VALIRON DEFICIENT VALUES FOR MEROMORPHIC FUNCTIONS IN THE PLANE

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1. Introduction

1.1. *The function-theoretic problem.* This paper is concerned with a problem in the Nevanlinna theory of functions meromorphic in $|z| < \infty$ (referred to in the sequel simply as functions). We shall assume acquaintance with the standard notation of the Nevanlinna theory (cf. [3], [5] or [10])

$$T(r, f), \quad N(r, a), \quad m(r, a)$$

and with Nevanlinna's fundamental theorems (see [5] pp. 5, 31 or [10] pp. 168, 243). The *Valiron deficiency* $\Delta(a, f)$ of a value a for the function $f = f(z)$ is, by definition,

$$\Delta(a, f) = \limsup_{r \to \infty} \frac{m(r, a)}{T(r, f)} = \limsup_{r \to \infty} \frac{T(r, f) - N(r, a)}{T(r, f)}.$$

If $\Delta(a, f) > 0$ for a particular value of a, then that value is said to be *Valiron deficient* for the function f.

We here investigate the size of the set of Valiron deficient values, when the function f is given.

To say that a function value a is Valiron deficient means, roughly, that it is assumed significantly less often in $|z| \leq r$ than are other function values, for some sequence $r = r_n \to \infty$. If this is the case for all sufficiently large r-values, then a is Nevanlinna deficient.

1.2. *Some known results.* A meromorphic function can have at most countably many values a which are Nevanlinna deficient ($\liminf m(r, a)/T(r, f) > 0$), but Valiron has constructed an example of an entire function of order one such that the set of (what is now called) Valiron deficient values has "effectivement la puissance du continu" and hence is non-countable (cf. [12] pp. 263–266).

The set of Valiron deficient values, however, is a nullset and is also known to have (logarithmic) capacity zero (cf. [10] p. 280 and [4]). Therefore (via connections between set functions, [2] p. 28 or [10] pp. 281, 151) the set of Valiron deficient values has Hausdorff measure zero with respect to all measure functions $h(t)$ for which

$$\int_0 h(t)\, t^{-1} dt < \infty. \tag{1}$$

1.3. *Some results presented here.* The set of Valiron deficient values (for a meromorphic function of finite order in $|z| < \infty$) is a point set of Hausdorff measure zero for all measure functions $h(t)$ for which

$$\int_0 h(t)\, t^{-1} (-\log t)^{-1}\, dt < \infty. \tag{2}$$

This follows from (b) $\Rightarrow$ (a) of Theorem 1, since (2) means

$$\sum_{n=1}^{\infty} h(\exp(-\exp(nk)) < \infty$$

for given $k > 0$. (For complementary results, see [9].)

Estimates for sets of Valiron deficient values are given by means of a sequence of circles, the upper limit of which contains the point set considered.

We give both necessary and sufficient conditions in order that a given point set be contained in a set of Valiron deficient values.

Dan F. Shea has suggested that our methods from [7] might apply to the present problem of Valiron deficient values (oral communication, La Jolla, 1966).

2. Theorem on Valiron deficient values

2.1. We now state our main result.

THEOREM 1. *Given a point set U in the complex plane, then the following three conditions are equivalent:*

(a) *There exist a positive number k and an infinite sequence $a_1, a_2, \dots$ of complex numbers so that each element a of U satisfies the inequality*

$$|a - a_n| < \exp(-\exp(nk))$$

for an infinity of values of n.

(b) *There exist a real number x, $0 < x < 1$, and a meromorphic function $f(z)$ of finite order in $|z| < \infty$ so that*

$$\Delta(a, f) > x$$

for every a in U.

