilon}{2}\right| > 0$. Hence by (6.11)

$$|\psi(t, \lambda)| \leq \begin{cases} c\, r^{-1} e^{-tr|\sin\gamma|}, & \sin\alpha \neq 0 \\ c\, e^{-tr|\sin\gamma|}, & \sin\alpha = 0. \end{cases}$$

From the definition of $I(t, \lambda)$ it now immediately follows that

$$|I(t, \lambda)| \leq \frac{c\int_0^t |f|\, dx}{|\lambda|}, \quad t \leq b < b_0, \ \lambda \text{ in (6.8)}, \ |\lambda| > R.$$

The property 3° of No. 5 hence follows.

[1] TITCHMARSH, p. 20.

7. Proof of the Property 4° of No. 5.

If $n \neq m$ then $\lambda_n \neq \lambda_m$ and $\lambda_n^* \neq \lambda_m^*$. Further it holds that $\lambda_n \neq \lambda_m^*$ for all n and m. For otherwise there must be two solutions $\psi_n(x)$ and $\psi_m^*(x)$, linearly independent owing to the fact that $\psi_n = r_n^{\frac{1}{2}} \phi(x, \lambda_n)$ and $\psi_m^* = r_n^{*\frac{1}{2}} \phi_n^*(x, \lambda_n^*)$[1], and belonging to $L^2(0, b_0)$. This is impossible in the limit point case. In the limit circle case we have

$$\psi^*(x, \lambda) = (a + c m^*(\lambda))[\theta(x, \lambda) + (b + d m^*)(a + c m^*)^{-1} \phi(x, \lambda)], \tag{7.1}$$

where a, b, c, d are constants, independent of λ (defined by $\theta^*(x, \lambda) = a\theta + b\phi$, $\phi^*(x, \lambda) = c\theta + d\phi$). Now

$$\vartheta^*(x, \lambda) = \theta^* + l^* \phi^* = (a + c l^*)[\theta + (b + d l^*)(a + c l^*)^{-1} \phi]$$

satisfy the boundary condition $\vartheta^*(b, \lambda) \cos\beta + \vartheta^{*\prime}(b, \lambda) \sin\beta = 0$, i. e. the quotient $(b + d l^*)(a + c l^*)^{-1}$ is a point of the circle C_b as well as $l(\lambda)$.[2] The boundary conditions (R) and (R^*) mean in the limit circle case that as $b \to b_0$, $l(\lambda)$ converges to a certain point $m(\lambda)$ of the limit circle and that the quotient $(b + d l^*)\cdot(a + c l^*)^{-1}$ converges to the same point. At the same time $l^*(\lambda)$ converges to $m^*(\lambda)$, a point of the limit circle of the circles C_b^*. Thus

$$m(\lambda) = (b + d m^*(\lambda))(a + c m^*(\lambda))^{-1}. \tag{7.2}$$

If $\lambda_n = \lambda_m^*$, we must have $c = 0$, i. e. $\phi^*(x, \lambda) \equiv d\phi(x, \lambda)$, which is impossible, owing to the fact that $\alpha^* \not\equiv \alpha \pmod{\pi}$.

So the poles of $\Phi(t, \lambda) = \int\limits_0^{b_0} \Gamma(x, t, \lambda) f(x)\, dx$ are simple. We will determine the residue in one of them, for instance $\lambda = \lambda_n$.

$$\begin{aligned} \lim_{\lambda=\lambda_n} (\lambda - \lambda_n) \int\limits_0^{b_0} \Gamma(x, t, \lambda) f(x)\, dx &= \lim_{\lambda=\lambda_n} (\lambda - \lambda_n) \int\limits_0^{b_0} 2\gamma\gamma^* f\, dx + \\ &+ \lim_{\lambda=\lambda_n} (\lambda - \lambda_n) \int\limits_0^{t} 2 g(x, t, \lambda)(\gamma + \gamma^*) f\, dx. \end{aligned} \tag{7.3}$$

From (6.3) we obtain for the first limit

$$2\psi_n(t) \sum_{m=1}^{\infty} \frac{\psi_m^*(t)}{\lambda_n - \lambda_m^*} \int\limits_0^{b_0} \psi_n(x) \psi_m^*(x) f(x)\, dx = 2\psi_n(t) \int\limits_0^{b_0} \gamma^*(x, t, \lambda_n) \psi_n(x) f(x)\, dx. \tag{7.4}$$

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[1] TITCHMARSH, p. 24. [2] Cf. No. 2.

