ON TWO PROBLEMS CONCERNING LINEAR TRANSFORMATIONS IN HILBERT SPACE.

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Introduction.

1. Let H be a Hilbert space and T and T^* two adjoined transformations, both determined throughout H. Let Φ_λ be the set of eigenelements of T, corresponding to λ, i. e. the solutions $\varphi \neq 0$ of the equation $T\varphi = \lambda\varphi$; and Φ the sum of all Φ_λ. Firstly we assume that

(A) *the set Φ is fundamental on H.*

We shall denote by C_f and C_g^* the closed linear manifolds spanned by $\{T^n f\}_0^\infty$ and $\{T^{*n} g\}_0^\infty$, respectively; f, g being elements in H.

This study is devoted to two general problems concerning the transformations T and T^* which we shall call the extinction problem and the closure problem. We shall say that T has *an extinction theorem* if, for every $f \neq 0$, it is true that the manifold C_f contains at least one eigenelement $\varphi \neq 0$. In the case

$$f = \sum_{\nu=0}^{n} c_\nu \varphi_\nu, \quad \varphi_\nu \in \Phi_{\lambda_\nu},$$

where $\lambda_\nu \neq \lambda_\mu$ for $\nu \neq \mu$, it is obvious that all φ_ν belong to C_f. By (A), every f may certainly be approximated arbitrarily closely by linear combinations of eigenelements; but this does by no means imply that the extinction theorem is a consequence of (A).

By the closure problem we mean the characterizing of the elements g, for which $C_g^* = H$, by the behaviour of the scalar product (φ, g), when φ runs through Φ. From the relations

$$(\varphi_\lambda, T^{*n} g) = (T^n \varphi_\lambda, g) = \lambda^n (\varphi_\lambda, g), \quad n \geq 0,$$

it immediately follows that the condition

$$(\varphi, g) \neq 0, \quad \varphi \in \Phi, \tag{1}$$

is *necessary* for closure. If this condition also is sufficient we shall, on account of a well known analogy in harmonic analysis, say that the transformation T^* will possess a *Wiener closure theorem.*

It seems to be a very difficult undertaking to decide, in general non-trivial cases, whether these two theorems are true or not and if they are equivalent. However, under the assumptions already made, it is always true that the extinction theorem implies the Wiener closure theorem. For, if C_g^* is a proper subset of H, an element $f \neq 0$ exists, such that

$$0 = (f, T^{*n} g) = (T^n f, g), \quad n \geq 0.$$

Thus g is orthogonal to every $h \in C_f$, and hence to the eigenelement φ, which, according to the extinction theorem, must belong to C_f.

2. If we, in addition to the postulate (A), also assume that T is isometric, the extinction theorem holds and is a simple consequence of v. Neumann's ergodic theorem which we state in the following generalized form, due to F. Riesz[1] and G. Birkhoff:

If T is a linear isometric transformation, or a contraction ($\|Tf\| \leq \|f\|$), of a uniformly convex Banach space, then the limit

$$\lim_{n=\infty} \frac{1}{n} \sum_0^{n-1} T^\nu f = S(f)$$

will exist for every element f.

Let us first give the following complement of this theorem. We shall say that f is orthogonal to g, or $f \perp g$, if

$$\|f + c g\| \geq \|f\|$$

for every complex number c. It ought to be observed that in general Banach spaces the property $f \perp g$ does not imply $g \perp f$.

If T has a fixelement $\varphi_0 = T \varphi_0$ that is not orthogonal to f, then the limit $S(f)$ will be different from the null element.

By the definition of orthogonality, there exists a constant c such that $\|\varphi_0 + cf\| < \|\varphi_0\|$. From this and the relations $\|S(g)\| \leq \|g\|$ and $S(\varphi_0) = \varphi_0$, it follows that

[1] See [4] in the References and G. Birkhoff, *The mean ergodic Theorem,* Duke vol. 5 (1939).

$$\| S(cf) \| = \| S(\varphi_0 + cf) - \varphi_0 \| \geq \| \varphi_0 \| - \| \varphi_0 + cf \| > 0,$$

i. e. $S(f) \neq 0$. We thus get the following general extinction theorem:

Let T be a linear isometric transformation of a uniformly convex Banach space, such that the set Φ of eigenelements of T has the property:

(A') $\varphi \perp f, \quad \varphi \in \Phi$ *implies* $f = 0$.

Then for every $f \neq 0$ the manifold C_f will contain at least one eigenelement $\varphi \neq 0$.

