ON INJECTIVE BANACH SPACES AND THE SPACES $L^\infty(\mu)$ FOR FINITE MEASURES μ

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Contents

Introduction

We are interested here in the linear topological properties of those Banach spaces associated with injective Banach spaces. We study in particular detail, the spaces $L^\infty(\mu)$ for finite measures μ, and obtain applications of this study to problems concerning injective Banach spaces in general.(2) (Throughout the rest of this introduction, "μ" and "ν" denote arbitrary finite measures on possibly different unspecificed measureable spaces).

For example, we classify the spaces $L^\infty(\mu)$ themselves up to isomorphism (linear homeomorphism) in § 3, and all their conjugate spaces $((L^\infty(\mu))^*$, $(L^\infty(\mu))^{**}$, $(L^\infty(\mu))^{***}$, etc.)

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(2) It is easily seen that if λ is a σ-finite measure, then there exists a finite measure μ with $L^p(\lambda)$ isometric to $L^p(\mu)$ for all p, $1 \leq p \leq \infty$. Thus all of our results concerning finite measures generalize immediately to σ-finite measures.

up to isomorphism in § 5. In § 2, we give a short proof of a result (Corollary 2.2) which simultaneously generalizes the result of Pelczynski that $L^1(\mu)$ is not isomorphic to a conjugate space if μ is non-purely atomic [19], and the result of Gelfand that $L^1[0, 1]$ is not isomorphic to a subspace of a separable conjugate space [9]. We apply this result to demonstrate in Theorem 2.3 that an injective double conjugate space is either isomorphic to l^∞ or contains an isomorph of $l^\infty(\Gamma)$ for some uncountable set Γ, if it is infinite dimensional. (Henceforth, all Banach spaces considered in this paper are taken to be infinite dimensional.) In § 3, by applying a result of Gaifman [8], we obtain that there exists a $\mathcal{P}_1$ space which is not isomorphic to any conjugate Banach space. We also obtain there that a compact Hausdorff space S satisfies the countable chain condition if and only if every weakly compact subset of $C(S)$ is separable.

We now indicate in greater detail the organization and results of the paper. The interdependence of the sections is as follows: Sections 2, 3, and 4 depend on Section 1 (or more specifically, on Lemma 1.3). § 3 is independent of § 2 (with the exception of Corollary 3.2). Theorem 4.8 depends on § 3; all the other results of § 4 are independent of § 2 and § 3. Finally, § 5 is independent of all of the Sections 1–4. (§ 0 consists of definitions and notation, and § 6 of open problems).

The results 1.1 and 1.3 of § 1 yield various conditions that a Banach space contain a complemented subspace isomorphic to $l^1(\Gamma)$ for some uncountable set Γ. We also obtain there that if the conjugate Banach space X^* contains an isomorph of $c_0(\Gamma)$, then X contains a complemented isomorph of $l^1(\Gamma)$, thus generalizing the result of Bessaga and Pelczynski (Theorem 4 of [2]) that this holds for countable Γ.

We have indicated the main results of § 2; they are a consequence of Theorem 2.1, which shows that in a weakly compactly generated conjugate Banach space satisfying the Dunford–Pettis property, weak Cauchy sequences converge in norm (cf. § 0 for the relevant definitions).

If X is a normed linear space, dim X denotes the least cardinal number corresponding to a subset of X with linear span dense in X. The main classification result of § 3 is Theorem 3.5, which states that $L^\infty(\mu)$ is isomorphic to $L^\infty(\nu)$ if and only if $\dim L^1(\mu) = \dim L^1(\nu)$. (Theorem 5.1 has as one of its consequences that $(L^\infty(\mu))^*$ is isomorphic to $(L^\infty(\nu))^*$ if and only if $\dim L^\infty(\mu) = \dim L^\infty(\nu)$.) Theorems 3.5 and 3.6 contain results considerably stronger than this classification result; for example 3.6 shows that if A is a Banach space with A^* isomorphic to $L^\infty(\mu)$, then $L^1(\mu)$ is isomorphic to a quotient of A. We mention also the result Corollary 3.2, which shows that $L^\infty(\mu)$ is not isomorphic to a double conjugate space if $L^1(\mu)$ isn't separable. Many of the results of § 3 (including 3.2), hold for the spaces A^* as well as $L^\infty(\mu)$, where A is a subspace of $L^1(\nu)$ for some ν.

Let "S" denote a compact Hausdorff space. We determine in § 4 certain topological properties of S which yield linear topological invariants of the space $C(S)$. Thus we show in 4.1 that if S satisfies the C.C.C. (the countable chain condition) and if $C(S)$ is isomorphic to a conjugate space, then S carries a strictly positive measure (the relevant terms are defined at the beginning of § 4). We show in Theorem 4.5 that S satisfies the C.C.C. if and only if every weakly compact subset of $C(S)$ is separable, and that S carries a strictly positive measure if and only if $C(S)^*$ contains a weakly compact total subset. Corollary 4.4, which asserts the existence of an injective Banach space non-isomorphic to a conjugate space, is an immediate consequence of Theorem 4.1 and the results of [8]. Theorem 4.5 together with the results of [1] shows that a weakly compact subset of a Banach space satisfying the C.C.C. is separable (cf. Corollary 4.6). (Theorem 4.5 and Corollary 4.6 have suitable generalizations to spaces S satisfying the $\mathfrak{m}$-chain condition, as defined in the remark following Lemma 4.2; these generalizations are stated and proved in the remark following Corollary 4.6.) The main ingredients of the proof of 4.1 are Lemma 1.3 and the combinatorial Lemma 4.2; the proof of the latter has as a consequence that if S is Stonian and $c_0(\Gamma)$ is isomorphic to a subspace of $C(S)$, then $l^\infty(\Gamma)$ is isometric to a subspace of $C(S)$.

Theorem 4.5 has as a consequence that every weakly compact subset of $L^\infty(\mu)$ is separable; an alternate proof is provided by Proposition 4.7. The final result of § 4, 4.8, gives several necessary and sufficient conditions for an injective conjugate Banach space X to be isomorphic to a subspace of $L^\infty(\mu)$.

Let B denote one of the spaces $L^\infty(\mu)$ for some homogeneous μ or $l^\infty(\Gamma)$ for some infinite set Γ. The main result of § 5 is Theorem 5.1, which determines isometrically the space B^*, and isomorphically the spaces B^*, B^{**}, ... (cf. Remark 4 following Theorem 5.4). It also shows that if Y is an injective Banach space with $\dim Y \leqslant \dim B$, then Y is isomorphic to a quotient space of B; $\mathcal{P}_1$ quotient algebras of B are also determined. Thus, a particular case of 5.1 is as follows: let $\mu_{\mathfrak{c}}$ denote the homogeneous measure with $\dim L^1(\mu) = \mathfrak{c}$ (the continuum); then $(L^\infty(\mu_{\mathfrak{c}}))^*$ is isometric to $(l^\infty)^*$ and $L^\infty(\mu_{\mathfrak{c}})$ is algebraically isomorphic to a quotient algebra of l^∞.

The results 5.2–5.4 are concerned with the proof of Theorem 5.1. Theorem 5.5 yields a class of compact Hausdorff spaces K (including some non-separable ones) with $C(K)^*$ isometric to $C[0, 1]^*$. (A special case of Theorem 3.6 is that if B^* is isomorphic to l^∞, then B must be separable.) The final result of § 5, Theorem 5.6, shows that every injective Banach space of dimension the continuum, has its dual isomorphic to $(l^\infty)^*$; its proof uses critically the results of [23].

Some of the results given here have been announced in [24] and [27], with sketches of certain of the proofs. (The section numbers of [27] correspond to those of the present paper; Theorem 3.3 of [27] is proved here by Corollary 3.3, Theorem 3.6, and Theorem 3.7. The results of § 1 and § 2 of [24] are given here in § 4 and § 5 respectively.)

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0. Definitions, notation, and some standard facts

We follow [7] for the most part. The reader should note however that by a Banach (Hilbert) space, we shall mean an *infinite* dimensional complete real or complex normed (inner product) space.

A subspace B of a Banach space X is said to be *complemented* if there exists a bounded linear map P from X onto B with $P(b)=b$ for all $b\in B$. Such a map P is called a *projection* from X onto B.

If X and Z are Banach spaces and $T\colon X\to Y$ is a linear map, then T is said to be an *isomorphism* (resp. an *isometry*) if T is a one-to-one bicontinuous (resp. norm-preserving) map from X onto $T(X)$. Two Banach spaces are thus said to be isomorphic (resp. isometric) if there exists an isomorphism (resp. isometry) mapping one onto the other.

If X is a Banach space, X^* denotes its dual. The weak* topology on X^* is the X-topology on X^*, in the terminology of [7]; (the weak topology on X^* is then the X^{**} topology on X^*). Given X, χ denotes the canonical isometry imbedding X in X^{**}. If B is a subset of X or if Y is a subset of X^*,

$$B^\perp = \{f\in X^*\colon f(b)=0 \text{ for all } b\in B\}$$

$$Y^\perp = \{x\in X\colon y(x)=0 \text{ for all } y\in Y\}.$$

Y is said to be *total* if $Y^\perp=\{0\}$. If Y is a linear subspace of X^*, Y is said to be of *positive characteristic* if there exists a finite $K>0$ such that for all $x\in X$, $\|x\|\leqslant K \sup\{|y(x)|\colon y\in Y$ and $\|y\|\leqslant 1\}$. (Cf. [5] for equivalent definitions of this notion.)

A Banach space X is said to be weakly compactly generated, (X is WCG), if there exists a weakly compact subset of X whose linear span is dense in X. We note that since bounded linear operators are weakly continuous, complemented subspaces of a WCG Banach space are also WCG, and if X is WCG and Y is isomorphic to X, Y is WCG. (For further properties of WCG Banach spaces, see [16].)

A Banach space X is said to satisfy the *Dunford–Pettis* property, (X satisfies DP), if given a Banach space Y and $T\colon X\to Y$ a weakly compact operator, then T maps weak Cauchy sequences in X into convergent sequences in the norm topology of Y. We note that if X satisfies DP, so does any complemented subspace of X, and if Z is isomorphic to X, so does Z. Finally we recall the result of Grothendieck [10]: X satisfies DP if and only if given (x_n) and (x_n^*) a pair of sequences in X and X^* respectively with $x_n \to 0$ weakly and $x_n^* \to 0$ weakly, then $x_n^*(x_n) \to 0$.

By $L^p(\mu)$ we refer to the (real or complex) space $L_p(S, \Sigma, \mu)$ in the notation of [7], for any p with $1 \leqslant p \leqslant \infty$. (Thus the set S and Σ, the σ-algebra of subsets of S, are usually suppressed in our notation.) We recall that $L^1(\mu)$ satisfies DP (cf. [7]), and that if μ is a finite measure, $L^1(\mu)$ is WCG (since then $L^2(\mu)$ injects densely into $L^1(\mu)$); in this case, we also identify $(L^1(\mu))^*$ with $L^\infty(\mu)$. In accordance with our conventions, unless explicitly stated to the contrary, we shall take all measures μ to be such that $L^1(\mu)$ is of infinite dimension.

Given a compact Hausdorff space S, $C(S)$ denotes the space of scalar-valued continuous functions on S. We denote by $M(S)$ the space of all regular finite scalar-valued Borel measures on S; we identify $C(S)^*$ with $M(S)$ by the Riesz representation theorem. Moreover if μ is a positive member of $M(S)$, we identify $L^1(\mu)$ with the subspace of $M(S)$ consisting of all measures λ with λ absolutely continuous with respect to μ, by the Radon–Nikodym Theorem. Finally, if λ is an arbitrary member of $M(S)$, we denote by $d\lambda/d\mu$ that member of $L^1(\mu)$ such that $d\nu = (d\lambda/d\mu)\,d\mu - d\lambda$ is singular with respect to μ (i.e. $(d\lambda/d\mu)\,d\mu$ is the absolutely continuous part of λ with respect to μ, in the Lebesgue decomposition of μ).

Given a set Γ, $l^\infty(\Gamma)$ denotes the Banach space of all bounded scalar-valued functions defined on Γ under the supremum norm; $c_0(\Gamma)$ denotes the subspace of $l^\infty(\Gamma)$ consisting of all f such that for all $\varepsilon > 0$ there exists a finite subset F_ε of Γ with $|f(\gamma)| < \varepsilon$ for all $\gamma \notin F_\varepsilon$; $l^1(\Gamma)$ denotes the subset of $c_0(\Gamma)$ consisting of those f for which $\sum_{\gamma\in\Gamma}|f(\gamma)| < \infty$, under the norm $\|f\| = \sum_{\gamma\in\Gamma}|f(\gamma)|$. By the unit-vectors-basis of $l^1(\Gamma)$ (resp. $c_0(\Gamma)$) we refer to $\{e_\gamma\}_{\gamma\in\Gamma}$, where $e_\gamma(\alpha) = 1$ if $\gamma = \alpha$, $e_\gamma(\alpha) = 0$ if $\gamma \neq \alpha$, for all γ, $\alpha \in \Gamma$. A subset $\{b_\gamma\}_{\gamma\in\Gamma}$ of the Banach space B is said to be *equivalent* to the unit-vectors-basis of $l^1(\Gamma)$ (resp. $c_0(\Gamma)$) if the map T: $\{e_\gamma\}_{\gamma\in\Gamma} \to \{b_\gamma\}_{\gamma\in\Gamma}$ defined by $Te_\gamma = b_\gamma$ for all $\gamma \in \Gamma$, may be extended to an isomorphism of $l^1(\Gamma)$ (resp. of $c_0(\Gamma)$) with the closed linear span of $\{b_\gamma\}_{\gamma\in\Gamma}$.

We assume the notation and standard facts concerning cardinal numbers, as exposed in [28]. Given a set Γ, card Γ denotes the cardinality of Γ; $\mathfrak{c}$ denotes the cardinality of the reals, and $\aleph_0$ the cardinality of the integers. If card $\Gamma = \mathfrak{m}$, we denote $l^\infty(\Gamma)$ by $l^\infty_\mathfrak{m}$ and $l^1(\Gamma)$ by $l^1_\mathfrak{m}$. In the case of countable infinite Γ, $l^\infty(\Gamma)$, $l^1(\Gamma)$, and $c_0(\Gamma)$ are denoted by l^∞, l^1, and c_0, respectively.

Given a normed linear space X, dim X, the dimension of X, refers to the smallest cardinal number $\mathfrak{m}$ for which there exists a subset of cardinality $\mathfrak{m}$ with linear span norm-dense in X.

Given an indexed family $\{X_\alpha\}_{\alpha\in\Gamma}$ of Banach spaces, we denote by $(\Sigma_{\alpha\in\Gamma} \oplus X_\alpha)_1$ (resp.

by $(\sum_{\alpha\in\Gamma}\oplus X_\alpha)_\infty)$ the Banach space consisting of all functions $x=\{x_\alpha\}_{\alpha\in\Gamma}$ with $x_\alpha \in X_\alpha$ for all α, and $\Sigma_{\alpha\in\Gamma}\|x_\alpha\|_{X_\alpha}<\infty$ (resp. $\sup_{\alpha\in\Gamma}\|x_\alpha\|_{X_\alpha}<\infty$) under the obvious norm. If $X_\alpha = X$ for all α, and if $\mathfrak{m}=\text{card }\Gamma$, we denote $(\sum_{\alpha\in\Gamma}\oplus X_\alpha)_p$ by $(\sum_{\mathfrak{m}}\oplus X)_p$ for $p=1$ or ∞.

Finally, given finitely many Banach spaces $X_1, ..., X_n$, we denote $(X_1\oplus...\oplus X_n)_1$ simply as $X_1\oplus X_2\oplus...\oplus X_n$.

A Banach space X is said to be injective (resp. a $\mathcal{P}_1$ space) if given Y a Banach space, Y_1 a subspace of Y, and $T\colon Y_1\to X$ a bounded linear operator, then there exists a bounded linear operator $\tilde{T}\colon Y\to X$ with $\tilde{T}\mid Y_1 = T$ (and additionally $\|\tilde{T}\|=\|T\|$ in the case of a $\mathcal{P}_1$ space). For properties of injective and $\mathcal{P}_1$ Banach spaces, see [4] and [21].

1. Preliminary results

The main result of this Paragraph, Lemma 1.3, is a useful tool for the work in Paragraphs 2–4. The essential ingredient of its proof is the following lemma, which generalizes a result of Köthe (p. 185 of [15]).

LEMMA 1.1. *Let X and A be Banach spaces with $A\subset X$, Γ a set, $T\colon X\to l^1(\Gamma)$ a bounded linear operator, $\delta>0$, and $\mathfrak{m}$ an infinite cardinal number such that*

$$\text{card}\,\{\gamma\in\Gamma\colon \exists a\in A \textit{ with } \|a\|\leqslant 1 \textit{ and } |Ta(\gamma)|>\delta\}=\mathfrak{m}.$$

Then A contains a subspace Y isomorphic to $l^1_{\mathfrak{m}}$ and complemented in X, such that $T\mid Y$ is an isomorphism.

Proof. Set
$$K=\{Ta\colon a\in A,\ \|a\|\leqslant 1\}.$$

K is a symmetric convex bounded subset of $l^1(\Gamma)$, with the property that

$$\text{card}\,\{\gamma\in\Gamma\colon \exists\, k\in K \text{ such that } |k(\gamma)|\geqslant\delta\}=\mathfrak{m}. \tag{1}$$

We now divide the proof into two parts.

A. There exists a family Δ of pairwise disjoint finite subsets of Γ, with card $\Delta=\mathfrak{m}$, and for each $F\in\Delta$, an associated $k_F\in K$, so that

$$\|k_F\|\geqslant\frac{\delta}{4} \quad\text{and}\quad \sum_{\gamma\notin F}|k_F(\gamma)|<\frac{\delta}{32}.$$

B. $\{k_F\colon F\in\Delta\}$ is equivalent to the unit-vectors-basis of $l^1(\Delta)$, and letting Z be the closed linear span of $\{k_F\colon F\in\Delta\}$, then Z is complemented in $l^1(\Gamma)$ (where the k_F's are as in A).

Once A and B are proved, the proof is completed as follows: for each $F \in \Delta$, we choose $a_F \in A$ with $\|a_F\| \leqslant 1$ and $T(a_F) = k_F$. Then since T is a bounded linear operator, it follows easily that $\{a_F: F \in \Delta\}$ is equivalent to the unit-vectors-basis of $l^1(\Delta)$, and that setting Y to be the closed linear span of $\{a_F: F \in \Delta\}$, then $T|Y$ is an isomorphism mapping Y onto Z. Letting P be a bounded linear projection from $l^1(\Gamma)$ onto Z, a projection Q from X onto Y may then be defined by putting

$$Q = (T|Y)^{-1}PT$$

Proof of A. This is somewhat similar to the arguments of [15]. We first observe that we may choose a subset K' of K with card $K' \geqslant \mathfrak{m}$, such that for any two distinct members k_1 and k_2 of K',

$$\|k_1 - k_2\| \geqslant \delta/2. \tag{2}$$

Indeed, the family of all non-empty subsets M of K such that for any two distinct members k_1, k_2 of M, (2) holds, is closed under nested unions, so we choose K', a maximal subset of this family. Now suppose we had that card $K' < \mathfrak{m}$. For each $k \in K'$, choose F_k a finite set with $|k(\gamma)| < \delta/32$ for all $\gamma \notin F_k$. Then since $\mathfrak{m}$ is an infinite cardinal number, card $\bigcup_{k\in K'} F_k < \mathfrak{m}$. But then by (1), we may choose a $\gamma \notin \bigcup_{k\in K'} F_k$, and a $k_0 \in K$ with $|k_0(\gamma)| \geqslant \delta$. Then for any $k \in K'$, $\|k_0 - k\| \geqslant |k_0(\gamma) - k(\gamma)| \geqslant \delta - \delta/32 \geqslant \delta/2$, hence $K' \cup \{k_0\}$ satisfies (2) for all distinct k_1, k_2 belonging to it, contradicting the maximality of K'.