(c) *There exist a real number t, $0<t<1$, and an entire function $g(z)$ so that the entire function $g(z)\exp(az)$ is of lower order less than t for every a in U.*

2.2. *Comments.* A point set U which satisfies (a) is called a set of finite μ-measure. The μ-measure is a subadditive set function, its value is denoted by $\mu(U)$, and it can be defined as the lower bound of $1/k$, for those k for which the sequence in (a) of Theorem 1 exists (cf. [7], [8], [9]).

The equivalence (a)$\Leftrightarrow$(c) of Theorem 1 has been established already (cf. [7] Satz 1 and Satz 2, or [8] Theorem 7.1 with $p=1$). (Misprint in [8] Th. 7.1, delete $\overset{p}{\underset{f}{}}(f)$ in (17) of that theorem.)

Here we prove that (a) and (b) are equivalent, thereby completing the proof of Theorem 1.

The proof of (b)$\Rightarrow$(a) occupies the main part of this paper. This result is a "covering theorem" for the set of Valiron deficient values. The inverse result, (a)$\Rightarrow$(b), is an explicit construction. We also obtain some quantitative results, i.e. inequalities connecting the bounds for k and x in Theorem 1,

$$9\sqrt[3]{x}\geqslant k\geqslant -\frac{x}{8}\log\left(1-\frac{x}{2}\right).$$

2.3. *Proof of Theorem 1,* (b)$\Rightarrow$(a). The starting point in this proof is the second fundamental theorem in Nevanlinna's theory of meromorphic functions. Roughly, this asserts that most function values are taken reasonably often. It is used here in the following form (cf. [5] pp. 31–32, [10] p. 243).

THE SECOND FUNDAMENTAL THEOREM. *Let $f(z)$ be a given non-constant meromorphic function in $|z|<\infty$. Let $a_1, a_2, \ldots, a_q$, where $q>2$, be distinct finite complex numbers, $0<\delta<1$, and suppose that $|a_\mu-a_\nu|\geqslant\delta$ for $1\leqslant\mu<\nu\leqslant q$. Then*

$$\sum_{\nu=1}^{q} m(r,a_\nu)\leqslant 2\,T(r,f)+m\left(r,\frac{f'}{f}\right)+m\left(r,\sum_{\nu=1}^{q}\frac{f'}{f-a_\nu}\right)+q\log\frac{3q}{\delta}+\log 2-\log|f'(0)|, \qquad (3)$$

if there is no zero or pole of f or f' at the origin.

Otherwise, $-\log|f'(0)|$ is to be replaced by $+O(\log r)$.

In our application we need $2T(r,f)$ to be the dominant term in (3). Accordingly, we require some restriction on δ and a_ν and on the growth of f. If $|a_\nu|<r$, then the first fundamental theorem (cf. [5] p. 5, $r=R$) implies

$$T(r,f)-N(r,a_\nu)\leqslant m(r,a_\nu)+O(\log r).$$

In condition (b), the function $f(z)$ is assumed to be of finite order, i.e.,

$$\log T(r, f) = O(\log r).$$

Without loss of generality, we can assume that $f(z)$ is non-rational, i.e.,

$$\log r = o(T(r, f)).$$

Since f is of finite order, we can apply estimates from the ordinary theory of Nevanlinna deficient values for meromorphic functions (cf. [5] p. 36, Lemma 2.3 with $R=2r$),

$$m\left(r, \frac{f'}{f-a_\nu}\right) < 4 \log^+ T(2r, f) + O(\log r) = o(T(r, f)),$$

and
$$m\left(r, \frac{f'}{f}\right) = o(T(r, f)).$$

The proximity function m satisfies an inequality similar to the one satisfied by the logarithm, i.e.,

$$m\left(r, \sum_{\nu=1}^{q} g_\nu\right) \leqslant \log q + \sum_{\nu=1}^{q} m(r, g_\nu).$$

Thus some of the error terms in (3) can be absorbed in others. This permits writing the second fundamental theorem in the following form:

THE MODIFIED SECOND FUNDAMENTAL THEOREM. *Let $f(z)$ be a given non-rational meromorphic function of finite order in $|z| < \infty$. If $\varepsilon > 0$ is given, then for all sufficiently large r and all δ, $0 < \delta < 1$, the following holds.*

If $a_1, a_2, \ldots, a_q$ $(q > 2)$ are given so that

$$|a_\nu| < r, \quad 1 \leqslant \nu \leqslant q \tag{4}$$

and
$$|a_\mu - a_\nu| \geqslant \delta, \quad 1 \leqslant \mu < \nu \leqslant q, \tag{5}$$

then
$$\sum_{\nu=1}^{q} (T(r, f) - N(r, a_\nu)) < (2 + \varepsilon q)\, T(r, f) + q \log \frac{3q}{\delta} + \log q. \tag{6}$$

We shall use the following result:

LEMMA. *Let $f(z)$ be a given non-rational meromorphic function of finite order in $|z| < \infty$. Let ε and y be given, $0 < 5\varepsilon < y < 1$.*

(I) *Then for each sufficiently large r, the set of those complex numbers a in*

$$|a| < r \tag{7}$$

for which $$T(r, f) - N(r, a) \geqslant yT(r, f) \tag{8}$$

is a point set that can be covered by $\left[\frac{4}{y}\right]$ *circles of radius* $\exp\left(-\frac{y}{5}T(r, f)\right)$.

(II) *Moreover, for each sufficiently large r, the set of those complex numbers a in*

$$|a| < r \tag{7}$$

for which the inequality

$$T(\varrho, f) - N(\varrho, a) \geqslant 2yT(\varrho, f) \tag{9}$$

holds for at least one ϱ *such that*

$$T(r, f) \leqslant T(\varrho, f) \leqslant (1-y)^{-1}T(r, f) \tag{10}$$

can be covered by $\left[\frac{4}{y}\right]$ *circles of radius* $\exp\left(-\frac{y}{5}T(r, f)\right)$.

For the proof of (I), let $\delta = \exp(-\varepsilon T(r, f))$ in the modified second fundamental theorem. We want to find an upper bound for the number q when $a_1, ..., a_q$ is a solution of (4), (5) and (8) with $a = a_\nu$. This would then be an upper bound for the number of solutions a of (7), (8), having mutual distances δ or more. The inequalities (6), (8) give

$$qy \leqslant 2 + \varepsilon q + \varepsilon q + (q \log 3q + \log q)(T(r, f))^{-1}. \tag{11}$$

For a given q, if $qy > 4$, (11) could not hold for all $T(r, f)$. This gives the desired bound, $q \leqslant [4/y]$, for the number of circles needed in (I).

To prove (II), now that (I) is established, let a be one point in the set which is to be covered, i.e. one solution of (7), (9) for some ϱ in (10). The inequalities (9), (10) give (8):

$$N(r, a) \leqslant N(\varrho, a) \leqslant (1-2y)\,T(\varrho, f) \leqslant T(r, f)\,(1-2y)/(1-y) < (1-y)\,T(r, f).$$

Therefore (I) applied at the left end point of the interval (10) gives (II).

To prove that (b)$\Rightarrow$(a), now that (II) has been established, let $T(r, f) = (1-y)^{-m}$ in (II). For each sufficiently large integer m, let us consider the $[4/y]$ covering circles described in (II). This infinite sequence of circles has some useful properties:

Let $a \neq f(0), \infty$ be one Valiron deficient value for the function f, $\Delta(a, f) > x$, and let $x = 2y$. This complex number a is then a solution of (9) for an unbounded set of ϱ-values. It is therefore a solution of (9), (10) with $T(r, f) = (1-y)^{-m}$ for an unbounded set of m-values. As a consequence of (II), this complex number a is then contained in infinitely many of the circles considered.