By (A') a $\varphi = \varphi_\lambda$ must exist which is not orthogonal to f. Since T is isometric, we will have $|\lambda| = 1$ and the operator $T_\lambda = \lambda^{-1} T$ is consequently isometric too, and has φ_λ as fixelement. Thus

$$S_\lambda(f) = \lim_{n=\infty} \frac{1}{n} \sum_{0}^{n-1} T_\lambda^\nu f \neq 0,$$

and the theorem follows since $S_\lambda(f)$ belongs both to C_f and Φ.[2]

It is immediately seen, that the theorem holds true also for a contraction, provided that its eigenvalues lie on the unit circumference $|\lambda| = 1$.

3. Returning to the space H, it is now natural to consider the following case: T is a proper metric contraction, i. e.

(B) $$\| Tf \| \leq \| f \|, \quad \lim_{n=\infty} \| T^n f \| = 0^{3},$$

while T^* is isometric,

(C) $$\| T^* f \| = \| f \|.$$

As will be seen subsequently, the class of operators subject to the conditions (A), (B) and (C) is still too wide to admit general results concerning the extinction and closure problems. However, under the additional assumption

(D) *at least one eigenvalue is simple,*

the two problems may be completely discussed with the following principal results: neither the Wiener closure criterion (1), nor the extinction theorem are valid; but the Wiener closure theorem holds true in a modified form stating that the inner product (φ_λ, g) is different from zero and, for normalized φ_λ, not »too small» as $|\lambda| \to 1$.

Similar results will be obtained for the extinction problem.

[2] The interest of this theorem is chiefly due to the fact that the relevant orthogonality is $\varphi \perp f$ and not the converse but more natural $f \perp \varphi$. It should also be noted that at least in the ordinary L^p-spaces ($p > 1, \neq 2$) there are subsets M having the property (A') without being fundamental.

[3] This latter condition may be replaced by the following weaker assumption: the eigenvalues of T are of modulus < 1.

An Isomorphism.

4. By elementary arguments we obtain the following proposition:

The conditions (A), (B), (C), and (D), imply the existence of a complete orthonormal set $\{e_n\}_0^\infty$, *such that*

$$\begin{cases} T e_0 = 0 \\ T e_n = e_{n-1}, \quad n \geq 1, \end{cases} \tag{2}$$

$$T^* e_n = e_{n+1}, \quad n \geq 0. \tag{3}$$

The eigenvalues of T *are all simple and fill the open unit circle* $|\lambda| < 1$. *The corresponding eigenelements, normalized by the condition* $(\varphi_n, e_0) = 1$, *are of the form*

$$\varphi_\lambda = \sum_0^\infty \lambda^n e_n. \tag{4}$$

In accordance with (C),

$$(T T^* f, g) = (T^* f, T^* g) = (f, g),$$

for every pair of elements f and g, and thus,

$$T T^* = I = \text{the identity}. \tag{5}$$

By (D), there exists a simple eigenvalue $\lambda = \alpha$, which in view of (B) must be of modulus < 1. If φ_α is a corresponding eigenelement, then $e_0 = \varphi_\alpha - \alpha T^* \varphi_\alpha$ will be different from the null element, since by (C), $\|e_0\| \geq \|\varphi_\alpha\|(1 - |\alpha|)$. In the following, we will suppose that φ_α is normalized by the condition $\|e_0\| = 1$. We get by (5)

$$T e_0 = T \varphi_\alpha - \alpha T T^* \varphi_\alpha = \alpha \varphi_\alpha - \alpha \varphi_\alpha = 0; \tag{6}$$

hence e_0 is an eigenelement corresponding to $\lambda = 0$. Putting $e_n = T^{*n} e_0$, $n \geq 0$, it follows from (6) that, for $n > m \geq 0$,

$$(e_n, e_m) = (T^{*n} e_0, T^{*m} e_0) = (e_0, T^{n-m} e_0) = 0.$$

In view of the normalization $\|e_0\| = 1$ we then get

$$(e_n, e_m) = \begin{cases} 0, & n \neq m, \\ 1, & n = m. \end{cases}$$

By the definition of the set $\{e_n\}_0^\infty$ the relations (2) and (3) are satisfied, and thus φ_λ defined by the series (4), really represents an eigenelement. Putting $\lambda = \alpha$ in the series, we get back our original element φ_α.

It remains to prove that every eigenvalue is simple. If this is not true, there will exist a number β, $|\beta| < 1$, such that the equation $T\varphi = \beta\varphi$, besides the solution φ_β of formula (4), also has a solution $\varphi'_\beta \neq 0$, orthogonal to φ_β. Starting from φ'_β we obtain, in the same manner as before, an orthonormal set $\{e'_n\}_0^\infty$ with the same properties as $\{e_n\}_0^\infty$. Furthermore, $(e_0, e'_0) = 0$, from which follows that $(e_n, e'_m) = 0$, $n \geq 0$, $m \geq 0$. Consequently, the closed linear manifold spanned by the two sets are orthogonal. On the other hand this implies that, for every λ in the open unit circle, φ_λ of formula (4) and

$$\varphi'_\lambda = \sum_0^\infty \lambda^n e'_n,$$

will be linearly independent eigenelements.