We now use Zorn's Lemma to produce Δ satisfying the properties in A. Consider all pairs $(\mathfrak{F}, \varphi_{\mathfrak{F}})$ where $\mathfrak{F}$ is a non-empty family of finite pairwise disjoint subsets of Γ, and $\varphi_{\mathfrak{F}}$ is a function with $\varphi_{\mathfrak{F}}: \mathfrak{F} \to K$ such that for all $F \in \mathfrak{F}$,

$$\|\varphi_{\mathfrak{F}}(F)\| \geqslant \frac{\delta}{4} \quad \text{and} \quad \sum_{\gamma \notin F} |\varphi_{\mathfrak{F}}(F)(\gamma)| < \frac{\delta}{32}. \tag{3}$$

We order this family of pairs in the natural way by

$$(\mathfrak{F}, \varphi_{\mathfrak{F}}) \leqslant (\mathcal{G}, \varphi_{\mathcal{G}})$$

if $\mathfrak{F} \subset \mathcal{G}$ and $\varphi_{\mathcal{G}}|\mathfrak{F} = \varphi_{\mathfrak{F}}$. Again, every totally ordered subset of this family of pairs has a least upper bound, so we choose a maximal element (Δ, φ_Δ). We claim that card $\Delta \geqslant \mathfrak{m}$. Well, suppose this were not the case; i.e. that card $\Delta < \mathfrak{m}$. Then set $\Gamma_1 = \bigcup_{F\in\Delta} F$. Since each $F \in \Delta$ is a finite set and $\mathfrak{m}$ is an infinite cardinal, we would have that card $\Gamma_1 < \mathfrak{m}$. But then we claim that, setting $\varepsilon = \delta/32$, we could choose k_1 and k_2 distinct members of K' such that $\|k_1|\Gamma_1 - k_2|\Gamma_1\| < \varepsilon$ (where for $k \in l^1(\Gamma)$, $k|\Gamma_1$ denotes the restriction of the function k to the set Γ_1, and then of course $\|k|\Gamma_1\| = \sum_{\gamma\in\Gamma_1} |k(\gamma)|$). Indeed, let $K'' = \{k|\Gamma_1: k \in K'\}$. If card $K'' < \mathfrak{m}$, then since card $K' = \mathfrak{m}$, we could choose two distinct members

k_1 and k_2 of K' with $k_1|\Gamma_1 = k_2|\Gamma_1$. If card $K'' = \mathfrak{m}$, consider the family of spheres $\{S_k: k \in K''\}$, where for all $k \in K''$, $S_k = \{g \in l^1(\Gamma_1): \|g-k\| < \varepsilon/2\}$. Since dim $l^1(\Gamma_1) =$ card $\Gamma_1 < \mathfrak{m}$ and K'' is a bounded subset of $l^1(\Gamma_1)$, two of these spheres must have a non-empty intersection. Hence we would obtain two distinct k_1 and k_2 in K' with $\|k_1|\Gamma_1 - k_2|\Gamma_1\| < \varepsilon = \delta/32$.

Thus we could choose a finite subset F_0 of $\Gamma \sim \Gamma_1$ such that

$$\sum_{\gamma \notin F_0} |(k_1 - k_2)|(\gamma) < \frac{\delta}{16},$$

Now set $k_0 = \frac{1}{2}(k_1 - k_2)$, $\mathfrak{F} = \Delta \cup \{F_0\}$, and define $\varphi_{\mathfrak{F}}: \mathfrak{F} \to K$ by $\varphi_{\mathfrak{F}}(F) = \varphi_\Delta(F)$ for all $F \in \Delta$ and $\varphi_{\mathfrak{F}}(F_0) = k_0$. Then $(\mathfrak{F}, \varphi_{\mathfrak{F}})$ would satisfy (3) and $(\Delta, \varphi_\Delta) \leqslant (\mathfrak{F}, \varphi_{\mathfrak{F}})$ with $\Delta \neq \mathfrak{F}$, so the maximality of (Δ, φ_Δ) would be contradicted.

Now of course by passing to a subfamily of Δ if necessary, we may assume that card $\Delta = \mathfrak{m}$. For each $F \in \Delta$, we simply set $k_F = \varphi_\Delta(F)$, and A is thus proved.

Proof of B. (The proof is similar to arguments found in [2].) Define for each $F \in \Delta$, e_F by, for all $\gamma \in \Gamma$,

$$e_F(\gamma) = k_F(\gamma) \quad \text{if } \gamma \in F$$

$$e_F(\gamma) = 0 \qquad \text{if } \gamma \notin F.$$

By A, we have that $\|e_F\| \geqslant \delta/5$ for all $F \in \Delta$, and the e_F's are disjointly supported. Thus if we put W equal to the closed linear span of $\{e_F: F \in \Delta\}$, W is isometric to $l^1(\Delta)$, there exists a projection R of $l^1(\Gamma)$ onto W with $\|R\| = 1$, and of course $\{e_F: F \in \Delta\}$ is equivalent to the unit-vectors basis of $l^1(\Delta)$. But then $\{k_F: F \in \Delta\}$ also has this property. Moreover, $R|Z$ is an isomorphism mapping Z onto W. Thus, a bounded linear projection P from $l^1(\Gamma)$ onto Z may be obtained by setting $P = (R|Z)^{-1} R$. Q.E.D.

COROLLARY 1.2. *Let X be a Banach space and Γ an infinite set, and suppose that $c_0(\Gamma)$ is isomorphic to a subspace of X^*. Then $l^1(\Gamma)$ is isomorphic to a complemented subspace of X (and consequently $l^\infty(\Gamma)$ is isomorphic to a subspace of X^*).*

If Γ is countable, this result is known and due to Bessaga and Pelczynski (Theorem 4 of [2]).

Proof. By assumption, there exists an indexed family $\{e_\gamma\}_{\gamma \in \Gamma}$ of elements of X^*, equivalent to the unit-vectors-basis of $c_0(\Gamma)$, with $\|e_\gamma\| = 1$ for all $\gamma \in \Gamma$. Now define a map T from X into the bounded scalar-valued functions on Γ by $(Tx)(\gamma) = e_\gamma(x)$ for all $x \in X$ and $\gamma \in \Gamma$. Since $(c_0(\Gamma))^*$ may be identified with $l^1(\Gamma)$, we have that there exists a $k > 0$ so that for all $x \in X$, $Tx \in l^1(\Gamma)$ with $\|Tx\|_{l^1(\Gamma)} \leqslant k\|x\|$. T thus satisfies the hypotheses of Lemma 1.1, and so X contains a complemented subspace isomorphic to $l^1(\Gamma)$. Q.E.D.

The next result of this section is used in Sections 2, 3, and 4. It is a fairly simple consequence of 1.1 and the Radon–Nikodym Theorem.

LEMMA 1.3. *Let S be a compact Hausdorff space, and let A be a closed subspace of $M(S)$. Then either there exists a positive $\mu \in M(S)$ such that $A \subset L^1(\mu)$ (that is, every member of A is absolutely continuous with respect to μ), or A contains a subspace complemented in $M(S)$ and isomorphic to $l^1(\Gamma)$ for some uncountable set Γ.*

The two possibilities of 1.3 are mutually exclusive, since $l^1(\Gamma)$ is not WCG for any uncountable set Γ (See also the second remark below.)

Proof. Let $\mathfrak{J}$ be a maximal family of mutually singular positive finite regular Borel measures on S. (A family $\mathfrak{N}$ of such measures is called mutually singular if $\mu \neq \nu$, $\mu, \nu \in \mathfrak{N} \Rightarrow \mu \perp \nu$ (i.e. $d\mu/d\nu = 0$). Such families $\mathfrak{N}$ are closed under nested unions, and hence there exists a maximal one by Zorn's Lemma). It follows that for each $\nu \in M(S)$, $d\nu/d\mu = 0$ for all but countably many $\mu \in \mathfrak{J}$, $\mu_1, \mu_2, \ldots$ say, and that $d\nu = \sum (d\nu/d\mu_i)\, d\mu_i$, the series converging in the norm topology (and in fact absolutely) to ν. Now let $\Gamma = \{\mu \in \mathfrak{J}$: there is an $a \in A$, with $da/d\mu \neq 0\}$. If Γ is countable, say $\Gamma = \{\mu_1, \mu_2, \ldots\}$, then every $a \in A$ is absolutely continuous with respect to the finite regular measure $\mu = \sum_{n=1}^{\infty} \mu_n / (2^n \|\mu_n\|)$, so by the Radon–Nikodym Theorem we may regard A as being contained in $L^1(\mu)$.

Now suppose Γ is uncountable. For each $\gamma \in \Gamma$, choose a_γ with $da_\gamma/d\gamma \neq 0$; then choose $\varphi_\gamma \in L^\infty(\gamma)$ with

$$\int \frac{da_\gamma}{d\gamma}\, \varphi_\gamma\, d\gamma \neq 0, \qquad \|\varphi_\gamma\|_{L^\infty(\gamma)} = 1.$$

Now define $F_\gamma \in (M(S))^*$ by $F_\gamma(\nu) = \int (d\nu/d\gamma) \varphi_\gamma\, d\gamma$ for all $\nu \in M(S)$. Our observations about the family $\mathfrak{J}$ show that $\|F_\gamma\| = 1$ and in fact

$$\Sigma_{\gamma \in \Gamma} |F_\gamma(\nu)| \leqslant \|\nu\| \quad \text{for all } \nu \in C(S)^*.$$

Thus we may define T: $M(S) \to l^1(\Gamma)$ by $(T\nu)_\gamma = F_\gamma(\nu)$ for all $\nu \in M(S)$; the definition of Γ shows that $T(A)$ is non-separable, and T is of course a linear operator with $\|T\| \leqslant 1$.

Now if we set $\Gamma_n = \{\gamma \in \Gamma$: $\exists\, a \in A$, $\|a\| \leqslant 1$, with $|Ta(\gamma)| \geqslant 1/n\}$, then $\Gamma = \bigcup_{n=1}^{\infty} \Gamma_n$. Hence there exists an n with Γ_n uncountable. The fact that A contains a subspace isomorphic to $l^1(\Gamma_n)$ and complemented in $M(S)$ now follows from Lemma 1.1. Q.E.D.

Remarks. 1. A suitable version of 1.2 (with practically the same proof) holds for closed subspaces A of $L^1(\nu)$ for any (possibly infinite) measure ν. Precisely, *either there exists an*

$f\in L^1(\nu)$ such that $A\subset L^1(\lambda)$, where $d\lambda=f d\nu$, or A contains a subspace complemented in $L^1(\nu)$ and isomorphic to $l^1(\Gamma)$ for some uncountable set Γ.

2. It is a consequence of known results that if Γ is an uncountable set, then $l^1(\Gamma)$ is not isomorphic to a subspace of any WCG Banach space X. For by the results of [1], such an X, and hence any subspace of X, has an equivalent smooth norm, while $l^1(\Gamma)$ has no such equivalent norm. (This may also be seen from the fact that the unit ball of $(l^1(\Gamma))^*$ in its weak* topology, contains a separable nonmetrizeable subset).

The next and last result of this section is known. Its proof is almost identical to that of a result of Pelczynski's (Proposition 4 of [21]). However, the argument is so elegant and short that we include it here.

PROPOSITION 1.4. *Let p be fixed with $1\leqslant p\leqslant\infty$, and let X be a Banach space such that X is isomorphic to $(X\oplus X\oplus\ldots)_p$. Let Y be a Banach space such that Y and X are each isomorphic to a complemented subspace of the other. Then Y and X are isomorphic.*

Proof. Letting "$\sim$" denote "is isomorphic to", we have that there are Banach spaces A and B such that

$$X\sim Y\oplus B \text{ and } Y\sim X\oplus A.$$

Thus, $X\sim X\oplus A\oplus B$. Hence,

$$\begin{aligned}X&\sim(X\oplus X\oplus\ldots)_p\sim((X\oplus A\oplus B)\oplus(X\oplus A\oplus B)\oplus\ldots)_p\\&\sim(X\oplus X\oplus\ldots)_p\oplus(B\oplus B\oplus\ldots)_p\oplus(A\oplus A\oplus\ldots)_p\\&\sim(X\oplus X\oplus\ldots)_p\oplus(B\oplus B\oplus\ldots)_p\oplus(A\oplus A\oplus\ldots)_p\oplus A\\&\sim X\oplus A\sim Y.\end{aligned}$$

2. Conjugate Banach spaces isomorphic to complemented subspaces of $L^1(\mu)$, with an application to injective double conjugate spaces

Our first result generalizes a result of Grothendieck (cf. the first remark below).

THEOREM 2.1. *Let the Banach space X satisfy* DP. *Then if X is isomorphic to a subspace of a weakly compactly generated conjugate Banach space, every weak Cauchy sequence in X converges in the norm topology of X.*

Proof. We first observe that since X is assumed to satisfy DP, then given (x_n) and (f_n) sequences in X and X^* respectively such that $x_n\to 0$ weakly and (f_n) is weak-Cauchy, then $f_n(x_n)\to 0$. Indeed if not, we can assume by passing to a subsequence if necessary that

$f_n(x_n) \to L \neq 0$. Now we may choose $n_1 < n_2 < n_3 < \dots$ such that $\lim_{i\to\infty} f_i(x_{n_i}) = 0$. (Let $n_1 = 1$; having chosen n_{k-1}, then since $\lim_{l\to\infty} f_k(x_l) = 0$, simply choose $n_k > n_{k-1}$ with $|f_k(x_{n_k})| < 1/k$.) But then $f_k - f_{n_k} \to 0$ weakly, so by a result of Grothendieck (page 138 of [10]) $(f_k - f_{n_k})(x_{n_k}) \to 0$. Thus $f_{n_k}(x_{n_k}) \to 0$, a contradiction.

Now since the property DP is linear topological, we may suppose that there is a Banach space B with B^* WCG and $X \subset B^*$. Let (x_n) in X with $x_n \to 0$ weakly. We shall show that $\|x_n\| \to 0$. Suppose not; again by passing to a subsequence if necessary, we may assume there is a $\delta > 0$ with $\|x_n\| > \delta$ for all n.

Now choose $b_n \in B$ with $\|b_n\| = 1$ and $|x_n(b_n)| > \delta$ for all n. Since B^* is WCG, the unit ball of B^{**} is weak* sequentially compact (cf. Corollary 2 of [1] and also the second remark following our Proposition 4.7 below). There are thus a subsequence (b_{n_i}) of the b_n's and a b^{**} in B^{**} with $\lim_{i\to\infty} b^*(b_{n_i}) = b^{**}(b^*)$ for all $b^* \in B^*$. Thus (b_{n_i}) is a weak Cauchy sequence. Then defining $T: B \to X^*$ by $(Tb)(x) = x(b)$ for all $b \in B$ and $x \in X$, T is a continuous linear operator, and so (Tb_{n_i}) is a weak Cauchy sequence in X^*. Thus by our first observation,

$$\lim_{i\to\infty} (Tb_{n_i})(x_{n_i}) = 0 = \lim_{i\to\infty} x_{n_i}(b_{n_i}),$$

a contradiction.

The fact that every Cauchy sequence in X converges in norm, now follows from the observation that a sequence (x_n) in X is weak (norm) Cauchy if and only if for every pair of its subsequences (x_{n_i}) and (x_{m_i}), $x_{n_i} - x_{m_i} \to 0$ weakly (in norm). Q.E.D.

Remarks. 1. A very slight modification of the above argument shows that if X satisfies DP and X^* is isomorphic to a subspace of a WCG Banach space, then weak Cauchy sequences in X^* converge in norm. (One has to remark that the unit cell of X^{**} will then be weak* sequentially compact, since it will be the weak* continuous image of a weak* sequentially compact set.) This implies a result of Grothendieck (cf. Proposition 1.2 of [22]).

2. It follows from Eberlein's theorem and our Theorem 2.1 that if X satisfies all its hypotheses, then every weakly compact subset of X is norm-compact and thus separable. Thus if X is in addition assumed to be WCG, X must be separable. We conjecture that the separability of X should follow without this additional assumption.

Our next result generalizes the result of Gelfand that $L^1[0, 1]$ is not isomorphic to a subspace of a separable conjugate space [9], and the result of Pelczynski that $L^1(\mu)$ is not isomorphic to a conjugate space if μ is finite and not purely-atomic [19].

COROLLARY 2.2. *Let μ be a measure and X be a complemented subspace of $L^1(\mu)$. Then if μ is finite and X is isomorphic to a conjugate Banach space, or more generally if μ is arbitrary* *and X is isomorphic to a subspace of a* WCG *conjugate Banach space, weak Cauchy sequences in X are norm convergent and X is isometric to a complemented subspace of* $L^1[0, 1]$.

We conjecture that if the Banach space X satisfies the assumptions of 2.2, then X is isomorphic to l^1.

Proof of 2.2. Suppose first that μ is finite. Then $L^1(\mu)$ is WCG and satisfies DP, and consequently its complemented subspace X is also WCG and satisfies DP. Thus if X is isomorphic to a subspace of a WCG conjugate space, we have by Theorem 2.1 that weak Cauchy sequences converge in the norm topology of X, and consequently X is separable by Eberlein's theorem. Then we may choose a subspace of $L^1(\mu)$ containing X and isometric to $L^1(\nu)$ for some separable measure ν. But it follows easily from Theorem C, page 123 of [12], that for such a ν, $L^1(\nu)$ is isometric to a complemented subspace of $L^1[0, 1]$.

The case of a general μ now follows from the above considerations and Lemma 1.3, which shows (cf. the remarks following 1.3) that if X is isomorphic to a subspace of a WCG Banach space, then there exists a finite measure ν and a subspace Z of $L^1(\mu)$ with Z isometric to $L^1(\nu)$ and $X \subset Z$. Q.E.D.

Our final result gives information on injective double conjugate spaces. Its proof yields more examples of subspaces of $L^1(\mu)$ non-isomorphic to conjugate Banach spaces (cf. the next remark).

THEOREM 2.3. *Let B be an injective Banach space which is isomorphic to a double conjugate Banach space. Then either B is isomorphic to l^∞ or there exists an uncountable set Γ with $l^\infty(\Gamma)$ isomorphic to a subspace of B.*

Proof. Since B is injective, there exists a compact Hausdorff space S with B isometric to a complemented subspace of $C(S)$, and hence B^* is isomorphic to a complemented subspace of $C(S)^*$. Let A be a Banach space with A^{**} isomorphic to B. Thus A^{***} is isomorphic to a complemented subspace of $C(S)^*$, and hence since A^* is isometric to a complemented subspace of A^{***}, A^* is isomorphic to a complemented subspace Y of $C(S)^*$ which we identify with $M(S)$. Now by Lemma 1.3, either we may choose an uncountable set Γ with $l^1(\Gamma)$ isomorphic to a complemented subspace of Y, or we may choose a positive $\mu \in M(S)$ with $Y \subset L^1(\mu)$. If the first possibility occurs, $(l^1(\Gamma))^*$ is isomorphic to a subspace of Y^*, which means that $l^\infty(\Gamma)$ is isomorphic to a subspace of B. If the second possibility occurs, we have that Y, and hence A^*, is isomorphic to a complemented subspace of $L^1(\mu)$. But then by Corollary 2.2, A^* is separable. Thus A^{**}, i.e. B, is isomorphic to a subspace of l^∞, so by Corollary 6 of [21], B is isomorphic to l^∞. Q.E.D.

Remark. Applying the full strength of Corollary 2.2 and Lemma 1.3, we obtain the following result: *Let X be a Banach space such that X^* is injective and X is isomorphic to a conjugate Banach space. Then either there exists an uncountable set Γ such that $l^1(\Gamma)$ is isomorphic to a complemented subspace of X, or X^* is isomorphic to l^∞, X is isomorphic to a complemented subspace of $L^1[0, 1]$, and weak Cauchy sequences in X converge in norm.* Now suppose that μ is a finite measure and A is a non-separable subspace of $L^1(\mu)$ such that A^* is injective. Then by Lemma 4.3 of [19] (cf. also the second remark following Lemma 1.3 above), $l^1(\Gamma)$ is not isomorphic to a subspace of A if Γ is an uncountable set. *Hence A is not isomorphic to a conjugate Banach space.* An immediate consequence of the first result of the next section is that A^* is not isomorphic to a double conjugate Banach space (Corollary 3.2).

We conclude this section with the

CONJECTURE. *Let X be a complemented subspace of $L^1(\lambda)$ for some measure λ, and suppose that X is isomorphic to a conjugate Banach space. Then X contains a complemented subspace isomorphic to $l^1_{\mathfrak{m}}$ where $\mathfrak{m} = \dim X$.*

This conjecture would have as a consequence that if the injective Banach space B is isomorphic to a double conjugate space, then there exists a set Γ with B isomorphic to $l^\infty(\Gamma)$. For if B is isomorphic to A^{**}, then A^* is isomorphic to a complemented subspace of $L^1(\lambda)$ for some measure λ. Thus letting $\mathfrak{m} = \dim A^*$, B is isomorphic to a subspace of $l^\infty_{\mathfrak{m}}$. If this conjecture is correct, $l^\infty_{\mathfrak{m}}$ would be isomorphic to a subspace of A^{**} and thus to a subspace of B. But since $l^\infty_{\mathfrak{m}}$ is isometric to $(l^\infty_{\mathfrak{m}} \oplus l^\infty_{\mathfrak{m}} \oplus \ldots)_\infty$ (because a countable union of disjoint sets each of cardinality $\mathfrak{m}$ also has cardinality $\mathfrak{m}$), B would be isomorphic to $l^\infty_{\mathfrak{m}}$ by Proposition 1.4. Lemma 1.3 and Corollary 2.2 do imply the validity of the conjecture for the case when $\mathfrak{m} = \aleph_1$, the cardinal number corresponding to the first uncountable ordinal (and the conjecture is a known result for $\mathfrak{m} = \aleph_0$ without the assumption of X being isomorphic to a conjugate Banach space (cf. Corollary 4 of [21])). We thus obtain that *if B is an injective double conjugate space isomorphic to a subspace of $l^\infty_{\aleph_1}$, then either B is isomorphic to l^∞ or B is isomorphic to $l^\infty_{\aleph_1}$.*

3. Classification of the linear isomorphism types of the spaces $L^\infty(\mu)$ for finite measures μ

Our first result uses the notion of hyper-Stonian spaces for its proof, and is crucial for the main result of this section (Theorem 3.5). It generalizes the following (unpublished) result due jointly to W. Arveson and the author: if μ is a finite measure with $\dim L^1(\mu) > \aleph_0$, then $(L^\infty(\mu))^*$ is not separable in its weak* topology.