The radii of the circles in this sequence decrease faster than $\{\exp(-\exp(nk))\}_{n=1}^{\infty}$ in (a) of Theorem 1, provided k is small enough,

$$4k < -y \log(1-y), \quad 2y = x. \tag{12}$$

This proves the desired implication, (b) $\Rightarrow$ (a).

Remarks. The bound for $1/k$ gives (by the definition of the set function μ) a bound for the μ-measure for the set of function values of Valiron deficiency greater than x. By a more complicated method of proof, it is possible to get better quantitative results than (12).

The assumption in (b) that f be of finite order can be replaced by the assumption that $d \log T(r, f)/dr$ be bounded.

The bound (2) for the Hausdorff measure (or rather for the measure function) is a direct consequence of the proof of the above result, (b) $\Rightarrow$ (a).

2.4. *Proof of Theorem 1,* (a) $\Rightarrow$ (b). Given a point set U in the complex plane, such that there exist a number $k>0$ and a sequence $a_1, a_2, \ldots$ of complex numbers so that $|a-a_p| < \exp(-\exp(pk))$ for an infinity of p-values, whenever $a \in U$. Then (b), i.e. the existence of a certain meromorphic function $f(z)$, is to be proved. We now give an explicit construction of $f(z)$.

Given the number $k>0$. The number k in (a) can be replaced by any smaller positive number, and therefore we can assume that $5k<1$. Let $y = \exp(k/2)$. There is also given the sequence $a_1, a_2, \ldots$ and without loss of generality we may assume that $|a_p| < p$ for all p (cf. [7] p. 563). Define

$$c_n = a_p, \qquad \exp((p-1)k) < n \leqslant \exp(pk)$$

and let the function f be defined as

$$f(z) = e^{-z} \sum_{n=0}^{\infty} \frac{c_n z^n}{n!}.$$

There remains to prove that f satisfies (b) for the given point set U and for some $x>0$. Without loss of generality the function f just defined is assumed not to be constant. This corresponds to the non-trivial case that the point set U in (a) is large enough (at least two elements).

Now we must find a positive lower bound for $\Delta(a, f)$, when $a \in U$. Since $a \in U$, we have

$$|a - a_p| < \exp(-\exp(pk))$$

for an infinity of values of p. Let p be one of them. Write $\exp((p-1)k) = N$ and $yN = r$. Then for $z = re^{i\varphi}$,

$$|f(z)-a_p| = \left| e^{-z}\left(\sum_{n=0}^{N} + \sum_{n=Ny^2}^{\infty}\right)(c_n - a_p)\frac{z^n}{n!}\right| = |e^{-z}((ey)^N + (e/y)^{Ny^2})|\,O(1),$$

as $p\to\infty$. The assumptions just made on the constants in this proof imply that

$$(ey)^N < \exp\left(r\cos\frac{k}{3}\right), \quad (e/y)^{Ny^2} < \exp\left(r\cos\frac{k}{3}\right), \quad |a-a_p| < \exp(-r).$$

The resulting inequality is

$$\log|f(z)-a| < r\left(\cos\frac{k}{3} - \cos\varphi\right) + O(1).$$

For the proximity function $m(r, a)$, this gives

$$m(r,a) = \frac{1}{2\pi}\int_0^{2\pi}\log^+\left|\frac{1}{f(z)-a}\right| d\varphi \geqslant \frac{r}{\pi}\left(\sin\frac{k}{3} - \frac{k}{3}\cos\frac{k}{3}\right) - O(1) \geqslant \frac{rk^3}{300} - O(1).$$

For each $a\in U$, this inequality holds for some infinite set of p-values, i.e. for some unbounded set of r-values.

A trivial upper bound for the characteristic function for the entire function f is given by the logarithm of the maximum modulus,

$$T(r,f) < 2r + o(r).$$

The desired estimate for the Valiron deficiency then becomes

$$\Delta(a,f) \geqslant \frac{k^3}{600}.$$

The implication (a) $\Rightarrow$ (b) is therefore established.

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