From the preceding discussion, it will be clear that the conditions (A), (B) and (C) imply that the dimension number of the set Φ_λ is the same for all λ in the open unit circle, hence $= 1$ in view of (D). Since Φ is fundamental by assumption, and (4) represents all normalized eigenelements, the set $\{e_n\}_0^\infty$ must be complete, thus proving our proposition.

5. For every $f \in H$ we then have

$$f = \sum_0^\infty \bar{f}_n e_n, \quad f_n = (e_n, f),$$

$$\|f\|^2 = \sum_0^\infty |f_n|^2.$$

The below scalar product, where the parameter λ is replaced by z,

$$(\varphi_z, f) = \sum_0^\infty f_n z^n \equiv f(z), \quad |z| < 1, \tag{7}$$

thus transforms f into a function $f(z)$, holomorphic in the unit circle and satisfying the inequality

$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta \leq \|f\|^2, \quad 0 \leq r < 1.$$

According to well known properties, the radial limit

$$f(e^{i\theta}) = \lim_{r=1-0} f(re^{i\theta})$$

exist almost everywhere, and has a summable square. Furthermore, the class of Taylor series $f(z)$ constitutes a Hilbert space H with the scalar product

$$\frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\,\overline{g(e^{i\theta})}\,d\theta$$

and the norm $\|f\| = \sqrt{(f,f)}$. By means of (7) we obtain a unitary transformation of H into H.

The operator T takes the function $f(z)$ into

$$Tf(z) = \frac{f(z) - f(0)}{z},$$

while T^* takes $f(z)$ into

$$T^* f(z) = z f(z).$$

The eigenelements of T in the space H are obviously the functions

$$\frac{1}{1-\lambda z}, \qquad |\lambda| < 1.$$

The Closure and Extinction Problems.

6. In the space H we may formulate the closure problem in the following way:

For which functions $f(z)$ is it true that the set

$$\{z^n f(z)\}_0^\infty \tag{8}$$

is fundamental on H? If $f(z)$ does not possess this property, which functions are then contained in the closed linear manifold C_f^* spanned by the set (8)?

We already know that the Wiener criterion $f(z) \neq 0$, $|z| < 1$, is a necessary condition for closure. At first sight, this condition also seems to be sufficient. However, as we shall see, this is not true. On the other hand, an additional condition of the form

$$\overline{\lim_{r=1-0}}\ \frac{1}{2\pi}\int_0^{2\pi} \left|\frac{1}{f(re^{i\theta})}\right|^p d\theta < \infty, \quad p > 0,$$

proves to be sufficient but not necessary. By aid of a quantity $\delta(f)$ defined in the following manner

$$\delta(f) = \begin{cases} \frac{1}{2\pi}\int\limits_0^{2\pi} \log\left|\frac{f(e^{i\theta})}{f(0)}\right| d\theta & \text{if} \quad f(0) \neq 0 \\ +\infty & \text{if} \quad f(0) = 0 \end{cases} \tag{9}$$

the adequate condition takes the simple form $\delta(f) = 0$. We easily see that

$$\lim_{r=1-0} \frac{1}{2\pi}\int\limits_0^{2\pi} \log\left|\frac{f(e^{i\theta})}{f(r\,e^{i\theta})}\right| d\theta \geq 0 \tag{10}$$

holds true if $f \not\equiv 0$. The relation $\delta(f) = 0$ thus requires both that

$$\lim_{r=1-0} \frac{1}{2\pi}\int\limits_0^{2\pi} \log\left|\frac{f(r\,e^{i\theta})}{f(0)}\right| d\theta = 0$$

i. e. $f(z)$ has no zeros in the unit circle, and that the limit (10) vanishes, which means that $|f(z)|$ is not allowed to be very small as $|z| \to 1$.

7. By the proof, we shall avail ourselves of some well known properties, essentially due to Herglotz, F. and M. Riesz and R. Nevanlinna[4], concerning harmonic and analytic functions. Here we shall not express these results in their original scope, but in a modified form appropriate for our special purpose.