THEOREM 3.1. *Let A be a Banach space of dimension $\mathfrak{m}$, and suppose that A is isomorphic to a subspace of $L^1(\nu)$ for some finite measure ν. Let B be a closed subspace of A^{**} such that B is isomorphic to a subspace of some* WCG *Banach space. Then if B is weak* dense in A^{**}, then* $\dim B \geq \mathfrak{m}$.

Proof. By a result of Dixmier (Théorème 1 of [6]), there exists a compact Hausdorff space Ω and a finite regular positive Borel measure μ on Ω with the following properties:

1. For every non-empty open subset U of Ω, $\mu(U) = \mu(\bar{U}) > 0$, and $\bar{U}$ is open.

2. $C(\Omega) = L^\infty(\mu)$. By this, we mean that every bounded Borel-measureable function f is equal μ-almost everywhere to a continuous function on Ω.

3. $L^1(\mu)$ is isometric to $L^1(\nu)$.

(In Dixmier's terminology, Ω is a hyper-Stonian space and μ is a normal measure on Ω. Ω may be taken to be the maximal ideal space of the Banach algebra $L^\infty(\nu)$, and μ the unique member of $M(\Omega)$ corresponding to the linear functional on $L^\infty(\nu)$ induced by ν.)

Since all the properties being considered in Theorem 3.1 are linear-topological, we may assume that A is a closed subspace of $L^1(\mu)$. Since A^{**} may then be identified with $A^{\perp\perp} \subset L^1(\mu)^{**} = M(\Omega)$, we assume that $B \subset A^{\perp\perp}$, and that B is weak* dense in A^{**}.

This means that if $f \in C(\Omega)$ is such that $\int f db = 0$ for all $b \in B$, then $f \in A^\perp$. We now assume that $\dim B < \mathfrak{m}$, and argue to a contradiction.

First, since B is assumed isomorphic to a subspace of some WCG Banach space, it follows by Lemma 1.3 (cf. the second remark following its proof) that there is a positive $\nu_1 \in M(\Omega)$ such that $B \subset L^1(\nu_1)$. By the Lebesgue decomposition theorem, we may write $\nu_1 = \lambda + \varrho$, where λ and ϱ are regular positive Borel measures with ϱ absolutely continuous with respect to μ, and λ singular with respect to μ. Thus there is a Borel measurable set E such that $\mu(E) = 0$, $\lambda(\sim E) = 0$. But then $\mu(\bar{E}) = 0$ also. Indeed, by the regularity of μ, there exists a sequence $U_1, U_2, \ldots$ of open sets with $E \subset U_n$ for all n, and $\mu(U_n) \to 0$. Thus $\mu(\bar{E}) \leq \lim_{n\to\infty} \mu(\bar{U}_n) = \lim_{n\to\infty} \mu(U_n) = 0$ by property 1. (In particular, we see again by property 1 that $\bar{E}$ is nowhere dense.) Now since ϱ is absolutely continuous with respect to μ, $\lambda + \varrho = \nu_1$ is absolutely continuous with respect to $\lambda + \mu$, so we may assume that $B \subset L^1(\lambda + \mu)$. Since μ is regular and $\mu(\Omega) = \mu(\sim \bar{E}) = \|\mu\|$, we may choose an increasing sequence of clopen (closed and open) sets $U_1 \subset U_2 \subset \dots$ such that $U_i \subset \sim \bar{E}$ for all i, with $\lim_{i\to\infty}\mu(U_i) = \|\mu\|$.

Now for each i, set $A_i = \{\chi_{U_i} a: a \in A\}$ and $B_i = \{\chi_{U_i} b: b \in B\}$. Since $U_i \cap \bar{E}$ is empty, $B_i \subset L^1(\mu)$ for all i, and, of course, B_i and A_i are linear spaces with $\dim B_i \leqslant \dim B$ for all i. Now we claim that for some i, $\dim \bar{A}_i > \dim B$. For if this were not the case, we could choose for all i, a subset S_i of $\bar{A}_i$, with card $S_i \leqslant \dim B$, and with the linear span of S_i dense in $\bar{A}_i$. Then card $\bigcup_{i=1}^\infty S_i \leqslant \aleph_0 \cdot \dim B = \dim B$, so dim(closed linear span $\bigcup_{i=1}^\infty S_i$) = dim(closed linear span $\bigcup_{i=1}^\infty A_i) \leqslant \dim B$. But A is contained in the closed linear span of $\bigcup_{i=1}^\infty \bar{A}_i$; indeed for each $a \in A$, $\lim_{i\to\infty}\|a - a\cdot\chi_{U_i}\|_{L^1(\mu)} = 0$. Thus $\dim A \leqslant \dim B < \mathfrak{m}$, a contradiction.

Now fixing i such that $\dim \bar{A}_i > \dim B$, we have that $\dim \bar{A}_i > \dim \bar{B}_i$. Hence there exists an $a \in A$ such that $\chi_{U_i}\cdot a \notin \bar{B}_i$. By the Hahn–Banach theorem, we may choose a linear functional $F \in (L^1(\mu))^*$ such that $F(\chi_{U_i}\cdot a) \neq 0$, with $F(y) = 0$ for all $y \in B_i$. Since $(L^1(\mu))^* = L^\infty(\mu) = C(\Omega)$, we thus have that there is a continuous function f on Ω such that $\int f\cdot\chi_{U_i} a \, d\mu \neq 0$, while $\int f\cdot\chi_{U_i} b \, d\mu = \int f\cdot\chi_{U_i} b \, d(\mu+\lambda) = 0$ for all $b \in B$. Since U_i is clopen, $f\cdot\chi_{U_i}$ is a continuous function on Ω such that $f\cdot\chi_{U_i} \notin A^\perp$, yet $f\cdot\chi_{U_i} \in B^\perp$, contradicting the assumed weak* denseness of B. Q.E.D.

Remark. It follows from Theorem 3.1 and Lemma 1.3 that if A is a non-separable subspace of $L^1(\nu)$ for a not necessarily finite measure ν, then A^{**} is not weak* separable. (One uses the observation that if Γ is an uncountable set, $c_0(\Gamma)^* = l^1(\Gamma)$ is not weak* separable.)

An almost immediate consequence of 3.1 and the proof of Corollary 2.3 is

COROLLARY 3.2. *Let A be isomorphic to a non-separable subspace of $L^1(\mu)$ for some non-separable finite measure μ, and suppose that A^* is injective. Then A^* (and in particular, $L^\infty(\mu)$ itself) is not isomorphic to a double conjugate space.*

Proof. By Theorem 3.1, A^{**} is not weak* separable, but $(l^\infty)^*$ is weak* separable (since χl^1 is weak* dense therein). Hence A^* is not isomorphic to l^∞, and so by the proof of Corollary 2.3 (cf. the remark following 2.3), if A^* were isomorphic to a double conjugate Banach space, A would contain a complemented subspace isomorphic to $l^1(\Gamma)$ for some uncountable set Γ, which is impossible (cf. the second remark following Lemma 1.3).

Remarks. 1. If $L^1(\mu)$ is separable, then $L^\infty(\mu)$ is isomorphic to the double conjugate space l^∞ (cf. [21]). It follows easily from known results that l^∞ is not isomorphic to a triple conjugate space. (In fact Theorem 5.1 and the results of [21] imply that if $\mathfrak{m}$ is a cardinal number with $\mathfrak{m} < 2^{\mathfrak{c}}$, then $l^\infty_{\mathfrak{m}}$ is not isomorphic to A^{***} for any Banach space A.)

2. It follows from a result of Grothendieck [11] that if μ is a finite measure and if the Banach space A is such that A^* is isometric to $L^\infty(\mu)$, then A is isometric to $L^1(\mu)$. Consequently if μ is in addition non-purely atomic, $L^\infty(\mu)$ cannot be isometric to a double conjugate space.

COROLLARY 3.3. *Let A and B be Banach spaces isomorphic to subspaces of $L^1(\mu)$ and $L^1(\nu)$ for some finite measures μ and ν respectively. Then if* $\dim B < \dim A$, *there exists no one-to-one bounded linear operator T from A^* into B^*.*

Proof. Suppose to the contrary that $T: A^* \to B^*$ were one-to-one. Then $T^*(\chi B)$ would be a weak* dense subspace of A^{**}. But $\dim \overline{T^*(\chi(B))} \leqslant \dim B$, and $\overline{T^*(\chi(B))}$ is isomorphic to a subspace of some WCG Banach space, by Lemmas 1.1 and 1.3. (Indeed, were this false, $\overline{T^*(\chi(B))}$ would contain a subspace isomorphic to $l^1(\Gamma)$ for some uncountable set Γ, by Lemma 1.3; but then B would contain a subspace isomorphic to $l^1(\Lambda)$ for some uncountable set Λ by Lemma 4.2 of [19] (this also follows from our Lemma 1.1), which is impossible). Thus by Theorem 3.1, $\dim \overline{T^*(\chi(B))} \geqslant \dim A$, a contradiction.

Remark. When $A = L^1(\mu)$ and $B = L^1(\nu)$ above, the argument is slightly easier, for then $T^*(\chi(B))$ is a WCG subspace of $(L^1(\mu))^{**}$. An easier version of Theorem 3.1 and Lemma 1.3 (not requiring Lemma 1.1) then produces the desired contradiction.

Before proceeding to the next result, we need some preliminary definitions and facts concerning product measures. Given a non-empty set Γ, we let μ_Γ denote the product measure $\prod_{\alpha\in\Gamma} m_\alpha$ on $[0, 1]^\Gamma = \prod_{\alpha\in\Gamma}[0, 1]$, where for all α, m_α is Lebesgue measure on $[0, 1]$ with respect to the Lebesgue measureable subsets of $[0, 1]$. Of course, μ_Γ depends up to measure-isomorphism only on card Γ; thus given any infinite cardinal $\mathfrak{m}$, any set Γ with card $\Gamma = \mathfrak{m}$, and any p with $1 \leqslant p \leqslant \infty$, we shall denote the space $L^p(\mu_\Gamma)$ by $L^p[0, 1]^{\mathfrak{m}}$.

Now given Γ and Λ a proper non-empty subset of Γ, $\mu_\Gamma = \mu_\Lambda \times \mu_{\sim\Lambda}$ by the general theory of product measures. Thus by Fubini's theorem, we may define a map $p_\Lambda: L^1(\mu_\Gamma) \to L^1(\mu_\Lambda)$ as follows: for each $f \in L^1(\mu_\Gamma)$, set

$$(p_\Lambda f)(x) = \int_{[0,1]^{\sim\Lambda}} f(x_\Lambda \cup x_{\sim\Lambda})\, d\mu_{\sim\Lambda}(x_{\sim\Lambda}),$$

where for $S \subset \Gamma$ and $x \in [0, 1]^\Gamma$, $x_S \in [0, 1]^S$ is defined by $x_S(\alpha) = x(\alpha)$ for all $\alpha \in S$. We have that p_Λ is a linear projection of norm one, onto a subspace of $L^1(\mu_\Gamma)$ isometric to $L^1(\mu_\Lambda)$.

LEMMA 3.4. *Let $\Gamma = \bigcup_{i=1}^\infty \Gamma_i$ with $\Gamma_n \subset \Gamma_{n+1}$ for all n. Then $L^1(\mu_\Gamma)$ is isometric to a quotient space of the Banach space*

$$X = (L^1(\mu_{\Gamma_1}) \oplus L^1(\mu_{\Gamma_2}) \oplus \dots)_1.$$

Proof. For simplicity of notation, put $p_i = p_{\Gamma_i}$, and for each i, let T_i: $L^1(\mu_{\Gamma_i}) \to p_i(L^1(\mu_\Gamma))$ be the linear isometry onto the range of p_i given by $(T_i f)(x \cup y) = f(x)$ for all $f \in L^1(\mu_{\Gamma_i})$, $x \in [0, 1]^{\Gamma_i}$ and $y \in [0, 1]^{\Gamma \sim \Gamma_i}$. Then define T: $X \to L^1(\mu_\Gamma)$ by $Tx = \sum_{i=1}^\infty T_i x_i$ for all $x = (x_i)$ in X. Now $\|T\| = 1$; to see that $X/\ker T$ is isometric to $L^1(\mu_\Gamma)$, it suffices to show that TS is dense in U, where S and U refer to the unit cells of X and $L^1(\mu_\Gamma)$ respectively. Now $TS \supset \bigcup_{i=1}^\infty p_i(U)$. But given $f \in U$ and $\varepsilon > 0$, then by Theorem 24, page 207 of [7], we may choose π a finite subset of Γ such that $\|p_\pi f - f\| < \varepsilon/2$. Then there exists an N such that $\Gamma_N \supset \pi$. For any $n \geqslant N$, $p_n p_\pi = p_\pi$ and so $\|p_n f - f\| \leqslant \|p_n(p_\pi f - f)\| + \|p_\pi f - f\| < \varepsilon$. Hence $\lim_{n\to\infty} p_n f = f$, and thus $\bigcup_{i=1}^\infty p_i(U)$ is dense in U, showing that $\overline{TS} = U$. Q.E.D.

THEOREM 3.5. *Let μ and ν be finite measures. Then $L^\infty(\mu)$ and $L^\infty(\nu)$ are isomorphic if and only if* $\dim L^1(\mu) = \dim L^1(\nu)$.

Remark. It is easily seen that $\dim L^1[0, 1]^{\mathfrak{m}} = \mathfrak{m}$ for any infinite cardinal number $\mathfrak{m}$. Thus the spaces $L^\infty[0, 1]^{\mathfrak{m}}$ over all infinite cardinals $\mathfrak{m}$ form a complete set of linear topological types for the spaces $L^\infty(\mu)$ for finite measures μ, with $L^\infty[0, 1]^{\mathfrak{m}}$ not isomorphic to $L^\infty[0, 1]^{\mathfrak{n}}$ for $\mathfrak{n} \neq \mathfrak{m}$. Previous to our work, the classification of the isomorphism types of the spaces $L^p(\mu)$ for $1 \leqslant p < \infty$, $p \neq 2$ had been accomplished by Joram Lindenstrauss as follows: let μ be given, put $\mathfrak{m} = \dim L^1(\mu)$, and suppose $\mathfrak{m} > \aleph_0$. If $\mathfrak{m}$ is not the limit of a (denumerable) sequence of cardinals each less than $\mathfrak{m}$, then $L^p(\mu)$ is isomorphic to $L^p[0, 1]^{\mathfrak{m}}$. If $\mathfrak{m}$ is such a limit, there are two mutually exclusive alternatives:

(1) $L^p(\mu)$ is isomorphic to $L^p[0, 1]^{\mathfrak{m}}$.

(2) choosing a fixed sequence $\mathfrak{n}_1 < \mathfrak{n}_2 < \ldots$ of cardinals with $\mathfrak{m} = \lim_{k\to\infty} \mathfrak{n}_k$, then $L^p(\mu)$ is isomorphic to $(L^p[0, 1]^{\mathfrak{n}_1} \oplus L^p[0, 1]^{\mathfrak{n}_2} \oplus \ldots)_p$. (For $\mathfrak{m} = \aleph_0$, we have the known result that $L^p(\mu)$ is isomorphic to $L^p[0, 1]$ or l^p, and the latter two spaces are not isomorphic). To show that (1) and (2) are mutually exclusive, it is demonstrated that $L^p[0, 1]^{\mathfrak{m}}$ contains a subspace isomorphic to a Hilbert space of dimension $\mathfrak{m}$, while $(L^p[0, 1]^{\mathfrak{n}_1} \oplus L^p[0, 1]^{\mathfrak{n}_2} \oplus \ldots)_p$ contains no such subspace (where $\mathfrak{n}_k < \mathfrak{m}$ for all k).

Proof of Theorem 3.5. We have already shown the "only if" part in Corollary 3.3. Now let μ a finite measure be given with $\mathfrak{m} = \dim L^1(\mu)$. We shall show that $L^\infty(\mu)$ is isomorphic to $L^\infty[0, 1]^{\mathfrak{m}}$, thus completing the proof. We consider only the case $\mathfrak{m} > \aleph_0$, for the case $\mathfrak{m} = \aleph_0$ is known and due to Pelczynski (cf. [20] and also Corollary 6 of [21]).

By Maharam's theorem [17], there exists a set F, empty, finite, or countably infinite, and a finite or countably infinite sequence $\mathfrak{n}_1, \mathfrak{n}_2, \ldots$ of infinite cardinal numbers such that $L^1(\mu)$ is isometric to $(l^1(F) \oplus L^1[0, 1]^{\mathfrak{n}_1} \oplus L^1[0, 1]^{\mathfrak{n}_2} \oplus \ldots)_1$ (where $l^1(F) = \{0\}$ by definition, if F is empty). Since for any cardinal $\mathfrak{n}$, $L^1[0, 1]^{\mathfrak{n}}$ is isometric to $(L^1[0, 1]^{\mathfrak{n}} \oplus L^1[0, 1]^{\mathfrak{n}} \oplus \ldots)_1$, we may assume that $\mathfrak{n}_1 \leqslant \mathfrak{n}_2 \leqslant \ldots$ and that $\mathfrak{n}_k$ is defined for all positive integers k. We must then have that $\mathfrak{m}$ equals the least cardinal number $\mathfrak{a}$ such that $\mathfrak{n}_k \leqslant \mathfrak{a}$ for all k. Since $\mathfrak{m} > \aleph_0$ is assumed, there exists a set Γ with card $\Gamma = \mathfrak{m}$ and sets Γ_i with card $\Gamma_i = \mathfrak{n}_i$ for all i, with $\Gamma_1 \subset \Gamma_2 \subset \Gamma_3 \ldots$ and $\Gamma = \bigcup_{i=1}^{\infty} \Gamma_i$.

We thus obtain from Lemma 3.4 that $L^1[0,1]^{\mathfrak{m}}$ is isometric to a quotient space of $(L^1[0,1]^{\mathfrak{n}_1} \oplus L^1[0,1]^{\mathfrak{n}_2} \oplus \ldots)_1$, and hence $L^\infty[0,1]^{\mathfrak{m}}$ is isometric to a subspace of $L^\infty(\mu)$. It is also easily seen that $L^1(\mu)$ is isometric to a quotient space of $L^1[0,1]^{\mathfrak{m}}$, since this is true of the space $l^1(F)$ and each of the spaces $L^1[0,1]^{\mathfrak{n}_i}$. Consequently $L^\infty(\mu)$ is isometric to a subspace of $L^\infty[0,1]^{\mathfrak{m}}$. Since $(L^\infty[0,1]^{\mathfrak{m}} \oplus L^\infty[0,1]^{\mathfrak{m}} \oplus \ldots)_\infty$ is isomorphic to $L^\infty[0,1]^{\mathfrak{m}}$ and each of the spaces $L^\infty(\mu)$ and $L^\infty[0,1]^{\mathfrak{m}}$ are injective, it follows from Proposition 1.4 that they are isomorphic. Q.E.D.

Remark. It is fairly easy to see that for any infinite cardinal $\mathfrak{m}$, card $L^\infty[0,1]^{\mathfrak{m}} = \dim L^\infty[0,1]^{\mathfrak{m}} = \mathfrak{m}^{\aleph_0}$. Indeed, let Γ be a set of cardinality $\mathfrak{m}$, and for each countable subset Λ of Γ, let L^∞_Λ be the subspace of $L^\infty(\mu_\Gamma)$ given by all bounded measurable functions f which depend only on the coordinates Λ; (i.e., if $x, y \in [0,1]^\Gamma$ are such that $x(\alpha) = y(\alpha)$ for all $\alpha \in \Lambda$, then $f(x) = f(y)$). Then L^∞_Λ is isometric to $L^\infty[0,1]$, and card $L^\infty[0,1] = \mathfrak{c}$. Then $L^\infty(\mu_\Gamma) = \bigcup \{L^\infty_\Lambda : \Lambda$ is a countable subset of $\Gamma\}$, hence card $L^\infty(\mu_\Gamma) \leqslant \mathfrak{m}^{\aleph_0}$. A result considerably stronger than $\dim L^\infty[0,1]^{\mathfrak{m}} \geqslant \mathfrak{m}^{\aleph_0}$ is demonstrated in the proof of (d) of Theorem 5.1 below. It then follows that given any infinite cardinal number $\mathfrak{a}$, there exist cardinal numbers $\mathfrak{m}$ and $\mathfrak{n}$ greater than $\mathfrak{a}$ such that $\dim L^\infty[0,1]^{\mathfrak{m}} = \dim L^\infty[0,1]^{\mathfrak{n}}$, yet $L^\infty[0,1]^{\mathfrak{m}}$ and $L^\infty[0,1]^{\mathfrak{n}}$ are not isomorphic. For we simply let $\mathfrak{m}$ be the least cardinal greater than the sequence of cardinals $\mathfrak{n}_1, \mathfrak{n}_2, \ldots$ defined by $\mathfrak{n}_1 = \mathfrak{a}$; $\mathfrak{n}_k = 2^{\mathfrak{n}_{k-1}}$ for all $k > 1$ and then set $\mathfrak{n} = 2^{\mathfrak{m}}$ $(= \mathfrak{m}^{\aleph_0})$.