The last expression is obtained by the Parseval relation, noticing that $\lambda_n \neq \{\lambda_n^*\}$, i. e. that $\gamma^*(x, t, \lambda_n) < L^2(0, b_0)$.

Let as above r_n denote the residue of $m(\lambda)$ in $\lambda = \lambda_n$. Then, according to $\psi_n(x) = r_n^{\frac{1}{2}} \phi(x, \lambda_n)$[1], we obtain for the second limit

$$\begin{aligned}\lim (\lambda - \lambda_n) \int_0^t 2\, g(x, t, \lambda)(\gamma + \gamma^*) f(x)\, dx &= \lim_{\lambda=\lambda_n} (\lambda - \lambda_n) \int_0^t 2\, g(x, t, \lambda)\, \gamma(x, t, \lambda) f(x)\, dx \\ &= 2 \int_0^t g(x, t, \lambda_n)\, \psi_n(x)\, \psi_n(t) f(x)\, dx,\end{aligned} \tag{7.5}$$

for the interval $(0, t)$ is finite and $\phi(x, \lambda) \to \phi(x, \lambda_n)$, $g(x, t, \lambda) \to g(x, t, \lambda_n)$ uniformly in that interval.

Adding (7.4) and (7.5) and using the last of the relations (5.8) we get

$$2\, \psi_n(t)\, \phi_n^*(t, \lambda_n) \int_0^{b_0} \psi_n(x)\, \psi^*(x, \lambda_n) f(x)\, dx.$$

Now $\psi^*(x, \lambda_n)$ is a solution of $Ly - \lambda_n y = 0$, which belongs to $L^2(0, b_0)$. In the limit-point case this means that $\psi^*(x, \lambda_n) = c_n \psi_n(x)$ (c_n being independent of x). Using (7.1) and (7.2) we easily find that the same relation holds also in the limit circle case. Thus

$$\lim_{\lambda=\lambda_n} (\lambda - \lambda_n) \int_0^{b_0} \Gamma(x, t, \lambda) f(x)\, dx = 2\, c_n\, \psi_n(t)\, \phi^*(t, \lambda_n) \int_0^{b_0} \psi_n^2(x) f(x)\, dx.$$

The residue in $\lambda = \lambda_n^*$ will be computed in the same way. It has the corresponding form: $2\, c_n^*\, \psi_n^*(t)\, \phi(t, \lambda_n^*) \int_0^{b_0} \psi_n^{*2}(x) f(x)\, dx$. The property 4° of No. 5 is proved.

8. The Point $x = b_0$ ($b_0 < \infty$) is Regular or Regularly Singular.

In this case is

$$\begin{aligned}\psi(x, \lambda) &= (x - b_0)^{\frac{1}{2}+\sigma} O(1), \quad \operatorname{Re}\sigma \geq 0 \\ \left.\begin{matrix}\theta(x, \lambda)\\ \phi(x, \lambda)\end{matrix}\right\} &= (x - b_0)^{\frac{1}{2}-\sigma} O(1) \quad \operatorname{Re}\sigma \geq 0,\end{aligned} \tag{8.1}$$

i. e. $\psi^2(x, \lambda) < L^2(0, b_0)$. Then $\Phi(t, \lambda) = \int_0^{b_0} \Gamma(x, t, \lambda) f(x)\, dx$ has a sense also when

[1] TITCHMARSH, p. 24.

only $f(x) < L^2(0, b_0)$ is assumed. It is also easily proved that the theorem of No. 3 holds if, in the present case, the class of functions $M(0, b_0)$ is exchanged for the class $L^2(0, b_0)$. Only a few modifications in the proof above are needed. These depend upon the fact that the inequality, leading to the relation (6.7), does not generally hold. But instead we have equality in the relations (6.2), (6.2*), which gives all that is needed for the proof. The validity of these equalities follows from the fact that the functions $\psi_n(x)$ are in this case uniformly bounded and that $\Sigma \lambda_n^{-\frac{1}{2}-\varepsilon} < \infty$. These last properties can in their turn be traced back to the corresponding properties of the Bessel functions (of 1st kind). We need only write the eq. (L) in the form