Let $f(z) \not\equiv 0$ be holomorphic for $|z| < 1$ and subject to a Hardy condition

$$\overline{\lim_{r=1-0}} \frac{1}{2\pi}\int\limits_0^{2\pi} |f(r\,e^{i\theta})|^p\, d\theta < \infty, \quad p > 0. \tag{11}$$

The radial limit $f(e^{i\theta})$ then exists almost everywhere and $\log|f(e^{i\theta})|$ is summable. Let us put

$$f_1(z) = \exp\left\{\frac{1}{2\pi}\int\limits_0^{2\pi} \log|f(e^{i\theta})| \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\theta + i\,a\right\} \tag{12}$$

where a is the argument of the first nonvanishing Taylor coefficient of $f(z)$. The following important inequality,

$$|f(z)| \leq |f_1(z)|, \quad |z| < 1, \tag{13}$$

holds always true, and the function $f_0(z)$ defined by the relation

$$f(z) = f_0(z)\, f_1(z), \tag{14}$$

[4] See [3] Chap. VII, also for further references.

will then have the properties,

$$|f_0(z)| \leq 1, \qquad |z| < 1, \tag{15}$$

$$\lim_{r=1-0} |f_0(r e^{i\theta})| = 1 \quad \text{p. p.} \tag{16}$$

In view of the normalization of $f_1(z)$, the first nonvanishing Taylor coefficient of $f_0(z)$ will be real and positive. The general expression for a function of this type is

$$f_0(z) = \prod \frac{a_\nu - z}{1 - z\bar{a}_\nu} \frac{\bar{a}_\nu}{|a_\nu|} \exp\left\{-\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\alpha\right\}, \tag{17}$$

where

$$\Sigma(1 - |a_\nu|) < \infty, \quad |a_\nu| < 1,$$

and where $\alpha = \alpha(\theta)$ is a real nondecreasing bounded function, whose points of increase form a set of at most zero measure. If some of the a_ν's are zero, we define the corresponding factors in the Blaschke product as z. In special cases both the Blaschke product and the exponential factor may, of course, be reduced to the constant 1.

A function which can be expressed in the form (12), where $U(\theta) \equiv \log |f(e^{i\theta})|$ is summable, we shall call an *outer function*, whereas a function of the form (17) shall be called an *inner function.* The special functions f_1 and f_0 defined above, shall be termed the *outer factor* and the *inner factor* of f respectively. This decomposition is obviously uniquely determined if $f \not\equiv 0$, and will be referred to as the Factorization Lemma.

8. Let now

$$g_0(z) = \prod \frac{b_\nu - z}{1 - z\bar{b}_\nu} \frac{\bar{b}_\nu}{b_\nu} \exp\left\{-\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\beta\right\}$$

be another inner function. When g_0/f_0 is bounded in the unit circle, then it obviously is an inner function too, and we shall call f_0 a divisor of g_0. For this it is necessary and sufficient that $\{a_\nu\}$ is a subset of $\{b_\nu\}$ and that $\beta - \alpha$ is nondecreasing. In the general case we define the largest common factor of f_0 and g_0 as the inner function

$$h_0(z) = \prod \frac{c_\nu - z}{1 - z\bar{c}_\nu} \frac{\bar{c}_\nu}{c_\nu} \exp\left\{-\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\gamma\right\}$$

where $\{c_\nu\}$ is the intersection of the sets $\{a_\nu\}$ and $\{b_\nu\}$ and the connection between γ, α and β (considered as nonnegative and completely additive setfunctions) is such that γ is the largest common minorant of α and β. It is easily seen that h_0 has the following characteristic property: if z is a point in the unit circle such that $|f_0(z)| + |g_0(z)| \neq 0$, then $|h_0(z)| < |k_0(z)|$ for any inner function $k_0 \not\equiv h_0$ which is a divisor of both f_0 and g_0. In the case $h_0 \equiv 1$, we shall say that f_0 and g_0 are without common factor.

Again, supposing that f and g are two functions, satisfying a condition (11) and $\not\equiv 0$, we may write

$$\frac{f}{g} = \frac{f_0}{g_0} \cdot \frac{f_1}{g_1}.$$

If h_0 is the largest common factor of f_0 and g_0, we have $f_0 = h_0 F_0$ and $g_0 = h_0 G_0$ and obtain

$$\frac{f}{g} = \frac{F_0}{G_0} \cdot H_1 \tag{18}$$

where $H_1 = f_1/g_1$ is an outer function and F_0, G_0 are inner functions without common factor. Obviously these three functions are all uniquely determined.

Regarding the quantity $\delta(f)$, we have $\delta(f_1) = 0$ for every outer function. Hence

$$\delta(f) = \delta(f_0) + \delta(f_1) = \delta(f_0).$$

For inner function, on the other hand,

$$\delta(f_0) = \log \frac{1}{f_0(0)} = \sum \log \frac{1}{|a_\nu|} + \int_0^{2\pi} d\alpha \geq 0.$$

The relation $\delta(f) = 0$ thus implies that the inner factor of f reduces to the constant 1.