A special case of the next result is that if B^* is isomorphic to l^∞, then B must be separable (and in fact, isomorphic to a subspace of $L^1[0,1]$). In Proposition 5.5, we show that there exists a separable Banach space B_1 and a non-separable Banach space B_2 such that B_1^* is isometric to B_2^*.

THEOREM 3.6. *Let A be a subspace of $L^1(\mu)$ for some finite measure μ, and let $\mathfrak{m} = \dim A$. Then*

(a) *if B is a Banach space with B^* isomorphic to A^*, then B is isomorphic to a subspace of $L^1[0,1]^{\mathfrak{m}}$ and* $\dim B = \mathfrak{m}$;

(b) *if A^* is injective, then A^* is isomorphic to a subspace of $L^\infty[0,1]^{\mathfrak{m}}$.*

Proof. We first prove (a); assume that B^* is isomorphic to A^*. Then if Γ is an uncountable set, $l^1(\Gamma)$ is not isomorphic to a complemented subspace of B, for otherwise $l^\infty(\Gamma)$

would be isomorphic to a subspace of A^*, and hence $l^1(\Gamma)$ would be isomorphic to a complemented subspace of A by Corollary 1.2, which is impossible. But B is isomorphic to a subspace of A^{**}, which may be identified with $(A^\perp)^\perp \subset (L^1(\mu))^{**} = (L^\infty(\mu))^*$ which in turn may be identified with $M(S)$ for some compact Hausdorff space S. Thus by Lemma 1.3, B is isomorphic to a subspace of $L^1(\nu)$ for some finite measure ν. Since B^* is isomorphic to A^*, B is isomorphic to a weak* dense subspace of A^{**}, and so by Theorem 3.1, $\dim B \geqslant \dim A$. Also A is isomorphic to a weak* dense subspace of B^*, so again by 3.1, $\dim A \geqslant \dim B$, hence $\mathfrak{m} = \dim B$.

Now let Y be a subspace of $L^1(\nu)$ isomorphic to B. Then there exists a subspace Z with $Y \subset Z \subset L^1(\nu)$, such that Z is isometric to $L^1(\varrho)$ for some finite measure ϱ, with $\dim L^1(\varrho) = \mathfrak{m}$. Indeed, simply let D be a subset of Y of cardinality $\mathfrak{m}$, with linear span dense in Y. For each $d \in D$, choose a countable set $\mathfrak{F}_d$ of Borel measurable subsets $E_1^d, E_2^d, \ldots$ of S, such that d is in the closed-linear-span of $\{\chi_{E_j^d}: j = 1, 2, \ldots\}$ in $L^1(\nu)$. Now let Σ be the σ-subalgebra of the Borel subsets of S generated by $\bigcup_{d \in D} \mathfrak{F}_d$. Then the closed linear span of the characteristic functions of the members of Σ is isometric to $L^1(\varrho)$ where $\varrho = \mu | \Sigma$. Since card $\bigcup_{d \in D} \mathfrak{F}_d = \mathfrak{m}$, $\dim L^1(\varrho) = \mathfrak{m}$.

Finally, it follows from Maharam's theorem that $L^1(\varrho)$ is isometric to a subspace of $L^1[0, 1]^{\mathfrak{m}}$.

Proof of (*b*). Assuming that A^* is injective, there exists a compact Hausdorff space S such that A^* is isomorphic to a complemented subspace X of $C(S)$. Then there exists a subspace A_1 of $M(S)$ isomorphic to A, and a constant $K > 0$, such that for all $f \in X$.

$$\sup \left| \int_S f(s)\, d\lambda(s) \right| \leqslant K \|f\|_\infty,$$

the supremum being taken over all $\lambda \in A_1$ with $\|\lambda\| \leqslant 1$. By Lemma 1.3, there exists a positive $\nu \in M(S)$ with $A_1 \subset L^1(\nu)$, and hence as we showed in the proof of (a), there exists a subspace Z with $A_1 \subset Z \subset L^1(\nu)$ such that Z is isometric to $L^1(\varrho)$ for some finite measure ϱ, with $\dim L^1(\varrho) = \mathfrak{m}$. We now define $T: X \to Z^*$ by $(Tx)(z) = \int_S x(s) z(s)\, d\nu(s)$ for all $x \in X$ and $z \in Z$. Then T is an isomorphism between X and a closed subspace of Z^* and Z^* is isometric to $L^\infty(\varrho)$, which is in turn isomorphic to $L^\infty[0, 1]^{\mathfrak{m}}$ by Theorem 3.5. Thus A^* is isomorphic to a subspace of $L^\infty[0, 1]^{\mathfrak{m}}$. Q.E.D.

The next and final result of this section is considerably stronger than the main classification result, Theorem 3.5. Its proof uses the techniques of the proof of Theorem 3.1.

THEOREM 3.7. *Let μ be a finite measure with $\mathfrak{m} = \dim L^1(\mu)$, and suppose that X is a*

Banach space with X^* *isomorphic to* $L^\infty(\mu)$. *Then* $\dim X = \mathfrak{m}$, X *is isomorphic to a subspace of* $L^1[0, 1]^{\mathfrak{m}}$, *and* $L^1[0, 1]^{\mathfrak{m}}$ *is isomorphic to a quotient space of* X.

Proof. The fact that $\dim X = \mathfrak{m}$ and X is isomorphic to a subspace of $L^1[0, 1]^{\mathfrak{m}}$ follows immediately from the preceeding result.

Now by Theorem 3.5, $L^\infty(\mu)$ is isomorphic to $L^\infty[0, 1]^{\mathfrak{m}}$. Thus there exists a constant $K > 0$ and a subspace B of $(L^\infty[0, 1]^{\mathfrak{m}})^*$ isomorphic to X, such that for all $f \in L^\infty[0, 1]^{\mathfrak{m}}$,

$$\|f\|_\infty \leqslant K \sup \{|b(f)|: b \in B, \|b\| \leqslant 1\}. \tag{*}$$

Now letting ν be a finite measure with $L^1(\nu) = L^1[0, 1]^{\mathfrak{m}}$, and letting Ω be the Stone space of the measure algebra of ν, we may assume that the measure μ is the measure ν induces on Ω; i.e., μ is a regular finite positive Borel measure μ on Ω, and μ and Ω satisfy properties 1–3 of the proof of Theorem 3.1. Moreover, since ν is a homogeneous measure, we will have by Maharam's theorem

4. For each non-empty clopen subset U of Ω, $L^1(\mu | U)$ is isometric to $L^1(\mu)$, i.e., to $L^1[0, 1]^{\mathfrak{m}}$.

We identify $L^\infty[0, 1]^{\mathfrak{m}}$ with $C(\Omega)$, and consequently B with a subspace of $M(\Omega)$. Since B is isomorphic to a subspace of a WCG Banach space, we may choose, exactly as in the proof of Theorem 3.1, a positive $\lambda \in M(\Omega)$ with $\lambda \perp \mu$, such that $B \subset L^1(\mu + \lambda)$, a closed set E such that $\lambda(\sim E) = \mu(E) = 0$, and a clopen non-empty set $U \subset \sim E$. We now claim that the map T: $B \to L^1(\mu | U)$ defined by $T(b) = \chi_U \cdot b$ for all $b \in B$, is onto $L^1(\mu | U)$ (which we identify with $\{\chi_U \cdot f: f \in L^1(\mu)\}$). This will complete the proof, since then $L^1[0, 1]^{\mathfrak{m}}$ is isomorphic to $B/\ker T$, which in turn is isomorphic to a quotient space of X.

Let $W = \overline{\{Tb: b \in B, \|b\| \leqslant K\}}$ and let V denote the unit cell of $L^1(\mu | U)$. It suffices to prove that $W \supset V$ (by the usual proof given for the open-mapping theorem).

Now if this were false, since W is a closed convex set with $\alpha W \subset W$ for all scalars α with $|\alpha| \leqslant 1$, it follows by the Hahn–Banach theorem (cf. page 417 of [7]) that we could choose a $v \in V$, and a bounded linear functional F in $L^1(\mu)^*$, such that $\sup_{x \in W} |F(x)| < |F(v)|$.

Since $C(S) = L^\infty(\mu) = L^1(\mu)^*$, there thus exists a continuous function φ, such that

$$K \sup_{\substack{b \in B \\ \|b\| \leqslant 1}} \left| \int \varphi \chi_U b d(\mu + \lambda) \right| < \left| \int \varphi v d\mu \right| \leqslant \|\chi_U \varphi\|_\infty.$$

Setting $f = \chi_U \varphi$, $f \in C(S)$, and thus $(*)$ is contradicted. Q.E.D.

Remark. Let B be isomorphic to a subspace of a WCG Banach space, and a closed subspace of $(L^\infty[0, 1]^{\mathfrak{m}})^*$ for some cardinal $\mathfrak{m}$. The last part of the proof of Theorem 3.7 shows that if B is of positive characteristic (i.e., satisfies (*) for some constant K), then $L^1[0, 1]^{\mathfrak{m}}$ is isomorphic to a quotient space of B. If B satisfies the weaker hypothesis that it is weak* dense in $(L^\infty[0, 1]^{\mathfrak{m}})^*$, an easier argument than the one above shows that there exists a bounded linear operator from B onto a dense subspace of $L^1[0, 1]^{\mathfrak{m}}$.

4. Some linear topological invariants of injective Banach spaces and the spaces $C(S)$

We shall need the following definitions for this section: Given a compact Hausdorff space S and $\mu \in M(S)$, μ is called *strictly positive* if $\mu(U) > 0$ for all non-empty open $U \subset S$. We say that S *carries a strictly positive measure*, if there exists a strictly positive $\mu \in M(S)$.

Given a topological space X, we say that X satisfies the C.C.C. (*countable chain condition*) if every uncountable family of open subsets of X contains two sets with nonempty intersection.

Our first result together with a theorem of Gaifman shows that there exists a Stonian space S_G such that $C(S_G)$ is not isomorphic to a conjugate Banach space (see Corollary 4.4 below).

THEOREM 4.1. *Let S be a compact Hausdorff space satisfying the* C.C.C. *and suppose that $C(S)$ is isomorphic to a conjugate Banach space. Then S carries a strictly positive measure.*

Proof. Let B be a Banach space with B^* isomorphic to $C(S)$. Since B is isometric to a weak* dense subspace of B^{**}, B is isomorphic to a weak* dense subspace, A, of $C(S)^* = M(S)$. Now if there exists a positive $\mu \in M(S)$ with $A \subset L^1(\mu)$, we are done, for then $L^1(\mu)$ is weak* dense in $M(S)$, and this implies that μ is a strictly positive measure. Now suppose that there does not exist such a μ. We shall then show that S cannot satisfy the C.C.C., thus completing the proof.

By Lemma 1.3, there exists an uncountable set Γ such that A contains a complemented subspace isomorphic to $l^1(\Gamma)$. Since A^* is isomorphic to $C(S)$, we obtain that $C(S)$ contains a subspace isomorphic to $l^\infty(\Gamma)$, and consequently $C(S)$ has a subspace isomorphic to $c_0(\Gamma)$. Thus, we may choose a family $\{e_\gamma: \gamma \in \Gamma\}$ of functions in $C(S)$, with $\|e_\gamma\| = 1$ for all γ, and a constant $K > 0$ such that for all $\gamma_1, \ldots, \gamma_n$ in Γ, $\|\sum_{i=1}^n e_{\gamma_i}\| \leq K$. By multiplying each e_γ by a complex scalar of modulus one if necessary, we may assume that $\sup_{s \in S} \operatorname{Re} e_\gamma(s) = 1$ for all $\gamma \in \Gamma$, where $\operatorname{Re} e_\gamma$ denotes the real part of the function e_γ. Now for each $\gamma \in \Gamma$, let $U_\gamma = \{s \in S: |e_\gamma(s) - 1| < \frac{1}{2}\}$. Then if N is an integer with $N > 2K$, then if $\gamma_1, \ldots, \gamma_N$ are any N distinct members of Γ, $\bigcap_{i=1}^N U_{\gamma_i} = \varnothing$. For if there existed an $s \in \bigcap_{i=1}^N U_{\gamma_i}$, we would have that $|\sum_{i=1}^N e_{\gamma_i}(s)| \geq N/2 > K$, a contradiction. In particular, for each $\gamma_0 \in \Gamma$ there exist at most

N γ's in Γ with $U_\gamma = U_{\gamma_0}$. Since Γ is an uncountable set, we have that $\{U_\gamma: \gamma \in \Gamma\}$ is an uncountable family of open subsets of S, such that no point of S belongs to infinitely many members of the family. Thus S cannot satisfy the C.C.C. in virtue of the following

LEMMA 4.2. *Let S satisfy the* C.C.C., *and suppose that $\mathcal{F}$ is an uncountable family of open subsets of S. Then there exists an infinite sequence $F_1, F_2, \ldots$ of distinct members of $\mathcal{F}$ with* $\bigcap_{i=1}^{\infty} F_i \neq \emptyset$.

Proof. We first need some preliminaries. Given any family $\mathcal{A}$ of subsets of S and n a positive integer, let $\mathcal{A}_n$ denote the class of all sets of the form $F_1 \cap F_2 \cap \ldots \cap F_n$, where $F_1, \ldots, F_n$ are n distinct members of $\mathcal{A}$. Then put $\mathcal{A}^* = \bigcup_{n=1}^{\infty} \mathcal{A}_n$. $\mathcal{A}^*$ is, of course, the family of all finite intersections of members of $\mathcal{A}$; evidently if $\mathcal{A}$ is finite, so is $\mathcal{A}^*$; otherwise card $\mathcal{A} =$ card $\mathcal{A}^*$. We next observe that for all n, $(\mathcal{A}_n)_2 \subset \mathcal{A}^*_{n+1}$. Indeed, let A and B be distinct members of $\mathcal{A}_n$. We may choose $F_1, \ldots, F_n$ distinct members of $\mathcal{A}$ and $G_1, \ldots, G_n$ distinct members of $\mathcal{A}$ with $A = \bigcap_{i=1}^{n} F_i$ and $B = \bigcap_{i=1}^{n} G_i$. Since $A \neq B$, there must exist indices i, $1 \leqslant i \leqslant n$, such that $G_i \neq F_j$ for any j with $1 \leqslant j \leqslant n$. Let $i_1 < i_2 < \ldots < i_k$ be an enumeration of this set of indices; then for each r with $1 \leqslant r \leqslant k$, $F_1 \cap \ldots \cap F_n \cap G_{i_r}$ is a member of $\mathcal{A}_{n+1}$, and $A \cap B = \bigcap_{r=1}^{k} (F_1 \cap \ldots \cap F_n \cap G_{i_r})$, thus $A \cap B \in \mathcal{A}^*_{n+1}$.

Next, we observe that (assuming S satisfies the C.C.C.), if $\mathcal{A}$ is an uncountable family of open subsets of S, then

(*) either some non-empty member of $\mathcal{A}_2$ is contained in uncountably many members of $\mathcal{A}$, or $\mathcal{A}_2$ is uncountable.

To see this, let $\mathcal{H}$ denote the class of all sets F in $\mathcal{A}$ such that there exists a G in $\mathcal{A}$ with $G \neq F$ and $G \cap F \neq \emptyset$. Then $\mathcal{H}$ is uncountable. Indeed, $\mathcal{A} \sim \mathcal{H}$ is a disjoint family of open sets and is hence at most countable. Now for each $A \in \mathcal{A}_2$, let $\mathcal{A}_A$ denote the class of all sets $F \in \mathcal{A}$ with $F \supset A$. Then we have that $\mathcal{H} = \bigcup \{\mathcal{A}_A: A \in \mathcal{A}_2, A \neq \emptyset\}$. Thus if $\mathcal{A}_2$ is countable, $\mathcal{A}_A$ must be uncountable for some non-empty $A \in \mathcal{A}_2$, thus proving (*).

From (*) we easily deduce by induction that

(**) If $\mathcal{B}$ is an uncountable family of open subsets of S and n is a positive integer, then there are uncountably many distinct n-tuples $(B_1, \ldots, B_n)$ in $\mathcal{B}$ (i.e., $B_i \neq B_j$ if $i \neq j$) with $\bigcap_{i=1}^{n} B_i \neq \emptyset$.

To see this, let us assume that no non-empty member of $\mathcal{B}^*$ is contained in uncountably many members of $\mathcal{B}$ (since otherwise (**) holds automatically). We shall then show that $\mathcal{B}_n$ is uncountable for all n, from which (**) follows immediately. $\mathcal{B}_1$ is trivially uncountable. Suppose we have proved that $\mathcal{B}_n$ is uncountable. Then if $\mathcal{B}_{n+1}$ were countable, $\mathcal{B}^*_{n+1}$

and consequently $(\mathcal{B}_n)_2$ would also be countable by our preliminary observations. Thus by (*), there would exist A and B in $\mathcal{B}_n$ with $A \cap B$ non-empty and contained in uncountably many members of $\mathcal{B}_n$. But if $E \in \mathcal{B}_n$ and $A \cap B \subset E$, then E is a finite intersection of members of $\mathcal{B}$, each of which contains $A \cap B$. Hence $A \cap B$ would be contained in uncountably many members of $\mathcal{B}$, and of course $A \cap B \in \mathcal{B}^*$, so our assumption on $\mathcal{B}$ would be contradicted.

Thus (**) has been established by finite induction. Now let $\mathfrak{F}$ be as in the statement of 4.2, and for each positive integer n, let G_n be the set of all points in S which are contained in at most n distinct members of $\mathfrak{F}$, put G_n^0 equal to the interior of G_n, and let $\mathcal{G}_n = \{F \in \mathfrak{F}: F \cap G_n^0 \neq \varnothing\}$. Fixing n, we claim that $\mathcal{G}_n$ is at most countable. Indeed, denoting $\{F \cap G_n^0: F \in \mathcal{G}_n\}$ by $\mathcal{G}_n \cap G_n^0$, we have that no $n+1$ distinct elements of $\mathcal{G}_n \cap G_n^0$ have a point in common. Thus by (**), $\mathcal{G}_n \cap G_n^0$ is at most countable. But each member of $\mathcal{G}_n \cap G_n^0$ is contained in at most n members of $\mathcal{G}_n$, and $\mathcal{G}_n = \{F \in \mathfrak{F}: \exists A \in \mathcal{G}_n \cap G_n^0 \text{ with } F \supset A\}$.

Thus since $\mathcal{G}_n$ is countable for all n, $\bigcup_{n=1}^\infty \mathcal{G}_n$ is countable, so there exists a non-empty $F \in \mathfrak{F}$ with $F \notin \bigcup_{n=1}^\infty \mathcal{G}_n$. It is easily seen that G_n is closed for all n, and thus there exists an $s \in F$ with $s \notin \bigcup_{n=1}^\infty G_n \sim G_n^0$ by the Baire category theorem. Then $s \notin \bigcup_{n=1}^\infty G_n$ by the definition of F, so s belongs to infinitely many members of $\mathfrak{F}$. Q.E.D.

Remark. Let $\mathfrak{m}$ be a fixed cardinal number with $\mathfrak{m} > \aleph_0$. We say that the topological space X satisfies the $\mathfrak{m}$-chain condition if every disjoint family of open subsets of X has cardinality less than $\mathfrak{m}$. (Thus the C.C.C. is the $\aleph_1$-chain condition.) A slight modification of the proof of Lemma 4.2 shows that its conclusion holds if we replace the hypothesis that S satisfies the C.C.C. by the hypotheses that S satisfies the $\mathfrak{m}$-chain condition and that card $\mathfrak{F} = \mathfrak{m}$. In fact, the proof of 4.2 shows that the following more general result holds (using the notation introduced at the beginning of the proof of 4.2):

Let X satisfy the $\mathfrak{m}$-chain condition, and suppose that $\mathfrak{F}$ is a family of open subsets of X with card $\mathfrak{F} = \mathfrak{m}$. *Then either there exists a non-empty member of $\mathfrak{F}^*$ contained in uncountably many members of $\mathfrak{F}$, or for all positive integers n,* card $\mathfrak{F}_n = \mathfrak{m}$. *Moreover, if X is a Baire space, there exists a point in X belonging to infinitely many members of $\mathfrak{F}$. In fact, there exists a fixed set E of the first category in X, and an $\mathfrak{F}' \subset \mathfrak{F}$ with* card $(\mathfrak{F} \sim \mathfrak{F}') < \mathfrak{m}$, *such that for every non-empty $F \in \mathfrak{F}'$, every point of $F \sim E$ belongs to infinitely many members of $\mathfrak{F}$.* (X is called a Baire space if every countable union of closed nowhere dense subsets of X has void interior.)