$$y(x) = (b_0 - x)^{\frac{1}{2}} J_\nu[(b_0 - x)\sqrt{\lambda}] + \int_{b_0}^{x} g_1(x, t, \lambda)\left[q(t) - \frac{a}{(b_0 - t)^2}\right] y(t)\, dt,$$

where $g_1(x, t, \lambda)$ is the function $g(x, t, \lambda)$ of (5.8), but now belonging to the eq. $\frac{d^2 y}{d x^2} + \left(\lambda - \frac{a}{(b_0 - x)^2}\right) y = 0$, and where, further, $a = \nu^2 - \frac{1}{4}$, and chosen so that $q(x) - \frac{a}{(b_0 - x)^2}$ has no pole of order 2 at $x = b_0$. Then we apply a Liouville-Birkhoff estimation of $y(x)$ to find the results above. We do not enter upon details. The method is often used by R. E. Langer.

9. On the Minimality of the Set $\{\psi_n^2(x), \psi_m^{*2}(x)\}$.

A set $\{u_n(x)\}$ is said to be minimal ($u_n < L^2(0, b_0)$) in $L^2(0, b_0)$, if no function of the set can be approximated arbitrarily exactly by sums of the others. It especially holds, that $\{u_n(x)\}$ is minimal, if there is a set $\{v_n(x)\}$ ($v_n < L^2(0, b_0)$) of such a kind that $\{u_n(x); v_n(x)\}$ is a biorthogonal and normal set.

The set $\{\psi_n^2(x), \psi_n^{*2}(x)\}$ is not orthogonal. Therefore it is of interest to know something about the question of minimality. We shall prove

If (L) is regular or regularly singular at $x = b_0$, then the set $\{\psi_n^2(x), \psi_m^{*2}(x)\}$ is minimal after the exclusion of at most two eigenfunction-squares.

We will briefly sketch the proof, which is a generalization of an earlier one.[1] Put

[1] Cf. G. Borg, Acta math. 78 (1945), p. 57 f.

$$u_{2n-1}(x) = \psi_n^2(x), \ u_{2n}(x) = \psi_n^{*2}(x), \qquad n = 1, 2, 3, \ldots$$

We shall form a set $\{v_n(x)\}$, of such a kind that $(u_n(x); v_n(x))$ $(n = 3, 4, 5, \ldots)$ is biorthogonal and normal. We state that the set may be defined as follows:

$$v_{2m-1}(x) = \frac{d}{dx}(c_m \psi_m(x) \phi^*(x, \lambda_m) - d_m \psi_1(x) \phi^*(x, \lambda_1) - e_m \psi_1^*(x) \phi(x, \lambda_1^*))$$

$$v_{2m}(x) = \frac{d}{dx}(c_m^* \psi_m^* \phi(x, \lambda_m^*) - d_m^* \psi_1(x) \phi^*(x, \lambda_1) - e_m^* \psi_1^*(x) \phi(x, \lambda_1^*))$$

$(m = 2, 3, 4, \ldots)$, if the constants c_m, c_m^* etc. are appropriately determined. It may be observed that the principal term of $v_{2m-1}(x)$, $c_m \psi_m(x) \phi^*(x, \lambda_m)$, is proportional to the function $v_n(x)$, entering into the principal part of $\Phi(x, \lambda)$ at the pole $\lambda = \lambda_n$ (cf. No. 7 and 4° of No. 5). Owing to (8.1), all these functions belong to $L^2(0, b_0)$. We need further the relation (27) of the paper mentioned above[1], which here takes the form

$$(\lambda_u - \lambda_v) \int_0^{b_0} (u v' - u' v)\, dx = (\lambda_u - \lambda_v)\left(2 \int_0^{b_0} u v'\, dx - \Big/_0^{b_0} u v\right) = \begin{vmatrix} y, & \eta \\ y', & \eta' \end{vmatrix} \cdot \begin{vmatrix} y, & \zeta \\ y', & \zeta' \end{vmatrix} \Big/_0^{b_0},$$

if we put $u = y^2(x)$, $v = \eta(x) \zeta(x)$ and the functions y, η, ζ satisfy the eqs.

$$L y - \lambda_u y = 0, \ L \eta - \lambda_v \eta = 0, \ L \zeta - \lambda_v \zeta = 0.$$