9. The two problems, raised at the beginning of this chapter, will now be completely solved by the following:

Theorem I. *Let $f, g \in H$ and be $\not\equiv 0$. Then g will belong to the manifold C_f^* when, and only when, the inner factor of f is a divisor of the inner factor of g.*

The stated closure criterion $\delta(f) = 0$ is obviously a consequence of this theorem, since the property $C_f^* = H$ demands that the inner factor f_0 of f is a divisor of any inner function which can only be true if $f_0 \equiv 1$, i. e. if $\delta(f) = 0$.

Let us first prove that $g \in C_f^*$ if f_0 is a divisor of g_0. To this end, it is sufficient to prove that to every $\varepsilon > 0$, a polynominal p may be found such that

$$\|pf - g\| < \varepsilon.$$

But $|f_0(e^{i\theta})| = 1$ almost everywhere, and $g_0/f_0 = h_0$ is an inner function, hence

$$\|pf - g\| = \|pf_1 - h_0 g_1\|$$

and it is thus sufficient to prove that $C_{f_1}^* = H$ i. e. that the equations

$$(T^{*n} f_1, k) = 0, \quad n \geq 0 \tag{19}$$

have no solution $k \in H$ other than $k \equiv 0$. Let us put

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} f_1(e^{i\theta}) \overline{k(e^{i\theta})} \, e^{-in\theta} \, d\theta.$$

By (19), $c_n = 0$, $n \leq 0$, and consequently

$$f_1(e^{i\theta}) \overline{k(e^{i\theta})} \sim \sum_1^\infty c_n e^{in\theta}, \tag{20}$$

where $\sim$ is the common symbol in the theory of Fourier series. Since the left hand member in (20) is a summable function, the Taylor series

$$\psi(z) = \sum_1^\infty c_n z^n, \quad |z| < 1, \tag{21}$$

has the following well known properties,

$$\psi(e^{i\theta}) = \lim_{r=1-0} \psi(re^{i\theta}) = f_1(e^{i\theta}) \overline{k(e^{i\theta})} \quad \text{p. p.} \tag{22}$$

$$\lim_{r=1-0} \frac{1}{2\pi} \int_0^{2\pi} |\psi(e^{i\theta}) - \psi(re^{i\theta})| \, d\theta = 0. \tag{23}$$

According to the Factorization Lemma we get, if k_1 and ψ_1 denote the outer factors of k and ψ, respectively,

$$|\psi(z)| \leq |\psi_1(z)| = |f_1(z)| \, |k_1(z)|, \quad |z| < 1.$$

Hence

$$\left|\frac{\psi(z)}{f_1(z)}\right| \leq |k_1(z)|, \quad |z| < 1,$$

and the function ψ/f_1 then belongs to H and vanishes at the origin. It then follows from (22) that

$$(24)\qquad 0=\frac{1}{2\pi}\int_0^{2\pi}\frac{\psi(e^{i\theta})}{f_1(e^{i\theta})}e^{in\theta}\,d\theta=\frac{1}{2\pi}\int_0^{2\pi}\overline{k(e^{i\theta})}\,e^{in\theta}\,d\theta,\qquad n\geq 0.$$

Again, since $k\in H$,

$$0=\frac{1}{2\pi}\int_0^{2\pi}k(e^{i\theta})\,e^{in\theta}\,d\theta,\qquad n\geq 1.$$

By taking the conjugate value of (24), we see that all the Fourier coefficients of $k(e^{i\theta})$ vanish; thus $k\equiv 0$ and $C^*_{f_1}=H$.

We still have to prove that g cannot belong to C^*_f if g_0 is not divisible by f_0. To this end, let $m(r)$ denote the minimum of $|f_0(z)|$ on the circle $|z|=r<1$, and let r be a fixed value such that $m(r)=m>0$. Under the assumption $g\in C^*_f$ there will, for every $\varepsilon>0$, exist a polynomial $p=p_{r,\varepsilon}$, such that $\|pf-g\|<\varepsilon m$. Hence for $|z|=\varrho$, $0\leq\varrho\leq 1$,

$$(25)\qquad \frac{1}{2\pi}\int_0^{2\pi}|p(z)f(z)-g(z)|^2\,d\theta<\varepsilon^2m^2.$$

Putting $g_0/f_0=h$, we get on the circle $|z|=r$

$$(26)\qquad \frac{1}{2\pi}\int_0^{2\pi}|p(z)f_1(z)-h(z)\,g_1(z)|^2\,d\theta<\varepsilon^2$$

and obtain, for $|z|=r$, by Minkowsky's inequality,

$$(27)\qquad \frac{1}{2\pi}\int_0^{2\pi}|h(z)\,g_1(z)|^2\,d\theta<(\|g\|+\varepsilon(1+m))^2.$$

From this it is obvious that the function hg_1 must belong to H, and since $|h(e^{i\theta})|=1$ almost everywhere, g_1 must be the outer factor of hg_1, and thus h its inner factor. Then g_0 is divisible by f_0, which ends the proof.