Of course, it then follows as in the proof of Theorem 4.1, that if the compact Hausdorff space S satisfies the $\mathfrak{m}$-chain condition and if Γ is a set with card $\Gamma = \mathfrak{m}$, then $c_0(\Gamma)$ is not isomorphic to a subspace of $C(S)$. (Cf. the remark following 4.6 below for a stronger result.)

It follows from the above remark that if S is a Stonian space and Γ is an infinite set with $c_0(\Gamma)$ isomorphic to a subspace of $C(S)$, then $l^\infty(\Gamma)$ is isometric to a subspace of $C(S)$. For there exists a family Γ_1 of pairwise disjoint non-empty open subsets of S with card $\Gamma_1 =$ card Γ. (If card $\Gamma = \aleph_0$, this is obvious; if card $\Gamma > \aleph_0$. then S does not satisfy the $\mathfrak{m}$-chain condition where $\mathfrak{m} = \text{card}\,\Gamma$.) Since S is Stonian, S is totally disconnected, so we may assume that each $U \in \Gamma_1$ is a clopen (closed and open) set. The set of all $f \in C(S)$ with f supported on $\overline{\bigcup \Gamma_1}$ and $f|U$ constant for all $U \in \Gamma_1$, is then the desired subspace of $C(S)$ isometric to $l^\infty(\Gamma)$. In virtue of the known fact that every $\mathcal{P}_1$ space is isometric to $C(S)$ for some Stonian S (cf. [4]) and Corollary 1.2, we obtain immediately the

COROLLARY 4.3. *Let Γ be an infinite set, and let the Banach space B contain a subspace isomorphic to $c_0(\Gamma)$. Then if B is either isomorphic to a $\mathcal{P}_1$ space or isomorphic to a conjugate Banach space, B contains a subspace isomorphic to $l^\infty(\Gamma)$.* (1)

It follows from a result of Gaifman [8] via the theory of Boolean algebras (cf. [29]) that there exists a Stonian space S_G such that S_G satisfies the C.C.C. and carries no strictly positive measure. [Gaifman constructs a Boolean algebra $\mathcal{A}$ satisfying the C.C.C. (the σ-chain condition in the terminology of [29]) for which there is no strictly positive finite measure. Now as he points out on page 68 of [8], there exists a complete Boolean algebra $\mathcal{B}$ containing $\mathcal{A}$ as a dense subalgebra (cf. § 35 of [29]), and $\mathcal{B}$ will automatically also satisfy the C.C.C. and carry no strictly positive measure. We then simply let S_G be the Stone space of $\mathcal{B}$.] Since $C(S_G)$ is a $\mathcal{P}_1$ space (cf. [4]), we thus obtain immediately from Gaifman's result and Theorem 4.1 the

COROLLARY 4.4. *There exists a $\mathcal{P}_1$ space which is not isomorphic to any conjugate Banach space.*

The techniques we used in proving 4.1 yield linear topological invariants of injective Banach spaces and the spaces $C(S)$ which we shall now explore. In particular, letting S_G be as above, our next result implies that every weakly compact subset of $C(S_G)$ is separable, and that if μ is a finite measure, then there exists no one-to-one bounded linear operator mapping $C(S_G)$ into $L^\infty(\mu)$. We mention also that the algebra $\mathcal{A}$ constructed by Gaifman in [8] has cardinality at most the continuum; since $\mathcal{A}$ satisfies the C.C.C., it follows easily that a complete Boolean algebra containing $\mathcal{A}$ has cardinality the continuum, and hence dim $C(S_G) = \mathfrak{c}$. (Indeed, it is not difficult to show that if S is an infinite totally-disconnected compact Hausdorff space and $\mathcal{B}$ its family of closed-and-open subsets, then dim $C(S) =$

(1) We have recently proved that 4.3 holds for any space B isomorphic to a complemented subspace of a conjugate Banach space. *Added in proof:* This result and also a generalization of Lemma 1.1 will appear in *Studia Mathematica* in a paper by the author entitled "On relatively disjoint families of measures, with some applications to Banach space theory".

card $\mathcal{B}$.) Thus by Theorem 5.1 below, $C(S_G)$ is isometric to a quotient algebra of l^∞, i.e., S_G is homeomorphic to a subset of βN.

THEOREM 4.5. *Let S be a compact Hausdorff space. Then*

(a) *S satisfies the* C.C.C. *if and only if every weakly compact subset of $C(S)$ is separable (if and only if $C(S)$ contains no isomorph of $c_0(\Gamma)$ for any uncountable set Γ).*

(b) *S carries a strictly positive measure if and only if $C(S)^*$ contains a weakly compact total subset.*

Proof of 4.5(b). If $\mu \in M(S)$ is strictly positive, then $L^1(\mu)$ is a WCG total subspace of $C(S)^*$. (Thus the unit cell of $L^2(\mu)$ injects into $C(S)^*$ as a weakly compact total subset). On the other hand, if K is a weakly compact total subset of $C(S)^*$, then the closed linear span of K (in the norm topology) is a WCG subspace of $C(S)^*$; hence by Lemma 1.3, there exists a positive $\mu \in M(S)$ with $K \subset L^1(\mu)$. Since K is total, μ must be strictly positive.

Proof of 4.5(a). We first observe that if S fails to satisfy the C.C.C., then there exists an uncountable set Γ such that $c_0(\Gamma)$ is isometric to a subspace of $C(S)$. Indeed, we may choose an uncountable family $\{U_\gamma: \gamma \in \Gamma\}$ of pairwise disjoint non-empty open subsets of S, with $U_\gamma \neq U_{\gamma'}$ if $\gamma \neq \gamma'$, $\gamma, \gamma' \in \Gamma$. Then for each $\gamma \in \Gamma$, choose $e_\gamma \in C(S)$ with $\|e_\gamma\| = 1$ and $e_\gamma \equiv 0$ on $\sim U_\gamma$. Then the closed linear span of $\{e_\gamma: \gamma \in \Gamma\}$ is isometric to $c_0(\Gamma)$, and of course $\{0\} \cup \{e_\gamma: \gamma \in \Gamma\}$ is a weakly compact non-separable subset of $C(S)$.

Now suppose that S satisfies the C.C.C. It then follows from our proof of Theorem 4.1 that for no uncountable set Γ is $c_0(\Gamma)$ isomorphic to a subspace of $C(S)$. Thus the remaining (and only non-trivial) assertion to be proved is that every weakly compact set in $C(S)$ is separable.

Now suppose there exists a non-separable weakly compact subset of $C(S)$. Then by the Krein–Smulian theorem (cf. page 434 of [7]) the closed convex hull of this set is weakly compact, and consequently the set obtained by multiplying the latter by all scalars of modulus one, is also weakly compact. Thus we have established that there exists a symmetric convex non-separable weakly compact subset K of $C(S)$. By a result of Corson (see Proposition 3.4 of [16]), K contains a subset homeomorphic in its weak topology to the one-point compactification of an uncountable set. This means in virtue of the symmetry of K, that there exists an uncountable set Γ_1 of non-zero elements of K such that every sequence of distinct elements of Γ_1 converges weakly to zero. We may then choose a $\delta > 0$ such that $\Gamma = \{\gamma \in \Gamma_1: \|\gamma\| > \delta\}$ is uncountable, since $\Gamma_1 = \bigcup_{n=1}^\infty \{\gamma \in \Gamma_1: \|\gamma\| > 1/n\}$.

Now for each $\gamma \in \Gamma$, let $U_\gamma = \{s \in S: |\gamma(s)| > \delta/2\}$. Then there exists an infinite sequence $\gamma_1, \gamma_2, \ldots$ of distinct elements of Γ such that $\bigcap_{i=1}^\infty U_{\gamma_i}$ is non-empty. Indeed, if $\{U_\gamma: \gamma \in \Gamma\}$

is countable this is obvious; otherwise this follows from Lemma 4.2. Now $\gamma_i \to 0$ weakly, hence $\gamma_i(s) \to 0$ for all $s \in S$. But choosing $s \in \bigcap_{i=1}^{\infty} U_{\gamma_i}$, $|\gamma_i(s)| > \delta/2 > 0$ for all i, a contradiction. Q.E.D.

Remark. It follows immediately from Theorem 4.5(b) that if S is a compact Hausdorff space, then either there exists a finite measure μ such that $C(S)$ is isometric to a subspace of $L^\infty(\mu)$, or there is no bounded linear operator mapping $C(S)$ one-to-one into $L^\infty(\mu)$ for any finite measure μ. Lemma 1.3 may also be employed to show that if X is an injective Banach space, then X is isomorphic to a subspace of $L^\infty(\mu)$ for some finite measure μ if and only if X^* contains a WCG subspace of positive characteristic.

We obtain as an immediate consequence of Theorem 4.5 (a) and the results of [1], the

COROLLARY 4.6. *Let K be a weakly compact subset of a Banach space, and suppose that K satisfies the* C.C.C. *Then K is separable.*

Proof. By Theorem 4.5 (a), every weakly compact subset of $C(K)$ is separable. By a result of Amir and Lindenstrauss [1], $C(K)$ is a WCG Banach space. Hence $C(K)$ is separable, and thus K is metrizable. Q.E.D.

Remark. The density character of a compact Hausdorff space S is defined to be the smallest cardinal number $\mathfrak{m}$ such that there exists a dense subset of S, of cardinality $\mathfrak{m}$. Using the terminology introduced in the remark following Lemma 4.2, we note the following generalization of Theorem 4.5 (a):

THEOREM. *Let $\mathfrak{m} > \aleph_0$. The compact Hausdorff space S satisfies the $\mathfrak{m}$-chain condition if and only if every weakly compact subset of $C(S)$ has density character less than $\mathfrak{m}$ (if and only if $C(S)$ contains no isomorph of $c_0(\Gamma)$ for any set Γ of cardinality $\mathfrak{m}$).*

Now it is not difficult to show that if K is a compact Hausdorff space and if L is a weakly compact total subset of $C(K)$, then the density character of L equals the density character of K. It thus follows from the above Theorem and the results of [1] that *if K is a weakly compact subset of a Banach space with the density character of K equal to $\mathfrak{m}$, then K contains a family of pairwise disjoint open subsets, of cardinality $\mathfrak{m}$*. This, of course, generalizes Corollary 4.6.

For the sake of completeness, we give the proof of this Theorem. We first need a lemma which follows from a general result of Tarski concerning Boolean algebras (Theorem 4.5 of [30]).

LEMMA. *Let X be a topological space. Suppose there exists an increasing sequence of cardinal numbers, $\aleph_0 < \mathfrak{n}_1 < \mathfrak{n}_2 < \mathfrak{n}_3 < \dots$ and families $\mathfrak{F}_1$, $\mathfrak{F}_2$, ... of pairwise disjoint open subsets* *of* X, *such that* card $\mathcal{F}_k = \mathfrak{n}_k$ *for all* k. *Then there exists a family* $\mathcal{F}$ *of pairwise disjoint open subsets of* X *with* card $\mathcal{F} = \mathfrak{m}$, *where* $\mathfrak{m} = \lim_{k\to\infty} \mathfrak{n}_k$.

To prove the Lemma, define $\mathcal{G} \underline{\cap} A = \{G \cap A: G \in \mathcal{G}\}$ for any $A \subset X$ and $\mathcal{G}$ a family of subsets of X; we may assume (with no loss of generality) that $\mathfrak{n}_k$ is a successor cardinal for all k.

If there exists an n, $F_1, F_2, \ldots$ an infinite sequence of distinct members of $\mathcal{F}_n$, and $l_1 < l_2 < l_3 \ldots$ such that card $(\mathcal{F}_{l_i} \underline{\cap} F_i) = \mathfrak{n}_{l_i}$ for all i, then $\mathcal{F} = \bigcup_{i=1}^\infty \mathcal{F}_{l_i} \underline{\cap} F_i$ satisfies the conclusion of the Lemma. So suppose that there exists no n with these properties. Then for each n, there exists an integer $l(n)$ so that for all $m \geq l(n)$,

$$\text{card } \{F \in \mathcal{F}_n: \text{card } \mathcal{F}_m \underline{\cap} F = \mathfrak{n}_m\} < \aleph_0.$$

By removing from each $\mathcal{F}_n$ a countable subfamily if necessary, we may assume that if $m \geq l(n)$, then for all $F \in \mathcal{F}_n$, card $(\mathcal{F}_m \underline{\cap} F) < \mathfrak{n}_m$. Now choose $(a(n))$ a strictly increasing sequence of positive integers such that $a(i) \geq l(a(j))$ for all i and j with $1 \leq j < i$.

Then for each such i and j, and for any $F \in \mathcal{F}_{a(j)}$, card $(\mathcal{F}_{a(i)} \underline{\cap} F) < \mathfrak{n}_{a(i)}$. Now fix $i \geq 2$, and define

$$\mathcal{G}_i = \bigcup_{j=1}^{i-1} \bigcup_{F \in \mathcal{F}_{a(j)}} \{G \in \mathcal{F}_{a(i)} : G \cap F \neq \emptyset\}.$$

Since $\mathfrak{n}_{a(i)}$ is a successor cardinal and $\mathcal{F}_{a(i)}$ is a disjoint family of sets, card $\mathcal{G}_i < \mathfrak{n}_{a(i)}$. Then $\mathcal{F} = \bigcup_{i=2}^\infty \mathcal{F}_{a(i)} \sim \mathcal{G}_i$ satisfies the conclusion of the Lemma.

To prove the Theorem, we let $\mathfrak{n}$ be the smallest cardinal number $\mathfrak{m} > \aleph_0$ such that S satisfies the $\mathfrak{m}$-chain condition, and let K be a weakly compact subset of $C(S)$. We now show that the density character of K is less than $\mathfrak{n}$, thus proving the only non-trivial assertion of the Theorem. Suppose that the density character of K is greater than or equal to $\mathfrak{n}$; then by the Krein–Smulian theorem and the proof of Proposition 3.4 of [16], the closed convex circled hull of K contains a set Γ_1 of non-zero elements with card $\Gamma_1 = \mathfrak{n}$, such that every sequence of distinct elements of Γ_1 converges weakly to zero. The Lemma and the definition of $\mathfrak{n}$ imply that $\mathfrak{n}$ is not equal to the limit of an increasing sequence of smaller cardinals. Thus there exists a positive integer j, such that $\Gamma = \{\gamma \in \Gamma_1: \|\gamma\| > 1/j\}$ is a set of cardinality $\mathfrak{n}$. Using the remark following Lemma 4.2, we complete the proof exactly as in the last paragraph of the proof of Theorem 4.5 (a). Q.E.D.

Theorem 4.5 (a) has as one of its consequences, that if μ is a finite measure, then every weakly compact subset of $L^\infty(\mu)$ is separable. This is because $L^\infty(\mu)$ is isometric to $C(S)$ where S is the Stone space of the measure algebra of μ, and the finiteness of μ then implies that S satisfies the C.C.C. In view of our interest here in the spaces $L^\infty(\mu)$, we prefer to give the following simpler and more intrinsic proof of this fact:

PROPOSITION 4.7. *Let the Banach space B be* WCG *and satisfy* DP. *Then every weakly compact subset of B^* is (norm) separable. In particular, if μ is a finite measure, every weakly compact subset of $L^\infty(\mu)$ (and hence every* WCG *subspace of $L^\infty(\mu)$) is separable.*

Proof. We first observe that if K is a weakly-compact subset of the Banach space X, then the map T: $X^* \to C(K)$ defined by $Tx^*(k) = x^*(k)$ for all $x^* \in X^*$ and $k \in K$, is weakly compact. (This is an immediate consequence of Theorem 1 page 490 of [7] and the definitions involved.)

Now let K be a weakly compact subset of B^*. Then setting $X = B^*$ and letting T: $B^{**} \to C(K)$ as above, the map $T \circ \chi$: $B \to C(K)$ is also weakly compact (where χ: $B \to B^{**}$ is the canonical isometric imbedding). Now let G be a weakly compact subset of B, generating B. Since B satisfies DP, $T \circ \chi(G)$ is a compact subset of $C(K)$, hence a separable subset. Since G generates B, it follows that $T \circ \chi(B)$ is a separable subspace of $C(K)$; hence letting A be the smallest closed subalgebra of $C(K)$ containing $T \circ \chi(B)$ and the constants, A is also separable. But $T \circ \chi(B)$ separates the points of K; hence so does A, and so by the Stone–Weierstrass theorem, $A = C(K)$; hence K is metrizable in its weak topology. Thus K is separable. Q.E.D.

Remarks. 1. We say that a subset G of a Banach space B is pre-weakly compact if given any sequence (g_n) in G, there exists a weak-Cauchy subsequence (g_{n_i}) of G. Using the equivalent definitions of the property DP, the same proof as above shows that if B is generated by a pre-weakly compact set G and satisfies DP, then every weakly compact subset of B^* is separable.

2. Letting X, K, and T be as in the first sentence of the proof of Proposition 4.7 and letting S^* be the unit ball of X^* in the weak* topology, then it follows that T is continuous from S^* into $T(S^*)$ in the weak topology of $C(K)$. If moreover K generates X, then T is one-to-one, and hence one obtains the result of Amir and Lindenstrauss [1] that if X is WCG, S^* in its weak* topology is homeomorphic to a weakly compact subset of a Banach space, namely $T(S^*)$.

The final result of this section gives several necessary and sufficient conditions for an injective conjugate Banach space to be imbeddable in $L^\infty(\mu)$ for some finite measure μ. The proof is nothing but a summary of our preceding results.

THEOREM 4.8. *Let B be an injective Banach space that is isomorphic to a conjugate Banach space. Then the following conditions are equivalent:*

1. *B is isomorphic to a subspace of $L^\infty(\mu)$ for some finite measure μ.*
2. *If Γ is an uncountable set, then $l^\infty(\Gamma)$ is not isomorphic to a subspace of B.*

3. *Every weakly compact subset of B is separable.*

4. *B^* contains a weakly compact total subset.*

5. *There exists a finite measure μ and a closed subspace A of $L^1(\mu)$ such that B is isomorphic to A^*.*

Moreover, suppose one and hence any of the above conditions occur, and suppose that A_0 is a Banach space with B isomorphic to A_0^* and $\dim A_0 = \mathfrak{m}$. *Then A_0 is isomorphic to a subspace of $L^1[0, 1]^{\mathfrak{m}}$, B is isomorphic to a subspace of $L^\infty[0, 1]^{\mathfrak{m}}$, and if $\mathfrak{n}$ is a cardinal number with $\mathfrak{n} < \mathfrak{m}$, then no bounded linear operator from B into $L^\infty[0, 1]^{\mathfrak{n}}$ can be one-to-one.*

Proof. $5 \Rightarrow 1$ is a special case of 3.6 (b). $1 \Rightarrow 3$ follows from the preceding result, and $3 \Rightarrow 2$ is obvious. To see that $2 \Rightarrow 5$, suppose that 5 does not hold. Now it is assumed that there is a Banach space X with B isomorphic to X^*. The assumption that B is injective implies that X is isomorphic to a subspace of $M(S)$ for some compact Hausdorff space S; hence by Lemma 1.3, there exists an uncountable set Γ with $l^1(\Gamma)$ isomorphic to a complemented subspace of X, and so $l^\infty(\Gamma)$ is isomorphic to a subspace of B. This establishes the equivalence of the conditions 1, 2, 3, 5. Now $1 \Rightarrow 4$. Indeed, condition 1 implies that B^* is weak* isomorphic to a (weak*) quotient space of $(L^\infty(\mu))^*$, and thus B^* contains a weakly compact total subset since $(L^\infty(\mu))^*$ does (namely χU, where $U = \{f \in L^1(\mu) : f \in L^2(\mu) \text{ and } \|f\|_2 \leq 1\}$). To complete the proof of the equivalences of the five conditions, we show that $4 \Rightarrow 2$. Suppose that 2 doesn't hold. Letting Γ be an uncountable set with $l^\infty(\Gamma)$ isomorphic to a subspace of B, then if 4 holds, $(l^\infty(\Gamma))^*$ would contain a total weakly compact set by the same argument as $1 \Rightarrow 4$. But then letting $\beta\Gamma$ denote the Stone–Cech compactification of the discrete set Γ, $\beta\Gamma$ would contain a strictly positive measure by Theorem 4.5 (b), which is of course absurd, since $\beta\Gamma$ does not satisfy the C.C.C. Hence 4 doesn't hold.

The remaining assertions of 4.8 follow immediately from Theorem 3.6 and Corollary 3.3. Q.E.D.

Remarks. 1. The $\mathcal{P}_1$ space $C(S_G)$ of our Corollary 4.4 fails conditions 1, 4, and 5 of Theorem 4.8 but satisfies conditions 2 and 3. Thus the assumption that B is isomorphic to a conjugate Banach space is essential in the statement of 4.8. (This was used critically in the proof that $2 \Rightarrow 5$).