Now let $u = u_n(x)$ and

$$v = c_m \psi_m(x) \phi^*(x, \lambda_m), \ = - d_m \psi_1(x) \phi^*(x, \lambda_1), \ = - e_m \psi_1^*(x) \phi(x, \lambda_1^*)$$

successively and $\lambda_u \neq \lambda_m, \lambda_1, \lambda_1^*$. All right hand members $= 0$ according to the boundary conditions. Dividing by $(\lambda_u - \lambda_m)$, $(\lambda_u - \lambda_1)$ and $(\lambda_u - \lambda_1^*)$ respectively and adding, we get ($V_{2m-1} = c_m \psi_m \phi^* - d_m \psi_1 \phi^* - e_m \psi_1^* \phi$)

$$2 \int_0^{b_0} u_n(x) v_{2m-1}(x)\, dx - \Big/_0^{b_0} u_n(x) V_{2m-1}(x) = 0.$$

It is now a simple matter to determine the constants d_m and e_m so that $\Big/_0^{b_0} u_n(x) V_{2m-1}(x) = 0$ independently of the values of n; for we only need to solve

[1] Cf. G. Borg, Acta math. 78, p. 40 f.

the system of two linear eqs.

$$d_m \psi_1(x) \phi^*(x, \lambda_1) + e_m \psi_1^*(x) \phi(x, \lambda_1^*) = c_m \psi_m(x) \phi^*(x, \lambda_m) \tag{9.1}$$
$$\text{for } x = b_0 \text{ and } x = 0.$$

They always have finite solutions. In the same way d_m^* and e_m^* are determined Thus

$$\int_0^{b_0} u_n(x) v_m(x) \, dx = 0, \qquad n \neq m.$$

At last, after simple calculations[1], using (9.1), we get

$$\int_0^{b_0} u_{2n-1}(x) v_{2n-1}(x) \, dx = \frac{c_n}{2} W(\psi_n(x), \phi^*(x, \lambda_n)) \int_0^{b_0} \psi_n^2(x) \, dx$$

and analogous for even indices. Hence the constants c_n and c_n^* can always be chosen so as to make the set $\{u_n(x); v_n(x)\}$ normalized.

This proves the theorem.

10. On the Completeness of the Set $\{\psi_n^2(x)\}$.

The set $\{\sin nx\}$ is complete in $L^2(0, \pi)$; the set of squares $\{\sin^2 nx\}$ is complete in $L^2\left(0, \frac{\pi}{2}\right)$, i. e, in "half" the former space. We shall prove the following generalization of this property[2]

If the eq. (L) *is regular or regularly singular at* $x = b_0$ $(b_0 < \infty)$ *then the set of eigenfunction-squares of* (L, R) *is complete in the space* $L^2\left(\frac{b_0}{2}, b_0\right)$, *i. e. in the Hilbert space, belonging to half the original interval* $(0, b_0)$.

For the proof we will use the theorem of No. 3, but still one clause is needed. We assume $f(x) < L^2\left(\frac{b_0}{2}, b_0\right)$ and

$$\int_{b_0/2}^{b_0} \psi_n^2(x) f(x) \, dx = 0, \qquad n = 1, 2, 3, \ldots$$

The consequence wanted is $f(x) \equiv 0$.

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[1] Cf. G. Borg, loc. cit. p. 43.

[2] This theorem is a result of discussions with Prof. A. Beurling, Uppsala. I take the opportunity to thank him for this and for valuable advice during the preparation of this paper.

Let $m(\lambda) = \frac{T(\lambda)}{N(\lambda)}$, where $T(\lambda)$ and $N(\lambda)$ are integral functions without a common divisor, and put

$$\Psi(\lambda) = \int_{b_0/2}^{b_0} N(\lambda)\,\psi^2(x, \lambda) f(x)\,dx.$$

In the present case, $N(\lambda)$ and $T(\lambda)$ are integral functions of a middle type of order $1/2$ ($\Sigma \lambda_n^{-\frac{1}{2}-\varepsilon} < \infty$, cf. the argument of No. 6 p. 275). This property is the real foundation of the proof. We will only carry it through in the following special case.