In the preceding we have seen that the inner factor of f is of decisive importance for the properties of the set C^*_f. Thus it follows from Theorem I that a function generates the same manifold as its inner factor, i. e. $C^*_f=C^*_{f_0}$. More generally, C^*_f and C^*_g are identical when, and only when f_0 is a divisor of g_0, and conversely, g_0 a divisor of f_0, which will occur only when $f_0=g_0$.

10. Now we shall see that the inner factor, together with the quantity $\delta(f)$, besides their function-theoretical definition, also may be characterized by certain minimum properties in the manifold C_f^*. If e denotes the unit element $(e(z) \equiv 1)$, we have

$$T^{*n} e = z^n, \quad n = 0, 1, 2, \ldots \tag{28}$$

and, if d is the distance from e to C_f^*, then the condition $d = 0$ is obviously both necessary and sufficient for closure. In this case we have $f_0 = e$; in the general case the following holds true:

Theorem II. *The projection of e on C_f^* coincides with the inner factor f_0 of f multiplied by the constant $\sqrt{1 - d^2}$. The quantities d and $\delta = \delta(f)$ are connected by the relation*

$$d^2 = 1 - e^{-2\delta}. \tag{29}$$

If $f(z)$ vanish with its first $p - 1$ derivatives at the origin then the projection of $T^{*p} e = z^p$ on C_f^* falls on $\sqrt{1 - d_p^2}\, f_0$, where d_p is the distance from z^p to C_f^*.

By the proof of the first part of the theorem, we disregard the case $f(0) = 0$ as being trivial, since $d = 1$, $\delta = \infty$. Let us then assume $f(0) \neq 0$ and let g be the projection of e on C_f^*. Obviously $g - e$ must be orthogonal to all elements of C_f^*, hence in particular, to $\{T^{*n} g\}_0^\infty$, which yields,

$$\frac{1}{2\pi}\int_0^{2\pi} \overline{(g(e^{i\theta}) - 1)}\, g(e^{i\theta})\, e^{in\theta}\, d\theta = 0, \quad n \geq 0, \tag{30}$$

$$\frac{1}{2\pi}\int_0^{2\pi} |g(e^{i\theta})|^2\, e^{in\theta}\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} g(e^{i\theta})\, e^{in\theta}\, d\theta = \begin{cases} g(0), & n = 0 \\ 0\;, & n \geq 1. \end{cases} \tag{31}$$

By taking the conjugate part of this integral, we see that the Fourier series of $|g(e^{i\theta})|^2$ reduces to the constant term $g(0)$ and therefore,

$$|g(e^{i\theta})|^2 = g(0) \quad \text{p. p.}$$

from which follows,

$$\delta(g) = \tfrac{1}{2} \log \frac{1}{g(0)} \tag{32}$$

$$d^2 = 1 - g(0) = 1 - e^{-2\delta(g)}. \tag{33}$$

Let us now compare $g(z)$ with the function $h(z) = f_0(0) f_0(z)$; f_0 being the inner factor of f. A simple computation yields

$$\|h - e\|^2 = 1 - f_0^2(0) = 1 - e^{-2\delta(f_0)}. \tag{34}$$

Since f_0 must be a divisor of the inner factor of g, $\delta(g) \geq \delta(f_0)$; hence, by (33) and (34),

(35) $$\|h - e\| \leq \|g - e\| = d.$$

On the other hand, it follows from the definition of g that

$$\|h - e\| \geq \|g - e\|.$$

Then the sign of equality must hold in (35), which implies $h = g$, since the manifold C_f^* is linear and the Hilbert space is uniformly convex. Thus,

$$g(z) = f_0(0) f_0(z) = \sqrt{1 - d^2}\, f_0(z),$$

and the first part of the theorem is established.

As for the latter part of the theorem, let H_p be the subset of H consisting of all functions that vanish with their $p - 1$ first derivatives at the origin. The transformation defined as a multiplication by z^{-p} is then isometric and determined throughout H_p. From this it is clear, that the second part of the theorem is a consequence of the properties already proved. Let us only note that if $f_0(z) = a_p z^p + \ldots$, $a_p \geq 0$, then

(36) $$d_p^2 = 1 - a_p^2,$$

and $g_p(z) = a_p f_0(z)$, g_p being the projection of z^p on C_f^*.

Theorem III. *The closed linear manifold $C_{f,g}^*$ spanned by the sets*

$$\{z^n f(z)\}_0^\infty, \quad \{z^n g(z)\}_0^\infty,$$

where $f, g \not\equiv 0$, is identical with $C_{h_0}^*$ generated by the largest common divisor h_0 to the inner factors of f and g.