2. It follows from 4.8 and 3.1 that if B satisfies the hypotheses of 4.8 and B^* is weak* separable, then B is isomorphic to l^∞. For if B^* is weak* separable, then condition 4 of 4.8 is satisfied. Hence by 4.8 there exists a finite measure μ and a subspace A of $L^1(\mu)$ such that A^* is isomorphic to B. But letting Y be a separable subspace of B^* which is weak*

dense, we have by Theorem 3.1 that dim $Y \geqslant$ dim A, hence A is separable. Thus A^* is isomorphic to a subspace of l^∞, and hence to l^∞ by a result of Pelczynski [21]. (We do not know if the above holds if we omit the hypothesis that B is isomorphic to a conjugate Banach space).

5. Quotient algebras and conjugate spaces of $L^\infty(\mu)$ for a finite measure μ

We shall regard $L^\infty(\mu)$ as a commutative B^* algebra, and use elementary results from the theory of commutative B^* algebras (as exposed, for example in part II of [7]). If S is a compact Hausdorff space, we shall mean by a *subalgebra* of $C(S)$ a *conjugation closed*, uniformly closed subalgebra of $C(S)$ containing the constants. If $A \subset C(S)$ is a subalgebra and K is a compact Hausdorff space, then $\varphi: A \to C(K)$ is called a *homomorphism* if φ is linear and for all f and g in A, $\varphi(f \cdot g) = \varphi(f)\varphi(g)$ and if moreover, in the case of complex scalars, $\varphi(\bar{f}) = \overline{\varphi(f)}$ where $\bar{f}$ denotes the complex conjugate of f.

If X and Y are isomorphic Banach spaces, we define the distance coefficient of X and Y, denoted $d(X, Y)$, to be inf $\{\|T\|\|T^{-1}\|: T$ is an isomorphism from X onto $Y\}$.

We recall from Paragraph 3 that given $\mathfrak{m}$ an infinite cardinal number and $1 \leqslant p \leqslant \infty$, $L^p[0, 1]^{\mathfrak{m}}$ denotes the space $L^p(\mu_\Gamma)$, where Γ is any set with card $\Gamma = \mathfrak{m}$ and μ_Γ is the product Lebesgue measure on $\prod_\Gamma [0, 1] = [0, 1]^\Gamma$. For the sake of convenience, we denote the one-dimensional space of scalars by $L^1[0, 1]^0$. Also, given an indexed family $\{Y_\alpha: \alpha \in I\}$ of Banach spaces, we denote $(\sum_{\alpha \in I} \oplus Y_\alpha)_1$ by $\sum_{\alpha \in I} \oplus Y_\alpha$; if $Y = Y_\alpha$ for all $\alpha \in I$, then $\sum_{\alpha \in I} \oplus Y_\alpha$ is denoted by $\sum_{\mathfrak{m}} \oplus Y$, where $\mathfrak{m} = $ card I.

The following theorem is the main result of this section, and gives complete information concerning the conjugate spaces and $\mathcal{P}_1$ quotients of $L^\infty(\mu)$ for a finite measure μ.

THEOREM 5.1. *Let $\mathfrak{m}$ be an infinite cardinal number. Let B denote one of the Banach algebras $L^\infty(\mu)$ for some finite homogeneous measure μ, $l^\infty(\Delta)$ for some infinite set Δ, or $C(G^{\mathfrak{m}})$ where G denotes the closed unit interval with end-points identified; suppose* dim $B = \mathfrak{m}$. *Then*

(a) *B^* is isomorphic to $\sum_{2^{\mathfrak{m}}} \oplus L^1[0, 1]^{\mathfrak{m}}$ with $d(B^*, \sum_{2^{\mathfrak{m}}} \oplus L^1[0, 1]^{\mathfrak{m}}) \leqslant 9$.*

(b) *B^{**} is isomorphic to $l^\infty_{2^{\mathfrak{m}}}$.*

(c) *Let $\mathcal{C}_{\mathfrak{m}}$ denote the set of infinite cardinal numbers less than or equal to $\mathfrak{m}$, and for each $\mathfrak{n} \in \mathcal{C}_{\mathfrak{m}}$, let $\Lambda_{\mathfrak{n}}$ be a set of cardinality $2^{\mathfrak{m}}$, with $\Lambda_{\mathfrak{n}}$ disjoint from $\Lambda_{\mathfrak{n}'}$ for $\mathfrak{n} \neq \mathfrak{n}'$. Then B^* is isometric to $l^1_{2^{\mathfrak{m}}} \oplus \sum_{\mathfrak{n} \in \mathcal{C}_{\mathfrak{m}}} \sum_{\alpha \in \Lambda_{\mathfrak{n}}} \oplus (L^1[0, 1]^{\mathfrak{n}})_\alpha$.*

(d) *$C(G^{\mathfrak{m}})$ is algebraically isometric to a subalgebra of B.*

(e) *If Ω is a Stonian space with* dim $C(\Omega) \leqslant \mathfrak{m}$, *then $C(\Omega)$ is algebraically isometric to a quotient algebra of B.*

(f) *If Y is an injective Banach space with* $\dim Y \leqslant \mathfrak{m}$, *then Y is isomorphic to a quotient space of B.*

The assertions of this theorem that are somewhat difficult are (d) (for the case $B = L^\infty(\mu)$ for a finite homogeneous measure μ) and (c). To prove 5.1 we first show that (a) ⇒ (b), (d) ⇒ (f), and (c) ⇒ (a). We then prove (d); then the case of (c) for $B = C(G^\mathfrak{m})$ is proved in the slightly more general Proposition 5.2. Next is Lemma 5.3, which is used in proving Theorem 5.4, a rather general result. Theorem 5.4 together with (d) implies (immediately) both (c) and (e), thus completing the proof of 5.1. (Our 5.3 and 5.4 yield a slightly stronger result than (d) ⇒ (e); (d) ⇒ (e) is actually an immediate consequence of known results in the theory of Boolean algebras (cf. the second remark following 5.4).) We note also that 5.1 holds for finite non-homogeneous measures as well (cf. the first remark following 5.4).

Proof of Theorem 5.1. (a) ⇒ (b). It suffices to prove, setting $Y = (\sum_{2^\mathfrak{m}} \oplus L^1[0, 1]^\mathfrak{m})_1$, that Y^* is isomorphic to $l^\infty_{2^\mathfrak{m}}$. Now letting S denote the unit cell of Y, we have that card $S = 2^\mathfrak{m}$, and thus $l^\infty(S) = l^\infty_{2^\mathfrak{m}}$.(1) Thus Y^* is isometric to a subspace of $l^\infty_{2^\mathfrak{m}}$; since Y^* is isometric to $(\sum_{2^\mathfrak{m}} \oplus L^\infty[0, 1]^\mathfrak{m})_\infty$, $l^\infty_{2^\mathfrak{m}}$ is isometric to a subspace of Y^*. Since $l^\infty_{2^\mathfrak{m}}$ and Y^* are injective and $X = l^\infty_{2^\mathfrak{m}}$ satisfies the hypotheses of Proposition 1.4, $l^\infty_{2^\mathfrak{m}}$ and Y^* are isomorphic (cf. also Proposition (∗) of [20]).

(d) ⇒ (*f*). We first observe that $C(G^\mathfrak{m})$ contains a subspace isometric to $l^1_\mathfrak{m}$. Indeed, let Λ be a set of cardinality $\mathfrak{m}$, and for each $\lambda \in \Lambda$, let e_λ be the continuous function on G^Λ defined by

$$e_\lambda(x) = \exp(i2\pi x_\lambda) \quad \text{for all } x = \{x_\lambda\}_{\lambda \in \Lambda} \text{ in } G^\Lambda.$$

Then the uniform closure of the linear span of $\{e_\lambda : \lambda \in \Lambda\}$ (or of the real parts of the functions in this set in the case of the real scalar field) is isometric to $l^1(\Lambda) = l^1_\mathfrak{m}$.

But now we note that if Z is any Banach space containing a subspace Z_1 isomorphic to $l^1_\mathfrak{m}$, then if Y is an injective Banach space with $\dim Y \leqslant \mathfrak{m}$, Y is isomorphic to a quotient space of Z. For Y is isometric to a quotient space of $l^1_\mathfrak{m}$. Thus there exists $T: Z_1 \to Y$ a bounded linear operator mapping Z_1 onto Y. Hence by the injectivity of Y, there exists a bounded linear operator $\tilde{T}: Z \to Y$ with $\tilde{T} | Z_1 = T$. Thus $Y = \tilde{T}(Z)$, so Y is isomorphic to Z/B where B equals the kernel of $\tilde{T}$. (If Z_1 is isometric to $l^1_\mathfrak{m}$ and Y is a $\mathcal{P}_1$ space with $\dim Y \leqslant \mathfrak{m}$, we also obtain that Y is isometric to a quotient space of Z.)

(c) ⇒ (*a*). We first observe that if $\mathfrak{n}$ is a cardinal number with $\mathfrak{n} < \mathfrak{m}$, then $d(L^1[0, 1]^\mathfrak{n} \oplus L^1[0, 1]^\mathfrak{m}, L^1[0, 1]^\mathfrak{m}) \leqslant 9$ and also $d(l^1(F) \oplus L^1[0, 1]^\mathfrak{m}, L^1[0, 1]^\mathfrak{m}) \leqslant 9$, where F is any finite or

(1) To see that card $S = 2^\mathfrak{m}$, observe that card $L^1[0, 1]^\mathfrak{m} \leqslant 2^\mathfrak{m}$ and thus there are at most $2^\mathfrak{m}$ functions f from a set of cardinality $2^\mathfrak{m}$ into $L^1[0, 1]^\mathfrak{m}$ such that $f(x) \neq 0$ for at most countably many x.

countably infinite set. Indeed if X is a subspace of $L^1[0, 1]^{\mathfrak{m}}$ such that X is isometric to $X \oplus X$ and such that there is a projection of $L^1[0, 1]^{\mathfrak{m}}$ onto X of norm 1, then there exists a subspace A of $L^1[0,1]^{\mathfrak{m}}$ such that $d(X \oplus A, L^1[0,1]^{\mathfrak{m}}) \leqslant 3$, thus $d(X \oplus A \oplus X, L^1[0,1]^{\mathfrak{m}} \oplus X) \leqslant 3$, and hence $d(L^1[0, 1]^{\mathfrak{m}}, L^1[0, 1]^{\mathfrak{m}} \oplus X) \leqslant 9$. Thus the result follows if X is isometric to $L^1[0, 1]^{\mathfrak{n}}$ or l^1.(1) But if X is isometric to l^1, then $X \oplus l^1(F)$ is isometric to X, and consequently since $d(X \oplus A \oplus l^1(F),\ L^1[0, 1]^{\mathfrak{m}} \oplus l^1(F)) \leqslant 3$, again $d(L^1[0, 1]^{\mathfrak{m}} \oplus l^1(F), L^1[0, 1]^{\mathfrak{m}}) \leqslant 9$.

Now let $\Gamma = \mathcal{C}_{\mathfrak{m}} \cup \{0\}$ and let $X_{\mathfrak{n}} = \sum_{2^{\mathfrak{m}}} \oplus L^1[0, 1]^{\mathfrak{n}}$ for each $\mathfrak{n} \in \Gamma$. Then since $l^1_{2^{\mathfrak{m}}}$ is isometric to X_0, $X = l^1_{2^{\mathfrak{m}}} \oplus \sum_{\mathfrak{n} \in \mathcal{C}_{\mathfrak{m}}} \sum_{\alpha \in \Lambda_{\mathfrak{n}}} \oplus (L^1[0, 1]^{\mathfrak{n}})_\alpha$ is isometric to $\sum_{\mathfrak{n} \in \Gamma} \oplus X_{\mathfrak{n}}$. Of course, to show that (c) $\Rightarrow$ (a), it suffices to show that X is isomorphic to $X_{\mathfrak{m}}$, with $d(X, X_{\mathfrak{m}}) \leqslant 9$. Now card $\Gamma \leqslant \mathfrak{m}$, hence $2(\text{card}\,\Gamma)\, 2^{\mathfrak{m}} = 2^{\mathfrak{m}}$, hence $\sum_{\text{card}\,\Gamma} \oplus X_{\mathfrak{m}}$ is isometric to $X_{\mathfrak{m}}$. Thus $\sum_{\mathfrak{n} \in \Gamma} \oplus X_{\mathfrak{n}}$ is isometric to $\sum_{\mathfrak{n} \in \Gamma} \oplus (X_{\mathfrak{n}} \oplus X_{\mathfrak{m}})$. But for each $\mathfrak{n} \in \Gamma$, $d(X_{\mathfrak{n}} \oplus X_{\mathfrak{m}}, X_{\mathfrak{m}}) \leqslant 9$ since $d(L^1[0, 1]^{\mathfrak{n}} \oplus L^1[0, 1]^{\mathfrak{m}},\ L^1[0, 1]^{\mathfrak{m}}) \leqslant 9$. Thus $d(\sum_{\mathfrak{n} \in \Gamma} \oplus (X_{\mathfrak{n}} \oplus X_{\mathfrak{m}}),\ \sum_{\text{card}\,\Gamma} \oplus X_{\mathfrak{m}}) \leqslant 9$. Thus $d(X, X_{\mathfrak{m}}) \leqslant 9$, proving that (c) $\Rightarrow$ (a).

Proof of (d). To prove (d), it suffices to prove (by Maharam's theorem) that if $\mathfrak{n}$ is an infinite cardinal number, then

I. $C(G^{2^{\mathfrak{n}}})$ is algebraically isometric to a subalgebra of $l^\infty_{\mathfrak{n}}$.

II. $C(G^{\mathfrak{n}^{\aleph_0}})$ is algebraically isometric to a conjugation-closed subalgebra of $L^\infty[0, 1]^{\mathfrak{n}}$.

For dim $l^\infty_{\mathfrak{n}} = 2^{\mathfrak{n}}$, and II together with the fact that card $L^\infty[0, 1]^{\mathfrak{n}} \leqslant \mathfrak{n}^{\aleph_0}$ (cf. the remark following the proof of Theorem 3.5) shows that dim $L^\infty[0, 1]^{\mathfrak{n}} = \mathfrak{n}^{\aleph_0}$.

We shall consider in both cases the space D rather than G, where D denotes the two-point set in the discrete topology. This is legitimate, for given an infinite cardinal $\mathfrak{a}$, there exists a continuous map φ from $D^{\mathfrak{a}}$ onto $G^{\mathfrak{a}}$, since the Cantor function may be used to map $D^{\aleph_0}$ onto G, and $\aleph_0 \cdot \mathfrak{a} = \mathfrak{a}$; thus $C(G^{\mathfrak{a}})$ is algebraically isometric to a subalgebra of $C(D^{\mathfrak{a}})$.

To see I, by a result of Hewitt ([13], cf. also page 40 of [29]), there exists a dense subset Δ of $D^{2^{\mathfrak{n}}}$ of cardinality $\mathfrak{n}$; the map $f \to f|\Delta$ is then an algebraic isometry of $C(D^{2^{\mathfrak{n}}})$ into $l^\infty(\Delta) = l^\infty_{\mathfrak{n}}$. (Thus I is a known result.)

Proof of II. Fix Γ a set with card $\Gamma = \mathfrak{n}$ and let $\lambda = \mu_\Gamma$, the product measure on $[0, 1]^\Gamma$. We shall prove that there exists a family $\mathcal{G}$ of measureable subsets of $[0, 1]^\Gamma$, such that card $\mathcal{G} = \mathfrak{n}^{\aleph_0}$, and such that if k and l are any positive integers and $F_1, \ldots, F_k$ and $G_1, \ldots, G_l$ are any $k + l$ distinct members of $\mathcal{G}$, then $\lambda(\bigcap_{i=1}^{l} G_i \cap \bigcap_{i=1}^{k} \sim F_i) > 0$. It then follows that in the Boolean algebra of measureable subsets of $[0, 1]^\Gamma$ modulo the sets of measure zero, the

(1) To see that $L^1[0, 1]^{\mathfrak{n}}$ and l^1 satisfy the conditions stated on X, use Maharam's theorem and a suitable projection p_λ as defined before Lemma 3.4 for $L^1[0, 1]^{\mathfrak{n}}$; l^1 is obviously isometric to $l^1 \oplus l^1$ and the closed linear span in $L^1[0, 1]^{\mathfrak{m}}$ of the characteristic functions of a sequence of disjoint sets of positive measure is isometric to l^1 and the range of a norm-one "averaging" projection.

subsets of $\mathcal{G}$ generate a free Boolean algebra with $\mathfrak{n}^{\aleph_0}$ free generators, which is consequently isomorphic to the algebra of all clopen subsets of $D^{\mathfrak{n}^{\aleph_0}}$ (page 39 of [29]). Thus the closed linear span of the characteristic funtions of the elements of $\mathcal{G}$ (in $L^\infty[0, 1]^\Gamma$), is algebraically isometric to $C(D^{\mathfrak{n}^{\aleph_0}})$.

We first choose $\mathfrak{F}$ a family of infinite countable subsets of Γ, with card $\mathfrak{F}=\mathfrak{n}^{\aleph_0}$, such that if F_1 and F_2 are any two distinct members of $\mathfrak{F}$, then $F_1 \cap F_2$ is finite or empty. (The existence of such a family follows from the following known argument: let Γ' denote the set all finite sequences of Γ; for each infinite sequence $\gamma=(\gamma_n)$ of Γ, let F_γ denote the subset of Γ' consisting of all finite sequences which start γ, i.e., $F_\gamma=\{(\beta_1, ..., \beta_k): k$ is a positive integer and $\beta_i=\gamma_i$ for all $1 \leqslant i \leqslant k\}$. Then $\mathfrak{F}'=\{F_\gamma: \gamma$ is an infinite sequence of $\Gamma\}$ has the desired properties, and since card $\Gamma'=$ card Γ, the existence of $\mathfrak{F}$ follows.)

Next, let $(r_1, r_2, ...)$ be a fixed sequence of distinct real numbers in the open unit interval, with $\prod_{i=1}^\infty r_i>0$, and for each $F \in \mathfrak{F}$, let φ_F be a function mapping F one-to-one onto $\{r_n: n=1, 2, ...\}$. Finally, for each $F \in \mathfrak{F}$, let M_F be the subset of $[0, 1]^\Gamma$ equal to $\prod_{\alpha\in\Gamma} Y_\alpha$, where for all α, $Y_\alpha=[0, 1]$ if $\alpha \notin F$, and $Y_\alpha=[0, \varphi_F(\alpha)]$ if $\alpha \in F$. We claim that $\mathcal{G}=\{M_F: F\in\mathfrak{F}\}$ has the desired properties. Since card $\mathfrak{F}=\mathfrak{n}^{\aleph_0}$, it suffices to show that given k and l, and $F_1, ..., F_k; G_1, ..., G_l$ any $k+l$ distinct members of $\mathfrak{F}$, then $P=\bigcap_{i=1}^l M_{G_i} \cap \bigcap_{i=1}^k \sim M_{F_i}$ has positive measure.

For each $\alpha \in \bigcup_{i=1}^l G_i$, set $x_\alpha=\min\{\varphi_{G_i}(\alpha): 1 \leqslant i \leqslant l$ and $\alpha \in G_i\}$. Note that there are at most finitely many α's, say m of them, belonging to more than one set G_i, and for each such α, $x_\alpha \geqslant \inf\{r_n: n=1, 2, ...\} \geqslant \prod_{n=1}^\infty r_n$. Hence $\prod\{x_\alpha: \alpha \in \bigcup_{i=1}^l G_i\} \geqslant (\prod_{n=1}^\infty r_n)^{m+1}$. Now for each j with $1 \leqslant j \leqslant k$, choose α_j belonging to the infinite set $F_j \sim (\bigcup_{\substack{1\leqslant i \leqslant k\\ i\neq j}} F_i \cup \bigcup_{i=1}^l G_i)$, and set $a_j=\varphi_{F_j}(\alpha_j)$. Note that if $y \in [0, 1]^\Gamma$ is such that $y_{\alpha_j} \in (a_j, 1]$, then $y \notin M_{F_j}$. Finally, define Z_α for all $\alpha \in \Gamma$ as follows: If $\alpha \in \bigcup_{i=1}^l G_i$, set $Z_\alpha=[0, x_\alpha]$; if $\alpha=\alpha_j$ for some j, set $Z_\alpha=(a_j, 1]$; for all other α, set $Z_\alpha=[0, 1]$.

Then $\prod_{\alpha\in\Gamma} Z_\alpha$ is a subset of P of measure at least $(\prod_{n=1}^\infty r_n)^{m+1}\prod_{j=1}^k(1-a_j)$, a positive number. Thus (d) is proved.

In the next result, the terms $C_\mathfrak{m}$ and $\Lambda_\mathfrak{n}$ have the same meaning as in 5.1 (c).