Let $x = b_0$ be a regular point, and $y(0) = y(b_0) = 0$ be the boundary conditions (R). Then it holds that

$$\psi(x, \lambda) = \frac{\sin \sqrt{\lambda}(b_0 - x)}{\sqrt{\lambda}} \cdot \left\{\frac{\sin \sqrt{\lambda}\, b_0}{\sqrt{\lambda}}\right\}^{-1} \left(1 + O\left(\frac{1}{\sqrt{\lambda}}\right)\right)$$

$$N(\lambda) = \frac{\sin \sqrt{\lambda}\, b_0}{\sqrt{\lambda}} \left(1 + O\left(\frac{1}{\sqrt{\lambda}}\right)\right)$$

and thus

$$\Psi(\lambda) = \int_{b_0/2}^{b_0} \frac{\sin^2 \sqrt{\lambda}(b_0 - x)}{\sqrt{\lambda} \sin \sqrt{\lambda}\, b_0} \left(1 + O\left(\frac{1}{\sqrt{\lambda}}\right)\right) f(x)\,dx.$$

Since the integrand is $\leq C \cdot |f(x)|\,|\lambda|^{-\frac{1}{2}} \exp\left\{\left(2 \cdot \frac{b_0}{2} - b_0\right) |\operatorname{Im}(\sqrt{\lambda})|\right\}$ for all λ:s in the region (6.8), we get $|\Psi(\lambda)| \leq \frac{C}{\sqrt{|\lambda|}}$, λ in the region (6.8). Further $\Psi(\lambda)$ is meromorphic of a finite order and the principal parts at the poles $\lambda = \lambda_n$ are of the form

$$(\lambda - \lambda_n)^{-1} c_n \int_{b_0/2}^{b_0} \psi_n^2(x) f(x)\,dx,$$

whence, according to the assumption above, they are all $= 0$, and $\Psi(\lambda)$ is an integral function of a finite order. As in the case of theorem of No. 3 we then get $\Psi(\lambda) \equiv 0$, i. e.

$$\int_{b_0/2}^{b_0} \psi^2(x, \lambda) f(x)\,dx \equiv 0.$$

From this we conclude that $f(x)$ is orthogonal to any set of eigenfunction-squares, belonging to (L) and boundary conditions $y\left(\frac{b_0}{2}\right) \cos \alpha + y'\left(\frac{b_0}{2}\right) \sin \alpha = 0$;

$y(b_0) = 0$. Applying the theorem of No. 3 (in the version of No. 8) for the interval $(b_0/2, b_0)$, we get $f(x) \equiv 0$ as stated.

Further we state without detailed proof that

The set $\{\psi_n^2(x)\}$ is after the exclusion of at most one eigenfunction-square minimal in every space $L^2\left(\frac{b_0}{2} - \varepsilon, b_0\right)$ $(\varepsilon > 0)$.

The proof depends again upon a construction of a biorthogonal set. We choose boundary values (R^*) and form the biorthogonal set $\{u_n(x); v_n(x)\}$ of No. 9. This can be done in such a way that at most one eigenfunction-square is superfluous, viz. the function $\psi_1^2(x)$. Then the set $\{\psi_n^2(x), v_{2n-1}(x)\}$ $(n = 2, 3, 4, \ldots)$ is biorthogonal in $L^2(0, b_0)$, and all functions $v_{2m}(x)$ are orthogonal to all $\psi_n^2(x)$ $(n = 2, 3, 4, \ldots)$. Further it is possible to prove that the set $\{v_{2m}(x)\}$ is complete in $L^2\left(0, \frac{b_0}{2} - \varepsilon\right)$. The proof is analogous to that above in this No. Then for every n we can approximate $v_{2n-1}(x)$ by a sum $\Sigma c_{n\nu} v_{2\nu}(x)$ in $L^2\left(0, \frac{b_0}{2} - \varepsilon\right)$. Putting

$$\varphi_n \sim v_{2n-1}(x) - \Sigma c_{n\nu} v_{2\nu}(x)$$

we get

$$\int_0^{b_0} \psi_n^2(x)\, \varphi_m(x)\, dx = \int_{\frac{b_0}{2}-\varepsilon}^{b_0} \psi_n^2(x)\, \varphi_m(x)\, dx$$

and the first member also equals δ_{nm} (Kronecker δ) according to the biorthogonality property of the set $\{u_n(x); v_n(x)\}$.

The applications of the theorems above to Bessel functions, belonging to a boundary value problem over a finite interval, and to Legendre functions are immediate.

In a later paper I shall further unfold the results and apply them to inverse boundary-value problems.

Uppsala, oct. 1947.