Firstly, let us prove that $h_0 \in C_{f,g}^*$; i. e. for every $\varepsilon > 0$ polynomials p and q exist such that $\|pf + qg - h_0\| < \varepsilon$. Putting

$$f = f_0 f_1 = h_0 F_0 f_1 \equiv h_0 F$$
$$g = g_0 g_1 = h_0 G_0 g_1 \equiv h_0 G,$$

F_0 and G_0 will, by assumption, be inner functions without common factor. Hence

$$\|pf + qg - h_0\| = \|pF + qG - 1\|$$

and it is then sufficient to prove that $C_{F,G} = H$.

If this is not true an $h \not\equiv 0$ will exist, orthogonal to $\{z^n F(z)\}_0^\infty$, $\{z^n G(z)\}_0^\infty$. Putting

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} \overline{h(e^{i\theta})}\, F(e^{i\theta})\, e^{-in\theta}\, d\theta,$$

$$b_n = \frac{1}{2\pi}\int_0^{2\pi} \overline{h(e^{i\theta})}\, G(e^{i\theta})\, e^{-in\theta}\, d\theta,$$

we thus have $a_n = b_n = 0$, $n \leq 0$, and the functions

$$\varphi(z) = \sum_1^\infty a_n z^n, \quad |z| < 1,$$

$$\psi(z) = \sum_1^\infty b_n z^n, \quad |z| < 1,$$

will possess the same properties as $\psi(z)$ of formula (21). Since the radial limits $F(e^{i\theta})$ and $G(e^{i\theta})$ may vanish only on sets of measure zero, we have almost everywhere,

$$\lim_{r=1-0} \frac{\varphi(r e^{i\theta})}{F(r e^{i\theta})} = \overline{h(e^{i\theta})},$$

$$\lim_{r=1-0} \frac{\psi(r e^{i\theta})}{G(r e^{i\theta})} = \overline{h(e^{i\theta})},$$

and thus the quotients φ/F and ψ/G will represent one and the same meromorphic function $m(z)$ in the unit circle. According to (18) we may now write,

$$m = \frac{k_0}{l_0} m_1,$$

where k_0 and l_0 are inner functions without common factor, and m_1 an outer function such that $|m_1(e^{i\theta})| = |h(e^{i\theta})|$. Putting $\varphi = \varphi_0 \varphi_1$ and $\psi = \psi_0 \psi_1$, we thus obtain

$$\frac{\varphi_0 \varphi_1}{F_0 f_1} = \frac{\psi_0 \psi_1}{G_0 g_1} = \frac{k_0}{l_0} m_1.$$

Hence

$$F_0 = \frac{\varphi_0 l_0}{k_0}, \quad G_0 = \frac{\psi_0 l_0}{k_0}.$$

The inner functions φ_0 and ψ_0 must then be divisible by k_0. This implies that l_0 is a common factor of F_0, G_0, and therefore, l_0 must be $\equiv 1$. Accordingly, the function m not only is holomorphic in the unit circle, but it in addition also belongs to the space H, which, as has been previously shown, implies that $h \equiv 0$.

The function h_0 therefore belongs to $C^*_{f,g}$. Then, by Theorem I, $C^*_{h_0} < C^*_{f,g}$. Let us now assume that $k = k_0 k_1$ is an arbitrary function belonging to $C^*_{f,g}$. Then for every $\varepsilon > 0$, we can determine two polynomials p and q such that $\|pf + qg - k\| < \varepsilon$. In the same way as in Section 9, this leads to the boundedness of the quotient k_0/h_0; i.e. k_0 is divisible by h_0, which according to Theorem I, implies $C^*_{f,g} < C^*_{h_0}$. Thus, the two sets must be identical, and the theorem is proved.

11. Now, we shall consider the fully general case of a closed linear subset C^* of H with the property

$$T^* C^* < C^*, \tag{37}$$

i.e. $T^* f \in C^*$ when $f \in C^*$.