Proposition 5.2. *Let S be a compact Hausdorff space with* dim $C(S)=\mathfrak{m}$, *and suppose for each infinite cardinal $\mathfrak{n} \leqslant \mathfrak{m}$ there exists a family $\mathfrak{F}_\mathfrak{n}$ of closed subsets of S, with the following properties:*

(i) card $\mathfrak{F}_\mathfrak{n}=2^\mathfrak{m}$

(ii) *For each $F \in \mathfrak{F}_\mathfrak{n}$, there exists a positive $m_F \in M(S)$ with m_F supported on F (i.e., $m_F(\sim F)=0$) and $L^1(m_F)$ isometric to $L^1[0, 1]^\mathfrak{n}$.*

Suppose further that distinct members of $\bigcup_{\mathfrak{n}\in C_{\mathfrak{m}}}\mathfrak{F}_{\mathfrak{n}}$ *are disjoint. Then* $C(S)^*$ *is isometric to the space*

$$X = l^1_{2^{\mathfrak{m}}} \oplus \sum_{\mathfrak{n}\in C_{\mathfrak{m}}} \sum_{\alpha\in\Lambda_{\mathfrak{n}}} \oplus (L^1[0,1]^{\mathfrak{n}})_\alpha.$$

In particular, $S = G^{\mathfrak{m}}$ *satisfies these properties, so* $C(G^{\mathfrak{m}})^*$ *is isometric to* X.

Proof of 5.2. We first remark that since dim $C(S) = \mathfrak{m}$, card $C(S)^* \leqslant 2^{\mathfrak{m}}$; our hypotheses thus imply that card $S = 2^{\mathfrak{m}}$.

Next, we observe that if $\mathfrak{a}$ is an infinite cardinal number with $L^1[0,1]^{\mathfrak{a}}$ isomorphic to a subspace of $C(S)^*$, then $\mathfrak{a} \leqslant \mathfrak{m}$. Indeed, $L^1[0,1]^{\mathfrak{a}}$ contains a subspace H isomorphic to a Hilbert space of dimension $\mathfrak{a}$ (cf. Proposition 1.5 of [25]); by Proposition 1.2 of [25], a reflexive subspace of $C(S)^*$ is automatically weak* closed, and thus H^* is isomorphic to a quotient space of $C(S)$. Hence $\mathfrak{a} = \dim H = \dim H^* \leqslant \dim C(S) = \mathfrak{m}$.

Now for each $s \in S$, let δ_s be the measure assigning mass one to any set containing s, and let $\mathcal{S} = \{m_F\colon F \in \mathfrak{F}_{\mathfrak{n}},\ \mathfrak{n} \in C_{\mathfrak{m}}\} \cup \{\delta_s\colon s \in S\}$. $\mathcal{S}$ is thus a family of mutually singular measures on S; moreover, we notice that if μ is a regular Borel measure singular with respect to all the measures in $\mathcal{S}$, then $\mu\{s\} = 0$ for all $s \in S$; the regularity of μ then implies that the measure space corresponding to μ is atomless.

Thus by Zorn's Lemma and Maharam's theorem, there exists a maximal family $\mathcal{S}' \supset \mathcal{S}$ of mutually singular positive members of $M(S)$, where each $m \in \mathcal{S}' \sim \mathcal{S}$ is such that $L^1(m)$ is isometric to $L^1[0,1]^{\mathfrak{a}}$ for a unique infinite cardinal $\mathfrak{a}$ with $\mathfrak{a} \leqslant \mathfrak{m}$; i.e., for some $\mathfrak{a} \in C_{\mathfrak{m}}$. Moreover, since $C(S)^*$ has cardinality $2^{\mathfrak{m}}$, card $\mathcal{S}' = 2^{\mathfrak{m}}$. Now $\sum_{m\in\mathcal{S}'} \oplus L^1(m)$ is isometric to $C(S)^*$. But for each $\mathfrak{n} \in C_{\mathfrak{m}}$, if we set $\mathcal{S}_{\mathfrak{n}} = \{m \in \mathcal{S}'\colon L^1(m)$ is isometric to $L^1[0,1]^{\mathfrak{n}}\}$, then card $\mathcal{S}_{\mathfrak{n}} = 2^{\mathfrak{m}}$ since card $\mathcal{S}_{\mathfrak{n}} \cap \mathcal{S} = 2^{\mathfrak{m}}$. Thus

$$\sum_{m\in\mathcal{S}'} \oplus L^1(m) = \Big(\sum_{s\in S} \oplus L^1(\delta_s)\Big) \oplus \Big(\sum_{\mathfrak{n}\in C_{\mathfrak{m}}} \sum_{m\in\mathcal{S}_{\mathfrak{n}}} \oplus L^1(m)\Big),$$

and the right side of this equality is isometric to X.

To see that $G^{\mathfrak{m}}$ possesses such a family of $\mathfrak{F}_{\mathfrak{n}}$'s, let Γ be a set of cardinality $\mathfrak{m}$, and regard G^{Γ} as a compact abelian group, with $(x+y)_\alpha = (x_\alpha + y_\alpha)$ mod 1 for all $x, y \in G^{\Gamma}$ and $\alpha \in \Gamma$. Now since $2\mathfrak{m} = \mathfrak{m}$, we may choose Γ_1 and Γ_2 disjoint subsets of Γ with $\Gamma = \Gamma_1 \cup \Gamma_2$ and card $\Gamma_i = \mathfrak{m}$ for $i = 1, 2$. For each $x \in G^{\Gamma_2}$, let $H_x = \{g \in G^{\Gamma}\colon g_\gamma = x_\gamma$ for all $\gamma \in \Gamma_2\}$. Then $\mathfrak{F} = \{H_x\colon x \in G^{\Gamma_2}\}$ is a pairwise disjoint family of closed subsets of G^{Γ}, each homeomorphic to G^{Γ}, with card $\mathfrak{F} = 2^{\mathfrak{m}}$. Since card $C_{\mathfrak{m}} \leqslant \mathfrak{m} \leqslant 2^{\mathfrak{m}}$ and $2^{\mathfrak{m}} 2^{\mathfrak{m}} = 2^{\mathfrak{m}}$, we may choose a family $\{\mathfrak{F}_{\mathfrak{n}}\colon \mathfrak{n} \in C_{\mathfrak{m}}\}$ such that for all $\mathfrak{n}, \mathfrak{n}' \in C_{\mathfrak{m}}$, $\mathfrak{F}_{\mathfrak{n}} \subset \mathfrak{F}$ and $\mathfrak{F}_{\mathfrak{n}} \cap \mathfrak{F}_{\mathfrak{n}'} = \emptyset$ if $\mathfrak{n} \neq \mathfrak{n}'$. Thus $\bigcup_{\mathfrak{n}\in C_{\mathfrak{m}}}\mathfrak{F}_{\mathfrak{n}}$ is a pairwise disjoint family of closed sets.

Now for each cardinal $\mathfrak{n} \in C_{\mathfrak{m}}$, there exists a positive $\mu \in M(G^{\mathfrak{m}})$ with $L^1(\mu)$ isometric to $L^1[0,1]^{\mathfrak{n}}$. (Choose $\Gamma_{\mathfrak{n}} \subset \Gamma$ with card $\Gamma_{\mathfrak{n}} = \mathfrak{n}$; then let μ be Haar measure on $\{g \in G^{\Gamma}\colon g_\alpha = 0$

for all $\alpha \notin \Gamma_n\}$, a compact subgroup of G^Γ). Thus given $F \in \mathfrak{J}_n$, we may choose a positive $m_F \in M(G^{\mathfrak{m}})$, supported on F, with $L^1(m_F)$ isometric to $L^1[0, 1]^n$. Since $\dim C(G^{\mathfrak{m}}) = \mathfrak{m}$, the proof of Proposition 5.2 is now complete.

Remark. The argument given at the beginning of the above proof shows that if X is a Banach space with $\dim X \leqslant \mathfrak{m}$ such that X^* is isomorphic to $L^1(\lambda)$ for some (not necessarily finite) measure λ, then X^* is isomorphic to a complemented subspace of $(\sum_{2^{\mathfrak{m}}} \oplus L^1[0, 1]^{\mathfrak{m}})_1$. For by Maharam's theorem and the Kakutani representation theorem (cf. [17] and [14] respectively), X^* is isomorphic to $(\sum_{m \in S} \oplus L^1(m))_1$, where S is a family of measure spaces such that for each $m \in S$, $L^1(m)$ is either one-dimensional or isomorphic to $L^1[0, 1]^{\mathfrak{a}}$ for some infinite cardinal $\mathfrak{a}$. Fixing $m \in S$, we have by the argument of 5.2 that $\dim L^1(m) \leqslant \mathfrak{m}$. Thus $L^1(m)$ is isomorphic to a complemented subspace of $L^1[0, 1]^{\mathfrak{m}}$. Hence X^* is isomorphic to a complemented subspace of $(\sum_{\text{card } S} L^1[0, 1]^{\mathfrak{m}})_1$. Since $\dim X \leqslant \mathfrak{m}$, card $X^* \leqslant 2^{\mathfrak{m}}$, whence card $S \leqslant 2^{\mathfrak{m}}$, and thus the result follows.

LEMMA 5.3. *Let Ω be an extremely disconnected compact Hausdorff space, S a compact Hausdorff space, A a uniformly-closed conjugation-closed subalgebra of $C(S)$, and φ: $A \to C(\Omega)$ an algebraic homomorphism. Then there exists $\tilde{\varphi}$: $C(S) \to C(\Omega)$ an algebraic homomorphism, with $\tilde{\varphi} | A = \varphi$.*

Proof. We first observe, using the theory of Boolean algebras, that there exists an algebraic homomorphism τ mapping $l^\infty(\Omega)$ onto $C(\Omega)$ (we regard $C(\Omega) \subset l^\infty(\Omega)$). Indeed, the Boolean algebra $\mathcal{A}_0$ of clopen subsets of Ω may be regarded as a subalgebra of $\mathcal{P}(\Omega)$, the Boolean algebra of all subsets of Ω. Let i: $\mathcal{A}_0 \to \mathcal{A}_0$ denote the identity homomorphism. Since $\mathcal{A}_0$ is complete, by a theorem of Sikorski (page 112 of [29]), there exists a homomorphism $\tilde{i}$: $\mathcal{P}(\Omega) \to \mathcal{A}_0$ such that $\tilde{i} | \mathcal{A}_0 = i$. Thus $\mathcal{A}_0$ is isomorphic to a quotient algebra of $\mathcal{P}(\Omega)$. Again from the theory of Boolean algebras, this means that the Stone space of $\mathcal{A}_0$ (which is homeomorphic to Ω) is homeomorphic to a subset of the Stone space of $\mathcal{P}(\Omega)$ (which is homeomorphic to $\beta(\Omega_d)$, Ω_d denoting the set Ω in the discrete topology), from which the existence of τ follows by the Tietze extension theorem.

Now for each $\gamma \in \Omega$, $a \to (\varphi(a))(\gamma)$ defines a multiplicative linear functional on A. Thus by the general theory of Banach algebras (cf. part II of [7]) there exists a multiplicative linear functional M_γ on $C(S)$ such that $M_\gamma(a) = (\varphi(a))(\gamma)$ for all $a \in A$.

Now define T: $C(S) \to l^\infty(\Omega)$ by $(Tf)(\gamma) = M_\gamma(f)$ for all $f \in C(S)$ and $\gamma \in \Omega$; finally set $\tilde{\varphi} = \tau \circ T$. Since T and τ are each algebraic homomorphisms, so is $\tilde{\varphi}$, and of course $\tilde{\varphi} | A = \varphi$. Q.E.D.

Our next result completes the proof of Theorem 5.1.

THEOREM 5.4. *Suppose that* $\mathfrak{m} \geqslant \mathfrak{c}$, *let* S *be a compact Hausdorff space such that* $C(G^{\mathfrak{m}})$ *is algebraically isometric to a subalgebra of* $C(S)$, *and let* Ω *be a Stonian compact Hausdorff space such that* $\dim C(\Omega) \leqslant \mathfrak{m}$. *Then* Ω *is homeomorphic to a subset of* S. *If moreover* $\dim C(S) = \mathfrak{m}$ and $\mathfrak{m}^{\aleph_0} = \mathfrak{m}$, *then* $C(S)^*$ *is isometric to* $C(G^{\mathfrak{m}})^*$.

Before beginning the proof, we wish to make two observations. The first is that if we replace the condition "$\mathfrak{m}^{\aleph_0} = \mathfrak{m}$" by the condition "$\mathfrak{n}^{\aleph_0} \leqslant \mathfrak{m}$ for all $\mathfrak{n} < \mathfrak{m}$", the conclusion that $C(S)^*$ is isometric to $C(G^{\mathfrak{m}})^*$ still holds. (See Remark 6 following the proof of 5.4; note also that by Proposition 5.2, $C(G^{\mathfrak{m}})^*$ is isometric to the space given in (c) of 5.1.) If one assumes the generalized continuum hypothesis, then all cardinals $\mathfrak{m}$ satisfy the latter condition.

The second observation is that 5.4 together with (d) implies (c) and (e) of 5.1, thus completing its proof. For letting B and Ω be as in 5.1, and identifying the Banach algebra B with a $C(S)$, we have by (d) and 5.4 that Ω is homeomorphic to a subset of S, whence $C(\Omega)$ is algebraically isometric to a quotient algebra of $C(S)$, i.e., of B. Moreover, if $B = L^\infty(\mu)$ or $l^\infty(\Lambda)$ where μ, Λ, and $\mathfrak{m}$ are as in 5.1, then by our proof of (d), $\mathfrak{m}^{\aleph_0} = \mathfrak{m}$, thus by the final statement of 5.4, (c) holds.

Proof of 5.4. Let us first show that Ω is homeomorphic to a subset of S (assuming the hypotheses of the first sentence of 5.4). Since $\dim C(\Omega) \leqslant \mathfrak{m}$, Ω is homeomorphic to a subset of $G^{\mathfrak{m}}$. (Choose Γ a dense subset of the set of members of $C(\Omega)$ with values in $[0, \frac{1}{2}]$, with card $\Gamma = \dim C(\Omega)$. Then $\tau: \Omega \to G^\Gamma$ given by $(\tau(x))\gamma = \gamma(x)$ for all $x \in \Omega$ and $\gamma \in \Gamma$, is a homeomorphism of Ω with $\tau(\Omega)$). Thus there exists an algebraic homomorphism mapping $C(G^{\mathfrak{m}})$ onto $C(\Omega)$; whence by our assumptions and Lemma 5.3, there exists an algebraic homomorphism mapping $C(S)$ onto $C(\Omega)$. Thus Ω is homeomorphic to a subset of S, by the theory of commutative B^* algebras.

Now assume that $\mathfrak{m}^{\aleph_0} = \mathfrak{m}$. By the proof of 5.2, we may choose a family $\{\mathcal{F}'_{\mathfrak{n}}: \mathfrak{n} \in \mathfrak{C}_{\mathfrak{m}}\}$ such that $\bigcup_{\mathfrak{n} \in \mathfrak{C}_{\mathfrak{m}}} \mathcal{F}'_{\mathfrak{n}}$ is a pairwise disjoint family of subsets of $G^{\mathfrak{m}}$, with card $\mathcal{F}'_{\mathfrak{n}} = 2^{\mathfrak{m}}$ and each member of $\mathcal{F}'_{\mathfrak{n}}$ homeomorphic to $G^{\mathfrak{m}}$, for all $\mathfrak{n} \in \mathfrak{C}_{\mathfrak{m}}$. Now our assumptions imply that there exists a continuous function ψ mapping S onto $G^{\mathfrak{m}}$. For each $\mathfrak{n} \in \mathfrak{C}_{\mathfrak{m}}$, let $\mathcal{F}_{\mathfrak{n}} = \{\psi^{-1}(F): F \in \mathcal{F}'_{\mathfrak{n}}\}$. We shall show that $\{\mathcal{F}_{\mathfrak{n}}: \mathfrak{n} \in \mathfrak{C}_{\mathfrak{m}}\}$ satisfies the hypotheses of 5.2 for the space S, thus completing the proof by Proposition 5.2. In turn, it is immediate that the $\mathcal{F}_{\mathfrak{n}}$'s satisfy all the desired properties except possibly (ii) of 5.2.

Fix $\mathfrak{n} \in \mathfrak{C}_{\mathfrak{m}}$ and $F \in \mathcal{F}_{\mathfrak{n}}$; choose $F' \in \mathcal{F}'_{\mathfrak{n}}$ with $F = \psi^{-1}(F')$. Let $\Omega_{\mathfrak{n}}$ be the Stonian space such that $C(\Omega_{\mathfrak{n}})$ may be identified with $L^\infty[0, 1]^{\mathfrak{n}}$. Since F' is homeomorphic to $G^{\mathfrak{m}}$, $C(G^{\mathfrak{m}})$ is isomorphic to a subalgebra of $C(F)$, and $\dim C(\Omega_{\mathfrak{n}}) = \mathfrak{n}^{\aleph_0} \leqslant \mathfrak{m}$. Thus by the first part of our proof, $\Omega_{\mathfrak{n}}$ is homeomorphic to a subset of F. Now simply choose $m_F \in M(S)$ with m_F

supported on a homeomorphic image of $\Omega_\mathfrak{n}$ in F, with $L^1(m_F)$ isometric to $L^1[0, 1]^\mathfrak{n}$. Thus the $\mathcal{F}_\mathfrak{n}$'s satisfy (ii) of 5.2, so the proof is complete. Q.E.D.

Remark 1. Rovert Solovay has recently proved the following profound generalization of our proof of (d) of Theorem 5.1: *Let $\mathcal{B}$ be an infinite complete Boolean algebra satisfying the countable chain condition, and let $\mathfrak{m}$ equal the cardinality of the set of elements of $\mathcal{B}$. Then there exists a free Boolean subalgebra of $\mathcal{B}$ with $\mathfrak{m}$ free generators.* He also proved that if $\mathcal{B}$ is any infinite complete Boolean algebra, then $(\text{card } \mathcal{B})^{\aleph_0} = \text{card } \mathcal{B}$. It thus follows that if S is a Stonian space satisfying the C.C.C., then if $\mathfrak{m} = \dim C(S)$, $C(D^\mathfrak{m})$ and hence $C(G^\mathfrak{m})$ are algebraically isometric to a subalgebra of $C(S)$, and hence setting $B = C(S)$, all of the properties (a) through (f) of Theorem 5.1 hold for B, by our proof of 5.1.

In particular, we have that 5.1 holds with the word homogeneous omitted from its statement, since the measure algebra of a finite measure is a complete Boolean algebra satisfying the C.C.C. We shall indicate the argument for this special consequence of Solovay's result. Let (Y, Σ, μ) be a finite measure space with $\dim L^1(\mu) = \mathfrak{n}$. We wish to show that $C(G^{\mathfrak{n}^{\aleph_0}})$ is algebraically isometric to a subalgebra of $L^\infty(\mu)$. If $\mathfrak{n} = \aleph_0$, then l^∞ is algebraically isometric to a subalgebra of $L^\infty(\mu)$, and so this case follows from (d). Moreover, if there exists a measurable set E with $L^1(\mu \mid E)$ isometric to $L^1[0, 1]^\mathfrak{n}$, we are again done by (d). If none of these possibilities occur, we may choose by Maharam's theorem a sequence of pairwise disjoint measureable sets $E_1, E_2, \ldots$ and infinite cardinals $\mathfrak{n}_1 < \mathfrak{n}_2 < \ldots$ with $L^1(\mu \mid E_i)$ isometric to $L^1[0, 1]^{\mathfrak{n}_i}$ for all i and $\mathfrak{n} = \lim_{i\to\infty} \mathfrak{n}_i$.

We now produce a family $\mathcal{G}$ of measureable subsets satisfying the same conditions as in our proof of II of (d), to complete the argument. Call such a family Boolean independent; we first observe that there exists a Boolean independent family of cardinality at least $\mathfrak{n}$. Indeed, by our proof of (d), we may choose for each i a Boolean independent family $\mathcal{G}_i$ of measureable subsets of E_i with card $\mathcal{G}_i = \mathfrak{n}_i^{\aleph_0}$. Now choose by Zorn's Lemma a subset $\mathcal{F}$ of $\prod_{i=1}^\infty \mathcal{G}_i$ maximal with respect to the property that if a and b are distinct elements of $\mathcal{F}$, then $a_i = b_i$ for at most finitely many i's. Then card $\mathcal{F} \geq \mathfrak{n}$; for each $a \in \mathcal{F}$, set $M_a = \bigcup_{i=1}^\infty a_i$; then $\{M_a : a \in \mathcal{F}\}$ is a Boolean independent family of cardinality at least $\mathfrak{n}$.

Finally, we may choose pairwise disjoint sets $F_1, F_2, \ldots$ such that for each i, $L^1(\mu \mid F_i)$ is isometric to $L^1(\mu)$. We have just demonstrated that for each i we may choose $\mathcal{G}_i'$ a Boolean independent family of measureable subsets of F_i with card $\mathcal{G}_i' = \mathfrak{n}$. Now if Γ is a set with card $\Gamma = \mathfrak{n}$, we know there exists a family of countable infinite subsets of Γ of cardinality $\mathfrak{n}^{\aleph_0}$ any distinct pair of which intersect in at most a finite set. We may then choose $\mathcal{F}' \subset \prod_{i=1}^\infty \mathcal{G}_i'$ with card $\mathcal{F}' = \mathfrak{n}^{\aleph_0}$ such that if a and b are distinct members of $\mathcal{F}'$, $a_i = b_i$ for at most finitely many i. Then setting $\mathcal{G} = \{\bigcup_{i=1}^\infty a_i : a \in \mathcal{F}'\}$, $\mathcal{G}$ is a Boolean independent family of measureable subsets of Y of cardinality $\mathfrak{n}^{\aleph_0}$.