Theorem IV. *Every closed linear manifold C^* having the property* (37), *and not identical with the null element, contains a uniquely determined inner function f_0 that generates C^* in the sense*

$$C^* = C^*_{f_0}. \tag{38}$$

Let p be the least integer ≥ 0 such that C^* contains a function whose pth order derivative is $\neq 0$ at the origin. As is easily seen, the distance d_p from $T^{*p} e = z^p$ to C^*, is then < 1, and we may define a function f_0 by the relation:

$$\sqrt{1 - d_p^2}\, f_0 = \text{the projection of } z^p \text{ on } C^*.$$

In the same way as in Section 10, we find that f_0 is an inner function,

$$f_0(z) = a_p z^p + a_{p+1} z^{p+1} + \cdots,$$

such that

$$d_p^2 = 1 - a_p^2.$$

It then follows that $C^*_{f_0} < C^*$ Furthermore, if (38) were not true, there would exist an inner function $g_0 \in C^*$ which would not be divisible by f_0. In view of Theorem III, the largest common factor

$$h_0(z) = b_p z^p + b_{p+1} z^{p+1} + \cdots$$

of f_0, g_0 would also belong to C^*. Then

$$\left|\frac{f_0(z)}{h_0(z)}\right| < 1, \quad |z| < 1,$$

which implies $a_p < b_p$, thus leading to the contradiction

$$\|z^p - b_p h_0(z)\|^2 = 1 - b_p^2 < d_p^2,$$

which ends the proof.

From Theorem IV we immediately get the following corrolary which in an essential point is equivalent to the theorem itself: Every non-empty set of inner functions $\{f_0\}$, enumerable or not, has a uniquely determined largest common factor h_0 defined by the following properties: h_0 is an inner function which is a divisor of every $f_0 \in \{f_0\}$; whereas every k_0 with this property is a divisor of h_0.

12. Regarding the extinction problem, we shall content ourselves with the following result:

Theorem V. *Let C be a closed linear subset of H with the property*

$$TC < C, \tag{39}$$

and not identical with the null element. Then C will contain, either at least one eigenelement

$$\varphi_\lambda(z) = \frac{1}{1-\lambda z}, \quad |\lambda| < 1$$

or, otherwise, a function of the form

$$\psi(z) = 1 - \exp\left\{-\int_0^{2\pi} \frac{1}{1 - z e^{i\theta}} d\mu\right\} \not\equiv 0 \tag{40}$$

where $\mu = \mu(\theta)$ is a nondecreasing and bounded function whose points of increase form a set of at most zero measure.

Let us denote by C^* the orthogonal complement of C, and let $f \in C$, $g \in C^*$. In view of (39), we have,

$$0 = (f, g) = (T^n f, g) = (f, T^{*n} g), \quad n \geq 0$$

which implies that $T^* C^* < C^*$. As the theorem is evident in the case $C = H$ we can assume that C is a proper subset of H, and consequently, that C^* contains functions $\not\equiv 0$. According to Theorem IV, there will exist an inner function h generating C^*, and the condition

$$(f, T^{*n} h) = 0, \quad n \geq 0,$$

is then both necessary and sufficient for f to belong to C. In particular, an eigenfunction φ_λ belongs to C when, and only when $(\varphi_\lambda, h) = 0$. Putting

$$h(z) = \sum_0^\infty c_n z^n, \quad |z| < 1,$$

we obtain

$$(h, \varphi_\lambda) = \sum_0^\infty c_n \bar{\lambda}^n = h(\bar{\lambda}).$$

Then $\varphi_\lambda \in C$ only when $h(\bar{\lambda}) = 0$.

On the other hand, a general property of inner functions h, is that $1 - h(0)\,h(z)$ is orthogonal to $\{z^n h(z)\}_0^\infty$. Thus, $1 - h(0)\,h(z)$ belongs to C, and, in the case $h(z) \neq 0$, $|z| < 1$, we have

$$h(z) = \exp\left\{-\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\alpha\right\} \not\equiv 1,$$

$$1 - h(0)\,h(z) = 1 - \exp\left\{-\int_0^{2\pi} \frac{2}{1 - z e^{-i\theta}}\, d\alpha\right\} \not\equiv 0,$$

i. e. a function of the form (40), which proves the theorem.

Finally, let us point out that through slight modifications of the argument, the results obtained may be extended to the space H^p, $p > 1$, of holomorphic functions $f(z)$ subject to a Hardy condition (11) and with the norm

$$\|f\| = \left\{\frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^p\, d\theta\right\}^{\frac{1}{p}}.$$

However, a case of considerably greater interest is offered by the metric

$$\|f\| = \left\{\sum_0^\infty |a_n|^p\right\}^{\frac{1}{p}}, \quad p \geq 1,$$

a_n being the Taylor coefficients of $f(z)$. When $p = 1$ we arrive at a case included in Wieners original results concerning the closure of translations of functions. The eigenvalues now fill the closed unit circle and the Wiener closure criterion $f(z) \neq 0$, $|z| \leq 1$, holds true. If, however, the closure condition $\delta(f) = 0$ is relevant in the cases $1 < p < 2$, is an open question.[5]

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[5] For the semi-group of translations $T_\tau f \equiv f(x + \tau)$, $\tau \geq 0$, applied to functions defined over $0 \leq x < \infty$, a corresponding study will be published by B. Nyman.