Remark 2. Letting B denote one of the Banach algebras $L^\infty(\mu)$ or $l^\infty(\Delta)$ where μ is a finite measure and Δ is an infinite set, and letting S denote the maximal ideal space of B, then it follows from (e) and the above remark that if Ω is a Stonian space with $\dim C(\Omega) \leqslant \dim B$, then Ω is homeomorphic to a subset of S. Actually, if $B = l^\infty(\Delta)$, this is a known result. For it is known that if $\mathfrak{n} = \text{card}\, \Delta$, then $\mathcal{P}(\Delta)$, the Boolean algebra of all subsets of Δ, contains $2^{\mathfrak{n}}$ independent elements (cf. page 40 of [29]). Consequently if $\mathcal{B}$ is a complete Boolean algebra with card $\mathcal{B} \leqslant 2^{\mathfrak{n}}$, then $\mathcal{B}$ is isomorphic to a quotient Boolean algebra of $\mathcal{P}(\Delta)$, which means that the Stone space Ω of $\mathcal{B}$ is homeomorphic to a subset of $\beta(\Delta)$, the Stone space of $\mathcal{P}(\Delta)$. (Use § 14 and § 33 of [29]). Thus it is known, but worth stating, that *if* Ω *is a Stonian space with* $\dim C(\Omega) = \mathfrak{c}$, *then* Ω *and* βN *are each homeomorphic to a subset of the other* (cf. [18] for a special case of this.)

Remark 3. It follows from the proof of 5.1 that if S is any compact Hausdorff space with $\dim C(S) = \mathfrak{c}$ and βN homeomorphic to a subset of S, then $C(S)^*$ is isometric to $(l^\infty)^*$.

Remark 4. Given a Banach space X, set $X^{(1^*)} = X^*$ and inductively define the nth conjugate space of X for $n > 1$ by $X^{(n^*)} = (X^{((n-1)^*)})^*$. Now let B be as in the statement of 5.1, with $\mathfrak{m} = \dim B$. Define $\mathfrak{m}_1 = \mathfrak{m}$ and $\mathfrak{m}_{n+1} = 2^{\mathfrak{m}_n}$ for all positive integers n. Then Theorem 5.1 shows that for all positive n,

$$B^{((2n-1)^*)} \text{ is isomorphic to } \sum_{\mathfrak{m}_{2n}} \oplus L^1[0,1]^{\mathfrak{m}_{2n-1}}$$

and

$$B^{(2n^*)} \quad \text{is isomorphic to } l^\infty_{\mathfrak{m}_{2n}}.$$

We remark also that if B_1 and B_2 are as in 5.1, then if $\dim B_1 \neq \dim B_2$, B_1^* is not isomorphic to B_2^*. Indeed one can prove that if $\mathfrak{m}$ and $\mathfrak{n}$ are infinite cardinals with $\mathfrak{m} < \mathfrak{n}$, then $(\sum_{2^{\mathfrak{m}}} \oplus L^1[0,1]^{\mathfrak{m}})_1$ contains no subspace isomorphic to a Hilbert space of dimension $\mathfrak{n}$, while $(\sum_{2^{\mathfrak{n}}} \oplus L^1[0,1]^{\mathfrak{n}})_1$ contains such a subspace.

Remark 5. (a) of 5.1 (without the distance coefficient assertion) may be generalized as follows: *let* X *be a Banach space such that* X^* *is isomorphic to* $L^1(\lambda)$ *for some not necessarily finite measure* λ, *with* $l^1_{\mathfrak{m}}$ *isomorphic to a subspace of* X *and* $\dim X = \mathfrak{m}$. *Then* X^* *is isomorphic to* $(\sum_{2^{\mathfrak{m}}} \oplus L^1[0,1]^{\mathfrak{m}})_1$. The proof is obtained by showing that each of X^* and $(\sum_{2^{\mathfrak{m}}} \oplus L^1[0,1]^{\mathfrak{m}})_1$ is isomorphic to a complemented subspace of the other from which this follows (by the argument of Proposition (*) of [20], since each of these spaces is isomorphic to its own square by Maharam's theorem). We have already remarked following Proposition 5.2 that X^* is isomorphic to a complemented subspace of $(\sum_{2^{\mathfrak{m}}} \oplus L^1[0,1]^{\mathfrak{m}})_1$; in turn by Proposition 3.3 of [19], we have that $C(\{0,1\}^{\mathfrak{m}})^*$ is isomorphic to a subspace of X^*, and we know that

$C(\{0, 1\}^{\mathfrak{m}})^*$ is isomorphic to $(\sum_{2^{\mathfrak{m}}} \oplus L^1[0, 1]^{\mathfrak{m}})_1$ by our Proposition 5.2 above. The remainder of the proof follows from the Kakutani representation theorem and Maharam's theorem.

Remark 6. Suppose S and $\mathfrak{m}$ satisfy the assumptions of the first sentence of 5.4, and suppose that dim $C(S) = \mathfrak{m}$, with $\mathfrak{n}^{\aleph_0} \leqslant \mathfrak{m}$ for all $\mathfrak{n} < \mathfrak{m}$. Then $C(S)^*$ is isometric to $C(G^{\mathfrak{m}})^*$. To see this we construct families $\mathfrak{F}_{\mathfrak{n}}$ and their associated measures as in the proof of 5.4, for all $\mathfrak{n} < \mathfrak{m}$. We also define $\mathfrak{F}_{\mathfrak{m}}$ as in that proof. Then fixing $F \in \mathfrak{F}_{\mathfrak{m}}$ there exists an atom-free $\varrho \in M(F)$ so that $L^1[0, 1]^{\mathfrak{m}}$ is isometric to a subspace of $L^1(\varrho)$. (This follows from Proposition 1.4 of [25] and the fact that by definition, $G^{\mathfrak{m}}$ is a continuous image of F.) Now dim$L^1(\varrho) \leqslant \mathfrak{m}$ by Propositions 1.2 and 1.5 of [25]. Moreover, by Maharam's theorem, there exists a countable decomposition of F into Borel subsets $S_1, S_2, \ldots$ and infinite cardinals $\mathfrak{n}_1, \mathfrak{n}_2, \ldots$ such that $L^1(\varrho | S_i)$ is isometric to $L^1[0, 1]^{\mathfrak{n}_i}$. Thus $\mathfrak{n}_i \leqslant \mathfrak{m}$ for all i; since $L^1[0,1]^{\mathfrak{m}}$ is isometric to a subspace of $L^1(\varrho)$, we may choose an i such that $\mathfrak{n}_i = \mathfrak{m}$, by a recent result of Lindenstrauss. We then set $m_F = \varrho | S_i$. Thus $\{\mathfrak{F}_{\mathfrak{n}}: \mathfrak{n} \in \mathfrak{C}_{\mathfrak{m}}\}$ fulfills the conditions of Proposition 5.2.

Through the end of the next remark, X denotes the space $(\sum_{\mathfrak{c}} \oplus L^1[0, 1])_1 \oplus l^1_{\mathfrak{c}}$. Theorem 5.1 shows that $C(G^{\aleph_0})^*$ is isometric to X which is in turn isomorphic to $\sum_{\mathfrak{c}} \oplus L^1[0, 1]$ by the proof of (c) $\Rightarrow$ (a) of 5.1. However, an argument simpler than that for 5.1 shows that for every perfect compact metric space S, $C(S)^*$ is isometric to X. In fact, we have

PROPOSITION 5.5. *Let K be a weakly compact subset of a Banach space, such that* card $K = \mathfrak{c}$, *and such that K contains an infinite perfect subset. Then $C(K)^*$ is isometric to X.*

Before indicating the proof of this, we note that there are non-metrizable compact sets K satisfying the hypotheses of Proposition 5.5; for example, let K be the unit cell of a Hilbert space H of dimension $\mathfrak{c}$, in its weak topology. We then obtain that there exists a non-separable Banach space, $C(K)$, such that $C(K)^*$ is isometric to $C[0, 1]^*$, the dual of the separable Banach space $C[0, 1]$. Professor Pelczynski has shown us the following simpler example of this phenomenon: let Ω be the one-point compactification of an uncountable set of cardinality the continuum, and let $K = [0, 1] \cup \Omega$. A simple example of a perfect K may be obtained by taking the one-point compactification of the locally compact space $\bigcup\{[0, 1] \times \{\alpha\}: \alpha \in \Omega\}$ (in which $[0,1] \times \{\alpha\}$ is declared open for all α). (These are both special cases of 5.5; however, it is easily seen directly that they have the desired properties.)

Proof of Proposition 5.5. We first observe that if μ is a positive member of $C(K)^*$, then $L^1(\mu)$ is separable (and consequently isometric to a subspace of $L^1[0, 1]$). Indeed, by a result of Grothendieck (cf. Theorem 4.3 of [16]), μ must have metrizable support, call it L. But then $C(L)$ is separable, and injects densely into $L^1(\mu)$, from which this follows. It also shows that μ is a weak* limit of a sequence of finitely supported measures (i.e., of linear combinations of point masses). Thus the cardinality of $C(K)^*$ is less than or equal to the cardinality of all sequences of finitely supported measures, which equals $\mathfrak{c}^{\aleph_0}=\mathfrak{c}$.

Since K contains an infinite perfect subset, there exists a closed subset K_1 of K and a continuous map φ of K_1 onto $D^{\aleph_0}$ (cf. Proposition 1.3 of [25]; $D^{\aleph_0}$ is, of course, homeomorphic to the Cantor set). Regarding $D=\{0, 1\}$ as a group under addition modulo 2, the argument at the end of Proposition 5.2 shows that there exists a family $\mathfrak{F}$ of pairwise disjoint closed subsets of $D^{\aleph_0}$ each homeomorphic to $D^{\aleph_0}$, with card $\mathfrak{F}=2^{\aleph_0}=\mathfrak{c}$. Letting λ be the product measure on $D^{\aleph_0}$, we may choose for each $F\in\mathfrak{F}$, a positive atomless (i.e., continuous) measure $m_F\in M(\varphi^{-1}(F))$ with $L^1(\lambda)$ isometric to a subspace of $L^1(m_F)$ (cf. Proposition 1.4 of [25]). Since $L^1(m_F)$ is separable, we must have that $L^1(m_F)$ is isometric to $L^1[0, 1]$. The remainder of the argument, using Zorn's Lemma, is completed as in the proof of Proposition 5.2. (Indeed, Proposition 5.2 remains valid if one replaces the hypothesis "dim $C(S)=\mathfrak{m}$" in its statement by the hypotheses "dim $C(S)^*=2^{\mathfrak{m}}$ and such that if μ is a positive member of $C(S)^*$, then dim $L^1(\mu)\leqslant\mathfrak{m}$".) Q.E.D.

Remark. It is not difficult to see that if S is a compact Hausdorff space, then $C(S)^*$ is isometric to X if and only if $C(S)^*$ is isomorphic to X if and only if the following three conditions are all satisfied:

1. card $C(S)^*=\mathfrak{c}$.
2. $C(S)$ contains an infinite-dimensional reflexive subspace.
3. Every reflexive subspace of $C(S)^*$ is separable.

An example of a space $C(S)$ satisfying 1–3 and non-isomorphic to any of the spaces $C(K)$ of Proposition 5.5 is obtained by letting $C(S)$ be the Banach algebra, under supremum norm, of all bounded functions on the closed unit interval which are right continuous and whose limits from the left exist at every point. (This space was introduced by Corson in [3].) S denotes the maximal ideal space of this Banach algebra; $C[0, 1]$ may be considered as a subspace of $C(S)$, from which 2 follows. $C(S)/C[0, 1]$ is isometric to $c_0(\Gamma)$, where Γ is a set with card $\Gamma=\mathfrak{c}$, from which 1 follows. Moreover, since $C[0, 1]^\perp$ is thus isometric to $l^1(\Gamma)$, every reflexive subspace of $C(S)^*$ must have a finite-codimensional subspace isomorphic to a subspace of $C[0, 1]^*$, by Corollary 3 of [26]. Thus 3 holds. Finally, S is separable since the rational numbers of the unit interval imbed densely in it. Thus $C(S)$ is isometric to a subspace of l^∞, and $C(S)$ is, of course, non-separable. Hence $C(S)$ is not weakly-compactly-generated, and thus is not isomorphic to $C(K)$ for any K homeomorphic to a weakly compact subset of a Banach space (cf. [16]).

It follows from Theorem 5.1 that if Y is an injective Banach space with dim $Y=\mathfrak{c}$, then Y is isomorphic to a quotient space of l^∞. (dim $Y<\mathfrak{c}$ is impossible, for it is shown in

[23] that any injective Banach space must contain a subspace isomorphic to l^∞.) Our final result shows that Y^* is then isomorphic to $(l^\infty)^*$.

THEOREM 5.6. *Let the Banach space Y be isomorphic to a quotient space of l^∞ and a complemented subspace of $C(S)$ for some compact Hausdorff space S. Then Y^* is isomorphic to $(l^\infty)^*$.*

Proof. Set $E = (\sum_{2^{\mathfrak{c}}} \oplus L^1[0, 1]^{\mathfrak{c}})_1$; by Theorem 5.1, we know that E^* is isomorphic to $(l^\infty)^*$. We shall show that each of the Banach spaces Y^* and E are isomorphic to a complemented subspace of the other. Since E is isomorphic to $(E \oplus E \oplus \dots)_1$, it follows by Proposition 1.4 that E and Y^* are isomorphic.

Now our assumptions imply that there exists a compact Hausdorff space S_1 such that $\dim C(S_1) \leqslant \mathfrak{c}$, and such that Y is isomorphic to a complemented subspace of $C(S_1)$. Indeed, $\dim Y \leqslant \mathfrak{c}$, since $\dim l^\infty = \mathfrak{c}$ and Y is a continuous linear image of l^∞. Now letting Z be a complemented subspace of $C(S)$ isomorphic to Y, let Z_1 be the closed conjugation-closed subalgebra of $C(S)$ generated by Z. Then $\dim Z_1 \leqslant \mathfrak{c}$, Y is isomorphic to a complemented subspace of Z_1, and Z_1 is isometric to $C(S_1)$ where S_1 is the maximal ideal space of Z_1.

Thus Y^* is isomorphic to a complemented subspace of $C(S_1)^*$, which is in turn isomorphic to a complemented subspace of E by the remark following the proof of Proposition 5.2.

Now Y is not reflexive since we assume always that Y is infinite dimensional, and it is a theorem of Grothendieck [10] that no infinite-dimensional complemented subspace of a $C(S)$ is reflexive. But then by Theorem 2 of [23], Y contains a subspace isomorphic to l^∞. Since l^∞ is injective, it follows that $(l^\infty)^*$ and hence E is isomorphic to a complemented subspace of Y^*. Q.E.D.

Remark. It follows easily from the fact that $l^1_{\mathfrak{c}}$ is isomorphic to a subspace of l^∞, that the following two statements are equivalent for any Banach space Y:

1. $\dim Y \leqslant \mathfrak{c}$ and Y is isomorphic to a quotient space of some injective Banach space.
2. Y is isomorphic to a quotient space of l^∞.

6. Open problems

We summarize here the conjectures and problems stated above, and mention some additional questions also. Throughout, "X" denotes a Banach space (of infinite dimension).

1. Conjecture. *Let X satisfy the Dunford–Pettis property, and suppose that X is isomorphic to a subspace of a weakly compactly generated conjugate Banach space. Then X is separable.*

(Theorem 2.1 implies that if X satisfies these hypotheses and is WCG, then X is separable.)

2. Conjecture. *Let X be a complemented subspace of $L^1[0, 1]$ and suppose that X is isomorphic to a conjugate Banach space (or less restrictively, to a subspace of a* WCG *conjugate Banach space). Then X is isomorphic to l^1.* (Cf. Corollary 2.2.)

3. Conjecture. *Let X be a complemented subspace of $L^1(\lambda)$ for some measure λ, and suppose that X is isomorphic to a conjugate Banach space. Then X contains a subspace isomorphic to $l^1(\Gamma)$, where Γ is a set with* card $\Gamma = \dim X$.

As we remarked at the end of § 2, this conjecture would have as a consequence that every injective double conjugate Banach space is isomorphic to $l^\infty(\Gamma)$ for some set Γ.

4. Suppose that X is injective, isomorphic to a conjugate Banach space, and is such that every weakly compact subset of X is separable. Does there exist a finite measure μ such that X is isomorphic to $L^\infty(\mu)$?

Theorem 4.9 states our present knowledge concerning X's satisfying these three properties; we mention also that if μ is a finite measure, then by Proposition 4.7 and known results, $L^\infty(\mu)$ is such an X. Finally, we note that the answer to this question is affirmative if and only if the answer to the following question is affirmative:

4′. Let μ be a finite measure, and let A be a closed subspace of $L^1(\mu)$ with $\dim A = \dim L^1(\mu)$. Suppose further that A^* is injective. Is $L^1(\mu)$ isomorphic to a quotient space of A?

(Theorem 3.7 shows that the answer to 4′ is affirmative if A^* is isomorphic to $L^\infty(\mu)$. Theorem 4.9 shows that if A satisfies these hypotheses, then A^* is isomorphic to a subspace of $L^\infty(\mu)$. Thus if the answer to 4′ were yes, $L^\infty(\mu)$ would be isomorphic to a subspace of A^*, and hence A^* would be isomorphic to $L^\infty(\mu)$ by Proposition 1.4.)

5. Suppose that X is injective and X^* is weak* separable. Is X isomorphic to l^∞?

(If X satisfies the additional hypothesis that X is isomorphic to a conjugate Banach space, then the answer is affirmative; cf. the second remark following Theorem 4.8 above.) Of course, the answer is affirmative if these hypotheses imply that X is isomorphic to a subspace of l^∞. However, we don't even know if these hypotheses imply that X is isomorphic to a subspace of $L^\infty(\mu)$ for some finite measure μ. (The latter implication holds if X is isomorphic to $C(S)$ for some compact Hausdorff space S, by Theorem 4.5(b).)

6. For each infinite cardinal number $\mathfrak{m}$, let $L_\mathfrak{m} = (\sum_{2^\mathfrak{m}} \oplus L^1[0, 1]^\mathfrak{m})$. Let L_0 denote the

one-dimensional space of scalars. We conjecture that l^1-sums of the spaces $L_{\mathfrak{m}}$ (over $\mathfrak{m}=0$ or $m \geq \aleph_0$) exhaust the isomorphism types of L^1-spaces isomorphic to conjugate spaces.

CONJECTURE. *Let μ be a measure. If $L^1(\mu)$ is isomorphic to a conjugate Banach space, then there exists an infinite set Γ and a function φ from Γ to a set of cardinal numbers with $\varphi(\gamma)=0$ or $\varphi(\gamma) \geq \aleph_0$ for all $\gamma \in \Gamma$, such that $L^1(\mu)$ is isomorphic to the space*

$$X = (\sum_{\gamma \in \Gamma} \oplus L_{\varphi(\gamma)})_1.$$

Theorem 5.1 shows that conversely any space X of the above form is isomorphic to $C(S)^*$ for some compact Hausdorff space S. Indeed, by 5.1, X is isomorphic to

$$Y = (\sum_{\gamma \in \Gamma} \oplus (C(G^{\varphi(\gamma)}))^*)_1,$$

where G^0 equals the one-point space. There exists a locally compact Hausdorff space S_1 possessing a family $\{U_\gamma : \gamma \in \Gamma\}$ of pairwise disjoint compact and open subsets with their union dense in S_1, and with U_γ homeomorphic to $G^{\varphi(\gamma)}$ for all $\gamma \in \Gamma$. Then Y is isometric to $C(S)^*$, where S is the one-point compactification of S_1.

7. Let X be an injective Banach space, and let $\mathfrak{m} = \dim X$.

(a) Is $\mathfrak{m}^{\aleph_0} = \mathfrak{m}$?
(b) Let Γ be a set with card $\Gamma = \mathfrak{m}$. Is $l^1(\Gamma)$ isomorphic to a subspace of X?
(c) Is X^* isomorphic to $(\sum_{2^{\mathfrak{m}}} \oplus L^1[0,1]^{\mathfrak{m}})_1$?

It follows from the known characterization of $\mathcal{P}_1$ spaces and an unpublished result of Solovay that the answer to 7(a) is affirmative if X is isomorphic to a $\mathcal{P}_1$ space. If X is a given injective space such that X^* is isomorphic to $L^1(\lambda)$ for some measure λ, then if the answer to 7(b) is affirmative for X, so is the answer to 7(c) (cf. the fifth remark following Theorem 5.4). The results of § 5 and of [23] show that the answers to 7(b) and 7(c) are affirmative if $\mathfrak{m} = \mathfrak{c}$, and, of course, 5.1 and the first remark following 5.4 give special cases of $\mathcal{P}_1$ spaces X for which the answers are affirmative.

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